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CONSTITUTIVE THEORY IN GENERAL RELATIVITY: BASIC FIELDS, STATE SPACES AND THE PRINCIPLE OF MINIMAL COUPLING

Abstract. A scheme is presented how to describe material properties under the influence of gravitation. The relativistic dissipation inequality is exploited by LIU's procedure. As an example an ideal spinning fluid is considered in the given framework.

1. Introduction

We investigate how constitutive properties can be introduced into Einstein's gravitation theory. Starting out with the balances of particle number density, spin and energy - momentum, Einstein's field equations and the relativistic dissipation inequality we consider constitutive equations and state spaces in 3-1- decomposition determining classes of materials. The set of possible constitutive equations compatible with the balances, the state space and the dissipation inequality is found out by LIU's exploitation of the dissipation inequality [1], [2].

2. Balances

We start out with the balances of particle number density, energy - momentum and spin in Einstein's gravitation theory, that means in Riemann geometry of a curved space without torsion:

$$N^{\alpha}_{;\alpha} = 0, \quad \text{1 equation,} \tag{1}$$

$$T^{\alpha\beta}{}_{;\beta} = 0, \quad \text{4 equations,} \tag{2}$$

$$S^{\alpha\beta}_{;\beta} = 0, \quad \text{3 equations.} \tag{3}$$

Here the particle flux is defined by $N^{\alpha} = nu^{\alpha}$ with the particle density n and the 4-velocity u^{α}. First of all the energy - momentum tensor is proposed to be not symmetric $T^{\alpha\beta} \neq T^{\beta\alpha}$. The spin density $S^{\alpha\beta}$ is antisymmetric $S^{\alpha\beta} = -S^{\beta\alpha}$ and satisfies the so-called Frenkel condition $u_{\alpha}S^{\alpha\beta} = 0 = S^{\alpha\beta}u_{\beta}$ which expresses that the spin tensor is purely spatial.

Because we want to describe material under the influence of gravitation in Riemann geometry we need Einstein's field equations

$$\tilde{R}_{\alpha\beta} - (1/2)g_{\alpha\beta}\tilde{R} = \kappa T_{(\alpha\beta)}, \quad \text{10 equations.} \tag{4}$$

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Here are $\tilde{R}_{\alpha\beta}$ the Ricci tensor and $\tilde{R}$ the curvature scalar due to the Riemann geometry. They are marked with a tilde because we although examine material under the framework of other geometries, so we have to distinguish between different geometric quantities. Due to the symmetry of the left hand side only the symmetric part of the energy- momentum tensor appears in the field equations.

Finally we have to take into account the dissipation inequality

$$S^{\alpha}_{;\alpha} \quad = \quad \sigma \geq 0. \tag{5}$$

Here $S^{\alpha} = su^{\alpha} + s^{\alpha}$, introducing the entropy density s and the entropy flux density s^{α}.

The 18 equations (1) to (4) and the dissipation inequality contain more fields than equations are. Therefore the set of equations is underdetermined. This is due to the fact, that (1) to (5) are valid for all materials and up to now no special material was taken into consideration. Hence we have to split the 37 fields appearing in (1) to (5) into the basic fields which we are looking for and into the constitutive equations describing the considered material or the class of materials. In more detail the 37 fields are:

$$\begin{array}{rl} N^{\alpha}, & \text{4 fields,} \\ T^{\alpha\beta}, & \text{16 fields,} \\ S^{\alpha\beta}, & \text{3 fields,} \\ g^{\alpha\beta}, & \text{10 fields,} \\ S^{\alpha}, & \text{4 fields.} \end{array}$$

