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HOLONOMIC AND SEMI-HOLONOMIC GEOMETRIES

by

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Abstract. — Holonomic and semi-holonomic geometries modelled on a homogeneous space G/P are introduced as reductions of the holonomic or semi-holonomic frame bundles respectively satisfying a straightforward generalization of the partial differential equation characterizing torsion–free linear connections. Under a suitable regularity assumption on the model space G/P we establish an equivalence of categories between Cartan geometries and semi-holonomic geometries modelled on G/P.

Résumé **(Géométries holonomes et semi–holonomes).** — On introduit les géométries holonomes et semi–holonomes modelées sur un espace homogène G/P comme réductions des fibrés de repères holonomes et semi–holonomes vérifiant une généralisation de l'équation aux dérivées partielles caractérisant les connexions linéaires sans torsion. Sous certaines conditions de régularité sur l'espace modèle G/P, nous établissons une équivalence de catégories entre les géométries de Cartan et les géométries semi–holonomes modelées sur G/P.

1. Introduction

The study of geometric structures with finite dimensional isometry groups has ever made up an important part of differential geometry and is intimately related with the notions of connections and principal bundles, coined by Cartan in order to give an interpretation of Lie's ideas on geometry. Principal bundles are undoubtedly useful in the study of geometric structures on manifolds, nevertheless one should not fail to notice the problematic and somewhat paradox aspect of their use. In fact the frame bundles of a manifold M are defined as jet bundles, with a single projection to M, say the target projection, but we have to keep track of the source projection, too. From the point of view of exterior calculus on principal bundles there is a natural way to work around this problem, needless to say it was Cartan who first treated the classical examples of geometric structures along these lines of thought, which have by now become standard. The paradox itself however remains and its impact is easily

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noticed when turning to more general geometric structures, say geometries modelled on homogeneous spaces G/P.

Analysis on homogeneous spaces G/P is well understood and it is tempting to generalize this analysis to curved analogues of the flat model space G/P. In particular the extension problem for invariant differential operators studied in conformal and more general parabolic geometries only makes sense in this context. Cartan's original definition [**C**] of Cartan geometries as curved analogues of homogeneous spaces G/P relies on the existence of an auxiliary principal bundle $\mathcal{G}$ on a manifold M. Unless we are content with studying pure Cartan geometries we need to discover the geometry first in order to establish the existence of the principal bundle. In fact most Cartan geometries arise via Cartan's method of equivalence in the process of classifying underlying geometric structures interesting in their own right. In this respect the work of Tanaka [**T**] has been most influential, who introduced parabolic Cartan geometries to classify regular differential systems with simple automorphism groups.

An alternative, but essentially equivalent definition of a curved analogue of a homogeneous space is introduced in this note. Holonomic and semi-holonomic geometries modelled on a homogeneous space G/P will be reductions of the holonomic or semi-holonomic frame bundles $\mathbf{GL}^{d}M$ or $\overline{\mathbf{GL}}^{d}M$ of M satisfying a suitable partial differential equation, which is a straightforward generalization of the partial differential equation characterizing torsion–free linear connections as reductions of $\mathbf{GL}^{2}M$ to the structure group $\mathbf{GL}^{1}\mathbb{R}^n \subset \mathbf{GL}^{2}\mathbb{R}^n$. The critical step in the formulation of this partial differential equation is the construction of a map similar to

$$\mathcal{J}: \quad \mathbf{O}\,\mathbb{R}^n\backslash\mathbf{GL}^{2}\mathbb{R}^n \;\longrightarrow\; \mathrm{Jet}^1_0(\mathbf{O}\,\mathbb{R}^n\backslash\mathbf{GL}^{1}\mathbb{R}^n)$$

in Riemannian and

$$\mathcal{J}: \quad \mathbf{CO}\,\mathbb{R}^n \ltimes \mathbb{R}^{n*}\backslash\mathbf{GL}^{2}\mathbb{R}^n \;\longrightarrow\; \mathrm{Jet}^1_0(\mathbf{CO}\,\mathbb{R}^n\backslash\mathbf{GL}^{1}\mathbb{R}^n)$$

in conformal geometry. The classical construction of $\mathcal{J}$ applies only for affine geometries, i. e. geometries modelled on quotients of the form $P \ltimes \mathfrak{u}/P$, where the semidirect product is given by some linear representation of P on $\mathfrak{u}$. In non–affine geometries the straightforward map $\mathbf{GL}^{d+1}\mathbb{R}^n \longrightarrow \mathrm{Jet}^1_0\mathbf{GL}^{d}\mathbb{R}^n$ fails in general to descend to quotients. In particular this problem arises in split geometries, which are of particular interest in differential geometry. Split geometries are modelled on homogeneous spaces G/P, such that some subgroup $U \subset G$ acts simply transitively on an open, dense subset of G/P. A couple of talks at the conference in Luminy centered about parabolic geometries, which form a class of examples of split geometries interesting in its own right due to the existence of the Bernstein–Gelfand–Gelfand resolution [**BE**], [**CSS**].

Without loss of generality we will assume that the model space G/P is connected, i. e. every connected component of G meets P. However G/P will have to satisfy a technical regularity assumption in order to be able to construct holonomic and semi-holonomic geometries modelled on G/P. Choose a linear complement $\mathfrak{u}$ of $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ and consider the corresponding exponential coordinates of G/P:

$$\exp: \quad \mathfrak{u} \;\longrightarrow\; G/P, \quad v \;\longmapsto\; e^{v}P$$

The action of the isotropy group P of $\exp(0) = eP$ in these exponential coordinates gives rise to a group homomorphism $\Phi_{\mathfrak{u}} : P \longrightarrow \mathbf{GL}^k\mathfrak{u}$ from P to the group $\mathbf{GL}^k\mathfrak{u}$ of k–th order jets of diffeomorphisms of $\mathfrak{u}$ into itself fixing $0 \in \mathfrak{u}$. We require that the image of P is closed in $\mathbf{GL}^k\mathfrak{u}$ for all $k \geq 1$, a condition evidently independent of the choice of $\mathfrak{u}$. This regularity assumption is certainly met by all pairs of algebraic groups, but it does not hold in general, perhaps the simplest counterexample is the affine geometry modelled on $\mathbb{R} \ltimes (\mathbb{C} \oplus \mathbb{C})/\mathbb{R}$ with $\mathbb{R}$ acting on $\mathbb{C} \oplus \mathbb{C}$ by an irrational line in $S^1 \times S^1$. In general neither of the homomorphisms $P \longrightarrow \mathbf{GL}^k\mathfrak{u}$, $k \geq 1$, needs be injective, however the intersection of all their kernels is a closed normal subgroup P_∞ of P called the isospin group of P in G. Alternatively P_∞ can be characterized as the kernel of the homomorphism $G \longrightarrow \operatorname{Diff} G/P$.

