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GENERALIZED DENSITIES AND DISTRIBUTIONAL ADJOINTS OF NATURAL OPERATORS

Abstract. Distributional adjoints of the main operators arising in fundamental field theory are examined, with particular regard to their direct geometric construction, in the context of generalized densities valued in a vector bundle; this notion extends the usual one of 'section-distributions'.

1. Introduction

Generalized maps (in the distributional sense) have been usually treated in a purely analytical setting, all constructions being based on $\mathbb{R}^n$. However, the notion of generalized sections of a vector bundle, in a geometrical setting, is certainly important in fundamental physics. This notion has been briefly considered by some authors [10, 6], and some specific cases [8] have been examined in greater detail.

Let us consider an arbitrary finite dimensional vector bundle, with no fixed background structure; I will argue that, in this setting, the notion of section-distribution has to be introduced as that of 'generalized density' valued in the bundle. All other cases, including currents and generalized half-densities, can be seen as particular cases of this one. If a volume form on the base manifold is fixed, then one recovers the notion of section-distribution.

The notion of distributional adjoint of a differential operator can be readily introduced in the geometrical setting. Actually, the local coordinate expression of the adjoint can be written by the standard methods. However, the intrinsic geometric meaning of this expression may not be evident.

The main goal of this paper is to examine the distributional adjoints of the main operators arising in fundamental field theory, with particular regard to their geometric construction. First, I will consider natural operators on scalar-valued currents. Then I will argue that, in some cases, the Frölicher-Nijenhuis bracket yields a natural operator on bundle-valued currents, and examine these. Next, the covariant derivative of a generalized section along a given vector field, relatively to a given connection, yields a non-trivial case. The codifferential and Laplacian are considered in a setting which generalizes Lichnerowicz' [8]. Finally, the distributional adjoint of the Dirac operator on curved background confirms the importance of spacetime torsion in this context.

2. Generalized densities

By a *classical manifold* I mean a Hausdorff paracompact smooth real manifold of finite dimension. By $\mathsf{p} : \mathbf{E} \rightarrowtail \mathbf{M}$ I will denote a real or complex vector bundle over an oriented classical manifold $\mathbf{M}$ without boundary; $\dim \mathbf{E} = m+n$, $\dim \mathbf{M} = m$. I will write $\mathbb{Y} := (\wedge^m \mathrm{T}\mathbf{M})^+$, so that $\mathbb{Y}^* \rightarrowtail \mathbf{M}$ is the bundle of all positively oriented volume forms on $\mathbf{M}$.

By $\mathcal{D}_\circ(\mathbf{M}, \mathbf{E}^*)$ I will denote the vector space of all C^∞ sections $\mathbf{M} \to \mathbf{E}^*$ which have compact support. If $\mathbf{X} \subset \mathbf{M}$ is an open set and $\mathbf{E}_\mathbf{X} := \mathsf{p}^{-1}(\mathbf{X})$ is trivializable, then the subspace $\mathcal{D}_\circ(\mathbf{X}, \mathbf{E}^*_\mathbf{X}) \subset \mathcal{D}_\circ(\mathbf{M}, \mathbf{E}^*)$ can be given the usual test map topology [10, 6] by choosing any trivialization $(\mathsf{p}, \mathsf{y}) : \mathbf{E}_\mathbf{X} \to \mathbf{X} \times \mathbb{R}^n$. This topology is actually independent of the chosen map y, and it turns out that a bundle atlas yields a topology on the whole space $\mathcal{D}_\circ(\mathbf{M}, \mathbf{E}^*)$ of 'test sections'. I will denote its topological dual space * by

$$\mathcal{D}(\mathbf{M}, \mathbb{Y}^* \underset{\mathbf{M}}{\otimes} \mathbf{E}) := \mathcal{D}'_\circ(\mathbf{M}, \mathbf{E}^*) ,$$

and call it the space of *generalized* $\mathbf{E}$*-valued densities*.

An ordinary $\mathbf{E}$-valued density $\theta : \mathbf{M} \to \mathbb{Y}^* \otimes_\mathbf{M} \mathbf{E}$ is said to be *locally integrable* if the ordinary density $\mathbf{M} \to \mathbb{Y}^* : x \mapsto \langle u(x), \theta(x) \rangle$ is integrable for all $u \in \mathcal{D}_\circ(\mathbf{M}, \mathbf{E}^*)$. Such θ can be identified with an element of $\mathcal{D}(\mathbf{M}, \mathbb{Y}^* \otimes_\mathbf{M} \mathbf{E})$; if ϕ is an arbitrary element, then I will also write $\phi : \mathbf{M} \rightsquigarrow \mathbb{Y}^* \otimes_\mathbf{M} \mathbf{E}$. Note that a generalized $\mathbf{E}$-valued density has a coordinate expression just like an ordinary section, its components being generalized functions.

