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RANDOM VARIABLES RELATED TO DISTRIBUTIONS OF SUMS MODULO AN INTEGER

Abstract. Starting from known results concerning distributions of sums modulo an integer, we build two random variables, $\xi_{(E,v)}$ (with $\emptyset \neq E \subseteq \mathcal{P},\ v \in \mathbb{N},\ v \geq 3$, $\mathcal{P}$ denoting the set of prime numbers) and $\eta_{(E,v)}$ (with $\emptyset \neq E \subseteq \mathcal{P}\backslash\{2\},\ v \in \mathbb{N},\ v \geq 2$) and find upper bounds for their expected values; the probability functions related to such random variables are also shown.

1. The functions ℓ_E and $\mathcal{L}_E$ $(E \subseteq \mathcal{P},\ E \neq \emptyset)$

There is a wide literature about congruence equations (see, for example, [1]) and in the last twenty years interesting related formulas and functions have been derived: among these, expressions giving the number of solutions of linear congruences. Counting such solutions has also nice relations with statistic and probabilistic problems like the distribution of the values taken by particular sums in $\mathbb{Z}_r$ $(r \in \mathbb{N})$, as we are going to see.

Two arithmetic functions ℓ_E and $\mathcal{L}_E$, labelled by a generic non-empty subset E of the set $\mathcal{P}$ of prime numbers, are known (see [4]; for a detailed discussion of the particular case $E = \mathcal{P}\backslash\{2\}$ see [3]) with the following properties. For given positive integers k and r, with all prime divisors of r lying in E, we set $\mathcal{D} = \{d \in \mathbb{N} : d|r\}$, consider two generic elements $\Delta = (d_1, d_2, ..., d_k) \in \mathcal{D}^k$ and $H = (h_1, h_2, ..., h_k) \in (\mathbb{Z}_r)^k$ and define the function

$$\begin{cases} \mathcal{S} = \mathcal{S}_{r,k}^{(H,\Delta)} : & \mathcal{D}_1 \times \mathcal{D}_2 \times ... \times \mathcal{D}_k & \to \mathbb{Z}_r \\ & (x_1, x_2, ..., x_k) & \mapsto \sum_{j=1}^{k} h_j x_j \ (mod\ r) \end{cases},$$

with $\mathcal{D}_j = \{x \in \mathbb{Z}_r : (x, r) = d_j\}$ for $j = 1, 2, ..., k$. For any fixed positive integer v we denote by $\mathcal{S}_v$ the set of functions $\mathcal{S}$ (each of which corresponds to a particular choice of r, k, H, Δ) such that for every prime $p \in \mathcal{D}$ we have $\sharp(\{j, 1 \leq j \leq k : p \nmid h_j d_j\}) \geq v$. For any $\mathcal{S} \in \mathcal{S}_v$ and $a \in \mathbb{Z}_r$, the integer $N_{\mathcal{S},a}$ denotes the number of solutions of the congruence equation $\mathcal{S}(x_1, x_2, ..., x_k) \equiv a\ (mod\ r)$ (formulas for $N_{\mathcal{S},a}$ in particular cases can be found in [2], [5] and [6]), while for each $\mathcal{S}$ we pose $\bar{N}_{\mathcal{S}} = \frac{1}{r} \sum_{a \in \mathbb{Z}_r} N_{\mathcal{S},a}$. The numbers $\ell_E = \ell_E(v)$ and $\mathcal{L}_E = \mathcal{L}_E(v)$ are then defined as the lower and the upper bounds, respectively, of the ratio $\frac{N_{\mathcal{S},a}}{\bar{N}_{\mathcal{S}}}$ for $\mathcal{S}$ ranging over $\mathcal{S}_v$

and a over $\mathbb{Z}_r$. The following equalities have been proven in [4]:

$$\ell_E(v) = \prod_{p\in E}\left[1 - \frac{1}{(p-1)^{s(v)}}\right]; \quad \mathcal{L}_E(v) = \prod_{p\in E}\left[1 + \frac{1}{(p-1)^{t(v)}}\right],$$

where, for each v, $s(v)$ and $t(v)$ are, respectively, the greatest even integer and the greatest odd integer not higher than v.

