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ABOUT THE INFLUENCE OF OSCILLATIONS ON STRICHARTZ-TYPE DECAY ESTIMATES

Abstract. Starting with the well-known Strichartz decay estimate for the wave equation we are interested in a similar estimate for wave equations with a time dependent coefficient. The model under consideration is the strictly hyperbolic equation $u_{tt} - a(t)\triangle u = 0$. By the aid of an example we illustrate the deteriorating influence of oscillations in $a = a(t)$ on decay estimates. Moreover we prove, that in the case of slow oscillations one gets Strichartz-type decay estimates with a decay rate similar to the classical one.

1. Introduction

To prove global existence results (in time) of small data solutions for the Cauchy problem for nonlinear wave equations Strichartz decay estimate [9] plays an important role. That is the following estimate: there exist constants C and L depending on p and n such that

$$\|u_t(t,\cdot)\|_{L_q} + \|\nabla_x u(t,\cdot)\|_{L_q} \leq C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\psi\|_{W_p^L}\,, \tag{1}$$

where $1 \leq p \leq 2$, $1/p + 1/q = 1$ and $u = u(t,x)$ is the solution to

$$u_{tt} - \triangle u = 0\,, \quad u(0,x) = 0\,, \quad u_t(0,x) = \psi(x) \in C_0^\infty\left(\mathbb{R}^n\right).$$

If one is interested in such estimates for $u_{tt} - a(t) \triangle u = 0$, then one has to explain properties of $a = a(t)$ which take influence on Strichartz-type decay estimates.

In [5] the weakly hyperbolic Cauchy problem

$$u_{tt} - t^{2l} \triangle u = 0\,, \quad u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x)\,,$$

was studied and the decay estimates are derived there imply the hypothesis that *increasing parts in $a = a(t)$ have an improving influence on decay estimates.*

To feel the influence of oscillating parts in $a = a(t)$ let us consider

$$u_{tt} - (1+\varepsilon\sin t)^2 \triangle u = 0\,, \quad u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x)\,, \tag{2}$$

with a sufficiently small ε. Although the coefficient is near to 1 in usual topologies, decay estimates are in general not valid. Is this a surprise or not? Independent of the point of view we feel, that *oscillating parts in $a = a(t)$ have a deteriorating influence on decay estimates.*

For (2) this follows from Theorem 1 will be proved in Section 2. In this regard, we have found it necessary from the results of [10] to resort to some specialized representation of the coefficient $a = a(t)$ relating to increasing or oscillating part:

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u = 0\,, \tag{3}$$

where

- $\lambda = \lambda(t)$ describes the increasing part,
- $b = b(t)$ describes the oscillating part.

In Section 3 we give the definition of slow oscillations which describes a special interplay between both parts. For the case of slow oscillations Strichartz-type decay estimates will be proved. The main result (Theorem 2) can be applied to (3), where possible λ and b are given in the next examples.

EXAMPLE 1 (LOGARITHMIC GROWTH). $\lambda(t) = \ln(e+t)$, $b(t) = 2+\sin\left((\ln(e+t))^{\beta+1}\right)$, $\beta \in [0, 1)$.

EXAMPLE 2 (POTENTIAL GROWTH). $\lambda(t) = (1+t)^{\alpha}$, $\alpha > 0$, $b(t)$ as in Example 1.

EXAMPLE 3 (EXPONENTIAL GROWTH). $\lambda(t) = \exp(t^{\alpha})$, $\alpha > 1/2$, $b = b(t)$ is periodic, positive, non-constant and smooth.

EXAMPLE 4 (SUPEREXPONENTIAL GROWTH). $\lambda(t) = \exp(\exp(t^{\alpha}))$, $\alpha > 0$, $b = b(t)$ as in Example 3.

2. Wave equations with a periodic coefficient

The bad influence of oscillations in the coefficient $a = a(t)$ on decay estimates will be clear by employing Theorem 1. We claim:

THEOREM 1. *Consider the Cauchy problem*

$$u_{tt} - b(t)^2 \triangle u = 0\,, \quad u(0,x) = \varphi(x)\,, \quad u_t(0,x) = \psi(x)\,, \tag{4}$$

where $b = b(t)$ defined on $\mathbb{R}$, is a 1-periodic, non-constant, smooth, and positive function. Then there are no constants q, p, C, L and a nonnegative function f defined on $\mathbb{N}$ such that for every initial data $\varphi, \psi \in C_0^{\infty}(\mathbb{R})$ the estimate

$$\|u_t(m,\cdot)\|_{L_q} + \|\nabla_x u(m,\cdot)\|_{L_q} \le C\, f(m) \left(\|\varphi\|_{W_p^L} + \|\psi\|_{W_p^L}\right) \tag{5}$$

is fulfilled for all $m \in \mathbb{N}$, while $f(m) \to \infty$, $\ln f(m) = o(m)$ as $m \to \infty$.

REMARK 1. The conditions for $f = f(m)$ are very near to optimal ones. Indeed it is well-known, that due to Gronwall's inequality one can prove the energy estimate

$$\|u_t(t,\cdot)\|_{L_2} + \|\nabla_x u(t,\cdot)\|_{L_2} \le C\, \exp(C_0 t) \left(\|\varphi\|_{W_2^1} + \|\psi\|_{L_2}\right), \quad t \in [0,\infty)\,,$$

for the solution of (4). Choosing $t = m$, $m \in \mathbb{N}$, $p = q = 2$, $L = 1$, we obtain an inequality like (5) with $\ln f(m) = O(m)$ as $m \to \infty$.