A few special cases are of particular interest. By replacing † $\mathbf{E}$ with $\mathbb{Y}^{1/2} \otimes_\mathbf{M} \mathbf{E}$ one obtains generalized $\mathbf{E}$-valued *half-densities* $\mathbf{M} \rightsquigarrow \mathbb{Y}^{-1/2} \otimes_\mathbf{M} \mathbf{E}$, where $\mathbb{Y}^{-1/2} \equiv (\mathbb{Y}^{1/2})^*$. If a smooth Hermitian structure in the fibres of $\mathbf{E}$ is given (in particular, in the case of scalar generalized half-densities $\mathbf{M} \rightsquigarrow \mathbb{C} \otimes \mathbb{Y}^{-1/2}$), then the test space $\mathcal{D}_\circ(\mathbf{M}, \mathbb{Y}^{-1/2} \otimes_\mathbf{M} \mathbf{E}^*)$ is naturally included into its dual $\mathcal{D}(\mathbf{M}, \mathbb{Y}^{-1/2} \otimes_\mathbf{M} \mathbf{E})$; moreover one has a well-defined notion of square-integrable sections, and a Hilbert space $\mathcal{H}$ can be introduced; finally, one gets a so-called 'rigged Hilbert space' $\mathcal{D}_\circ \subset \mathcal{H} \subset \mathcal{D}$ [1]. In the other cases there exist no such natural inclusions, unless one has a fixed volume form and an isomorphism $\mathbf{E} \cong \mathbf{E}^*$.

By replacing $\mathbf{E}$ with $\wedge^{m-r} \mathrm{T}\mathbf{M} \otimes_\mathbf{M} \mathbf{E}$, because of the isomorphism $\mathbb{Y}^* \otimes \wedge^{m-r} \mathrm{T}\mathbf{M} \cong \wedge^r \mathrm{T}^*\mathbf{M}$, one gets $\mathbf{E}$-valued r*-currents* $\mathbf{M} \rightsquigarrow \wedge^r \mathrm{T}^*\mathbf{M} \otimes_\mathbf{M} \mathbf{E}$; this includes 0-currents which can be identified with the usual section-distributions [6]. In particular, a locally integrable ordinary section $\alpha : \mathbf{M} \to \wedge^r \mathrm{T}^*\mathbf{M}$ acts as a functional on test sections $\beta \in \mathcal{D}_\circ(\mathbf{M}, \wedge^{m-r} \mathrm{T}^*\mathbf{M})$ by the rule $\langle \alpha, \beta \rangle := \int_\mathbf{M} \alpha \wedge \beta$.

*The most usual notation is $\mathcal{D}$ for the space of test maps and $\mathcal{D}'$ for the corresponding distributional space. This, however, could generate some confusion in the present context, since the two mutually dual spaces live in different bundles.

†The 'square root' bundle $\mathbb{Y}^{1/2}$ is characterized, up to isomorphism, by $\mathbb{Y}^{1/2} \otimes \mathbb{Y}^{1/2} \cong \mathbb{Y}$ [4, 5].

3. Differential operators and their distributional adjoints

Let $v : \mathbf{M} \to \mathrm{T}\mathbf{M}$ be a smooth vector field. If $\alpha : \mathbf{M} \to \wedge^r \mathrm{T}^*\mathbf{M}$ is a smooth ordinary current and $\beta : \mathbf{M} \to \wedge^{m-r} \mathrm{T}^*\mathbf{M}$ is a test current, then $\langle v.\alpha, \beta\rangle = -\langle \alpha, v.\beta\rangle$, where $v.$ denotes the Lie derivative operator. This formula can be taken as the *definition* of $v.\alpha$ for an arbitrary current α, and it turns out that the mapping $\alpha \mapsto v.\alpha$ is a continuous linear operator in $\mathcal{D}(\mathbf{M}, \wedge^r \mathrm{T}^*\mathbf{M})$. A similar construction can be carried out for scalar generalized half-densities [4]. In the domain of a coordinate chart, in particular, on has the *partial derivatives* of a scalar current or generalized half-density.

Consider now $\mathbf{E}$-valued currents (the following argument is quite similar for $\mathbf{E}$-valued generalized half-densities). Given a frame of $\mathbf{E}$, the components of any such current are scalar currents, whose partial derivatives with respect to the base coordinates are well-defined. If D is a differential operator acting on smooth currents, which can be locally expressed in terms of partial derivatives, then D has a unique continuous extension to the whole space of currents, and this extension maintains the same coordinate expression. More specifically, one considers a *polynomial derivation operator of order* $k \in \mathbb{N}$, which means that its coordinate expression is of the type $(D\alpha)^i = \sum_{|p|\leq k} c^{\,i}_{p\,j}\,\partial^p \alpha^j$; here, $p = \{p_1, .., p_m\}$ is a multi-index of order $|p| := \sum_a p_a$, and $c^{\,i}_{p\,j} : \mathbf{M} \to \mathbb{C}$ are smooth functions.

Because the operator D can be expressed in terms of Lie derivatives, it yields a *distributional adjoint* operator D', fulfilling $\big\langle D'\phi, \theta\big\rangle = \langle \phi, D\theta\rangle$; this acts in the dual space, is still a polynomial operator of degree k, and extends to the whole, appropriate space of generalized sections. In practice, its coordinate expression is usually found by assuming ϕ to be C^k, expressing $\langle \phi, D\theta\rangle$ as an integral and applying integration by parts (boundary terms disappear).