2. The random variable $\xi_{(E,v)}$ ($\emptyset \neq E \subseteq \mathcal{P},\ v \in \mathbb{N},\ v \geq 3$)

After fixing a non-empty subset E of $\mathcal{P}$ and an integer $v \geq 3$, for each $p \in E$ we can consider the number $\mathcal{L}_{\{p\}}(v) = 1 + \dfrac{1}{(p-1)^{t(v)}}$ and define the function

$$\begin{cases} \xi_{(E,v)}: & E \;\; \to \mathbb{R} \\ & p \;\; \mapsto [\ln \mathcal{L}_{\{p\}}(v)]/[\ln \mathcal{L}_E(v)] \end{cases}.$$

The following property holds.

PROPOSITION 1. *The function $\xi_{(E,v)}$ is a random variable over the set E.*

Proof. For any $p \in E$ we have $\mathcal{L}_E(v) \geq \mathcal{L}_{\{p\}}(v) > 1$; it follows that $\forall\, p \in E$, $\xi_{(E,v)}(p) = [\ln \mathcal{L}_{\{p\}}(v)]/[\ln \mathcal{L}_E(v)] > 0$. Furthermore, the following equalities are verified:

$$\sum_{p\in E} \xi_{(E,v)}(p) = \frac{\sum_{p\in E} \ln \mathcal{L}_{\{p\}}(v)}{\ln \mathcal{L}_E(v)} = \frac{\ln\left[\prod_{p\in E} \mathcal{L}_{\{p\}}(v)\right]}{\ln \mathcal{L}_E(v)} = 1.$$

Then $\xi_{(E,v)}$ is a random variable over E.

□

A function $Pr_{(\xi,E,v)}$, related to $\xi_{(E,v)}$, can be defined with range equal to the set of all subsets of E, by posing

$$\forall\, I \subseteq E,\ Pr_{(\xi,E,v)}(I) = \sum_{p\in I} \xi_{(E,v)}(p),$$

where the empty sum is considered equal to 0. The following relations are satisfied:

1. $\forall\, I \subseteq E,\ Pr_{(\xi,E,v)}(I) \geq 0$;
2. $\forall\, I,\ J \subseteq E\ with\ I \cap J = \emptyset,\ Pr_{(\xi,E,v)}(I \cup J) = Pr_{(\xi,E,v)}(I) + Pr_{(\xi,E,v)}(J)$;
3. $Pr_{(\xi,E,v)}(E) = 1$.

Then the function $Pr_{(\xi,E,v)}$ is a probability over the set E.

EXAMPLE 1. Let us take $E = \{2, 3, 5, 7\}$ and $v = 3$. We have:

$$\mathcal{L}_{\{2\}}(3) = 1 + \frac{1}{(2-1)^3} = 2; \quad \mathcal{L}_{\{3\}}(3) = 1 + \frac{1}{(3-1)^3} = \frac{9}{8};$$
$$\mathcal{L}_{\{5\}}(3) = 1 + \frac{1}{(5-1)^3} = \frac{65}{64}; \quad \mathcal{L}_{\{7\}}(3) = 1 + \frac{1}{(7-1)^3} = \frac{217}{216};$$

$$\mathcal{L}_E(3) = \prod_{p \in E} \mathcal{L}_{\{p\}}(3) = 2 \cdot \frac{9}{8} \cdot \frac{65}{64} \cdot \frac{217}{216} = \frac{14105}{6144}.$$

From such equalities we deduce that:

$$\xi_{(E,3)}(2) = \frac{\ln 2}{\ln(14105/6144)} \approx 0.8341; \quad \xi_{(E,3)}(3) = \frac{\ln(9/8)}{\ln(14105/6144)} \approx 0.1417;$$

$$\xi_{(E,3)}(5) = \frac{\ln(65/64)}{\ln(14105/6144)} \approx 0.0187; \quad \xi_{(E,3)}(7) = \frac{\ln(217/216)}{\ln(14105/6144)} \approx 0.0056.$$