From the energy - momentum tensor we can see, that parts of it belong to the constitutive equations, namely the 3 - stress tensor, and other parts, namely the energy density, belong to the basic fields. Therefore we perform as usual the following 3-1 decomposition

$$\begin{array}{rcll} \epsilon & := & T_{\alpha\beta}\frac{1}{c^2}u^{\alpha}u^{\beta}, & \text{energy density,} \\ t^{\alpha\beta} & := & h^{\alpha\gamma}T_{\gamma\sigma}h^{\sigma\beta}, & \text{stress tensor,} \\ q^{\alpha} & := & -h^{\alpha\sigma}T_{\gamma\sigma}u^{\gamma}, & \text{heat flux density,} \\ p^{\beta} & := & h^{\alpha\sigma}T_{\sigma\gamma}u^{\gamma}, & \text{momentum density.} \end{array}$$

Here $h^{\alpha\beta}$ is the projection tensor perpendicular to the 4-velocity:

$$h^{\alpha\gamma} := g^{\alpha\gamma} + \frac{1}{c^2}u^{\alpha}u^{\gamma} = h^{\gamma\alpha}.$$

Now we introduce the 18 basic fields:

$$\{\epsilon, n, u_{\alpha}, g_{\alpha\beta}, S_{\alpha\beta}\}(x^{\alpha}),$$

and the remaining 19 constitutive equations:

$$\{t_{\alpha\beta}, q_{\alpha}, p_{\beta}, S_{\alpha}\}(x^{\alpha}).$$

Dealing with Riemann geometry one finally have to satisfy some constraints:

$$\begin{array}{rcll} u^{\alpha}u_{\alpha} & = & -c^2, & \text{normalisation of the the 4-velocity,} \\ g_{\alpha\beta} & \stackrel{!}{=} & g_{\beta\alpha}, & \text{symmetry of the metric,} \\ g_{\alpha\beta;\gamma} & \stackrel{!}{=} & 0, & \text{vanishing of the non-metricity,} \\ \{^{\alpha}_{\beta\gamma}\} & = & F(g_{\alpha\beta}, g_{\alpha\beta,\gamma}), & \text{symmetric connection as a function of the metric} \\ & & & \text{and the first partial derivative of the metric,} \\ A_{\alpha;\beta} & = & A_{\alpha,\beta} - \{^{\sigma}_{\alpha\beta}\}A_{\sigma}, & \text{covariant derivative according to the geometry.} \end{array}$$

3. State Space

Because the system of equations (1) to (4) is underdetermined one has to introduce the constitutive equations which depend on of state space variables, which are characterizing the material.

By introducing the state space one get balances on the state space in the following form:

$$A^{\beta}_{;\beta}(Z) = a, \quad \text{or} \quad A^{\alpha\beta}_{;\beta}(Z) = a^{\alpha}.$$

Here the symbol Z represents all state space variables.

If we want to to describe material under the influence of gravitation we have to introduce a state space inducing variables which describe gravitational effects. Hence in general the state space looks like

$$Z = Z(Z_{therm}, Z_{grav})$$

Here Z_{grav} is the set of variables which describes effects of gravitation, and Z_{therm} are all other variables [3, 4, 5].

3.1. First derivative state space

First of all we have a look on state spaces which containes first derivatives:

$$Z^{I} = Z(Z^{therm}, Z^{therm}_{;\alpha}, Z^{grav}, Z^{grav}_{;\alpha}) = Z(Z^{therm}, Z^{therm}_{;\alpha}, g_{\alpha\beta}, \underbrace{g_{\alpha\beta;\gamma}}_{\equiv 0})$$

This chosen state space consists of covariant quantities. If we decompose the covariant derivative of a tensor of first order

$$A_{\alpha;\beta} = A_{\alpha,\beta} - \{^{\sigma}_{\alpha\beta}\} A_{\sigma},$$

and similary for tensors of higher order the state space writes

$$Z^{I} = Z(Z^{therm}, Z^{therm}_{,\alpha}, g_{\alpha\beta}, \{^{\alpha}_{\beta\gamma}\}).$$

Here the state space is spanned by non-covariant quantities, but the constitutive equations on it depend on covariant combinations of these non-covariant state space variables.