REMARK 2. We will construct a special sequence $\{(\varphi_M, \psi_M)\}_{M \in N}$ of data for which (5) is violated at least for large M. An interesting question is that for a classification of data: data which allow, don't allow respectively, a decay estimate for the solution of (4).

Proof. The proof is divided into several steps.

1. One lemma for ordinary differential equations with a periodic coefficient
Consider the ordinary differential equation with a periodic coefficient

$$w_{tt} + \lambda\, b(t)^2 w = 0\,. \tag{6}$$

Let the matrix-valued function $X = X(t, t_0)$ depending on λ be the solution of the Cauchy problem

$$d_t X = \begin{pmatrix} 0 & -\lambda\, b(t)^2 \\ 1 & 0 \end{pmatrix} X\,, \quad X(t_0, t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Thus, $X = X(t, t_0)$ gives a fundamental solution to (6). The matrix $X(t+1, t)$ is independent of $t \in \mathbb{N}$. Set

$$X(1, 0) =: \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

The following lemma follows from the considerations in [1] and provides an important tool for our proof.

LEMMA 1 ([10]). *If $b = b(t)$ is a non-constant, positive, smooth function on $\mathbb{R}$ which is 1-periodic, then there exists a positive λ_0 such that the corresponding matrix $X(t_0+1, t_0)$ of (7) has eigenvalues μ_0 and μ_0^{-1} with $|\mu_0| > 1$.*

2. Lower bound for the energy
We use the periodicity of b and the eigenvalue μ_0 of $X(1, 0)$ to construct solutions of (6) with prescribed values on a discrete set. Thus we get lower bounds for the energy of the solution on this set.

LEMMA 2. *Let $w = w(t)$ be the solution of (6) with $\lambda = \lambda_0$ and initial data $w(0) = 1$, $w_t(0) = b_{12}/(\mu_0 - b_{11})$. Then for every positive integer number $M \in \mathbb{N}$ the solution satisfies $w(M) = \mu_0^M$.*

Proof. For the solution $w = w(t)$ we have

$$\begin{pmatrix} d_t w(M) \\ w(M) \end{pmatrix} = (X(1, 0))^M \begin{pmatrix} d_t w(0) \\ w(0) \end{pmatrix}.$$

The matrix

$$B := \begin{pmatrix} \frac{b_{12}}{\mu_0 - b_{11}} & 1 \\ 1 & \frac{b_{21}}{\mu_0^{-1} - b_{22}} \end{pmatrix}$$

is a diagonalizer for $X(1, 0)$, that is,

$$X(1, 0)B = B \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_0^{-1} \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} d_t w(M) \\ w(M) \end{pmatrix} = \mu_0^M \begin{pmatrix} \frac{b_{12}}{\mu_0 - b_{11}} \\ 1 \end{pmatrix}, \qquad w(M) = \mu_0^M\,.$$

□

3. Construction of unstable solutions

Let us construct a family of solutions $\{u_M\}$ of (4) with data $\{(\varphi_M, \psi_M)\}$. These functions will violate (5) for sufficiently large M. They will be called *unstable solutions*. With a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) = 1$ when $|x| \le 1$, $\chi(x) = 0$ when $|x| \ge 2$, let us choose the initial data

$$\varphi_M(x) = e^{ix\cdot y}\chi\left(\frac{x}{M^2}\right), \qquad \psi_M(x) = e^{ix\cdot y}\chi\left(\frac{x}{M^2}\right)\frac{b_{12}}{\mu_0 - b_{11}}. \tag{8}$$

Here y is chosen such that $|y|^2 = \lambda_0$. Using the theory of the Cauchy problem for strictly hyperbolic equations there exists a unique solution $u_M = u_M(t,x)$ to (4), (8), where $u_M(t,\cdot)$ has compact support for every given $t \in [0,\infty)$. Let $B_1(0) \subset \mathbb{R}^n$ be the ball of radius 1 centered at the origin. Assuming (5) we have

$$\begin{aligned} \|\partial_t u_M(M,\cdot)\|_{L_q(B_1(0))} + \|\nabla_x u_M(M,\cdot)\|_{L_q(B_1(0))} &\le \|\partial_t u(M,\cdot)\|_{L_q} + \|\nabla_x u(M,\cdot)\|_{L_q} \\ &\le C\ f(M)\left\|e^{ix\cdot y}\chi\left(\frac{x}{M^2}\right)\right\|_{W_p^L} \\ &\le C_L\ f(M)\ M^{2n/p}. \end{aligned} \tag{9}$$

4. The role of the cone of dependence

If we take into account the cone of dependence for the Cauchy problem (4), then the solution u_M is representable in $B_1(0)$ at time $t = M$ as

$$u_M(M,x) = e^{ix\cdot y}w(M), \qquad \partial_t\, u_M(M,x) = e^{ix\cdot y}w_t(M), \tag{10}$$

where w is taken from Lemma 2. Indeed, for the set $\{(x,t);\ |x| \le 1,\ t = M\}$ let us calculate the lower base at $t = 0$ of the truncated cone with the slope $\max_{t\in[0,1]} b(t)$ and height M. For sufficiently large M this lower base is contained in the ball

$$B_{\text{depend},\,M} := \left\{x \in \mathbb{R}^n : |x| \le 2M \max_{t\in[0,1]} b(t)\right\}$$

on $t = 0$. If $x \in B_{\text{depend},\,M}$, then $\left|\frac{x}{M^2}\right| \le 1$ for large M. Consequently, $\chi\left(\frac{x}{M^2}\right) = 1$ on $B_{\text{depend},\,M}$ for large M.

The function $e^{ix\cdot y}w(t)$ solves (4) and takes at $t = 0$ the data (8) if $|x| \le M^2$. The cone of dependence property yields (10).