In the case of scalar-valued currents one has some important instances. Let $v : \mathbf{M} \to \mathrm{T}\mathbf{M}$ be a smooth vector field, and $\omega : \mathbf{M} \to \wedge \mathrm{T}^*\mathbf{M}$ a smooth form; indicating by $|\theta|$ the degree of the current θ, by simple exterior algebra calculations and taking into account $\partial\mathbf{M} = \emptyset$ one obtains

$$
\begin{aligned}
&\langle v|\alpha, \beta\rangle = (-1)^{|\alpha|+1}\langle \alpha, v|\beta\rangle\,, && |\beta| = m - |\alpha| + 1\,.\\
&\langle \omega\wedge\alpha, \beta\rangle = (-1)^{|\alpha|\cdot|\omega|}\langle \alpha, \omega\wedge\beta\rangle\,, && |\beta| = m - |\alpha| - |\omega|\,.\\
&\langle \mathrm{d}\alpha, \beta\rangle = (-1)^{|\alpha|+1}\langle \alpha, \mathrm{d}\beta\rangle\,, && |\beta| = m - |\alpha| - 1\,.\\
&\langle \mathrm{d}(v|\alpha), \beta\rangle = -\langle \alpha, v|(\mathrm{d}\beta)\rangle\,, && |\beta| = m - |\alpha|\,.\\
&\langle v|(\mathrm{d}\alpha), \beta\rangle = -\langle \alpha, \mathrm{d}(v|\beta)\rangle\,, && |\beta| = m - |\alpha|\,.\\
&\langle v.\alpha, \beta\rangle = -\langle \alpha, v.\beta\rangle\,, && |\beta| = m - |\alpha|\,.\\
&\langle *\alpha, \beta\rangle = (-1)^{|\alpha|(m+1)}\langle \alpha, *\beta\rangle\,, && |\beta| = |\alpha|\,.\\
&\langle \delta\alpha, \beta\rangle = (-1)^{|\alpha|}\langle \alpha, \delta\beta\rangle\,, && |\beta| = m - |\alpha| + 1\,.\\
&\langle \mathrm{d}\delta\alpha, \beta\rangle = \langle \alpha, \delta\mathrm{d}\beta\rangle\,, && |\beta| = m - |\alpha|\,.\\
&\langle (\mathrm{d}\delta + \delta\mathrm{d})\alpha, \beta\rangle = \langle \alpha, (\mathrm{d}\delta + \delta\mathrm{d})\beta\rangle\,, && |\beta| = m - |\alpha|\,.
\end{aligned}
$$

In the four last formulas, $*$ stands for the Hodge isomorphism relatively to a given non-degenerate (pseudo)metric on $\mathbf{M}$. The operators $\delta := *\mathrm{d}*$ and $\mathrm{d}\delta + \delta\mathrm{d}$ are, up to sign, the usual codifferential and Laplacian of scalar-valued currents; in general, the latter is different from the connection-induced Laplacian defined on $\mathbf{E}$-valued currents (§6).

4. Operators induced by the Frölicher-Nijenhuis bracket

Let $\mathbf{N}$ be a classical manifold. Let us recall that the Frölicher-Nijenhuis bracket [7, 9] of two differentiable tangent-valued forms $F : \mathbf{N} \to \wedge^p \mathrm{T}^*\mathbf{N} \otimes_{\mathbf{N}} \mathrm{T}\mathbf{N}$, $G : \mathbf{N} \to \wedge^r \mathrm{T}^*\mathbf{N} \otimes_{\mathbf{N}} \mathrm{T}\mathbf{N}$, is a tangent-valued form

$$[F, G] : \mathbf{N} \to \wedge^{p+r} \mathrm{T}^*\mathbf{N} \underset{\mathbf{N}}{\otimes} \mathrm{T}\mathbf{N} .$$

This operation can be carried on fibred manifolds; in particular consider the case when $\mathbf{N} \equiv \mathbf{E}$, the total space of our vector bundle. A smooth current $\alpha : \mathbf{M} \to \wedge^r \mathrm{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E}$ can also be seen as a tangent-valued r-form on $\mathbf{E}$ (because of the natural inclusions $\mathrm{T}^*\mathbf{M} \subset \mathrm{T}^*\mathbf{E}$ and $\mathbf{E} \subset \mathbf{E} \otimes_{\mathbf{M}} \mathbf{E} \equiv \mathrm{V}\mathbf{E} \subset \mathrm{T}\mathbf{E}$). Then we can consider the F-N bracket $[\omega, \alpha]$, whenever ω is a smooth tangent-valued form on $\mathbf{E}$. In practice, one is interested mainly in *basic* forms $\omega : \mathbf{M} \to \wedge^p \mathrm{T}^*\mathbf{M} \otimes_{\mathbf{E}} \mathrm{T}\mathbf{E}$ which are *projectable* over $\mathrm{T}\mathbf{M}$-valued forms. Taking fibred coordinates $(\mathsf{x}^a, \mathsf{y}^i) : \mathbf{E}_{\mathbf{X}} \to \mathbb{R}^m \times \mathbb{R}^n$, the projectability condition for ω becomes $\partial_j \omega^b_{a_1\dots a_p} = 0$, and one obtains the local expression

$$\begin{aligned}[\omega, \alpha] = \Big(&\omega^b_{a_1\dots a_p}\, \partial_b \alpha^i_{a_{p+1}\dots a_{p+r}} + (-1)^{pr} r\, \partial_{a_r} \omega^b_{a_{r+1}\dots a_{r+p}}\, \alpha^i_{a_1\dots a_{r-1}\, b} + \\ &+ (-1)^{pr+1} \partial_j \omega^i_{a_{r+1}\dots a_{r+p}}\, \alpha^j_{a_1\dots a_r}\Big) \mathrm{d}\mathsf{x}^{a_1} \wedge \dots \wedge \mathrm{d}\mathsf{x}^{a_{p+r}} \otimes \partial\mathsf{y}_i\ .\end{aligned}$$

