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A COSSERAT DETECTOR FOR DYNAMIC GEOMETRY

Abstract. It is proposed to explore the interaction of weak gravitational fields with slender elastic materials in order to assess the viability of achieving enhanced laser interferometric sensitivities for the detection of gravitational waves with frequencies between 10^{-4} and 1 Hz. The aim is the design of novel gravitational antennae in interplanetary orbit. The implementation of these ideas would be complimentary to existing programmes of gravitational wave research but exploiting a current niche in the frequency spectrum.

The dynamics of slender structures, several km in length, are ideally suited to analysis by the simple theory of Cosserat rods. Such a description offers a clean conceptual separation of the vibrations induced by bending, shear, twist and extension and the coupling between eigen-modes due to tidal accelerations can be reliably estimated in terms of the constitutive properties of the structure. The detection of gravitational waves in the 1 Hz region would provide vital information about stochastic backgrounds in the early Universe and the relevance of supermassive black holes to the processes that lead to processes in the centre of galaxies.

1. Introduction

One of the most striking predictions of Einstein's theory of gravitation follows from solutions describing gravitational waves. Such solutions are thought to be produced by astrophysical phenomena ranging from the coalescence of orbiting binaries to violent events in the early Universe. Their detection would herald a new window for the observation of natural phenomena. Great ingenuity is being exercised in attempts to detect such waves in the vicinity of the earth using either laser interferometry or various resonant mass devices following Weber's pioneering efforts with aluminum cylinders. Due to the masking effects of competing influences and the weakness of gravitation compared with the electromagnetic interactions the threshold for the detection of expected gravitationally induced signals remains tantalisingly close to the limits set by currently technology. In order to achieve the signal to noise ratios needed for the unambiguous detection of gravitational waves numerous alternative strategies are also under consideration. These include more sophisticated transducer interfaces, advanced filtering techniques and the use of dedicated arrays of antennae. Earth based gravitational wave detectors require expensive vibration insulation in order to discriminate the required signals from the background. This is one reason why the use of antennae in space offer certain advantages. It is argued here that the gravitationally induced elastodynamic vibrations of slender material structures in space offer other advantages that do not appear to have been considered. Multiple structures of such continua possess attractive properties when used as coincidence detectors of gravitational disturbances with a dominant

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spectral content in the 10^{-4} to 1 Hz region. Furthermore this window can be readily extended to lower frequencies and higher sensitivities by enlarging the size of the structures.

Newtonian elastodynamics [1] is adequate as a first approximation if supplemented by the *tidal stresses* generated by the presence of spacetime curvature that is small in comparison with the detector. The latter are estimated from the accelerations responsible for spacetime geodesic deviations. Since the constituents of material media owe their elasticity to primarily non-gravitational forces their histories are non-geodesic. The geodesic motions of particles offer a reference configuration and the geodesic deviation of neighbours in a geodesic reference frame provide accelerations that are additionally resisted in a material held together by elastic forces. Since in practical situations the re-radiation of gravitational waves is totally negligible the computation of the stresses induced by the tidal tensor of a background incident gravitational wave offers a viable means of exploring the dynamical response of a material domain to a fluctuating gravitational field.

Surprisingly little recent attention seems to have been devoted to problems in gravito-elastodynamics beyond the recognition that the shape, size and density of a material may be tuned in order to expedite the excitation of particular normal modes of vibration by resonance [2, 3, 4, 5]. There appear to be no detailed studies of the dynamic response of interplanetary material structures to gravitational wave phenomena. Current resonant mode detectors are designed to permit reconstruction of the direction and polarisation of gravitational waves that can excite resonances. Clearly such detectors are designed to respond to a narrow spectral window of gravitational radiation and are not particularly good at determining the temporal profile of incident gravitational pulses. A significant advantage of space-based antennae based on slender material structures is that they can be designed to respond to transient gravitational pulses, to polarised uni-directional gravitational waves or omni-directional unpolarised waves.