3.2. Second derivative state space and the principle of minimal coupling

Next a second derivative state space is discussed:

$$Z^{II} = Z(Z^{therm}, Z^{therm}_{;\alpha}, Z^{therm}_{;\alpha\beta}, g_{\alpha\beta}, \underbrace{g_{\alpha\beta;\gamma}}_{\equiv 0}, g_{\alpha\beta;\gamma\delta}, \tilde{R}_{\alpha\beta\gamma\delta}).$$

With respect to $A_{\alpha;[\beta\gamma]} = \tilde{R}^{\sigma}_{\alpha\beta\gamma} A_{\sigma}$ one can replace the skew-symmetric part of the second covariant derivatives by the Riemann curvature tensor and the quantity itself. Consequently we obtain

$$Z^{II} = Z(Z^{therm}, Z^{therm}_{;\alpha}, \underbrace{Z^{therm}_{;(\alpha\beta)}}_{\text{symmetric part}}, g_{\alpha\beta}, \tilde{R}_{\alpha\beta\gamma\delta}).$$

As done before one can rewrite this state space containing only partial derivatives

$$Z^{II} = Z(Z^{therm}, Z^{therm}_{,\alpha}, Z^{therm}_{(,\alpha\beta)}, g_{\alpha\beta}, \{^{\alpha}_{\beta\gamma}\}, \{^{\alpha}_{\beta\gamma}\}_{,\delta}).$$

By introducing this second derivative state space we are in trouble with the *principle of minimal coupling*. It states that second derivative state spaces must not include the curvature tensor or the the partial derivative of the connection (or equivalently in Riemann geometry the second derivative of the metric). The *principle of minimal coupling* guarantees that the equivalence principle holds. So we have to remove $\tilde{R}_{\alpha\beta\gamma\delta}$, $Z^{therm}_{;[\beta\gamma]}$ and $\{^{\alpha}_{\beta\gamma}\}_{,\delta}$ from the state space, and we get a second derivative state space obeying the *principle of minimal coupling*

$$Z^{II} = Z(Z^{therm}, Z^{therm}_{,\alpha}, Z^{therm}_{(,\alpha\beta)}, g_{\alpha\beta}, \{^{\alpha}_{\beta\gamma}\}).$$

With respect to the variables which are introduced to describe the effects of gravitation this state space looks like Z^I.

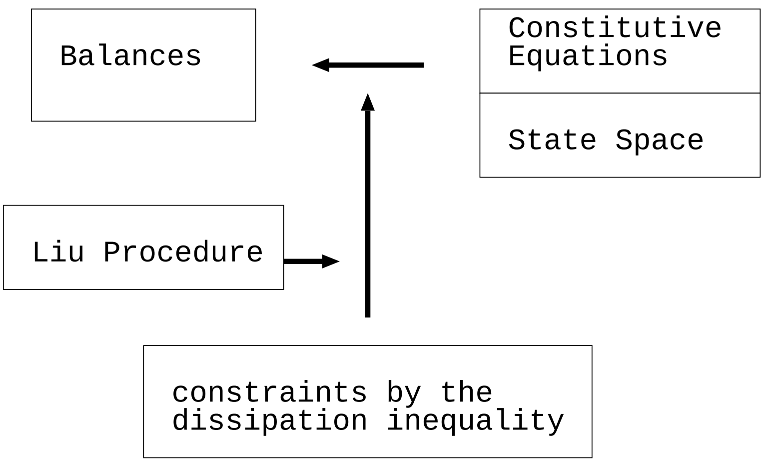
3.3. State space without derivatives

A state space which contains no derivatives is:

$$Z^0 = Z(Z^{therm}, g_{\alpha\beta}).$$

The higher (directional) derivatives belonging to this state space are $Z^{therm}_{,\alpha}$ and $\{^{\alpha}_{\beta\gamma}\}$ or $g_{\alpha\beta,\gamma}$, respectively. But the Ricci tensor and the curvature scalar in Einstein‘s field equations depend on the partial derivatives of the Christoffel symbols or the of the connection, respectively. These quantities are no higher derivatives with respect to Z^0. Consequently the Einstein equations do not determine higer derivatives of the chosen state space in this case. Therefore Z^0 does not fit to Einstein‘s equations and we cannot use it.
Now we have to exploit the dissipation inequality.