In order that $[\omega, \alpha]$ be still a current, then, one has to assume that ω is a linear morphism over $\mathbf{M}$, which in coordinates reads $\omega^i_{a_1\dots a_p} = \mathsf{y}^j \omega^i_{j\, a_1\dots a_p}$ with $\omega^i_{j\, a_1\dots a_p} : \mathbf{M} \to \mathbb{R}$. Now it is clear that the mapping $\alpha \mapsto [\omega, \alpha]$ extends naturally to a continuous operator acting on the distributional space of $\mathbf{E}$-valued currents. The coordinate expression of its distributional adjoint is readily written. We see that, in general, its geometrical interpretation involves viewing α as a $\mathrm{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E}$-valued $r-1$ current, leading to a rather involved situation. However, the really interesting cases for field theory applications are simpler: essentially, when ω is either a vertical-valued form or a connection.

Consider the first instance, namely $\omega : \mathbf{M} \to \wedge^p \mathrm{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathrm{V}\mathbf{E}$, so that $\omega^b_{a_1\dots a_p} = 0$. We obtain the distributional adjoint formula

$$\begin{aligned}\langle [\omega, \alpha], \beta \rangle &= (-1)^{pr+1} \int_{\mathbf{M}} \Big(\alpha^j_{a_1\dots a_r}\, \omega^i_{j\ a_{r+1}\dots a_{r+p}}\, \beta_{i\ a_{r+p+1}\dots a_m} \Big) \\ &\qquad\qquad \mathrm{d}\mathsf{x}^{a_1} \wedge \dots \wedge \mathrm{d}\mathsf{x}^{a_m} \otimes \mathrm{d}\mathsf{y}^j \\ &= (-1)^{pr+1} \big\langle \alpha, [\omega^*, \beta] \big\rangle ,\end{aligned}$$

where $\beta : \mathbf{M} \to \wedge^{m-p-r} \mathrm{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E}^*$ and $\omega^* : \mathbf{E}^* \to \wedge^p \mathrm{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E}^*$ is the transpose morphism over $\mathbf{M}$.

A similar result holds for a linear connection on the bundle $\mathbf{E} \rightarrowtail \mathbf{M}$; this can be defined [9] as a section $\gamma : \mathbf{E} \to \mathrm{T}^*\mathbf{M} \otimes_{\mathbf{E}} \mathrm{T}\mathbf{E}$, which is projected over the identity $1\!\!1 : \mathbf{M} \to \mathrm{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathrm{T}\mathbf{M}$ and is a linear morphism over $\mathbf{M}$. This means that its coordinate expression is $\gamma = \mathsf{d}\mathsf{x}^a \otimes \partial\mathsf{x}_a + \gamma_{a\,i}^{\;\,j}\mathsf{y}^i\, \mathsf{d}\mathsf{x}^a \otimes \partial\mathsf{y}_j$. We obtain

$$\begin{aligned} \langle[\gamma, \alpha], \beta\rangle &= (-1)^{r+1} \int_{\mathbf{M}} \alpha^i_{a_1\dots a_r} \Big(\partial_{a_{r+1}}\beta_{i\,a_{r+2}\dots a_m} + \gamma_{a_{r+1}\,i}^{\;\;\;\;\;j}\, \beta_{j\,a_{r+2}\dots a_m}\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{d}\mathsf{x}^{a_1} \wedge \dots \wedge \mathsf{d}\mathsf{x}^{a_m} \\ &= (-1)^{r+1}\big\langle\alpha, [\gamma^*, \beta]\big\rangle, \end{aligned}$$

where γ^* is the dual connection on $\mathbf{E}^*$. In particular if $r = 0$ then $[\gamma, \alpha] \equiv \nabla\alpha$, the standard covariant derivative of sections $\mathbf{M} \to \mathbf{E}$; the distributional adjoint of ∇ is now seen to be the F-N bracket of γ^* with $\mathbf{E}^*$-valued (generalized) densities.

5. Covariant derivative along a given vector field

In this section I assume an assigned linear connection γ on $\mathbf{E} \rightarrowtail \mathbf{M}$ and an assigned vector field $v : \mathbf{M} \to \mathrm{T}\mathbf{M}$. Consider the differential operator D' acting in $\mathcal{D}_\circ(\mathbf{M}, \mathbf{E})$ which, for each test section u, is given by $D'u := \nabla_v u$ or, in coordinates, $D'u = v^a(\partial_a u^i - \gamma_{a\,j}^{\;\,i} u^j)\partial\mathsf{y}_i$ (the choice of test maps as sections $\mathbf{M} \to \mathbf{E}$ is for notational convenience, and in what follows one could reverse the roles of $\mathbf{E}$ and $\mathbf{E}^*$).

The expression of the adjoint operator D, acting in $\mathcal{D}(\mathbf{M}, \mathbb{Y}^* \otimes_{\mathbf{M}} \mathbf{E}^*)$, turns out to be

$$D\phi = -\big[(\partial_a v^a)\phi_j + v^a(\partial_a\phi_j + \gamma_{a\,j}^{\;\,i}\phi_i)\big]\, \mathsf{d}\mathsf{x} \otimes \mathsf{d}\mathsf{y}^j \ ,$$

where $\mathsf{d}\mathsf{x} \equiv \mathsf{d}\mathsf{x}^1 \wedge \dots \wedge \mathsf{d}\mathsf{x}^m$. We can recover this operator by a direct geometrical construction as follows.