In 1985 V. Braginsky and K. Thorn proposed [6] an Earth-orbiting gravitational wave detector, called a “skyhook”, which could operate in the 10 to 1000 mHz band. It consisted of two heavy masses, one on each end of a long cable with a spring at its centre. The proposal was refined by R W R Drever who suggested that certain noise pollution could be reduced by increasing the rigidity of the design. These authors explored many of the competing noise perturbations and concluded that such devices offer an attractive, simple instrument with gravity-wave sensitivity in an interesting range where sources might exist. However these conclusions were based on a particular non-resonant radial string configuration in earth orbit and to our knowledge no detailed simulations of the elastic wave excitations in the connecting cable have ever been performed in this or more general scenarios. The proposal here, to use several material structures, differs in a number of important respects not the least of which is the fact that laser technology has advanced enormously since this original proposal. Furthermore the analysis of the original detector ignored the ability of a continuum structure to be tuned to the entire acceleration field of a gravitational wave. Resonant response to such circumferential excitations optimise power absorption from the wave. Such mechanisms deserve a more comprehensive analysis, not only to update the viability of the general skyhook concept but to exploit to advantage the detection of both axial, torsional and flexural elastic wave excitations *of the cable itself* by laser interferometry in much more general dynamical configurations than were originally envisaged.

The general mathematical theory of non-linear Newtonian elasticity is well established. The general theory of one-dimensional Newtonian Cosserat continua derived as limits of three-dimensional continua can be consulted in [1]. The theory is fundamentally formulated in the Lagrangian picture in which material elements are labelled by s. The behaviour of a Cosserat structure at time t may be described in terms of the motion $\mathbf{R}(s, t)$ in space of the line of centroids of its cross-sections and elastic deformations about that line. Such a structure is modelled mathematically by an elastic space-curve with structure. This structure defines the relative orientation of neighbouring cross-sections along the structure. Specifying a unit vector $\mathbf{d}_3$ (which may be identified with the normal to the cross-section) at each point along the structure enables the state of flexure to be related to the angle between this vector and the tangent to the space-curve. Specifying a second vector $\mathbf{d}_1$ orthogonal to the first vector (thereby placing it in the plane of the cross-section) can be used to encode the state of bending and twist along its length. Thus a field of two mutually orthogonal unit vectors along the structure provides three continuous dynamical degrees of freedom that, together with the continuous three degrees of freedom describing a space-curve relative to some arbitrary origin in space, define a simple Cosserat model. It is significant for this proposal that the theory includes thermal variables that can be coupled to the dynamical equations of motion, compatible with the laws of thermodynamics. The theory is completed with equations that relate the deformation strains of the structure to the elastodynamic forces and torques. The simplest constitutive model to consider is based on Kirchoff relations with shear deformation and viscoelasticity. Such a Cosserat model provides a well defined six dimensional quasi-linear hyperbolic system of partial differential equations in two independent variables. It may be applied to the study of gravitational wave interactions by suitably choosing external body forces $\mathbf{f}$ to represent the tidal interaction with each element in the medium. A typical system might consist of at least two material structures orbiting in interplanetary space. Each structure would be composed of several km of hollow segmented pipe. A structure, several km in length , made up of transportable segments, could be conveyed to an orbiting station and the system constructed in space. The lowest quadrupole excitation of a steel circular structure would be about 0.4Hz and vary inversely as the (stress-free) length of the structure. Actuator and feedback instrumentation could be placed inside the pipe to "tune" the system to an optimal reference configuration. A series of laser beams from sources attached to the structure, deflected across its diameters from one side to another and along segments of the circumference of a polygon inscribed within the loop could form the structure of a laser interferometer system. In this manner vibrations induced by a quadrupole deformation of the structure (in which both the variations in length of orthogonal diameters and circumferential elements) contribute to a path difference for laser interference. The precise details of the density and elastic moduli needed to enhance the sensitivity of the receiver would result from an in-depth analytic analysis of the Cosserat equations for free motion. The ability to readily optimise the resonant behaviour for coupled axial, lateral and torsional vibrations by design is a major advantage over other mechanical antennae that have been proposed.