The related probability function $Pr_{(\xi,E,3)}$ is defined as follows:

$$\begin{aligned}
&Pr_{(\xi,E,3)}(\emptyset) = 0;\\
&Pr_{(\xi,E,3)}(\{2\}) = \xi_{(E,3)}(2) \approx 0.8341;\\
&Pr_{(\xi,E,3)}(\{3\}) = \xi_{(E,3)}(3) \approx 0.1417;\\
&Pr_{(\xi,E,3)}(\{5\}) = \xi_{(E,3)}(5) \approx 0.0187;\\
&Pr_{(\xi,E,3)}(\{7\}) = \xi_{(E,3)}(7) \approx 0.0056;\\
&Pr_{(\xi,E,3)}(\{2, 3\}) = \xi_{(E,3)}(2) + \xi_{(E,3)}(3) \approx 0.9758;\\
&Pr_{(\xi,E,3)}(\{2, 5\}) = \xi_{(E,3)}(2) + \xi_{(E,3)}(5) \approx 0.8527;\\
&Pr_{(\xi,E,3)}(\{2, 7\}) = \xi_{(E,3)}(2) + \xi_{(E,3)}(7) \approx 0.8396;\\
&Pr_{(\xi,E,3)}(\{3, 5\}) = \xi_{(E,3)}(3) + \xi_{(E,3)}(5) \approx 0.1604;\\
&Pr_{(\xi,E,3)}(\{3, 7\}) = \xi_{(E,3)}(3) + \xi_{(E,3)}(7) \approx 0.1473;\\
&Pr_{(\xi,E,3)}(\{5, 7\}) = \xi_{(E,3)}(5) + \xi_{(E,3)}(7) \approx 0.0242;\\
&Pr_{(\xi,E,3)}(\{2, 3, 5\}) = \xi_{(E,3)}(2) + \xi_{(E,3)}(3) + \xi_{(E,3)}(5) \approx 0.9944;\\
&Pr_{(\xi,E,3)}(\{2, 3, 7\}) = \xi_{(E,3)}(2) + \xi_{(E,3)}(3) + \xi_{(E,3)}(7) \approx 0.9813;\\
&Pr_{(\xi,E,3)}(\{2, 5, 7\}) = \xi_{(E,3)}(2) + \xi_{(E,3)}(5) + \xi_{(E,3)}(7) \approx 0.8583;\\
&Pr_{(\xi,E,3)}(\{3, 5, 7\}) = \xi_{(E,3)}(3) + \xi_{(E,3)}(5) + \xi_{(E,3)}(7) \approx 0.1659;\\
&Pr_{(\xi,E,3)}(\{2, 3, 5, 7\}) = \xi_{(E,3)}(2) + \xi_{(E,3)}(3) + \xi_{(E,3)}(5) + \xi_{(E,3)}(7) = 1.
\end{aligned}$$

In Example 1, the value $Pr_{(\xi,E,3)}(\{2\})$ is much larger than $Pr_{(\xi,E,3)}(\{3, 5, 7\})$. This is a particular case of a general characteristic of the probability functions $Pr_{(\xi,E,v)}$: from the definitions of $\xi_{(E,v)}$ and $Pr_{(\xi,E,v)}$, one can easily deduce that the values $\xi_{(E,v)}(p)$ decrease fast as p grows.

3. The expected value of $\xi_{(E,v)}$

E and v being fixed, the real number $H(\xi_{(E,v)}) = \sum_{p\in E}[p \cdot \xi_{(E,v)}(p)]$ is called expected value of the random variable $\xi_{(E,v)}$. For any E and v we have

$$H(\xi_{(E,v)}) = \sum_{p\in E}\left[p \cdot \frac{\ln\left(1+\frac{1}{(p-1)^{t(v)}}\right)}{\ln \mathcal{L}_E(v)}\right] = \sum_{p\in E}\frac{\ln\left[\left(1+\frac{1}{(p-1)^{t(v)}}\right)^{p}\right]}{\ln \mathcal{L}_E(v)}$$