4. LIU’s procedure



A general balance looks like

$$A_{\alpha\beta;\beta}(Z) = a_\alpha.$$

Applying the chain rule with respect to the chosen state space variables we obtain

$$\Longleftrightarrow \quad \underbrace{a_\alpha}_{\in \underline{B}} \quad = \quad \frac{\partial}{\partial Z}(A_{\alpha\beta})Z_{,\beta} - \underbrace{\{^{\sigma}_{\alpha\beta}\}A^{\beta}_{\sigma} - \{^{\beta}_{\sigma\beta}\}A^{\sigma}_{\alpha}}_{\in \underline{B}} .$$

Only the first term on the right hand side contains quantities which are higher higher derivatives than those included in the state space. So one can split the terms in such containing higher derivatives and such which do not. Rewriting the balances and the dissipation inequality we obtain then in the form

$$\begin{aligned} \underline{\underline{A}} \cdot \underline{y} - \underline{B} &= 0, \\ \underline{\alpha} \cdot \underline{y} - \beta &\geq 0. \end{aligned}$$

Here are $\underline{\underline{A}}$, $\underline{B}$, $\underline{\alpha}$ and β state functions depending on the constitutive properties and $\underline{y}$ represents the process direction in the chosen state space. The equations above are linear in $\underline{y}$.

DEFINITION 1. *All constitutive equations being compatible with the chosen state space and satisfying the balances and the dissipation inequality determine the* class of materials *([6], [7]).*

There exist two possibilities to find this class of materials:

- For fixed $\underline{\underline{A}}$, $\underline{B}$, $\underline{\alpha}$ and β certain $\underline{y}$ are excluded,
- For all possible $\underline{y}$ the $\underline{\underline{A}}$, $\underline{B}$, $\underline{\alpha}$ and β have to be determined in such a way, that the dissipation inequality is satisfied.

Starting out with the Coleman-Mizel formulation of the second law that all solutions of the balances are satisfying the dissipation inequality

$$\underline{\underline{A}} \cdot \underline{y} - \underline{B} = 0 \quad \Longrightarrow \quad \underline{\alpha} \cdot \underline{y} - \beta \geq 0,$$

Liu‘s proposition is valid:

$$\begin{aligned} \underline{\alpha}(Z) &= \underline{\Lambda}(Z) \cdot \underline{\underline{A}}(Z), \\ \underline{\Lambda}(Z) \cdot \underline{B}(Z) &\geq \beta(Z). \end{aligned}$$

This expresses that the entropy production $\sigma := \underline{\Lambda}(Z) \cdot \underline{B}(Z) - \beta(Z) \geq 0$ is independent of the process direction and so the second possibilty for finding the class of materials holds. These equations are the so-called LIU equations. By eliminating the lagrange parameters $\underline{\Lambda}$ from the LIU equations and inserting them into the dissipation inequality we obtain constraints restricting the possible materials.

5. Weyssenhoff fluid in Riemann geometry

A covariant description of a classical fluid with spin in Riemann spacetime can be obtained by generalising the work of Weyssenhofff and Raabe as it is done by by Obukhoy and Piskareva [8], [9] and [10].

Tensors of spin and energy-momentum for the Weyssenhoff dust are postulated to be:

$$\begin{aligned} S^{\alpha}_{\beta\gamma} &\overset{!}{=} u^{\alpha} S_{\beta\gamma}, \\ T_{\alpha\beta} = \hat{T}_{\alpha\beta} &\overset{!}{=} u_{\alpha} P_{\beta}. \end{aligned}$$