By contrast a broad band detector would consist of an open ended structure coiled into a spiral. For planar spirals with traction free open ends the spectral density of normal transverse and axial linearised modes increases with the density of the spiral winding number. They form an ideal broad band detector with directional characteristics. Such antennae can sustain non-resonant weakly dispersive axial travelling waves excited by incident gravitational pulses. Furthermore by coupling such a spiral at its outer end to a light mass by a short length of high-Q fibre (such as sapphire) one can tune such an extension to internal resonances, thereby amplifying the spiral elastic excitation. Such excitations offer new detection mechanisms based on the enhanced motion of the outer structure of the spiral.

2. Cosserat equations

The dynamical evolution of the structure with mass density, $s \in [0, L_0] \mapsto \rho(s)$, and cross-sectional area, $s \in [0, L_0] \mapsto A(s)$, is governed by Newton's dynamical laws:

$$\rho A \ddot{\mathbf{R}} = \mathbf{n}' + \mathbf{f}$$

$$\partial_t(\rho\,\mathbf{I}(\mathbf{w})) = \mathbf{m}' + \mathbf{R}' \times \mathbf{n} + \mathbf{l}$$

applied to a triad of orthonormal vectors: $s \in [0, L_0] \mapsto \{\mathbf{d_1}(s,t), \mathbf{d_2}(s,t), \mathbf{d_3}(s,t)\}$ over the space-curve: $s \in [0, L_0] \mapsto \mathbf{R}(s,t)$ at time t where $\mathbf{n}' = \partial_s \mathbf{n}$, $\dot{\mathbf{R}} = \partial_t \mathbf{R}$, $\mathbf{f}$ and $\mathbf{l}$ denote external force and torque densities respectively and $s \in [0, L_0] \mapsto \rho\mathbf{I}$ is a structure moment of inertia tensor. In these field equations the *contact forces* $\mathbf{n}$ and *contact torques* $\mathbf{m}$ are related to the *strains* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ by constitutive relations. The strains are themselves defined in terms of the configuration variables $\mathbf{R}$ and $\mathbf{d}_k$ for $k = 1, 2, 3$ by the relations:

$$\mathbf{R}' = \mathbf{v}$$

$$\mathbf{d}_k' = \mathbf{u} \times \mathbf{d}_k$$

$$\dot{\mathbf{d}}_k = \mathbf{w} \times \mathbf{d}_k.$$

The latter ensures that the triad remains orthonormal under evolution. The last equation identifies

$$\mathbf{w} = \frac{1}{2}\sum_{k=1}^{3} \mathbf{d}_k \times \dot{\mathbf{d}}_k$$

with the local angular velocity vector of the director triad. The general model accommodates continua whose characteristics (density, cross-sectional area, rotary inertia) vary with s. For a system of two coupled continua with different elastic characteristics on $0 \le s < s_0$ and $s_0 < s \le L_0$ respectively one matches the degrees of freedom at $s = s_0$ according to a junction condition describing the coupling.

To close the above equations of motion constitutive relations appropriate to the structure must be specified:

$$\begin{aligned} \mathbf{n}(s,t) &= \hat{\mathbf{n}}(\mathbf{u}(s,t), \mathbf{v}(s,t), \mathbf{u}_t(s), \mathbf{v}_t(s), \ldots, s) \\ \mathbf{m}(s,t) &= \hat{\mathbf{m}}(\mathbf{u}(s,t), \mathbf{v}(s,t), \mathbf{u}_t(s), \mathbf{v}_t(s), \ldots, s) \end{aligned}$$