$$= \sum_{p\in E}\frac{\left[\ln\left(1+\frac{1}{(p-1)^{t(v)}}\right)\right]+\left[\ln\left(\left(1+\frac{1}{(p-1)^{t(v)}}\right)^{p-1}\right)\right]}{\ln \mathcal{L}_E(v)}$$

$$= 1+\sum_{p\in E}\frac{\ln\left[\left(1+\frac{1}{(p-1)^{t(v)}}\right)^{p-1}\right]}{\ln \mathcal{L}_E(v)}$$

$$= 1+\sum_{p\in E}\frac{\ln \sqrt[(p-1)^{t(v)-1}]{\left(1+\frac{1}{(p-1)^{t(v)}}\right)^{[(p-1)^{t(v)}]}}}{\ln \mathcal{L}_E(v)}.$$

Then

$$1+\sum_{p\in E}\frac{\ln \sqrt[(p-1)^{t(v)-1}]{2}}{\ln \mathcal{L}_E(v)} \leq H(\xi_{(E,v)}) < 1+\sum_{p\in E}\frac{\ln \sqrt[(p-1)^{t(v)-1}]{e}}{\ln \mathcal{L}_E(v)},$$

i.e.

$$(1)\qquad 1+\frac{\left(\sum_{p\in E}\frac{1}{(p-1)^{t(v)-1}}\right)\cdot \ln 2}{\ln \mathcal{L}_E(v)} \leq H(\xi_{(E,v)}) < 1+\frac{\sum_{p\in E}\frac{1}{(p-1)^{t(v)-1}}}{\ln \mathcal{L}_E(v)}.$$

In particular

$$(2)\qquad H(\xi_{(E,v)}) < 1+\frac{\zeta(t(v)-1)}{\ln \mathcal{L}_E(v)},$$

ζ denoting Riemann's function. Relation (2) implies that for any choice of E and v the value $H(\xi_{(E,v)})$ is finite.

EXAMPLE 2. Let us take $E = \{2, 3, 5, 7\}$ and $v = 3$. By using appropriate approximations of the values taken by $\xi_{(E,3)}$, we can write $H(\xi_{(E,3)}) = 2 \cdot \xi_{(E,3)}(2) +$

$3 \cdot \xi_{(E,3)}(3) + 5 \cdot \xi_{(E,3)}(5) + 7 \cdot \xi_{(E,3)}(7) \approx 2.2255$. This result agrees with inequalities (1), which tell us that $H(\xi_{(E,3)}) > 1 + \dfrac{193/144}{\ln(14105/6144)} \cdot \ln 2 > 2.1178$ and that $H(\xi_{(E,3)}) < 1 + \dfrac{193/144}{\ln(14105/6144)} < 2.6128$. In this case, since the primes of E are small, the value $H(\xi_{(E,3)})$ is much closer to the lower bound established by (1) than to the corresponding upper bound.

If E is fixed with $2 \in E$, we can observe that for any $v \geq 3$ we have $\mathcal{L}_{\{2\}}(v) = 2$ and, if $E \neq \{2\}$, $\mathcal{L}_E(v) = \mathcal{L}_{\{2\}}(v) \cdot \mathcal{L}_{E\setminus\{2\}}(v) = 2 \cdot \mathcal{L}_{E\setminus\{2\}}(v) > 2$. Therefore, from inequality (2) we deduce

$$H(\xi_{(E,v)}) < 1 + \frac{\zeta(t(v)-1)}{\ln 2} \quad \forall\, v \in \mathbb{N},\ v \geq 3. \tag{3}$$