In the absence of isospin $P_\infty = \{1\}$ Morimoto [**M**] constructed a P–equivariant embedding of a Cartan geometry $\mathcal{G}$ on a manifold M into the infinite frame bundle $\mathcal{G} \longrightarrow \overline{\mathbf{GL}}^\infty M$. The main result of the current note is a generalization of this result, which provides a complete classification of Cartan geometries $\mathcal{G}$ on M modelled on G/P in terms of semi-holonomic geometries of sufficiently high order:

Theorem 1.1. — *Consider a connected homogeneous quotient G/P of a finite dimensional Lie group G by a closed subgroup P such that the image of P in $\mathbf{GL}^k\mathfrak{u}$ is closed for all $k \geq 1$. There exists an integer $d \geq 0$ depending only on the pair of Lie algebras $\mathfrak{g} \supset \mathfrak{p}$ such that every Cartan geometry $\mathcal{G}$ on M is an isospin P_∞–bundle over a unique semi-holonomic geometry $\mathcal{G}/P_\infty \subset \overline{GL}^{d+1}M$ of order $d+1$ modelled on G/P. The semi-holonomic geometry fixes the Cartan connection on $\mathcal{G}$ up to an affine subspace of isospin connections.*

Consequently in the absence of isospin $P_\infty = \{1\}$ there is a natural correspondence between Cartan geometries and semi-holonomic geometries of order $d+1$ on M establishing an equivalence of the respective categories. The actual proof of Theorem 1.1 is very simple once we forget everything we learned about the canonical connection etc. on frame bundles. The explanation for the need to introduce an auxiliary bundle in the original definition of Cartan geometries seems to be that people clinged to the concept of "canonical" translations, because it fitted so neatly with exterior calculus, instead of taking the problematic aspect of principal bundles in geometry at face value.

It is a striking fact that no classical example is known where the integer d in Theorem 1.1 is different from $d = 1$ or $d = 2$. In fact the relationship between Cartan geometries and holonomic geometries should become very interesting for examples with $d > 2$. A partial negative result in this direction is given in Lemma 4.4 showing that all examples with reductive G have $d \leq 2$.

Perhaps the most important aspect of Theorem 1.1 is that it associates a classifying geometric object and thus local covariants to any Cartan geometry without any artificial assumptions on the model space G/P. In particular the techniques available in the formal theory of partial differential equations or exterior differential systems [$\mathbf{BCG}^3$] can be used to describe the space of local solutions to the partial differential equation characterizing holonomic and semi-holonomic geometries. The most ambitious program is to derive the complete resolution of the space of local covariants and we hope to return to this project in [**W**]. The methods and results of Tanaka [**T**] and Yamaguchi [**Y**] for parabolic geometries will certainly find their place in the more general context of split geometries.

In the following section we will review the fundamentals of jet theory with particular emphasis on the delicate role played by the translations in order to construct the map $\mathcal{J}$ for all model spaces G/P. Moreover we will review the notion of torsion in this section, because similar to the map $\mathcal{J}$ the most intuitive definition of torsion depends on the choice of translations. This example is particularly interesting, because it contradicts the usual definition of torsion as the exterior derivative of the soldering form and may serve as a sample calculation showing the way the translations affect the relevant formulas in exterior calculus.

Using the map $\mathcal{J}$ we set up the partial differential equation characterizing holonomic and semi-holonomic reductions of the holonomic and semi-holonomic frame bundles $\mathbf{GL}^{d}M$ and $\overline{\mathbf{GL}}^{d}M$ respectively. In particular we will provide stable versions of these partial differential equations, a problem we thought about at the time of the conference in Luminy. Moreover we will discuss what kind of connections are associated with holonomic and semi-holonomic reductions. In the final section we prove Theorem 1.1 and thus establish an equivalence of categories between the category of Cartan geometries and the category of semi-holonomic geometries of sufficiently high order.

I would like to thank the organizers of the conference for inviting me to Luminy and giving me extra time to finish this note. Moreover the discussions with Jan Slovák and Lukáš Krump in Luminy turned my attention to the local covariant problem in pure Cartan geometry. My special thanks are due to Tammo Diemer, who introduced me to conformal geometry and the related extension problem for invariant differential operators.

2. Jet Theory and Principal Bundles

The language of jet theory will dominate the following sections, most of the ideas and definitions will emerge from this way of expressing calculus. Since there are numerous text books on this subject it is needless to strive for a detailed introduction, see [**KMS**], [**P**] for further reference. For the convenience of the reader we want to recall the basic concepts and definitions of jet theory and discuss its interplay with the theory of principal bundles. In particular we want to point out the problematic aspect of using principal bundles in the description of jets of geometric structures on manifolds. In order to get a well defined projection from a principal bundle to the base manifold we have to fix say the target of a jet, however we have to keep track of its source, too.

Perhaps the cleanest way around this problem is to discard principal bundles and turn to groupoid–like structures. In fact the description of geometric structures on manifolds using groupoids or better Lie pseudogroups has a long history originating from Lie and predating the concept of principal bundles by decades, see [**P**] for an enthusiastic and in parts rather polemical historical survey. On the other hand the use of principal bundles has a tremendous advantage over the use of groupoids, we really can do calculations without the need to resort to local coordinates and the powerful algebraic machinery of resolutions by induced modules becomes available in this context.

There is a standard recipe to deal with this dichotomy and it works remarkably well in affine and other important geometries. Moreover it links neatly with exterior calculus on principal bundles pioneered by Cartan. In this note we will explore variants of the standard recipe depending in geometrical language on the choice of translations. Although these variants may look somewhat artificial from the point of view of exterior calculus they allow us to deal easily not only with affine but with all split geometries. A striking example is Lemma 2.5, which essentially reproduces the definition of torsion in Cartan geometries without any reference to connections at all. The modifications in the definitions needed in general geometries modelled on homogeneous spaces G/P will appear in [**W**].