where $\mathbf{u}_t(s)$ etc., denote the history of $\mathbf{u}(s,t)$ up to time t. These relations specify a reference configuration (say at $t = 0$) with strains $\mathbf{U}(s)$, $\mathbf{V}(s)$ in such a way that $\hat{\mathbf{n}}(\mathbf{U}(s), \mathbf{V}(s), \ldots, s)$ and $\hat{\mathbf{m}}(\mathbf{U}(s), \mathbf{V}(s), \ldots, s)$ are specified. A reference configuration free of flexure has $\mathbf{R}_s(s,0) = \mathbf{d}_3(s,0)$, i.e. $\mathbf{V}(\mathbf{s}) = \mathbf{d}_3(s,0)$. If a standard configuration is such that $\mathbf{R}(s,0)$ is a space-curve with Frenet curvature κ_0 and torsion τ_0 and the standard directors are oriented so that $\mathbf{d}_1(s,0)$ is the unit normal to the space-curve and $\mathbf{d}_2(s,0)$ the associated unit binormal then $\mathbf{U}(s) = \kappa_0(s)\,\mathbf{d}_2(s,0) + \tau_0(s)\,\mathbf{d}_3(s,0)$.

For a "rod" of density ρ and cross-sectional area A in a weak plane gravitational wave background the simplest model to consider consists of the Newtonian Cosserat equations with a time dependent body force modelled by the tidal interaction $\mathbf{f} = \rho A\,\mathbf{R}_{\dot{C}\xi}\dot{C}$ where $\xi = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + z\,\frac{\partial}{\partial z}$ is the Newtonian vector locating an element of the structure in a Newtonian frame defined by the gravitational wave and $\mathbf{R}_{X\,Y}$ is the pseudo-Riemannian curvature operator. The plane gravitational wave metric in $\{x, y, z, t\}$ coordinates is expressed as

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3$$

where

$$\begin{aligned} e^0 &= \frac{d\,t + d\,z}{2} + \mathcal{F}(t,x,y,z)\,\frac{d\,(t-z)}{2} + \frac{d\,(t-z)}{2}, \\ e^1 &= d\,x, \\ e^2 &= d\,y, \\ e^3 &= \frac{d(t+z)}{2} + \mathcal{F}(t,x,y,z)\,\frac{d\,(t-z)}{2} - \frac{d\,(t-z)}{2} \end{aligned}$$

and serves to anchor the wave to a Minkowski spacetime when $\mathcal{F} = 0$. An exact plane gravitational wave is given by

$$\mathcal{F} = f_+(t-z)\frac{x^2 - y^2}{2} + f_\times(t-z)\frac{xy}{2}$$

for arbitrary profiles f_+ and $f_\times$. The vector $\dot{C}$ is the normalised 4-velocity associated with a time-like observer curve in a geodesic reference frame in the above metric. In the absence of elastic forces each element in the structure would then be governed by the equation of geodesic deviation. Additional stationary Newtonian gravitational fields add terms of the form $\rho A\mathbf{g}$ to $\mathbf{f}$ where $\mathbf{g}$ is the "effective local acceleration due to gravity". Post Newtonian gravitational fields (such as gravitomagnetic and Lens-Thirring effects) can be accommodated with a more refined metric background.

It should be stressed that unlike many current low frequency detectors (e.g. LISA), based on measuring the relative motion of a small number of discrete masses, the loop antennae under discussion provide a response from a vibrating mass continuum where every element of each structure can be made to respond *in an optimal manner* to the acceleration field of a gravitational perturbation. By choosing material with a critical ratio of the shear to Youngs modulus of elasticity, the quadrupole mode of the structure can be induced to absorb maximum power from a plane gravitational wave propagating orthogonal to a circular loop. In such a mode the deformation of each structure element remains along the acceleration vector in the tidal field. This important observation follows by linearising the Cosserat equations about a circular loop of length L_0 with the deformation fields