5. Completion of the proof

We obtained

$$\begin{aligned} &\|\partial_t u_M(M,\cdot)\|_{L_q(B_1(0))} + \|\nabla_x u_M(M,\cdot)\|_{L_q(B_1(0))} \\ &\qquad = \left(|w_t(M)| + \lambda_0^{1/2}|w(M)|\right)(\text{meas } B_1(0))^{1/q}. \end{aligned}$$

In view of (9) we have

$$|w(M)| + |w_t(M)| \le C_L\ f(M)M^{2n/p}$$

for all large M. But this is a contradiction to the statement of Lemma 2. Thus, Theorem 1 is completely proved. □

3. Wave equations with slow oscillations in the time-dependent coefficient

3.1. Classification of oscillations

DEFINITION 1. *Let us suppose that there exists a real* $\beta \in [0,1]$ *such that the following condition is satisfied:*

$$|d_t b(t)| \leq C \frac{\lambda(t)}{\Lambda(t)} (\ln \Lambda(t))^{\beta} \qquad \textit{for large} \quad t\,, \tag{11}$$

where $\Lambda(t) := \int_0^t \lambda(s)\,ds$. *If* $\beta \in [0,1)$, $\beta = 1$, *respectively, we call the oscillations* slow oscillations, fast oscillations, *respectively. If (11) is not satisfied for* $\beta = 1$, *then we call the oscillations* very fast oscillations.

REMARK 3. Very fast oscillations may destroy $L_p - L_q$ decay estimates. These oscillations give us an exact description of a fairly wide class of equations in which the oscillating part dominates the increasing one. In [6] it is shown that one can prove in this case a statement similar to Theorem 1.

REMARK 4. The case of fast oscillations is studied in [7]. We could derive $L_p - L_q$ decay estimates only for large dimension n. Moreover, the behaviour of b and λ and its first two derivatives has an influence on the decay rate.

The goal of the following considerations is to show that for slow oscillations ($\beta \in [0,1)$ in (11)) we have $L_p - L_q$ decay estimates similar to Strichartz decay estimate (1)

- for any dimension $n \geq 2$,
- with a decay rate which coincides with the classical decay rate,
- with the decay function $1 + \Lambda(t)$,
- without an essential influence of the oscillating part.

3.2. Main result and philosophy of our approach

Let us study

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u = 0\,, \quad u(0,x) = \varphi(x)\,, \quad u_t(0,x) = \psi(x)\,, \tag{12}$$

under the following assumptions for the positive coefficient $\lambda(t)^2 b(t)^2$:

- it holds
$$\Lambda(t) \to \infty \quad \text{as} \quad t \to \infty\,; \tag{13}$$
- there exist positive constants c_0, c_1 and c such that
$$c_0 \frac{\lambda(t)}{\Lambda(t)} \leq \frac{\lambda'(t)}{\lambda(t)} \leq c_1 \frac{\lambda(t)}{\Lambda(t)} \leq c(\ln \Lambda(t))^c \quad \text{for large} \quad t\,; \tag{14}$$
- there exist positive constants c_k such that for all $k = 2, 3, \ldots$ it holds
$$\left| d_t^k \lambda(t) \right| \leq c_k \left(\frac{\lambda(t)}{\Lambda(t)} \right)^k \lambda(t) \qquad \text{for large} \quad t\,; \tag{15}$$

- with two positive constants d_0 and d_1 we have

$$d_0 \le b^2(t) \le d_1 \qquad \text{for} \quad t \in [0,\infty)\,; \tag{16}$$

- there exist positive constants d_k such that for all $k = 2, 3, \dots$ it holds

$$\left| d_t^k b(t) \right| \le d_k \left(\frac{\lambda(t)}{\Lambda(t)} (\ln \Lambda(t))^{\beta} \right)^k \qquad \text{for large} \quad t\,. \tag{17}$$

THEOREM 2 (MAIN RESULT). *Assume that the conditions (13) to (17) are satisfied with* $\beta \in [0,1)$. *Then for every* $\varepsilon > 0$ *there exists a constant* C_ε *such that the decay estimate*

$$\|u_t(t,\cdot)\|_{L_q} + \|\lambda(t)\nabla_x u(t,\cdot)\|_{L_q} \le C_\varepsilon (1+\Lambda(t))^{1+\varepsilon-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|\varphi\|_{W_p^{L+1}} + \|\psi\|_{W_p^L} \right)$$

holds for the solution $u = u(t,x)$ *to (12). Here* $L = \left[n\left(\frac{1}{p}-\frac{1}{q}\right)\right] + 1$, $1 < p < 2$, $\frac{1}{p}+\frac{1}{q} = 1$.

Let us explain the philosophy of our approach. By $F,\ F^{-1}$ we denote the Fourier transform, inverse Fourier transform with respect to x, respectively. Applying F to (12) we get with $v = F(u)$ the Cauchy problem

$$v_{tt} + \lambda(t)^2 b(t)^2 |\xi|^2 v = 0\,, \quad v(0,\xi) = F(\varphi)\,, \quad v_t(0,\xi) = F(\psi)\,. \tag{18}$$

Setting $V = (V_1, V_2)^T := (\lambda(t)|\xi|v, D_t v)$, $D_t := d/dt$, the differential equation can be transformed to the system of first order

$$D_t V - \begin{pmatrix} 0 & \lambda(t)|\xi| \\ \lambda(t)b(t)^2|\xi| & 0 \end{pmatrix} V - \frac{D_t\lambda}{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = 0\,. \tag{19}$$