For the particular case $v = 3$ we obtain $H(\xi_{(E,3)}) < 1 + \dfrac{\zeta(2)}{\ln 2} < 3.3732$. For generic v, if we cannot calculate a sufficiently good approximation of $\zeta(t(v)-1)$, we can utilise the inequality $\zeta(r) < \dfrac{r}{r-1}$ (holding for any $r \in \mathbb{R}$, $r > 1$), and from inequalities (3) we derive the weaker relations

$$H(\xi_{(E,v)}) < 1 + \frac{t(v)-1}{(t(v)-2)\ln 2}.$$

Now let us fix $v \in \mathbb{N}$, $v \geq 3$ and a generic E with $\emptyset \neq E \subseteq \mathcal{P}\setminus\{2\}$; let us pose $m(E) = \min(E)$. Since $\mathcal{L}_E(v) \geq \mathcal{L}_{\{m(E)\}}(v)$, from the second of ineqs. (1) we can obtain

$$H(\xi_{(E,v)}) < 1 + \frac{\displaystyle\sum_{p \in E} \frac{1}{(p-1)^{t(v)-1}}}{\ln \mathcal{L}_E(v)} < 1 + \frac{\displaystyle\sum_{n=m(E)-1}^{+\infty} \frac{1}{n^{t(v)-1}}}{\ln \mathcal{L}_{\{m(E)\}}(v)} < 1 + \tag{4}$$

$$+ \frac{\displaystyle\int_{m(E)-2}^{+\infty} x^{1-t(v)}\,dx}{\ln \mathcal{L}_{\{m(E)\}}(v)} = 1 + \frac{1}{(t(v)-2) \cdot (m(E)-2)^{t(v)-2} \cdot \ln \mathcal{L}_{\{m(E)\}}(v)}.$$

Moreover, we recall the relation $\mathcal{L}_{\{m(E)\}}(v) = 1 + \dfrac{1}{(m(E)-1)^{t(v)}}$ and observe that, since for any positive $x \in \mathbb{R}$ the inequality $\ln(1+x) > \dfrac{x}{1+x}$ is verified, replacing x by $\dfrac{1}{(m(E)-1)^{t(v)}}$ gives rise to the relation $\ln \mathcal{L}_{\{m(E)\}}(v) > \dfrac{1}{(m(E)-1)^{t(v)}+1}$. This fact, together with relations (4), implies that

$$H(\xi_{(E,v)}) < 1 + \frac{(m(E)-1)^{t(v)}+1}{(t(v)-2)(m(E)-2)^{t(v)-2}}.$$

4. The random variable $\eta_{(E,v)}$ $(\emptyset \neq E \subseteq \mathcal{P}\backslash\{2\},\ v \in \mathbb{N},\ v \geq 2)$

For E and v fixed with $\emptyset \neq E \subseteq \mathcal{P}\backslash\{2\}$ and $v \in \mathbb{N}$, $v \geq 2$, we can also define a function $\eta_{(E,v)} : E \to \mathbb{R}$ related to the values $\ell_{\{p\}}(v)$ with $p \in E$. Let us set, for each prime $p \in E$,

$$\eta_{(E,v)}(p) = [\ln \ell_{\{p\}}(v)]/[\ln \ell_E(v)].$$

For any $p \in E$, since $\ell_E(v) \leq \ell_{\{p\}}(v) < 1$, we have $\ln \ell_E(v) \leq \ln \ell_{\{p\}}(v) < 0$, which implies $0 < \eta_{(E,v)}(p) \leq 1$. Moreover, we can write

$$\sum_{p \in E} \eta_{(E,v)}(p) = \frac{\sum_{p \in E} \ln \ell_{\{p\}}(v)}{\ln \ell_E(v)} = \frac{\ln \left[\prod_{p \in E} \ell_{\{p\}}(v) \right]}{\ln \ell_E(v)} = 1.$$

We have so proven that $\eta_{(E,v)}$ is a random variable over the set E.

A function $Pr_{(\eta,E,v)}$, defined on the set of all subsets of E, is obtained from $\eta_{(E,v)}$ by posing

$$\forall\, I \subseteq E,\ Pr_{(\eta,E,v)}(I) = \sum_{p \in I} \eta_{(E,v)}(p);$$

$Pr_{(\eta,E,v)}$ is easily verified to be a probability over E.