$$\begin{aligned}
\mathbf{R}(s,t) &= \left[\left(\frac{L_0}{2\pi} + \xi(s,t)\right)\cos\left(\frac{2\pi}{L_0}(s + \lambda(s,t))\right),\right.\\
&\qquad \left.\left(\frac{L_0}{2\pi} + \xi(s,t)\right)\sin\left(\frac{2\pi}{L_0}(s + \lambda(s,t))\right), 0\right]\\
\mathbf{d}_1 &= \left[\cos\left(\frac{2\pi}{L_0}s + \phi(s,t)\right),\ \sin\left(\frac{2\pi}{L_0}s + \phi(s,t)\right), 0\right]\\
\mathbf{d}_2 &= [0, 0, 1]\\
\mathbf{d}_3 &= \mathbf{d}_1 \times \mathbf{d}_2
\end{aligned}$$

and a tidal perturbation due to a plane gravitational wave in the transverse trace-free gauge. The linearised vibrations correspond to a spectrum of combined flexural and circumferential modes and the quadrupole mode alone can be excited into resonance by the passage of the entire gravitational wave. Such modes have no analogue in detectors composed of discrete masses. A further intriguing property of this system deserves further investigation. Loops can be endowed with a uniform longitudinal speed along their circumference and the damping of the induced resonant modes due to the viscoelasticity of the structure is thereby diminished. More generally by solving for the dynamical evolution of the structure, given $\mathbf{f}$, initial and boundary data and suitable constitutive relations the dynamical behaviour of the structure can be used as a guide to decide how to interface Fabry-Perot devices to the system in order to detect gravitational wave environments by laser interferometry.

3. Anelastic modelling

An important consideration of any modelling of Cosserat continua to low levels of excitation is the estimation of signal to noise ratios induced by anelasticity and temperature. To gain an insight into the former one may attempt to extend the established theory of linear anelasticity to a Cosserat structure. For a slender straight rod with uniform density ρ, static Young's modulus E and area of cross section A, the free damped motion in one dimension is modelled by the equation:

$$\rho A\, \partial_{tt} x(s,t) = n'(s,t)$$

where the axial strain $v(x,t) = \partial_s x(s,t) \equiv x'(s,t)$ and

$$n(s,t) = EA(v(s,t) - 1) - EA \int_{\infty}^{t} \phi(t-t')\, \dot{v}(t') dt'$$

for some viscoelastic model ϕ with $0 \le s \le L$. For free motion in the mode:

$$x(s,t) = s + \xi(t) \cos(\pi s/L)$$

the amplitude $\xi(t)$ satisfies

$$\ddot{\xi}(t) + \omega_0^2\, \xi(t) = \omega_0^2 \int_{\infty}^{t} \phi(t-t') \dot{\xi}(t') dt'$$

with $\omega_0^2 = \frac{\pi^2 E}{L^2 \rho}$ while for a forced harmonic excitation:

$$\ddot{\xi}(t) + \omega_0^2 \xi(t) = \omega_0^2 \int_{\infty}^{t} \phi(t-t') \dot{\xi}(t') dt' + F_0 \exp(-i\Omega t).$$

With $\xi(0) = \dot{\xi}(0) = 0$ the Laplace transformed amplitude of forced axial motion is then given in terms of the Laplace transform *

$$\bar{\phi}(\sigma) = \int_0^{\infty} \phi(t) e^{-\sigma t} d\, t$$

of the anelastic modelling function $\phi(t)$ as:

$$\bar{\xi}(\sigma) = \frac{F_0}{(\sigma - i\Omega)(\sigma^2 - \omega_0^2 \sigma \bar{\phi}(\sigma) + \omega_0^2)}.$$

To extend this approach in a simple manner to a 3-D Cosserat model of a slender rod with uniform static moduli E and G, geometric elements A, $K_{\alpha\alpha} = J_{11} + J_{22}$, $J_{\alpha\beta}$, one adopts the following constitutive relations for the local director components of the contact force $\mathbf{n}$ and torque $\mathbf{m}$ in