Our main object is the fundamental solution $X = X(t,\tau,\xi) \in C^\infty\big([\tau,\infty)\times\mathbb{R}^n\big)$ of (19), that is the solution of

$$D_t X - \begin{pmatrix} 0 & \lambda(t)|\xi| \\ \lambda(t)b(t)^2|\xi| & 0 \end{pmatrix} X - \frac{D_t\lambda}{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X = 0\,, \tag{20}$$

$$X(\tau,\tau,\xi) = I\,, \tag{21}$$

with $\tau \ge 0$. We prove that $X = X(t,0,\xi)$ can be represented in the form

$$\begin{aligned} X(t,0,\xi) &= X^+(t,0,\xi) \exp\left(i\int_0^t \lambda(s)b(s)\,ds|\xi| \right) \\ &\quad + X^-(t,0,\xi) \exp\left(-i\int_0^t \lambda(s)b(s)\,ds|\xi| \right), \end{aligned} \tag{22}$$

where X^+ and X^- have connections to symbol classes. Using this representation we obtain the solution of (12) in the form

$$u(t,x) = F^{-1}\left(\frac{\lambda(0)}{\lambda(t)} X_{11}(t,0,\xi)F(\varphi) + \frac{1}{\lambda(t)|\xi|} X_{12}(t,0,\xi)F(\psi) \right), \tag{23}$$

$$D_t u(t,x) = F^{-1}\big(\lambda(0)|\xi| X_{21}(t,0,\xi)F(\varphi) + X_{22}(t,0,\xi)F(\psi) \big), \tag{24}$$

where X_{jk} are the elements of X. For these Fourier multipliers $L_p - L_q$ decay estimates are derived in Section 3.5.

We intend to obtain the representation (22) in $\big\{(t,\xi) \in [0,\infty)\times\mathbb{R}^n \setminus \{0\}\big\}$ by splitting this set into two *zones*.

DEFINITION 2. *We define the* pseudodifferential zone $Z_{pd}(\beta, N)$ *by*

$$Z_{pd}(\beta, N) := \Big\{(t,\xi) \in [0,\infty) \times \big(\mathbb{R}^n \setminus \{0\}\big) : \big(e^4 + \Lambda(t)\big)|\xi| \le N\big(\ln\big(e^4 + \Lambda(t)\big)\big)^{\beta}\Big\},$$

the hyperbolic zone $Z_{\text{hyp}}(\beta, N)$ *by*

$$Z_{\text{hyp}}(\beta, N) := \Big\{(t,\xi) \in [0,\infty) \times \big(\mathbb{R}^n \setminus \{0\}\big) : \big(e^4 + \Lambda(t)\big)|\xi| \ge N\big(\ln\big(e^4 + \Lambda(t)\big)\big)^{\beta}\Big\}.$$

The positive constant N will be chosen later.

For $|\xi| \in (0, p_0]$, $p_0 = 4^{\beta}N/e^4$, we define the function $t_\xi = t(|\xi|)$ as the solution of $\big(e^4 + \Lambda(t_\xi)\big)|\xi| = N\big(\ln\big(e^4 + \Lambda(t_\xi)\big)\big)^{\beta}$.

LEMMA 3. *The derivatives $\partial^k_{|\xi|} t_\xi$ can be estimated in the following way:*

$$\left|\partial^k_{|\xi|} t_{|\xi|}\right| \le C_k |\xi|^{-k} \frac{\big(e^4 + \Lambda(t_\xi)\big)}{\lambda(t_\xi)} \qquad \textit{for all} \quad \xi \in \mathbb{R}^n\,,\ |\xi| \in (0, p_0]\,.$$

3.3. The fundamental solution in $Z_{pd}(\beta, N)$

Denoting

$$A(t, |\xi|) := \begin{pmatrix} 0 & \lambda(t)|\xi| \\ \lambda(t)b(t)^2|\xi| & 0 \end{pmatrix} + \frac{D_t\lambda}{\lambda}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the fundamental solution $X(t, 0, \xi)$ can be written explicitly in the form

$$X(t, 0, |\xi|) = I + \sum_{k=1}^{\infty} \int_0^t A(t_1, |\xi|) \cdots \int_0^{t_{k-1}} A(t_k, |\xi|)\, dt_k \cdots dt_1 \tag{25}$$

for $|\xi| \in (0, p_0]$. For a given positive number T let us distinguish two cases.

a) $t_\xi \le T$: in this case we have

$$\int_0^t \|A(s, |\xi|)\|\, ds \le C(T) \qquad \text{for all} \quad t \le t_\xi\,;$$

b) $T \le t_\xi$: in this case we have

$$\begin{aligned}
\int_0^t \|A(s, |\xi|)\|\, ds &\le C(T) + C_b \int_T^t \lambda(s)|\xi|\, ds + \int_T^t \frac{\lambda'(s)}{\lambda(s)}\, ds \\
&\le C(T) + C_b\, \Lambda(t)|\xi| + \ln\frac{\lambda(t)}{\lambda(T)} \\
&\le C(T) + C_b\, N\left(\ln\left(e^4 + \Lambda(t)\right)\right)^{\beta} + \ln\frac{\lambda(t)}{\lambda(T)} \\
&\le C(T) + C_b\, N\left(\ln\left(e^4 + \Lambda(T)\right)\right)^{\beta-1}\ln\left(e^4 + \Lambda(t)\right) + \ln\frac{\lambda(t)}{\Lambda(T)}
\end{aligned}$$

for all $t \le t_\xi$. Consequently,

$$\exp\left(\int_0^t \|A(s, |\xi|)\|\, ds\right) \le C(T)\, \lambda(t)\big(e^4 + \Lambda(t)\big)^{\frac{C_b N}{\left(\ln\left(e^4 + \Lambda(T)\right)\right)^{1-\beta}}}.$$

This gives the next statement:

LEMMA 4. *To each small positive ε there exists a constant $C_\varepsilon(N)$ such that for all $(t,\xi) \in Z_{pd}(\beta,N)$ it holds*

$$\|X(t,0,\xi)\| \le C_\varepsilon(N)\lambda(t)\big(e^4+\Lambda(t)\big)^\varepsilon , \qquad \|X(t,0,\xi)\| \le C_\varepsilon(N)\lambda(t)|\xi|^{-\varepsilon} ,$$

respectively.