EXAMPLE 3. Let us take $E = \{3,\ 5,\ 7,\ 11\}$ and $v = 2$. We have:

$$\ell_{\{3\}}(2) = 1 - \frac{1}{(3-1)^2} = \frac{3}{4}; \quad \ell_{\{5\}}(2) = 1 - \frac{1}{(5-1)^2} = \frac{15}{16};$$
$$\ell_{\{7\}}(2) = 1 - \frac{1}{(7-1)^2} = \frac{35}{36}; \quad \ell_{\{11\}}(2) = 1 - \frac{1}{(11-1)^2} = \frac{99}{100};$$

$$\ell_E(2) = \prod_{p \in E} \ell_{\{p\}}(2) = \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdot \frac{99}{100} = \frac{693}{1024}.$$

Such equalities imply that:

$$\eta_{(E,2)}(3) = \frac{\ln(3/4)}{\ln(693/1024)} \approx 0.7368; \quad \eta_{(E,2)}(5) = \frac{\ln(15/16)}{\ln(693/1024)} \approx 0.1653;$$

$$\eta_{(E,2)}(7) = \frac{\ln(35/36)}{\ln(693/1024)} \approx 0.0722; \quad \eta_{(E,2)}(11) = \frac{\ln(99/100)}{\ln(693/1024)} \approx 0.0257.$$

From all this we deduce that:

$Pr_{(\eta,E,2)}(\emptyset) = 0;$
$Pr_{(\eta,E,2)}(\{3\}) = \eta_{(E,2)}(3) \approx 0.7368;$
$Pr_{(\eta,E,2)}(\{5\}) = \eta_{(E,2)}(5) \approx 0.1653;$
$Pr_{(\eta,E,2)}(\{7\}) = \eta_{(E,2)}(7) \approx 0.0722;$
$Pr_{(\eta,E,2)}(\{11\}) = \eta_{(E,2)}(11) \approx 0.0257;$
$Pr_{(\eta,E,2)}(\{3,\ 5\}) = \eta_{(E,2)}(3) + \eta_{(E,2)}(5) \approx 0.9021;$
$Pr_{(\eta,E,2)}(\{3,\ 7\}) = \eta_{(E,2)}(3) + \eta_{(E,2)}(7) \approx 0.8090;$
$Pr_{(\eta,E,2)}(\{3,\ 11\}) = \eta_{(E,2)}(3) + \eta_{(E,2)}(11) \approx 0.7626;$
$Pr_{(\eta,E,2)}(\{5,\ 7\}) = \eta_{(E,2)}(5) + \eta_{(E,2)}(7) \approx 0.2374;$
$Pr_{(\eta,E,2)}(\{5,\ 11\}) = \eta_{(E,2)}(5) + \eta_{(E,2)}(11) \approx 0.1910;$
$Pr_{(\eta,E,2)}(\{7,\ 11\}) = \eta_{(E,2)}(7) + \eta_{(E,2)}(11) \approx 0.0979;$
$Pr_{(\eta,E,2)}(\{3,\ 5,\ 7\}) = \eta_{(E,2)}(3) + \eta_{(E,2)}(5) + \eta_{(E,2)}(7) \approx 0.9743;$
$Pr_{(\eta,E,2)}(\{3,\ 5,\ 11\}) = \eta_{(E,2)}(3) + \eta_{(E,2)}(5) + \eta_{(E,2)}(11) \approx 0.9278;$
$Pr_{(\eta,E,2)}(\{3,\ 7,\ 11\}) = \eta_{(E,2)}(3) + \eta_{(E,2)}(7) + \eta_{(E,2)}(11) \approx 0.8347;$
$Pr_{(\eta,E,2)}(\{5,\ 7,\ 11\}) = \eta_{(E,2)}(5) + \eta_{(E,2)}(7) + \eta_{(E,2)}(11) \approx 0.2632;$
$Pr_{(\eta,E,2)}(\{3,\ 5,\ 7,\ 11\}) = \eta_{(E,2)}(3) + \eta_{(E,2)}(5) + \eta_{(E,2)}(7) + \eta_{(E,2)}(11) = 1.$

Similarly with respect to example 1, the value $Pr_{(\eta,E,2)}(\{3\})$ found in example 3 is much larger than $Pr_{(\eta,E,2)}(\{5,\ 7,\ 11\})$, and in general the values $\eta_{(E,v)}(p)$ decrease fast when p grows.