First, since $\mathrm{V}^*\mathbf{E} \cong \mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^*$, from ϕ we obtain a section $\tilde{\phi} : \mathbf{E} \to \mathbb{Y}^* \otimes_{\mathbf{M}} \mathrm{V}^*\mathbf{E}$; then, using the connection form $\omega_\gamma := 1\!\!1_{\mathrm{T}\mathbf{E}} - \gamma : \mathrm{T}\mathbf{E} \to \mathrm{V}\mathbf{E}$ we obtain a section

$$\omega^*_\gamma\tilde{\phi} : \mathbf{E} \to \wedge^m\mathrm{T}^*\mathbf{M} \underset{\mathbf{E}}{\otimes} \mathrm{T}^*\mathbf{E} \hookrightarrow \wedge^m\mathrm{T}^*\mathbf{E} \underset{\mathbf{E}}{\otimes} \mathrm{T}^*\mathbf{E} \ ,$$

with coordinate expression $\omega^*_\gamma\tilde{\phi} = \phi_i\, \mathsf{d}\mathsf{x} \otimes (\mathsf{d}\mathsf{y}^i - \gamma_a^{\;i}\mathsf{d}\mathsf{x}^a)$. On turn, by anti-symmetrization this yields a section

$$\hat{\phi} : \mathbf{E} \to \wedge^{m+1}\mathrm{T}^*\mathbf{E} \ ,$$

in coordinates $\hat{\phi} = \phi_i\, \mathsf{d}\mathsf{x} \wedge \mathsf{d}\mathsf{y}^i$.

Let now $\hat{v} : \mathbf{E} \to \mathrm{T}\mathbf{E}$ be the lift of v by γ, and consider the $(m+1)$-form $\hat{v}.\hat{\phi}$ on $\mathbf{E}$. One checks immediately that $\mathrm{d}\hat{\phi} = 0$, thus $\hat{v}.\hat{\phi} = \mathrm{d}(\hat{v}|\hat{\phi})$; we obtain the coordinate expressions

$$\hat{v}|\hat{\phi} = v^a\,(\partial\mathsf{x}_a + \gamma_a^{\;i}\partial\mathsf{y}_i)|(\phi_j\, \mathsf{d}\mathsf{x} \wedge \mathsf{d}\mathsf{y}^j) = v^a\,\phi_i(\mathsf{d}\mathsf{x}_a \wedge \mathsf{d}\mathsf{y}^i + (-1)^m\,\gamma_a^{\;i}\, \mathsf{d}\mathsf{x}) \ ,$$

where $\mathsf{d}\mathsf{x}_a := \partial\mathsf{x}_a|\mathsf{d}\mathsf{x}$ and, finally,

$$\hat{v}.\hat{\phi} = \mathrm{d}(\hat{v}|\hat{\phi}) = [(\partial_a v^a)\phi_j + v^a(\partial_a\phi_j + \gamma_{a\,j}^{\;\,i}\phi_i)\big]\, \mathsf{d}\mathsf{x} \wedge \mathsf{d}\mathsf{y}^j \ .$$

Then, $\hat{v}.\hat{\phi}$ is in the image of the composition

$$\wedge^m \mathrm{T}^*\mathbf{M} \underset{\mathbf{E}}{\otimes} \mathrm{T}^*\mathbf{E} \hookrightarrow \wedge^m \mathrm{T}^*\mathbf{E} \underset{\mathbf{E}}{\otimes} \mathrm{T}^*\mathbf{E} \xrightarrow{\wedge} \wedge^{m+1}\mathrm{T}^*\mathbf{E} \ ,$$

which is a monomorphism over $\mathbf{E}$. Thus $\hat{v}.\hat{\phi}$ comes from a unique $\mathrm{V}^*\mathbf{E}$-valued density; since its components do not depend actually on $\mathbf{E}$, this can be identified with a $\mathbf{E}^*$-valued density, which is exactly our $-D\phi$.

Observe that D, the distributional adjoint of ∇_v, is not itself a covariant derivative. If κ is a connection on $\mathbb{Y}^* \rightarrowtail \mathbf{M}$ then we have a connection on $\mathbb{Y}^* \otimes_{\mathbf{M}} \mathbf{E}^*$ and the covariant derivative $\nabla_v\phi$ with coordinate expression

$$\nabla_v\phi = v^a(\partial_a\phi_j + \gamma_{a\,j}^{\ i}\phi_i + \kappa_a\phi_j)\mathsf{dx}\otimes\mathsf{dy}^j \ .$$

The difference between the two operators has the expression

$$-D\phi - \nabla_v\phi = (\partial_a v^a - \kappa_a v^a)\phi \ .$$

6. Codifferential and Laplacian

I considered the codifferential and Laplacian of scalar-valued currents in §3. A somewhat different notion of these operators can be introduced for $\mathbf{E}$-valued distributions, by taking a linear connection γ on $\mathbf{E} \rightarrowtail \mathbf{M}$, a non-degenerate (pseudo)metric g on $\mathbf{M}$ and a metric connection Γ on $\mathrm{T}\mathbf{M} \rightarrowtail \mathbf{M}$ (while the latter notion of codifferential coincides with the former one for scalar-valued currents, this needs not be true for the Laplacian). I will not assume the torsion T of Γ to vanish in general.