To continue the solution from $Z_{pd}(\beta,N)$ to $Z_{\text{hyp}}(\beta,N)$ for $|\xi| \in (0,p_0]$ and to study its properties in $Z_{\text{hyp}}(\beta,N)$ we need the behaviour of $\partial_t^k\partial_\xi^\alpha X(t,0,\xi)$, too. It is obtained among other things from (25) and (14).

THEOREM 3. *To each small positive ε and each k and α there exists a constant $C_{\varepsilon,k,\alpha}(N)$ such that*

$$\left\|\partial_t^k\partial_\xi^\alpha\, X(t,0,\xi)\right\| \le C_{\varepsilon,k,\alpha}(N)\lambda(t)\left(\lambda(t)|\xi| + \frac{\lambda(t)}{e^4+\Lambda(t)}\right)^k |\xi|^{-|\alpha|-\varepsilon}$$

for all $(t,\xi) \in Z_{pd}(\beta,N)$.

3.4. The fundamental solution in $Z_{\text{hyp}}(\beta,N)$

The hyperbolic zone $Z_{\text{hyp}}(\beta,N)$ can be represented as the union of the two sets $\big\{(t,\xi) : |\xi| \in (0,p_0] : \big(e^4+\Lambda(t)\big)|\xi| \ge N(\ln(e+\Lambda(t)))^\beta\big\}$ and $\{(t,\xi) \in [0,\infty)\times\{|\xi| \ge p_0\}\}$. We restrict ourselves to the first set and sketch at the end of this section the approach in the second set.
In $Z_{\text{hyp}}(\beta,N)$ we apply a diagonalization procedure to the first order system (19). To carry out this procedure we need the following classes of symbols.

DEFINITION 3. *For given real numbers m_1,m_2,m_3, $\beta \in [0,1)$ and for positive N we denote by $S_{\beta,N}\{m_1,m_2,m_3\}$ the set of all symbols $a = a(t,\xi) \in C^\infty\big(Z_{\text{hyp}}(\beta,N) : |\xi| \in (0,p_0]\big)$ satisfying there*

$$\left|\partial_t^k\partial_\xi^\alpha a(t,\xi)\right| \le C_{k,\alpha}|\xi|^{m_1-|\alpha|}\lambda(t)^{m_2}\left(\frac{\lambda(t)}{e^4+\Lambda(t)}\big(\ln\big(e^4+\Lambda(t)\big)\big)^\beta\right)^{m_3+k} .$$

These classes of symbols are related to the Definitions 1 and 2. To understand that the diagonalization procedure improves properties of the remainder (as usually) one takes into consideration the following rules of the symbolic calculus:

- $S_{\beta,N}\{m_1,m_2,m_3\} \subset S_{\beta,N}\{m_1+k,m_2+k,m_3-k\}$, $\quad k \ge 0$;
- $a \in S_{\beta,N}\{m_1,m_2,m_3\}$, $\quad b \in S_{\beta,N}\{n_1,n_2,n_3\}$, $\quad$ then $ab \in S_{\beta,N}\{m_1+n_1,m_2+n_2,m_3+n_3\}$;
- $a \in S_{\beta,N}\{m_1,m_2,m_3\}$, $\quad$ then $\partial_t a \in S_{\beta,N}\{m_1,m_2,m_3+1\}$;
- $a \in S_{\beta,N}\{m_1,m_2,m_3\}$, $\quad$ then $\partial_\xi^\alpha a \in S_{\beta,N}\{m_1-|\alpha|,m_2,m_3\}$.

Let us define the matrices

$$M^{-1}(t) := \frac{1}{\sqrt{\lambda(t)b(t)}}\begin{pmatrix} 1 & 1 \\ -b(t) & b(t)\end{pmatrix}, \qquad M(t) := \frac{1}{2}\sqrt{\frac{\lambda(t)}{b(t)}}\begin{pmatrix} b(t) & -1 \\ b(t) & 1\end{pmatrix}.$$

Substituting $X = M^{-1}Y$ some calculations transform (20) into the first order system

$$D_t\, Y - D\, Y + B\, Y = 0\,, \tag{26}$$

where

$$D(t,\xi) := \begin{pmatrix} \tau_1(t,\xi) & 0 \\ 0 & \tau_2(t,\xi) \end{pmatrix}, \qquad B(t,\xi) := -\frac{D_t(\lambda(t)b(t))}{2\lambda(t)b(t)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\tau_1(t,\xi) := -\lambda(t)b(t)|\xi| + \frac{D_t\lambda(t)}{\lambda(t)}, \qquad \tau_2(t,\xi) := \lambda(t)b(t)|\xi| + \frac{D_t\lambda(t)}{\lambda(t)}.$$

Without difficulties one can prove $D \in S_{\beta,N}\{1,1,0\}$, $B \in S_{\beta,N}\{0,0,1\}$. To prove $L_p - L_q$ decay estimates for the solution of (12) we need further steps of the diagonalization of (26). This is carried out in the next lemma.