Here is $S_{\beta\gamma}$ the skew symmetric spin density, satisfying the Frenkel condition $u^\beta S_{\beta\gamma} = 0 = u^\gamma S_{\beta\gamma}$, u_α the 4-velocity and P_β the 4-vector density of energy-momentum. The explicit form of P_β can be derived from the spin conservation law $\hat{T}_{[\alpha\beta]} = 2\tilde{\nabla}_\gamma S^\gamma_{\alpha\beta}$

$$\hat{T}_{[\alpha\beta]} = u_\alpha P_\beta - u_\beta P_\alpha = 2\tilde{\nabla}_\gamma S^\gamma_{\alpha\beta}. \tag{6}$$

Taking into account the definition $u^\alpha P_\alpha = \epsilon$ where ϵ is the the energy density we obtain from (6) by contracting with the 4-velocity u^β:

$$P_\alpha \quad = \quad \epsilon u_\alpha + 2u^\beta 2\tilde{\nabla}_\gamma S^\gamma_{\beta\alpha}. \tag{7}$$

Inserting (7) into (6) one obtain the motion of spin, which describes the rotational dynamics of the fluid.

Assuming that the elements of the medium interact in such a way that Pascal law is valid we get the model for an ideal spinning fluid. By doing this we have to modify the stress tensor given above for the description of dust by the contribution of the isotropic pressure

$$\hat{T}^\alpha_\beta \quad \stackrel{!}{=} \quad u^\alpha P_\beta - p(\delta^\alpha_\beta - u^\alpha u_\beta) = -p\delta^\alpha_\beta + u^\alpha[u_\beta(\epsilon + p) + 2u^\sigma \tilde{\nabla}_\gamma u^\gamma S_{\sigma\alpha}]. \tag{8}$$

Taking the divergence of the Einstein field equations (4) one gets after 3-1-decomposition with respect to (8) the equations for the tranlational dynamics in 3-1- decomposition:

$$0 \stackrel{!}{=} \hat{T}^{(\alpha}_{\beta);\alpha}$$

$$\left\{ \begin{array}{lcl} 0 & = & (p+\epsilon)u^\alpha(u_{\beta;\alpha}) + (-\delta^\alpha_\beta + u^\alpha u_\beta)p_{;\alpha} + \\ & & +2[u^\alpha S_{\beta\gamma} u^\sigma u^\gamma_{;\sigma}]_{;\alpha} + R_{\gamma\sigma\alpha\beta} S^{\gamma\sigma} u^\alpha \\ 0 & = & (p+\epsilon)u^\alpha_{;\alpha} + u^\alpha \epsilon_\alpha \end{array} \right.$$

As it is shown above the spin enters in the spatial part of the symmetric energy-momentum tensor. So the spin is over the energy connected to the Einstein gravitation equations and in this sense connected to the geometry. But there exist no direct geometric quantity with which the spin is coupled and further on any skew symmetric parts of the balances stands alone. So one can say that Einstein gravitation field theory can deal with the physical quantity spin but say nothing about the skew symmetric parts which appear in the balances.

6. Conclusions

As usual in constitutive theory the split of the fields into basic fields and constitutive equations is also possible, if gravitation is taken into account. The non-relativistic state space is extended by geometrical variables induced by curvature which describe its influence on constitutive properties. Although the choice of the state spaces is free in principle, some restrictions appear in Riemann geometry: Because Einstein‘s field equations contain the second derivatives of the metric, its first derivatives have to be included among the state variables in form of the Christoffel symbols (connection) or the partial derivatives of the metric itself. This involves that the state space is spanned by non-covariant quantities. But nevertheless constitutive properties are described by covariant combinations of these non-covariant quantities. A second consequence is , that state spaces containing only the metric as a geometrical variable cannot be used.

The second derivative state spaces have a speciality:
They have to obey the *principle of minimal coupling*. This principle runs as follows: second derivative state spaces do not include the curvature tensor or the partial derivatives of the

Christoffel symbols (or those of the connection). From a physical point of view this principle states, that there are no materials by which the curvature of space-time can be measured by observing constitutive properties. This principle of minimal coupling is related to the equivalence principle which states, that for free falling, non-rotating observers locally the curvature of space-time vanishes.

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