Let $u : \mathbf{M} \rightsquigarrow \mathrm{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E}$ (this will include also r-currents, seen as 1-currents valued in $\wedge^{r-1}\mathrm{T}^*\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E}$). We define its covariant codifferential as $\nabla u := \langle g^{-1}, \nabla u\rangle :$ $\mathbf{M} \to \mathbf{E}$, with coordinate expression

$$\nabla u = g^{ab}\nabla_a u_b^i\,\partial\mathsf{y}_i = g^{ab}(\partial_a u_b^i + \Gamma_{a\,b}^{\ c}\,u_c^i - \gamma_{a\,j}^{\ i}\,u_b^j)\,\partial\mathsf{y}_i \ .$$

Its distributional adjoint acts on $\mathbf{E}^*$-valued generalized densities $\phi : \mathbf{M} \rightsquigarrow \mathbb{Y}^* \otimes_{\mathbf{M}} \mathbf{E}^*$ to give $\mathrm{T}\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E}^*$-valued generalized densities; we find the coordinate expression

$$\nabla'\phi = \Big(-g^{ab}\partial_a\phi_i - (\partial_a g^{ab})\phi_i + g^{ac}\Gamma_{a\,c}^{\ b}\,\phi_i - \gamma_{a\,i}^{\ j}\,\phi_j\Big)\mathsf{dx}\otimes\partial\mathsf{x}_b\otimes\mathsf{dy}^i \ .$$

By introducing the 1-form $\breve{T} : \mathbf{M} \to \mathrm{T}^*\mathbf{M}$ obtained from the torsion by contraction, in coordinates $\breve{T}_a = T_{ac}^c = \Gamma_{c\,a}^{\ c} - \Gamma_{a\,c}^{\ c}$, after some calculations we arrive to

$$\nabla'\phi = -g^{ab}(\nabla_a\phi_i + \breve{T}_a\,\phi_i)\mathsf{dx}\otimes\partial\mathsf{x}_b\otimes\mathsf{dy}^i \ ,$$

or $\nabla'\phi = -g^{-1}\rfloor(\nabla\phi + \breve{T}\otimes\phi)$.

Next I consider the Laplacian operator in the same setting as above. If $u : \mathbf{M} \to \mathbf{E}$ is a smooth section, then its Laplacian $\Delta u := \langle g^{-1}, \nabla\nabla u\rangle$ has the coordinate expression

$$\Delta u = g^{ab}(\partial_a\partial_b u^i - 2\gamma_{a\,j}^{\ i}\,\partial_b u^j - \partial_a\gamma_{b\,j}^{\ i}\,u^j + \gamma_{a\,k}^{\ i}\gamma_{b\,j}^{\ k}\,u^j + \Gamma_{a\,b}^{\ c}\,\partial_c u^i - \Gamma_{a\,b}^{\ c}\,\gamma_{c\,j}^{\ i}\,u^j)\,\partial\mathsf{y}_i \ .$$

Let us denote by D, acting in $\mathcal{D}(\mathbf{M}, \mathbb{Y}^* \otimes_{\mathbf{M}} \mathbf{E}^*)$, the distributional adjoint of Δ. We obtain the coordinate expression

$$\begin{aligned}
\phi = \big[&\partial_a \partial_b (g^{ab}\,\phi_j) + 2\partial_b (g^{ab}\,\gamma_{a\,j}^{\;i}\,\phi_i) - g^{ab}\,(\partial_a \gamma_{b\,j}^{\;i})\,\phi_i + g^{ab}\,\gamma_{ak}^{\;i}\gamma_{b\,j}^{\;k}\,\phi_i \\
&- \partial_c (g^{ab}\,\Gamma_{a\,b}^{\;c}\,\phi_j) - g^{ab}\,\Gamma_{a\,b}^{\;c}\,\gamma_{c\,j}^{\;i}\,\phi_i\big]\,\mathsf{d}\mathsf{y}^j \otimes \mathsf{d}\mathsf{x} = \\
= g^{ab}\big(&\partial_a \partial_b \phi_j + 2\,\Gamma_{c\,a}^{\;c}\,\partial_b \phi_j + \Gamma_{a\,b}^{\;c}\,\partial_c \phi_j + \partial_a \Gamma_{c\,b}^{\;c}\,\phi_j + \Gamma_{d\,c}^{\;d}\,\Gamma_{a\,b}^{\;c}\,\phi_j + \Gamma_{c\,a}^{\;c}\,\Gamma_{d\,b}^{\;d}\,\phi_j \\
&+ \partial_a \gamma_{b\,j}^{\;i}\,\phi_i + 2\,\gamma_{a\,j}^{\;i}\,\partial_b \phi_i + 2\,\Gamma_{c\,a}^{\;c}\,\gamma_{b\,j}^{\;i}\,\phi_i + \Gamma_{a\,b}^{\;c}\,\gamma_{c\,j}^{\;i}\,\phi_i + \gamma_{ak}^{\;i}\,\gamma_{b\,j}^{\;k}\,\phi_i\big)\,\mathsf{d}\mathsf{y}^j \otimes \mathsf{d}\mathsf{x}\,.
\end{aligned}$$

With the considered structures we also have the Laplacian $\Delta\phi := \langle g^{-1}, \nabla\nabla\phi\rangle$; after some calculations we arrive to the coordinate-free expression

$$D\phi - \Delta\phi = \Big\langle g^{-1}, (2\,\check{T} \otimes \nabla\phi + \nabla\check{T} \otimes \phi + \check{T} \otimes \check{T} \otimes \phi)\Big\rangle\,.$$

7. Dirac operator

The language of 2-spinors turns out to be very convenient for an integrated approach to Einstein-Cartan-Maxwell-Dirac fields starting from minimal geometric assumptions, and for other important issues in field theories [3, 5]. Moreover it can be proved [2] that the more usual 4-spinor language is completely equivalent to it, provided that one assumes the basic structures needed for physics. In this section I will give a very brief account of that language, before examining the distributional adjoint of the Dirac operator; the reader is referred to the above cited papers for details and various developments concerning 2-spinors.