LEMMA 5. *For a given nonnegative integer M there exist matrix-valued functions $N_M = N_M(t,\xi) \in S_{\beta,N}\{0,0,0\}$, $F_M = F_M(t,\xi) \in S_{\beta,N}\{-1,-1,2\}$ and $R_M = R_M(t,\xi) \in S_{\beta,N}\{-M,-M,M+1\}$ such that the following operator-valued identity holds:*

$$(D_t - D + B)\, N_M = N_M\, (D_t - D + F_M - R_M)\,,$$

where F_M is diagonal while N_M is invertible and its inverse N_M^{-1} belongs as N_M to $S_{\beta,N}$ $\{0,0,0\}$.

REMARK 5. The invertibility of the diagonalizer $N_M = N_M(t,\xi)$ (mod $S_{\beta,N}\{-M,-M,M+1\}$) is essential. This property follows by a special choice of the positive constant N in Definition 2. We need only a finite number of steps of diagonalization (cf. proof of Theorem 2), thus N can be fixed after carrying out these steps.

Now let us devote to the system

$$(D_t - D + F_M - R_M)\, Z = 0\,, \qquad Z = Z(t,r,\xi)\,, \tag{27}$$

where $t_\xi \le r \le t$. Let $E_2 = E_2(t,r,\xi)$; $t, r \ge t_\xi$, is defined by

$$E_2(t,r,\xi) := \frac{\lambda(t)}{\lambda(r)} \begin{pmatrix} \exp\Big(-i\int_r^t \lambda(s)b(s)\,ds|\xi| - i\int_r^t F_M^{(1,1)}(s,\xi)\,ds\Big) & 0 \\ 0 & \exp\Big(i\int_r^t \lambda(s)b(s)\,ds|\xi| - i\int_r^t F_M^{(2,2)}(s,\xi)\,ds\Big) \end{pmatrix}$$

be the solution of the Cauchy problem $(D_t - D + F_M)Z = 0$, $Z(r,r,\xi) = I$. Let us denote

$$P_M(t,r,\xi) := E_2(r,t,\xi)R_M(t,\xi)E_2(t,r,\xi)\,.$$

By the aid of P_M we define the matrix-valued function

$$\begin{aligned} Q_M(t,r,\xi) := \sum_{k=1}^{\infty} i^k \int_r^t P_M(t_1,r,\xi) \int_r^{t_1} P_M(t_2,r,\xi) \cdots \\ \cdots \int_r^{t_{k-1}} P_M(t_k,r,\xi)\, dt_k \ldots dt_1\,. \end{aligned} \tag{28}$$

The function $Q_M = Q_M(t,r,\xi)$ solves the Cauchy problem

$$D_t Q - P_M Q - P_M = 0\,, \qquad Q(r,r,\xi) = 0 \quad \text{for} \quad t, r \ge t_\xi\,.$$

Using these auxiliary functions it is easy to prove the next result.

LEMMA 6. *The matrix-valued function $Z(t,r,\xi) = E_2(t,r,\xi)(I + Q_M(t,r,\xi))$ solves the Cauchy problem (27) for $t, r \geq t_\xi$.*

Now we can go back to (20), (21) and obtain as its solution

$$X(t,0,\xi) = M^{-1}(t) N_M(t,\xi) E_2\left(t, t_\xi, \xi\right)\left(I + Q_M\left(t, t_\xi, \xi\right)\right) \cdot \cdot N_M^{-1}\left(t_\xi, \xi\right) M\left(t_\xi\right) X\left(t_\xi, 0, \xi\right). \tag{29}$$

We write $\exp\left(-i\int_{t_\xi}^t \lambda(s)b(s)\,ds\right) = \exp\left(-i\int_0^t \lambda(s)b(s)\,ds - i\int_{t_\xi}^0 \lambda(s)b(s)\,ds\right)$ in correspondence with our goal (22) and include the second factor in the amplitudes. The matrices M and M^{-1} are given in an explicit form. The properties of N_M and N_M^{-1} are described by Lemma 5 using Definition 3. To estimate $X(t_\xi, 0, \xi)$ we use Theorem 3. Consequently, it remains to estimate $E_2(0, t_\xi, \xi)$ and $Q_M(t, t_\xi, \xi)$.

LEMMA 7. *For every positive small ε and every α the following estimate in $Z_{pd}(\beta, N)$ holds:*

$$\left|\partial_\xi^\alpha \exp\left(i\int_{t_\xi}^t \lambda(s)b(s)\,ds\right)\right| \leq C_{\varepsilon,\alpha}|\xi|^{-|\alpha|-\varepsilon},$$

where $C_{\varepsilon,\alpha} = C_{\varepsilon,\alpha}(\beta, N)$.

LEMMA 8. *For every positive small ε and every α the following estimate in $Z_{\mathrm{hyp}}(\beta, N)$ holds, $|\xi| \in (0, p_0]$:*

$$\left|\partial_\xi^\alpha \exp\left(-i\int_{t_\xi}^t F_M^{(k,k)}(s,\xi)\,ds\right)\right| \leq C_{\varepsilon,\alpha}|\xi|^{-|\alpha|-\varepsilon}, \qquad k = 1, 2,$$

where $C_{\varepsilon,\alpha} = C_{\varepsilon,\alpha}(\beta, N)$.