The main object of study in jet theory is of course a jet, which is a generalization of the concept of a Taylor series associated to a smooth map $\mathbb{R} \longrightarrow \mathbb{R}$ to arbitrary smooth maps between manifolds. Let $\mathfrak{u}$ be a fixed real vector space and $\mathcal{F}$ some differentiable manifold. Two smooth maps $f : \mathfrak{u} \longrightarrow \mathcal{F}$ and $\tilde{f} : \mathfrak{u} \longrightarrow \mathcal{F}$ defined in some neighborhood of $0 \in \mathfrak{u}$ are called equivalent $f \sim \tilde{f}$ up to order $k \geq 0$ if $f(0) = \tilde{f}(0)$ and their partial derivatives up to order k in some and hence every local coordinate system of $\mathcal{F}$ about $f(0) = \tilde{f}(0)$ agree in 0. The equivalence class of a smooth map f up to order k is called the k–th order jet $\mathrm{jet}^k_0 f$ of f and the set of all these equivalence classes is denoted by $\mathrm{Jet}^k_0 \mathcal{F} := \{\mathrm{jet}^k_0 f |\ f : \mathfrak{u} \longrightarrow \mathcal{F}\ \}$. For all $k \geq l \geq 0$ there is a canonical projection

$$\mathrm{pr}: \quad \mathrm{Jet}^k_0 \mathcal{F} \longrightarrow \mathrm{Jet}^l_0 \mathcal{F}, \quad \mathrm{jet}^k_0 f \longmapsto \mathrm{jet}^l_0 f$$

and the evaluation

$$\mathrm{ev}: \quad \mathrm{Jet}^k_0 \mathcal{F} \longrightarrow \mathcal{F}, \quad \mathrm{jet}^k_0 f \longmapsto f(0)$$

which strictly speaking is a special case of the projection since we may identify $\mathrm{Jet}^0_0 \mathcal{F} \cong \mathcal{F}$. We will use a different notation for this special case nevertheless to avoid the cumbersome indication of the source and target orders of the projections. If the manifold $\mathcal{F}$ comes along with a distinguished base point $\{*\}$ the jets of pointed smooth maps $f : \mathfrak{u} \longrightarrow \mathcal{F}$ make up the subset of all reduced or pointed jets ${}^*\mathrm{Jet}^k_0 \mathcal{F} = \{\mathrm{jet}^k_0 f |\ f(0) = *\} \subset \mathrm{Jet}^k_0 \mathcal{F}$, which is just the preimage $\mathrm{ev}^{-1}(*) = {}^*\mathrm{Jet}^k_0 \mathcal{F}$ of the base point.

Consider now the case that Q is a Lie group then so are both ${}^*\mathrm{Jet}^k_0 Q$ and $\mathrm{Jet}^k_0 Q$ under pointwise multiplication with Lie algebras ${}^*\mathrm{Jet}^k_0 \mathfrak{q}$ and $\mathrm{Jet}^k_0 \mathfrak{q}$ respectively. With the help of the exponential $\exp : \mathfrak{q} \longrightarrow Q$ we may identify ${}^*\mathrm{Jet}^k_0 Q$ and ${}^*\mathrm{Jet}^k_0 \mathfrak{q}$, making the vector space ${}^*\mathrm{Jet}^k_0 \mathfrak{q}$ an algebraic group with group structure given by the polynomial approximation of the Campbell–Baker–Hausdorff formula. The group

$\mathrm{Jet}_0^k\,Q$ is then a semidirect product $\mathrm{Jet}_0^k\,Q \;\cong\; Q \ltimes {}^*\mathrm{Jet}_0^k\,\mathfrak{q}$ by the split exact sequence:

$$1 \;\longrightarrow\; {}^*\mathrm{Jet}_0^k\,Q \;\longrightarrow\; \mathrm{Jet}_0^k\,Q \;\xrightarrow{\mathrm{ev}}\; Q \;\longrightarrow\; 1$$

Similarly we could define jets of maps $f:\ \mathfrak{u} \longrightarrow \mathcal{F}$ at points different from 0 and jets of maps between arbitrary manifolds. However for our purposes it is sufficient to "gauge" the pointwise definitions and constructions given above. Consider therefore the open subset $\mathbf{GL}^{\,k}M \;\subset\; \mathrm{Jet}_0^k\,M$ of all k–jets of local diffeomorphisms $m:\ \mathfrak{u} \longrightarrow M$ defined in a neighborhood of $0 \in \mathfrak{u}$ together with the open subset $\mathbf{GL}^{\,k}\mathfrak{u} \;\subset\; {}^*\mathrm{Jet}_0^k\,\mathfrak{u}$ of all k–jets of local diffeomorphisms $A:\ \mathfrak{u} \longrightarrow \mathfrak{u}$ fixing 0:

$$\begin{aligned}
\mathbf{GL}^{\,k}M \quad &:= \quad \{\,\mathrm{jet}_0^k\,m|\;\; m:\mathfrak{u}\longrightarrow M,\; m\,\text{local diffeomorphism}\,\}\\
\mathbf{GL}^{\,k}\mathfrak{u} \quad &:= \quad \{\,\mathrm{jet}_0^k\,A|\;\; A:\mathfrak{u}\longrightarrow \mathfrak{u},\; A(0)=0,\; A\,\text{local diffeomorphism}\,\}
\end{aligned}$$

Obviously the set $\mathbf{GL}^{\,k}\mathfrak{u}$ is a group under composition acting on $\mathbf{GL}^{\,k}M$ again by composition. In this way $\mathbf{GL}^{\,k}M$ becomes a principal $\mathbf{GL}^{\,k}\mathfrak{u}$–bundle over M with projection $\pi:\ \ \mathbf{GL}^{\,k}M \;\longrightarrow\; M$, $\mathrm{jet}_0^k\,m \;\longmapsto\; m(0)$, given by evaluation. Elements of $\mathbf{GL}^{\,k}M$ are called holonomic k–frames, because in the special case $k = 1$ the principal bundle $\mathbf{GL}^{\,1}M$ is just the usual frame bundle $\mathbf{GL}\,M := \{\,\mathrm{jet}_0^1\,m = m_{*,0}:\ \mathfrak{u} \xrightarrow{\cong} T_{m(0)}M\,\}$ on M.