Let $\mathbf{S}$ be a 2-dimensional complex vector space, and $\overline{\mathbf{S}}$ its conjugate space. Then $\wedge^2\mathbf{S}$ is 1-dimensional. The Hermitian subspace of $\wedge^2\mathbf{S} \otimes \wedge^2\overline{\mathbf{S}}$ is a real vector space with a distinguished orientation; its positively oriented semispace $\mathbb{L}^2$ has the square root semispace $\mathbb{L}$, called the space of *length units*. The fundamental 2-spinor space is defined to be $\mathbf{U} := \mathbb{L}^{-1/2} \otimes \mathbf{S}$. The space $\wedge^2\mathbf{U}$ is naturally endowed with a Hermitian metric, so that normalized symplectic forms $\varepsilon \in \wedge^2\mathbf{U}^\star$ (I indicate complex-dual spaces by a *) constitute a U(1)-space (any two of them are related by a phase factor). Then $\varepsilon \otimes \bar\varepsilon$ is unique and determines a natural Lorentz metric on the Hermitian subspace of $\mathbf{H} \subset \mathbf{U} \otimes \overline{\mathbf{U}}$; this real vector space is also naturally endowed with a Clifford map $\gamma : \mathbf{H} \to \mathrm{End}(\mathbf{W})$, where $\mathbf{W} := \mathbf{U} \oplus \overline{\mathbf{U}}^\star$ is the 4-spinor space.

If now $\mathbf{S} \rightarrowtail \mathbf{M}$ is a complex vector bundle with 2-dimensional fibers, a linear connection Ξ on it determines linear connections G on $\mathbb{L}$, A on $\wedge^2\mathbf{U}$ and $\tilde\Gamma$ on $\mathbf{H}$. Moreover Ξ can be expressed in terms of these connections as

$$\Xi_{a\,B}^{\;A} = (G_a + i\,A_a)\delta^A_{\;B} + \frac{1}{2}\tilde\Gamma_{a\;\;BA^{\cdot}}^{\;AA^{\cdot}},$$

where capital latin indices refer to a frame of $\mathbf{U}$, and dotted indices to the corresponding frame of $\overline{\mathbf{U}}$.

Let now $\mathbf{M}$ be 4-dimensional. A *tetrad* is defined to be a linear morphism $\Theta : \mathrm{T}\mathbf{M} \to \mathbb{L} \otimes \mathbf{H}$. A non-singular field theory can be constructed [3] even if Θ is not required to be invertible everywhere; this requirement, however, is needed in order to recover a theory of the Einstein-Cartan-Maxwell-Dirac type. One finds that an invertible tetrad determines, by pull-back, a Lorentz metric on $\mathbf{M}$ and a metric connection on $\mathrm{T}\mathbf{M} \rightarrowtail \mathbf{M}$, as well as a Dirac map.

The fields of the above said theory are the 2-spinor connection Ξ, the tetrad Θ, the electromagnetic field F and the Dirac field $\psi : \mathbf{M} \rightsquigarrow \mathbb{L}^{-3/2} \otimes \mathbf{W}$. The gravitational field is represented by by Θ and $\tilde{\Gamma}$. The connection G induced on $\mathbb{L}$ is assumed to have vanishing curvature, $\mathrm{d}G = 0$, so that I can find local charts such that $G_a = 0$; this amounts to 'gauging away' the conformal 'dilaton' symmetry. Coupling constants arise as covariantly constants sections of $\mathbb{L}^r$ (r rational). One writes a natural Lagrangian which yields all the field equations: the Einstein equation and the equation for torsion; the equation $F = 2\,\mathrm{d}A$ (thus A is identified with the electromagnetic potential) and the other Maxwell equation; the Dirac equation.

REMARK 1. In standard gravity theory, allowing a non-zero torsion is somewhat questionable, since it can be considered just as a further field, separated from the spacetime connection. In the above sketched Einstein-Cartan-Maxwell-Dirac context, however, the spacetime connection is not a fundamental field but derives from the 2-spinor connection and the tetrad. These yield also the torsion, which couples to the Dirac field in such a way to become unavoidable.