Proof. The statement for $|\alpha| = 0$ follows from

$$\begin{aligned}
\left|\int_{t_\xi}^t F_M^{(k,k)}(s,\xi)\,ds\right| &\leq \int_{t_\xi}^t \left|F_M^{(k,k)}(s,\xi)\right| ds \\
&\leq C_k \int_{t_\xi}^t \frac{\lambda(s)\big(\ln\left(e^4 + \Lambda(s)\right)\big)^{2\beta}}{|\xi|\big(e^4 + \Lambda(s)\big)^2}\,ds, \\
\int_{t_\xi}^t \frac{\lambda(s)\big(\ln\left(e^4 + \Lambda(s)\right)\big)^{2\beta}}{\big(e^4 + \Lambda(s)\big)^2}\,ds &\leq \frac{\left(\ln\left(e^4 + \Lambda\left(t_\xi\right)\right)\right)^{2\beta}}{e^4 + \Lambda\left(t_\xi\right)} \\
&\quad + \int_{t_\xi}^t \frac{2\beta}{\ln\left(e^4 + \Lambda(s)\right)} \frac{\lambda(s)\big(\ln\left(e^4 + \Lambda(s)\right)\big)^{2\beta}}{\big(e^4 + \Lambda(s)\big)^2}\,ds,
\end{aligned}$$

Definition 2 and

$$\frac{\left(\ln\left(e^4 + \Lambda\left(t_\xi\right)\right)\right)^\beta}{N} = \left(\ln\left(e^4 + \Lambda\left(t_\xi\right)\right)\right)^{\frac{1}{N\left(\ln\left(e^4 + \Lambda\left(t_\xi\right)\right)\right)^{1-\beta}}}.$$

By induction we prove the statement for $|\alpha| > 0$ by using $F_M \in S_{\beta,N}\{-1,-1,2\}$ and Lemma 3. □

More problems appear if we derive an estimate for Q_M. Here we refer the reader to [8].

LEMMA 9. *The matrix-valued function $P_M = P_M(t, t_\xi, \xi)$ satisfies for every l and α in $Z_{\text{hyp}}(\beta, N) \cap \{|\xi| \in (0, p_0]\}$ the estimates*

$$\begin{aligned}\left\|\partial_t^l \partial_\xi^\alpha P_M\left(t, t_\xi, \xi\right)\right\| \quad &\leq \quad C_{M,l,\alpha}(\lambda(t)|\xi|)^l \big(e^4 + \Lambda(t)\big)^{|\alpha|} \frac{\lambda(t)}{e^4 + \Lambda(t)} \cdot \\ &\cdot \big(\ln\big(e^4 + \Lambda(t)\big)\big)^\beta \left(\frac{\big(\ln\big(e^4 + \Lambda(t)\big)\big)^\beta}{\big(e^4 + \Lambda(t)\big)|\xi|}\right)^M .\end{aligned}$$

LEMMA 10. *For every positive small ε and every α, $|\alpha| \leq M - 1$, it holds the following estimate in $Z_{\text{hyp}}(\beta, N)$, $|\xi| \in (0, p_0]$:*

$$\left\|\partial_\xi^\alpha Q_M\left(t, t_\xi, \xi\right)\right\| \leq C_{\varepsilon,\alpha}(1 + \Lambda(t))^\varepsilon |\xi|^{-|\alpha|-\varepsilon} ,$$

where $C_{\varepsilon,\alpha} = C_{\varepsilon,\alpha}(\beta, N)$.

Proof. We use the representation (28) with $r = t_\xi$ and form the derivatives $\partial_\xi^\alpha Q_M(t, t_\xi, \xi)$. For $|\alpha| = 0$ the statement from Lemma 9 and similar calculations as in the proof of Lemma 8 imply the estimate for $\|Q_M(t, t_\xi, \xi)\|$. If we differentiate for $|\alpha| = 1$ inside of the integrals, then the estimate follows immediately. If we differentiate the lower integral bound in $\int_{t_\xi}^{t_{k-1}} P_M(t_k, t_\xi, \xi)\, dt_k$, then there appears a term of the form

$$P_M\left(t_\xi, t_\xi, \xi\right) \frac{\partial t_\xi}{\partial \xi_l} \int_{t_\xi}^t P_M\left(t_1, t_\xi, \xi\right) \cdots \int_{t_\xi}^{t_{k-2}} P_M\left(t_{k-1}, t_\xi, \xi\right) dt_{k-1} \ldots dt_1 .$$

Using Lemma 9 and Lemma 3 gives the desired estimate in this case, too. But we can only get estimates for $|\alpha| \leq M - 1$. In this case we can have an integrand of the form

$$\frac{\lambda(t)}{\big(e^4 + \Lambda(t)\big)^2 |\xi|} \big(\ln\big(e^4 + \Lambda(t)\big)\big)^{\beta(M+1)} .$$

The term $\big(\ln\big(e^4 + \Lambda(t)\big)\big)^{\beta M}$ can be estimated by $C(1 + \Lambda(t))^\varepsilon$, the other factor is integrable and can be estimated by C. An induction procedure yields the statement for $|\alpha| \leq M - 1$ (see [8]). □

Now we have all tools to get an estimate for (29).