*For a "Hudson" type solid :

$$\bar{E}(\sigma) = k\sigma^{\nu} = E(1 - \sigma\bar{\phi}(\sigma))$$

for some constants k and ν.

terms of the local strain vectors **v** and **u** and anelastic response functions ϕ_E and ϕ_G:

$$\begin{aligned}
n_3(s,t) &= EA\,(v_3(s,t)-1) - EA\int_{\infty}^{t}\phi_E(t-t')\,\dot v_3(s,t')\,dt' \\
n_1(s,t) &= GA\,v_1(s,t) - GA\int_{\infty}^{t}\phi_G(t-t')\,\dot v_1(s,t')\,dt' \\
n_2(s,t) &= GA\,v_2(s,t) - GA\int_{\infty}^{t}\phi_G(t-t')\,\dot v_2(s,t')\,dt' \\
m_3(s,t) &= K_{\alpha\alpha}G\,u_3(s,t) - K_{\alpha\alpha}G\int_{\infty}^{t}\phi_G(t-t')\,\dot u_3(s,t')\,dt' \\
m_\alpha(s,t) &= E\sum_{\beta=1}^{2}J_{\alpha\beta}\,u_\beta(s,t) - E\sum_{\beta=1}^{2}J_{\alpha\beta}\int_{\infty}^{t}\phi_E(t-t')\,\dot u_\beta(s,t')\,dt'
\end{aligned}$$

for $\alpha, \beta = 1, 2$.

4. Estimate of equilibrium thermal noise at resonance

The detailed effects of temperature on low levels of elastic excitation are more difficult to estimate. However order of magnitude estimates may be based on elementary considerations. Let x be the mean thermal R.M.S. displacement of the elastic medium in a thermal equilibrium state at temperature T, for an antenna of mass m resonating with angular frequency ω. Then

$$\frac{kT}{2} \simeq \frac{m\omega^2 x^2}{2}$$

where k is the Boltzman constant. For an antenna of effective length L, mass density ρ and Young's modulus E take

$$\omega \simeq \frac{\pi\sqrt{\frac{E}{\rho}}}{L}$$

and

$$m = \rho L A$$

in terms of the cross-section area A of the structure. If Q is the quality factor of the antenna one may estimate the gravitational signal from a harmonic gravitational wave with amplitude h and frequency ω to be given by $dl = hLQ$. If $S = dl/x$ is the signal to noise ratio:

$$h(T, Q, E, A, L, S) \simeq \frac{S}{Q\pi}\sqrt{\frac{kT}{LEA}}.$$

Choosing $Q = 1E6$, $E = 2.02E11$, $A = (0.01)^2$, $S = 5$ in MKS units and $T = 100K$ one sees that with $L = 1$km one could detect $h = 1E-21$ provided the signal $dl = 1E-12$ is detectable. This corresponds to a resonance at 0.2Hz. Using the above parameters one may estimate the mass and corresponding lowest resonant frequency for such a detector. These are illustrated in Figures 1 and 2.

5. Estimate of thermal fluctuation noise induced by laser measurements

The act of measuring ”small” elastic strains can induce thermal fluctuations that contribute to elastodynamic noise. According to Braginsky et al. [7], temperature fluctuations can be induced by laser measurement of strain. Such fluctuations act as a stochastic source for elastic deformations in the structure that compete with the strains induced by gravitational excitations. If one neglects the geometry of the structure and applies the estimates in [7] one finds a temperature dependence for the spectral density of the average induced noise $\chi(T,\omega)^2$ given by:

$$\chi(T,\omega)^2 = 4\frac{\sqrt{2}\alpha(T)^2(1+\sigma)^2 kT^2\tau}{\sqrt{\pi}\rho(T)C(T)r_o(1+\tau^2\omega^2)}$$

where the relaxation time $\tau = r_0^2\rho(T)C(T)/K(T)$, $K(T)$ is the thermal conductivity, σ the Poisson ratio, $\alpha(T)$ the coefficient of linear expansion and $C(T)$ the specific heat of the elastic medium. The radius r_0 is a measure of the spatial dimension of the laser spot used to excite the temperature fluctuations of the structure, taken here to be 0.03m.