Let us consider the Dirac equation with fixed background gravitational and electromagnetic fields. The Dirac Lagrangian (which we do not need here) yields the Euler-Lagrange operator $\mathcal{E}$, and the Dirac equation can be written as $\mathcal{E}\psi = 0$, where

$$\mathcal{E}\psi : \mathbf{M} \to \mathbb{L}^{3/2} \otimes \mathbb{Y}^* \underset{\mathbf{M}}{\otimes} \overline{\mathbf{W}}^{\star} .$$

Namely, the Euler-Lagrange operator can be viewed as sending generalized $\mathbb{L}^{-3/2} \otimes \mathbf{W}$-valued sections to generalized $\mathbb{L}^{3/2} \otimes \mathbf{W}^*$-valued densities (as we have a fixed background volume form η induced by the metric, the distinction between sections and densities is not essential here). It can be conveniently expressed through the *co-tetrad* $\breve{\Theta} := \Theta^{-1} \otimes \eta$ as

$$(\mathcal{E}\psi)_{\dot{A}} = \sqrt{2}\,\mathrm{i}\,\breve{\Theta}^a_{A\dot{A}}\,\nabla_a \psi^A - m\,\psi_{\dot{A}}\,\det\Theta + \tfrac{\mathrm{i}}{\sqrt{2}}\,T_{A\dot{A}}\,\psi^A ,$$

$$(\mathcal{E}\psi)^A = \sqrt{2}\,\mathrm{i}\,\breve{\Theta}^{aA\dot{A}}\,\nabla_a \psi_{\dot{A}} - m\,\psi^A\,\det\Theta + \tfrac{\mathrm{i}}{\sqrt{2}}\,T^{A\dot{A}}\,\psi_{\dot{A}} ,$$

where $m \in \mathbb{L}^{-1}$ and

$$T_{A\dot{A}} := \breve{\Theta}^a_{A\dot{A}}\,\breve{T}_a = \partial_a \breve{\Theta}^a_{A\dot{A}} + \tfrac{1}{2}\,(\breve{\Theta}^a_{B\dot{A}}\,\tilde{\Gamma}_a{}^{B\dot{C}}{}_{A\dot{C}} + \breve{\Theta}^a_{A\dot{B}}\,\tilde{\Gamma}_a{}^{C\dot{B}}{}_{C\dot{A}})$$

(spinor indices are raised and lowered through a normalized symplectic form ε, its inverse and their conjugates).

If $\phi \in \mathcal{D}_\circ(\mathbf{M}, \mathbb{L}^{3/2} \otimes \overline{\mathbf{W}})$ then $\langle \mathcal{E}\psi, \phi\rangle = \big\langle \psi, \mathcal{E}'\phi\big\rangle$, where $\mathcal{E}'\phi : \mathbf{M} \to \mathbb{L}^{3/2} \otimes \mathbb{Y}^* \otimes \mathbf{W}^\star$ has the local coordinate expression

$$(\mathcal{E}'\phi)_A = -\sqrt{2}\,\mathrm{i}\,\partial_a(\breve{\Theta}^a_{AA^\cdot}\phi^{A^\cdot}) - \sqrt{2}\,\mathrm{i}\,\breve{\Theta}^a_{AA^\cdot}\,\Xi_a{}^A{}_B\,\phi^{A^\cdot} - m\,\phi_A\,\det\Theta + \tfrac{\mathrm{i}}{\sqrt{2}}\,T_{AA^\cdot}\phi^{A^\cdot},$$

$$(\mathcal{E}'\phi)^{A^\cdot} = -\sqrt{2}\,\mathrm{i}\,\partial_a(\breve{\Theta}^{aAA^\cdot}\phi_A) + \sqrt{2}\,\mathrm{i}\,\breve{\Theta}^{aAA^\cdot}\,\bar{\Xi}_a{}^{B^\cdot}{}_{A^\cdot}\phi_A - m\,\phi^{A^\cdot}\det\Theta + \tfrac{\mathrm{i}}{\sqrt{2}}\,T^{AA^\cdot}\phi_A\,.$$

Now, our task is working on this expression in such a way to recognize it as representing an intrinsic object. Taking into account the above written expressions for $\Xi_a{}^A{}_B$ and $T_{AA^\cdot}$ we arrive, after some calculations, to

$$(\mathcal{E}'\phi)_A = -\sqrt{2}\,\mathrm{i}\,\breve{\Theta}^a_{AA^\cdot}\nabla_a\phi^{A^\cdot} - m\,\phi_A\,\det\Theta - \tfrac{\mathrm{i}}{\sqrt{2}}\,T_{AA^\cdot}\phi^{A^\cdot},$$

$$(\mathcal{E}'\phi)^{A^\cdot} = -\sqrt{2}\,\mathrm{i}\,\breve{\Theta}^{aAA^\cdot}\nabla_a\phi_A - m\,\phi^{A^\cdot}\det\Theta - \tfrac{\mathrm{i}}{\sqrt{2}}\,T^{AA^\cdot}\phi_A\,.$$

This formula has a straightforward interpretation. Consider the conjugate generalized section $\bar{\psi} : \mathbf{M} \rightsquigarrow \mathbb{L}^{3/2} \otimes \overline{\mathbf{W}}$ and note that the target bundle is the same as that of ϕ above (by the way, $\bar{\psi}$ can be identified with the 'Dirac adjoint' of ψ via the exchange map $\overline{\mathbf{W}} \equiv \overline{\mathbf{U}} \oplus \mathbf{U}^\star \to \mathbf{U}^\star \oplus \overline{\mathbf{U}} \equiv \mathbf{W}^\star$). Then

$$\mathcal{E}'\bar{\psi} = \bar{\mathcal{E}}\psi \equiv \overline{\mathcal{E}\psi},$$

namely $\mathcal{E}'$ is just the complex-conjugate operator of $\mathcal{E}$.

Note how this balance depends from the fact that the torsion is included in the operator. If one defines a new operator $\tilde{\mathcal{E}}$ by dropping the torsion terms, then one gets them doubled in the distributional adjoint $\tilde{\mathcal{E}}'$.

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