G. Milovanovic informed us that the solution to Problem 7 is known. See https://arxiv.org/abs/2104.02348v2 for a revised version of our paper, which contains the following text and references instead of Problem 7.

The Bernstein-type version of (41)/(43) was found by A. Guessab and G. V. Milovanovic [1] much earlier and actually in a stronger form: if $w(x) = (1+x)^\alpha(1-x)^\beta$, $\alpha, \beta > -1$, is a Jacobi weight, then

$$\left(\int_{-1}^{1} |\sqrt{1-x^2}P_n'(x)|^2 w(x)\,\mathrm{d}x\right)^{1/2} \leq \sqrt{n(n+1+\alpha+\beta)}\left(\int_{-1}^{1} |P_n(x)|^2 w(x)\,\mathrm{d}x\right)^{1/2},$$

with equality for the corresponding Jacobi polynomial of degree n. Remarkably, [1] also contains the analogue of this inequality for higher derivatives as well with precise constants for all n.

In the $\alpha = \beta = -1/2$ case this can be written in the somewhat less precise form

$$\left(\int_{-1}^{1} \left|\sqrt{1-x^2}P_n'(x)\right|^2 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x\right)^{1/2} \leq n(1+o(1))\left(\int_{-1}^{1} |P_n(x)|^2 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x\right)^{1/2}, \tag{1}$$

and this form has an extension to other L^p spaces and to several intervals (see [2]): let $E \subset \mathbf{R}$ be a compact set consisting of non-degenerate intervals. Then for $1 \leq p < \infty$ and for algebraic polynomials P_n of degree $n = 1, 2, \ldots$ we have

$$\left(\int_E \left|\frac{P_n'(x)}{\pi\omega_E(x)}\right|^p \omega_E(x)\,\mathrm{d}x\right)^{1/p} \leq n(1+o(1))\left(\int_E |P_n(x)|^p \omega_E(x)\,\mathrm{d}x\right)^{1/p},$$

and this is precise in the usual sense. Note that if $E = [-1, 1]$, then $\pi\omega_E(x) = 1/\sqrt{1-x^2}$, so in this case this inequality reduces to (1) for $p = 2$.

References

[1] A. Guessab and G. V. Milovanovic, Weighted L^2-analogous of Bernstein's inequality and classical orthogonal polynomials. *J. Math. Anal. Appl.*, **182**(1994), 244–249.

[2] B. Nagy and F. Toókos, Bernstein inequality in L^α norms. *Acta Sci. Math. (Szeged)*, **79**(2013), 129–174.