5. The expected value of $\eta_{(E,v)}$

After fixing E and v, let us consider the expected value of $\eta_{(E,v)}$, i.e. the real number $H(\eta_{(E,v)}) = \sum_{p\in E}[p \cdot \eta_{(E,v)}(p)]$. We have

$$H(\eta_{(E,v)}) = \sum_{p\in E}\left[p \cdot \frac{\ln\left(1 - \dfrac{1}{(p-1)^{s(v)}}\right)}{\ln \ell_E(v)}\right] = \sum_{p\in E}\frac{\ln\left[\left(1 - \dfrac{1}{(p-1)^{s(v)}}\right)^{p}\right]}{\ln \ell_E(v)}$$

$$= \sum_{p\in E}\frac{\left[\ln\left(1 - \dfrac{1}{(p-1)^{s(v)}}\right)\right] + \left[\ln\left(\left(1 - \dfrac{1}{(p-1)^{s(v)}}\right)^{p-1}\right)\right]}{\ln \ell_E(v)}$$

$$= 1 + \sum_{p\in E}\frac{\left|\ln\left[\left(1 - \dfrac{1}{(p-1)^{s(v)}}\right)^{p-1}\right]\right|}{\ln[\ell_E(v)^{-1}]}$$

$$= 1 + \sum_{p \in E} \frac{\left| \ln \sqrt[(p-1)^{s(v)-1}]{\left(1 - \frac{1}{(p-1)^{s(v)}}\right)^{[(p-1)^{s(v)}]}} \right|}{\ln[\ell_E(v)^{-1}]}.$$

Then

$$(5) \quad 1 + \frac{\sum_{p \in E} \frac{1}{(p-1)^{s(v)-1}}}{\ln[\ell_E(v)^{-1}]} < H(\eta_{(E,v)}) \leq 1 + \frac{4\ln(4/3) \cdot \left(\sum_{p \in E} \frac{1}{(p-1)^{s(v)-1}}\right)}{\ln[\ell_E(v)^{-1}]}.$$

In particular for $v \geq 4$ we have

$$(6) \qquad H(\eta_{(E,v)}) < 1 + \frac{4\ln(4/3) \cdot [\zeta(s(v)-1) - 1]}{\ln[\ell_E(v)^{-1}]}.$$

Inequality (6) (or, equivalently, the second of ineqs. (5)) proves that for any pair (E, v) with $v \geq 4$ the value $H(\eta_{(E,v)})$ is finite. For any (E, v) with $v \leq 3$, from relations (5) it follows that $H(\eta_{(E,v)})$ is finite if and only if the value of the series $\sum_{p \in E} \frac{1}{p-1}$ is finite.

EXAMPLE 4. For $E = \{3, 5, 7, 11\}$, let us calculate the value $H(\eta_{(E,2)})$. We obtain: $H(\eta_{(E,2)}) = 3 \cdot \eta_{(E,2)}(3) + 5 \cdot \eta_{(E,2)}(5) + 7 \cdot \eta_{(E,2)}(7) + 11 \cdot \eta_{(E,2)}(11) \approx 3.8251$. This result agrees with ineqs. (5), which tell us that $H(\eta_{(E,2)}) > 1 + \frac{61/60}{\ln(1024/693)} > 3.6038$ and $H(\eta_{(E,2)}) \leq 1 + \frac{4\ln(4/3) \cdot (61/60)}{\ln(1024/693)} < 3.9964$. In this case, the primes of E are small and hence the value $H(\eta_{(E,2)})$ is closer to the upper bound established by (5) than to the corresponding lower bound.

For any pair (E, v) with $v \geq 4$, starting from the second of ineqs. (5) and using a method like the one at the end of section 3, we can deduce that

$$H(\eta_{(E,v)}) < 1 + \frac{4\ln(4/3) \cdot (m(E)-1)^{s(v)}}{(s(v)-2)(m(E)-2)^{s(v)-2}},$$

where $m(E) = \min(E)$.

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