THEOREM 4. *The fundamental solution $X = X(t, 0, \xi)$ can be represented in $Z_{\text{hyp}}(\beta, N) \cap \{|\xi| \in (0, p_0]\}$ as follows:*

$$X(t, 0, \xi) = X^+(t, 0, \xi) \exp\left(i \int_0^t \lambda(s) b(s) ds|\xi|\right) + X^-(t, 0, \xi) \exp\left(-i \int_0^t \lambda(s) b(s) ds|\xi|\right),$$

where the matrix-valued amplitudes X^-, X^+ satisfy for all $|\alpha| \leq M - 1$ and all positive small ε the estimates

$$\left\|\partial_\xi^\alpha X^\pm(t, 0, \xi)\right\| \leq C_{M,\varepsilon} \sqrt{\lambda(t) \lambda\left(t_\xi\right)} |\xi|^{-|\alpha|-\varepsilon} . \tag{30}$$

REMARK 6. There are no new difficulties to derive a corresponding estimate to (30) in $\{(t,\xi) \in [0,\infty) \times \{|\xi| \geq p_0\}\}$ which belongs to $Z_{\text{hyp}}(\beta, N)$ completely. We obtain for all $|\alpha| \leq M-1$ the estimates

$$\left\| \partial_\xi^\alpha X^{\pm}(t,0,\xi) \right\| \leq C_M \sqrt{\lambda(t)} |\xi|^{-|\alpha|} . \tag{31}$$

3.5. Fourier multipliers

The statements of Theorems 3, 4 and Remark 6 imply together with (23), (24) representations of the solution of (12) and its derivatives by the aid of Fourier multipliers. To estimate $\lambda = \lambda(t_\xi)$ in (30) we use assumption (14), especially $\frac{\lambda(t)}{\Lambda(t)} \leq c(\ln \Lambda(t))^c$ for large t. This corresponds to the Examples 3 and 4. To study Examples 1 and 2 we can follow our strategies in the same way. To get $L_p - L_q$ decay estimates we divide our consideration into two steps in accordance with two completely different ideas:

Hardy-Littlewood inequality [2] and *Littman's lemma* [3].

Instead of $Z_{pd}(\beta, N)$ from Definition 2 we use now

$$Z_{pd}(\beta, N) = \Big\{ (t,\xi) \in [0,\infty) \times \big(\mathbb{R}^n \setminus \{0\}\big) : \big(e^4 + \Lambda(t)\big)|\xi| \leq 4N\big(\ln\big(e^4 + \Lambda(t)\big)\big)^\beta \Big\} .$$

It is clear that the statement of Theorem 3 remains unchanged.

Let us choose a function $\psi \in C^\infty(\mathbb{R})$ satisfying $\psi(s) \equiv 0$ for $|s| \leq 1/2$, $\psi(s) \equiv 1$ for $|s| \geq 1$ and $0 \leq \psi(\xi) \leq 1$. Moreover, we define

$$K(t) := \frac{2N\big(\ln\big(e^4 + \Lambda(t)\big)\big)^\beta}{e^4 + \Lambda(t)} .$$

Following the approach of [8] which generalizes that one of [4] one can prove the next two results.

a) **Fourier multipliers with amplitudes supported in the pseudodifferential zone**

THEOREM 5 (APPLICATION OF HARDY – LITTLEWOOD INEQUALITY). *Let us consider Fourier multipliers depending on the parameter* $t \in [0,\infty)$ *which are defined by*

$$I_1 := F^{-1}\left(e^{i \int_0^t \lambda(s) b(s)\, ds |\xi|} \left(1 - \psi\left(\frac{\xi}{K(t)} \right) \right) |\xi|^{-2r} a(t,\xi) F(u_0)(\xi) \right), \quad u_0 \in C_0^\infty\left(\mathbb{R}^n\right) .$$

Suppose that $|a(t,\xi)| \leq C_\varepsilon\, \lambda(t)|\xi|^{-\varepsilon}$ *in* $Z_{pd}(\beta, N)$*. Then we have*

$$\|I_1\|_{L_q} \leq C_{r,N,\varepsilon} \lambda(t) \big(e^4 + \Lambda(t)\big)^{2r+\varepsilon-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_0\|_{L_p}$$

provided that $1 < p < 2$, $1/p + 1/q = 1$, $2r + \varepsilon \leq n\left(\frac{1}{p} - \frac{1}{q}\right)$.

b) **Fourier multipliers with amplitudes supported in the hyperbolic zone**

THEOREM 6 (APPLICATION OF LITTMAN'S LEMMA). *Let us consider*

$$I_2 := F^{-1}\left(e^{i \int_0^t \lambda(s) b(s)\, ds |\xi|} \psi\left(\frac{\xi}{K(t)} \right) |\xi|^{-2r} a(t,\xi) F(u_0)(\xi) \right), \quad u_0 \in C_0^\infty\left(\mathbb{R}^n\right) .$$

Suppose that $\left|\partial_\xi^\alpha a(t,\xi)\right| \le C\sqrt{\lambda(t)}|\xi|^{-\frac{1}{2}-|\alpha|-\varepsilon}$ *for* $|\alpha| \le n$ *in* $Z_{\text{hyp}}(\beta, N)$ *from Definition 2. Then we have*

$$\|I_2\|_{L_q} \le C_{r,N,n,\varepsilon}\sqrt{\lambda(t)}\big(e^4+\Lambda(t)\big)^{2r+\frac{1}{2}+\varepsilon-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|u_0\|_{L_p}$$

provided that $1 < p < 2$, $1/p + 1/q = 1$, $\frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right) \le 2r$.

Proof of Theorem 2. Theorem 6 tells us that we have to carry out $M = n+1$ steps of the perfect diagonalization. Here we use $2r = \frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$. In Theorem 5 we use $2r = \left[n\left(\frac{1}{p}-\frac{1}{q}\right)\right]+1$. But from (24) we know that this is the supposed regularity for $\nabla\varphi$, ψ, respectively. Consequently, the regularity for φ and ψ from Theorem 2 can be understood. The decay function $1+\Lambda(t)$ with the rate $1+\varepsilon-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$ follows immediately from the statements of Theorems 5 and 6. Due to (23) the estimate for $\lambda(t)\nabla u$ coincides with that for $D_t u$. □

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