The square root $\chi(T,\omega)$ of the spectral density for a structure composed of a Fe-Ni alloy is plotted as a function of temperature for the series of frequencies, 0Hz, 0.001Hz, 0.01Hz, 1Hz in Fig. 3. The noise is limited by the top 0Hz curve. To make a comparison with the thermal noise $x(T)$ above at a resonance frequency Ω, one may integrate the spectral fluctuation noise induced by laser measurement over a band $[\omega_1, \omega_2]$ centered on the resonant frequency:

$$X(T)^2_{[\omega_1,\omega_2]} = \int_{\omega_1}^{\omega_2} \chi(T,\omega)^2\, d\omega.$$

If one takes the window $\omega_1 = 0$ and $\omega_2 = 2\Omega$ for a choice of Ω corresponding to the lowest resonance in structures with lengths of 0.01m, 1, 100m,1km and 10km, then the full curve in Fig. 4 is obtained for $X(T)_{[0,2\Omega]}$. On this scale $X(T)_{[0,2\Omega]}$ is insensitive to the choice of frequency window. By contrast the dotted curves indicate the resonant R.M.S. thermal noise $x(T)$ with the lowest curve corresponding to the shortest length and the others showing noise increasing with length. Clearly the thermal fluctuation noise induced by measurement is several orders of magnitude smaller compared with the thermal R.M.S. noise for a structure of several km in length resonating in the mHz region. The strong temperature dependence around $T = 60$ and $T = 250$ is due to the behaviour of the thermal expansion coefficient in these regions. It offers another means of tuning the antenna by careful selection of material properties. The thermal fluctuation noise may have more significance for the broad band detector configurations mentioned above.

6. Conclusions

Cosserat modelling offers a natural approach to analyse the interaction between gravitational fluctuations and slender elastic structures. Unlike current proposals for low frequency orbiting detectors based on measuring the relative motion of a small number of discrete masses, the antennae discussed above provide a response from an extended vibrating mass continuum where every element can be made to respond *in an optimal manner* to the acceleration field of a gravitational perturbation. Cosserat methods may be generalised to include a more detailed study of the thermo-mechanics of antennae constrained by the Clasius-Duhem inequality and promise a clearer picture of the competing effects of thermal noise on the detection of gravitational waves. If thermal noise is such that cryogenic cooling is not mandatory with current technology they may also provide a cheaper alternative.

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References

[1] ANTMAN S., *Non-linear problems in elasticity*, Applied Mathematical Sciences **107**, Springer-Verlag 1991.

[2] RAYNER C.B., *Elasticity in general relativity*, Proc. R. Soc. London **272** A (1963), 44–53.

[3] DYSON F.J., *Seismic response of the earth to a gravitational wave in the 1Hz band*, Astrophys. J. **156** (1969), 529–540.

[4] CARTER B. AND QUINTANA H., *Gravitational and acoustic waves in an elastic medium*, Phys. Rev **16** D (1977), 2928–2938.

[5] EHLERS J., *General Relativity*, in: "Relativity, astrophysics, and cosmology" (Ed. W. Israel), Reidel 1973, 1–20.

[6] BRAGINSKY V.B. AND THORN K.S., *Skyhook gravitational-wave detector*, Nature **316** (1985), 610–612.

[7] BRAGINSKY V.B. AND THORN K.S., *Thermodynamical fluctuations and photo-thermal shot noise in gravitational wave antennae*, Phys. Letts **264** A (1985), 1–10.

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