Given now an arbitrary principal bundle $\pi:\ \mathcal{G} \longrightarrow M$ over M with principal fibre Q and some Q-representation F there is an associated vector bundle $\mathcal{G}\times_Q F$ on M. Historically this construction goes back to the Cartan's idea of recovering the tangent bundle from the frame bundle $\mathbf{GL}\,M$ and the left representation of $\mathbf{GL}\,\mathfrak{u}$ on $\mathfrak{u}$. Although left and right Q–representations and more generally left and right Q–spaces are in bijective correspondence, it is certainly more natural to use right representations instead in order to recover the cotangent bundle. Hence we will always associate fiber bundles by right Q–actions

$$\mathcal{G}\times_Q\mathcal{F} \;:=\; \mathcal{G}\times\mathcal{F}/_{\sim} \qquad\qquad (g,\,f) \;\sim\; (g\star q,\; f\star q) \qquad\qquad [g,\,f] \;:=\; (g,\,f)/\sim$$

if not explicitly stated otherwise. The advantage of this choice becomes evident in explicit calculations, because inverting elements of $\mathbf{GL}^{\,k}\mathfrak{u}$ is not particularly easy in practice. By abuse of notation we will identity sections $f\in\Gamma(G\times_Q\mathcal{F})$ of $\mathcal{G}\times_Q\mathcal{F}$ and associated functions $f\in C^\infty(\mathcal{G},\mathcal{F})^Q$ on $\mathcal{G}$ with values in $\mathcal{F}$ satisfying $f(gq) = f(g)\star q$ via $f(\pi\,g) \;=\; [g, f(g)]$. General jet theory associates to any fibre bundle on M the family of its jet bundles. In the context of principal bundles and associated fibre bundles like $\mathcal{G}\times_Q\mathcal{F}$ the construction can be formulated naturally with the help of the principal bundle of holonomic k–frames of $\mathcal{G}$ over M and its structure group

$$\begin{aligned}
\mathbf{GL}^{\,k}(\mathcal{G}, M) \quad &:= \quad \{\mathrm{jet}_0^k\,g|\; g:\,\mathfrak{u}\longrightarrow\mathcal{G},\; \pi\circ g\,\text{local diffeomorphism}\}\\
\mathbf{GL}^{\,k}(Q,\,\mathfrak{u}) \quad &:= \quad \{\mathrm{jet}_0^k\,A|\; A:\,\mathfrak{u}\longrightarrow\mathfrak{u}\times Q,\; A_\mathfrak{u}(0)=0,\; A_\mathfrak{u}\,\text{local diffeomorphism}\}
\end{aligned}$$

with multiplication $\mathrm{jet}_0^k\,A\cdot\mathrm{jet}_0^k\,B \;:=\; \mathrm{jet}_0^k\,(A_\mathfrak{u}\circ B_\mathfrak{u},\; (A_Q\circ B_\mathfrak{u})\cdot B_Q)$ and right operation

$$\mathrm{jet}_0^k\,g \;\star\; \mathrm{jet}_0^k\,A \quad := \quad \mathrm{jet}_0^k\,((g\circ A_\mathfrak{u})\star A_Q)$$

Note that $\mathbf{GL}^{\,k}(\mathcal{G},\,M)$ and $\mathbf{GL}^{\,k}(Q,\,\mathfrak{u})$ project to $\mathbf{GL}^{\,k}M$ and $\mathbf{GL}^{\,k}\mathfrak{u}$ respectively. The jet operator from sections of $\mathcal{G}\times_Q\mathcal{F}$ to sections of $\mathbf{GL}^{\,k}(\mathcal{G}, M)\times_{\mathbf{GL}^{\,k}(Q,\mathfrak{u})}\mathrm{Jet}_0^k\mathcal{F}$

is just

$$\begin{array}{cccc} \mathrm{jet}^k : & C^\infty(G,\mathcal{F})^Q & \longrightarrow & C^\infty(\,\mathbf{GL}^{\,k}(G,\,M),\,\mathrm{Jet}_0^k\,\mathcal{F}\,)^{\mathbf{GL}^{\,k}(Q,\,\mathfrak{u})} \\ & f & \longmapsto & \mathrm{jet}^k f \end{array}$$

with $\mathrm{jet}^k f(\,\mathrm{jet}_0^k\,g\,) := \mathrm{jet}_0^k\,(f\circ g)$, which is equivariant over the right action of $\mathbf{GL}^{\,k}(Q,\mathfrak{u})$:

$$\mathrm{jet}_0^k\,f \,\star\, \mathrm{jet}_0^k\,A \quad := \quad \mathrm{jet}_0^k\,((f\circ A_{\mathfrak{u}})\star A_Q)$$

Besides the principal bundles $\mathbf{GL}^{\,k}M$ and $\mathbf{GL}^{\,k}(\mathcal{G},M)$ of holonomic k–frames we will consider the principal bundles $\overline{\mathbf{GL}}^{k}M$ and $\overline{\mathbf{GL}}^{k}(\mathcal{G},M)$ of semi-holonomic k–frames later on. The essential idea of their definition is to forget that partial derivatives commute although it is not particularly apparent from the actual definition. Say $\overline{\mathbf{GL}}^{k}(\mathcal{G},M)$ is defined as a principal subbundle of the k times iterated bundle of 1–frames of $\mathcal{G}$ over M

$$\overline{\mathbf{GL}}^{k}(\mathcal{G},M) \quad\subset\quad \mathbf{GL}^{\,1}(\mathbf{GL}^{\,1}(\mathbf{GL}^{\,1}(\ldots\mathbf{GL}^{\,1}(\mathcal{G},M)\ldots),M),M),M)$$

by the requirement that all of the k different evaluation maps of the k times iterated to the $k-1$ times iterated bundle of 1–frames of $\mathcal{G}$ over M agree on $\overline{\mathbf{GL}}^{k}(\mathcal{G},M)$. This condition is void for $k=1$ and so we have $\overline{\mathbf{GL}}^{1}(\mathcal{G},M)=\mathbf{GL}^{\,1}(\mathcal{G},M)$. The definition of the bundle of semi-holonomic k–frames $\overline{\mathbf{GL}}^{k}M$ of M and the corresponding structure groups $\overline{\mathbf{GL}}^{k}(Q,\mathfrak{u})$ and $\overline{\mathbf{GL}}^{k}\mathfrak{u}$ is more or less the same. Note that all k evaluation maps agree on $\overline{\mathbf{GL}}^{k}(\mathcal{G},M)$ by definition and so all of them provide us with the same projection map:

$$\mathrm{pr}: \quad \overline{\mathbf{GL}}^{k}(\mathcal{G},M) \;\longrightarrow\; \overline{\mathbf{GL}}^{k-1}(\mathcal{G},M)$$

The definitions given above depend only on the differentiable structure of the manifold M or the principal bundle $\mathcal{G}$ involved. More precisely although the functors $\mathbf{GL}^{\,k}$ and $\mathbf{GL}^{\,k}(\cdot,\cdot)$ from manifolds or principal bundles to principal bundles over the same base have different models, depending say on the choice of the vector space $\mathfrak{u}$, there are natural transformations between any two such models. However the natural transformations are neither unique nor canonical and this ambiguity is the problem with principal bundles in jet theory en nuce. In fact the various models for the functors $\mathbf{GL}^{\,k}$, $\mathbf{GL}^{\,k}(\cdot,\cdot)$ and Jet_0^k differ by an additional structure called the diagonal

$$\Delta:\;\; \mathbf{GL}^{\,k+l}(\,\mathcal{G},\,M\,) \;\longrightarrow \mathbf{GL}^{\,k}(\,\mathbf{GL}^{\,l}(\,\mathcal{G},\,M\,),\,M\,)$$

which is in general not preserved by the natural transformations between different models. In the construction of the diagonal we choose implicitly or explicitly the translations underlying the geometry we want to describe. Of course any formulation of calculus is equivalent to any other formulation and if we want to we may proceed even with an improper choice for the translations, but the counter terms needed to put everything straight again will soon get too complex.

Thus it is prudent to construct the diagonal with respect to the model space G/P of our geometry and we will describe this construction in detail, because it is fundamental for all calculations to come. Choose a linear complement $\mathfrak{u}$ of $\mathfrak{p}$ in $\mathfrak{g}=\mathfrak{u}\oplus\mathfrak{p}$. The exponential map $\exp : \mathfrak{g} \longrightarrow G$, $v \longmapsto e^v$, provides us with local diffeomorphisms of $P \times \mathfrak{u}$ and $\mathfrak{u} \times P$ with tube domains around $P \subset G$. On the intersection of the tubes the difference of these two diffeomorphisms gives rise to the commutator map:

Definition 2.1. — The commutator $\Phi : P \times \mathfrak{u} \to \mathfrak{u} \times P$, $(p, v) \mapsto (\Phi_{\mathfrak{u}}(p, v), \Phi_P(p, v))$ is uniquely defined in some tubular neighborhood of $P \times \{0\}$ in $P \times \mathfrak{u}$ by the requirement:

$$p\, e^v \quad = \quad e^{\Phi_{\mathfrak{u}}(p,v)}\, \Phi_P(p, v)$$

Whether or not is is defined outside this neighborhood is of no practical importance.

The component $\Phi_{\mathfrak{u}}$ describes the rotations of G/P induced by elements of P in exponential coordinates $\exp : \mathfrak{u} \longrightarrow G/P$, $v \longmapsto e^v P$, since $p \exp v = \exp \Phi_{\mathfrak{u}}(p, v)$. In particular the jet of $\Phi_{\mathfrak{u}}$ is a group homomorphism:

$$\Phi_{\mathfrak{u}} : \; P \;\longrightarrow\; \mathbf{GL}^k\mathfrak{u}, \quad p \;\longmapsto\; \mathrm{jet}^k_0 \Phi_{\mathfrak{u}}(p, \cdot)$$

It is less obvious that the jet of the commutator itself defines a group homomorphism:

$$\Phi : \; P \longrightarrow\; \mathbf{GL}^k(P, \mathfrak{u}), \quad p \;\longmapsto\; \mathrm{jet}^k_0(\Phi_{\mathfrak{u}}(p, \cdot), \Phi_P(p, \cdot))$$

The group homomorphism Φ splits the evaluation $\mathrm{ev} : \mathbf{GL}^k(P, \mathfrak{u}) \longrightarrow P$ and hence its image is always a closed subgroup of $\mathbf{GL}^k(P, \mathfrak{u})$. This is not true in general for the group homomorphism $\Phi_{\mathfrak{u}}$, e. g. it is not satisfied by $\mathbb{R} \ltimes (\mathbb{C} \oplus \mathbb{C})/\mathbb{R}$ with $\mathbb{R}$ acting as an irrational line $\mathbb{R} \subset S^1 \times S^1$ on $\mathbb{C} \oplus \mathbb{C}$. Let us therefore agree on the following regularity assumption on the model space G/P:

Definition 2.2. — A model space G/P is called admissible if the image of P under the group homomorphism $\Phi_{\mathfrak{u}} : P \longrightarrow \mathbf{GL}^k\mathfrak{u}$ is closed for all $k \geq 1$. Equivalently the quotient of $\mathbf{GL}^k\mathfrak{u}$ by the image of P is an analytic manifold for all $k \geq 1$. As confusion is unlikely to occur in this context we will denote the quotient by $P \backslash \mathbf{GL}^k\mathfrak{u}$ for short.

Actually we do not know how restrictive this assumption really is, but it is certainly no issue for an algebraic group G and an algebraic subgroup P. Besides the homomorphism Φ the most important part of the geometry of the model space G/P are the translations:

Definition 2.3. — The translations $t : \mathfrak{u} \times \mathfrak{u} \longrightarrow \mathfrak{u}$, $(v, \tilde{v}) \longmapsto t_v\tilde{v}$, are defined in a neighborhood of $(0, 0) \in \mathfrak{u} \times \mathfrak{u}$ by:

$$e^{t_v \tilde{v}}\, P \quad = \quad e^v\, e^{\tilde{v}}\, P$$

Evident properties of the translations are:

$$t_0 \;=\; \mathrm{id} \qquad\qquad t_v^{-1} \;=\; t_{-v} \qquad\qquad t_v 0 \;=\; v \qquad\qquad t_v\tilde{v} \;=\; v \,+\, \tilde{v} \,+\, O(v\tilde{v})$$

In the affine case the translations of the model space $P \ltimes \mathfrak{u}/P$ reduce to the obvious choice $t_v\tilde{v} := v + \tilde{v}$ and in fact this choice is the only one considered classically ([**K**] or much more recently [**KMS**]). Symbolic calculus is of course independent of the choice of translations and this is reflected by $t_v\tilde{v} = v + \tilde{v} + O(v\tilde{v})$.

Whereas the commutator describes the action of the isotropy group P in exponential coordinates and provides us with the geometrically motivated group homomorphism $\Phi_{\mathfrak{u}}$ from P to $\mathbf{GL}^{d}\mathfrak{u}$ the translations of the model space G/P enter the theory through the construction of the diagonal Δ. In general the definition of Δ is modelled on

$$\begin{array}{rrcl}\Delta: & \mathrm{Jet}_0^{k+l}\mathcal{F} & \longrightarrow & \mathrm{Jet}_0^k\mathrm{Jet}_0^l\mathcal{F}\\ & \mathrm{jet}_0^{k+l}f & \longmapsto & \mathrm{jet}_0^k[\upsilon\longmapsto \mathrm{jet}_\upsilon^l f := \mathrm{jet}_0^l(f\circ t_\upsilon)]\end{array}$$

with an important exception for the group $\mathbf{GL}^{k+l}\mathfrak{u}$ to make the map of principal bundles

$$\begin{array}{rrcl}\Delta: & \mathbf{GL}^{k+l}(\mathcal{G},M) & \longrightarrow & \mathbf{GL}^{k}(\mathbf{GL}^{l}(\mathcal{G},M),M)\\ & \mathrm{jet}_0^{k+l} & \longmapsto & \mathrm{jet}_0^k[\upsilon\longmapsto \mathrm{jet}_\upsilon^l g = \mathrm{jet}_0^l(g\circ t_\upsilon)]\end{array}$$

equivariant over the group homomorphism:

$$\begin{array}{rrcl}\Delta: & \mathbf{GL}^{k+l}(Q,\mathfrak{u}) & \longrightarrow & \mathbf{GL}^{k}(\mathbf{GL}^{l}(Q,\mathfrak{u}),\mathfrak{u})\\ & \mathrm{jet}_0^{k+l}(A_{\mathfrak{u}},A_Q) & \longmapsto & \mathrm{jet}_0^k[\upsilon\longmapsto (A_{\mathfrak{u}}(\upsilon),\,\mathrm{jet}_\upsilon^l A_{\mathfrak{u}},\,\mathrm{jet}_\upsilon^l A_Q)]\end{array}$$

Although as expected $\mathrm{jet}_\upsilon^l A_Q = \mathrm{jet}_0^l(A_Q\circ t_\upsilon)$ we have to set $\mathrm{jet}_\upsilon^l A_{\mathfrak{u}} := \mathrm{jet}_0^l[t_{-A_{\mathfrak{u}}(\upsilon)}\circ A_{\mathfrak{u}}\circ t_\upsilon]$ for all $A_{\mathfrak{u}}\in\mathbf{GL}^{k+l}\mathfrak{u}$. It is useful to think of M as a principal $Q=\{1\}$–bundle over M to get the definitions of the diagonal $\Delta:\ \mathbf{GL}^{k+l}M\longrightarrow\mathbf{GL}^{k}(\mathbf{GL}^{l}M,M)$ and the corresponding group homomorphism $\Delta:\ \mathbf{GL}^{k+l}\mathfrak{u}\longrightarrow\mathbf{GL}^{k}(\mathbf{GL}^{l}\mathfrak{u},\mathfrak{u})$ straight. In general the diagonals constructed above are not coassociative, the image of $\mathbf{GL}^{k+l+m}\mathfrak{u}$ in $\mathbf{GL}^{k}(\mathbf{GL}^{l}(\mathbf{GL}^{m}\mathfrak{u},\mathfrak{u}),\mathfrak{u})$ under successive diagonals will depend on whether we take the way over $\mathbf{GL}^{k+l}(\mathbf{GL}^{m}\mathfrak{u},\mathfrak{u})$ or $\mathbf{GL}^{k}(\mathbf{GL}^{l+m}\mathfrak{u},\mathfrak{u})$. In particular we lack a plausible way to think of $\mathbf{GL}^{k}\mathfrak{u}$ as a subgroup of $\overline{\mathbf{GL}}^{k}\mathfrak{u}$. Even more disastrous the naive prolongation of differential equations is impossible. Without coassociativity of the diagonals it simply seems impossible to proceed.

Coassociativity for the diagonals holds for all affine geometries $P\ltimes\mathfrak{u}/P$, although this property is too obvious to be spelt out explicitly in the classical literature [**K**]. However there is a class strictly larger than affine geometries, where coassociativity of the diagonals as introduced above holds true, namely split geometries. Split geometries are modelled on homogeneous spaces G/P, such that some subgroup $U\subset G$ acts simply transitively on an open, dense subset of G/P. If we choose the linear complement $\mathfrak{u}$ of $\mathfrak{p}$ in $\mathfrak{g}$ to be the Lie algebra of U, then the translations form a group $t_\upsilon\circ t_{\tilde\upsilon} = t_{t_\upsilon\tilde\upsilon}$ and coassociativity of the diagonals is restored. In this case the Campbell–Baker–Hausdorff formula for the group U allows us to expand the translations to arbitrary order $t_\upsilon\tilde\upsilon = \upsilon+\tilde\upsilon+\frac{1}{2}[\upsilon,\tilde\upsilon]+\cdots$.

The failure in general of coassociativity should be taken as an indication that the current definition of the diagonals is only a working and not a definite one. In fact there are other models for the functors $\mathbf{GL}^{k}$ and $\mathbf{GL}^{k}(\cdot,\cdot)$ eliminating this problem from its very roots, the details of this definite construction will be found in [**W**]. The proofs given below implicitly use this definite form of the diagonals, but the reader should have no problems checking the details, at least in the case of split geometries. In any case the definitions above reflect the state of our considerations at the time of the conference and are linked much closer to geometry with its flavor of translations than the abstract definitions.

The diagonals together with the commutator Φ fit into a commutative square, which will turn out to be the conditio sine qua non for the construction of holonomic and semi-holonomic geometries in the next section:

$$
\begin{tikzcd}
P \arrow[r, "\Phi"] \arrow[d, "\Phi_{\mathfrak{u}}"'] & \mathbf{GL}^{k}(P, \mathfrak{u}) \arrow[d, "\mathbf{GL}^{k}\Phi_{\mathfrak{u}}"'] \\
\mathbf{GL}^{k+l}\mathfrak{u} \arrow[r, "\Delta"] & \mathbf{GL}^{k}(\mathbf{GL}^{l}\mathfrak{u}, \mathfrak{u})
\end{tikzcd}
\tag{1}
$$

In fact rewriting the definition of the commutator as $e^{-\Phi_{\mathfrak{u}}(p,v)}\, p\, e^{v} \;=\; \Phi_P(p, v)$ we conclude

$$e^{-\Phi_{\mathfrak{u}}(p,v)}\, p\, e^{v}\, e^{\tilde{v}}\, P \quad = \quad \Phi_P(p,v)\, e^{\tilde{v}}\, P$$

for all $\tilde{v} \in \mathfrak{u}$ and consequently $\mathrm{jet}^{k}_{v}\, \Phi_{\mathfrak{u}}(p, \cdot) \;=\; \mathrm{jet}^{k}_{0}\, \Phi_{\mathfrak{u}}(\Phi_P(p, v), \cdot)$. The commutativity of the square (1) immediately implies that the orbit map

$$
\begin{aligned}
\mathcal{J}: \quad \mathbf{GL}^{k+l}\mathfrak{u} \,\subset\, \mathbf{GL}^{k}(\mathbf{GL}^{l}\mathfrak{u}, \mathfrak{u}) \;&\longrightarrow\; \mathrm{Jet}^{k}_{0}(\mathbf{GL}^{l}\mathfrak{u}),\\
A \;&\longmapsto\; \mathrm{jet}^{k}_{0}[v \longmapsto \mathrm{jet}^{l}_{0}\,\mathrm{id}] \,\star\, A
\end{aligned}
$$

through the basepoint $\mathrm{jet}^{k}_{0}[v \longmapsto \mathrm{jet}^{l}_{0}\,\mathrm{id}]$ of $\mathrm{Jet}^{k}_{0}(\mathbf{GL}^{l}\mathfrak{u})$ descends to quotients:

Corollary 2.4. — $\mathcal{J}: \quad {}_{P}\backslash\mathbf{GL}^{k+l}\mathfrak{u} \;\longrightarrow\; \mathrm{Jet}^{k}_{0}(\,{}_{P}\backslash\mathbf{GL}^{l}\mathfrak{u}\,)$.

In the introduction we remarked that this map is fundamental to define holonomic geometries in close analogy to Riemannian, conformal or projective geometry. Needless to say there is no apparent reason why the partial differential equation characterizing holonomic affine geometries should have no counterpart in more general circumstances, even if the classical construction of the map $\mathbf{GL}^{d}\mathfrak{u} \longrightarrow \mathrm{Jet}^{1}_{0}\mathbf{GL}^{d-1}\mathfrak{u}$ fails to descend to the right quotients. It was a decisive turning point in our line of thought, when we found remedy for this problem by judiciously choosing the translations. The conference in Luminy gave further impetus to reconsider the role played by the translations entirely in order to study pure Cartan geometries.

We want to close this section with a digression on the notion of torsion. In accordance with the general theme of this section we will review a classical argument [**K**] with particular emphasis on the role played by the choice of translations. Certainly the simplest and most intuitive definition of torsion is via the classifying section Ω_{tor} of the reduction

$$\mathbf{GL}^{2}M \;\xrightarrow{\Delta}\; \overline{\mathbf{GL}}^{2}M \;\xrightarrow{\Omega_{\mathrm{tor}}}\; {}_{\mathbf{GL}^{2}\mathfrak{u}}\backslash\overline{\mathbf{GL}}^{2}\mathfrak{u} \;\cong\; \Lambda^{2}\mathfrak{u}^{*}\otimes\mathfrak{u}$$

of the bundle $\overline{\mathbf{GL}}^{2}M$ of semi-holonomic 2–frames to the bundle $\mathbf{GL}^{2}M$ of holonomic 2–frames. This definition of torsion will depend on the choice of translations through the construction of the diagonal $\Delta: \;\; \mathbf{GL}^{2}M \longrightarrow \overline{\mathbf{GL}}^{2}M$ and a straightforward interpretation in terms of a torsion–free connection on the tangent bundle seems problematic.

In fact the concept of linear connections on the tangent bundle is intimately related and almost synonymous to the concept of affine geometries modelled on homogeneous spaces $P \ltimes \mathfrak{u}/P$. In this case the proper choice for the translations is the classical one using the affine structure of the vector space $\mathfrak{u}$ and the definition of torsion given above agrees with the definition of torsion via a linear connection. However the whole business with the translations is precisely about the fact that the affine structure on $\mathfrak{u}$ is induced by the group structure of the subgroup $\mathfrak{u} \subset P \ltimes \mathfrak{u}$ and should be replaced accordingly for more general geometries.

Recall that a linear connection on the tangent bundle of a manifold M is uniquely characterized by the $\mathbf{GL}^1\mathfrak{u}$–equivariant distribution of horizontal planes, i. e. linear subspaces $H \subset T_{\mathrm{jet}^1_0 m}\,\mathbf{GL}^1 M$ complementary to the space $\mathrm{Vert}_{\mathrm{jet}^1_0 m}\,\mathbf{GL}^1 M$ of vertical vectors. Note that every horizontal plane has a canonical identification with $\mathfrak{u}$ given by the soldering form θ. The set of all horizontal planes is readily identified with $\overline{\mathbf{GL}}^2 M$ as an affine bundle over $\mathbf{GL}^1 M$, namely every point $\overline{m} = \mathrm{jet}^1_0[v \longmapsto \mathrm{jet}^1_0[\tilde{v} \longmapsto \overline{m}_v(\tilde{v})]]$ in $\overline{\mathbf{GL}}^2 M$ defines a map

$$\mathfrak{u} \;\longrightarrow\; T_{\mathrm{pr}\,(\overline{m})}(\mathbf{GL}^1 M), \qquad X \;\longmapsto\; \frac{d}{dt}\Big|_0 \mathrm{jet}^1_0[\tilde{v} \longmapsto \overline{m}_{tX}(\tilde{v})]$$

whose image $H_{\overline{m}} \subset T_{\mathrm{pr}\,(\overline{m})}(\mathbf{GL}^1 M)$ is a horizontal plane, because the soldering form θ on $\mathbf{GL}^1 M$ provides an explicit inverse isomorphism $H_{\overline{m}} \longrightarrow \mathfrak{u}$. On the other hand every horizontal plane is the differential of a smooth local section $\overline{m} : \mathfrak{u} \longrightarrow \mathbf{GL}^1 M$ in $0 \in \mathfrak{u}$, whose first order jet is a point in $\overline{\mathbf{GL}}^2 M$.

Consider now a reduction $\mathcal{G} \subset \mathbf{GL}^1 M$ to the structure group $P \subset \mathbf{GL}^1\mathfrak{u}$ endowed with a connection, i. e. a P–equivariant distribution of horizontal planes $H \subset T_m\mathcal{G} \subset T_m(\mathbf{GL}^1 M)$, which we may think of as points $H = H_{\overline{m}}$ in $\overline{\mathbf{GL}}^2 M$. In this way the connection is described by a map $\mathcal{G} \longrightarrow \overline{\mathbf{GL}}^2 M$, $m \longmapsto \overline{m}$, equivariant over the homomorphism $P \longrightarrow \mathbf{GL}^1(P, \mathfrak{u}) \cap \overline{\mathbf{GL}}^2\mathfrak{u}$. Exterior calculus identifies the torsion of the associated linear connection on the tangent bundle $TM \cong \mathcal{G} \times_P \mathfrak{u}$ with the P–equivariant map

$$\mathcal{G} \;\longrightarrow\; \Lambda^2\mathfrak{u}^* \otimes \mathfrak{u}, \qquad\qquad m \;\longmapsto\; d\theta\,|_{H_{\overline{m}} \times H_{\overline{m}}}$$

sending a point $m \in \mathcal{G}$ to the restriction of the exterior derivative $d\theta$ of the soldering form θ to the horizontal space $H_{\overline{m}} \cong \mathfrak{u}$. However there is another P–equivariant map from G to $\Lambda^2\mathfrak{u}^* \otimes \mathfrak{u}$ given by the composition with Ω_{tor}:

$$\mathcal{G} \;\longrightarrow\; \overline{\mathbf{GL}}^2 M \;\xrightarrow{\Omega_{\mathrm{tor}}}\; \Lambda^2\mathfrak{u}^* \otimes \mathfrak{u}$$

A classical calculation for affine geometries shows that these two maps agree up to a normalization constant, thus relating Ω_{tor} to the torsion of a linear connection on TM [**K**]. Reconsidering this calculation in the context of split geometries leads to the following lemma:

Lemma 2.5. — $\left(d\theta + \frac{1}{2}\,[\theta \wedge \theta] \right)_{H_{\overline{m}} \times H_{\overline{m}}} \quad = \quad -2\,\Omega_{\mathrm{tor}}(\,\overline{m}\,)$

Proof. — According to symbolic calculus the right coset $\mathbf{GL}^2\mathfrak{u}\backslash\overline{\mathbf{GL}}^2\mathfrak{u}$ associated to a given $A\in\Lambda^2\mathfrak{u}^*\otimes\mathfrak{u}$ is represented by the element

$$\overline{A}:=\mathrm{jet}^1_0[\upsilon\longmapsto(\upsilon,\mathrm{jet}^1_0[\tilde{\upsilon}\longmapsto\tilde{\upsilon}+A(\upsilon,\tilde{\upsilon})])]$$

of $\overline{\mathbf{GL}}^2\mathfrak{u}$. Suppose $\overline{m}\in\overline{\mathbf{GL}}^2M$ is a semi-holonomic 2–frame with $\Omega_{\mathrm{tor}}(\overline{m})\;=\;\mathbf{GL}^2\mathfrak{u}\cdot\overline{A}$. This means that there is a local diffeomorphism $m:\;\mathfrak{u}\longrightarrow M$ satisfying:

$$\begin{aligned}\overline{m}\;&=\;\mathrm{jet}^1_0[\upsilon\longmapsto\mathrm{jet}^1_0[\tilde{\upsilon}\longmapsto(m\circ t_\upsilon)(\tilde{\upsilon})]]\star\overline{A}\\&=\;\mathrm{jet}^1_0[\upsilon\longmapsto\mathrm{jet}^1_0[\tilde{\upsilon}\longmapsto(m\circ t_\upsilon)(\tilde{\upsilon}+A(\upsilon,\tilde{\upsilon}))]]\end{aligned}$$

Being somewhat sloppy with notation for a moment we think of $\overline{m}$ as the local section $\overline{m}:\;\upsilon\longmapsto\mathrm{jet}^1_0[\tilde{\upsilon}\longmapsto(m\circ t_\upsilon)(\tilde{\upsilon}+A(\upsilon,\tilde{\upsilon}))]$ of $\mathbf{GL}^1M$. Recall that the translations in general satisfy $t_{\upsilon+tY}(0)=\upsilon+tY$ for all υ, $Y\in\mathfrak{u}$, hence in particular:

$$(\mathrm{ev}\circ\overline{m})_{*\upsilon}(Y)\;=\;\frac{d}{dt}\Big|_0\mathrm{ev}\left(\mathrm{jet}^1_0[\tilde{\upsilon}\longmapsto(m\circ t_{\upsilon+tY})(\tilde{\upsilon}+A(\upsilon+tY,\tilde{\upsilon}))]\right)\;=\;\frac{d}{dt}\Big|_0m(\upsilon+tY)$$

On the other hand we calculate

$$\begin{aligned}\mathrm{jet}^1_0[\tilde{\upsilon}\longmapsto(m\circ t_\upsilon)(\tilde{\upsilon}+\,A(\upsilon,\tilde{\upsilon}))]_{*0}(Y)&=\frac{d}{dt}\Big|_0(m\circ t_\upsilon)(tY+A(\upsilon,tY))\\&=\frac{d}{dt}\Big|_0m(\upsilon+tY+A(\upsilon,tY)+\tfrac{1}{2}\,[\upsilon,tY]+\cdots)\end{aligned}$$

where we have finally used the Campbell–Baker–Hausdorff formula to expand the translations $t_\upsilon(\widehat{\upsilon})=\upsilon+\widehat{\upsilon}+\frac{1}{2}[\upsilon,\widehat{\upsilon}]+\cdots$ for a general split geometry. We conclude that the pullback of the soldering form θ to $\mathfrak{u}$ via $\overline{m}:\;\mathfrak{u}\longrightarrow\mathbf{GL}^1M$ satisfies

$$(\overline{m}^*_\upsilon\theta)(Y)\;=\;Y\,-\,A(\upsilon,Y)\,-\,\tfrac{1}{2}\,[\upsilon,Y]\,-\,\cdots$$

in $\upsilon\in\mathfrak{u}$ up to terms of higher order. Consequently its exterior differential in $0\in\mathfrak{u}$ reads

$$(\overline{m}^*_0\,d\theta)(X,Y)\;=\;d(\overline{m}^*\theta)_0\,(X,Y)\;=\;-\,2\,A(X,Y)\,-\,[X,Y]$$

□

3. Holonomic and Semi-Holonomic Geometries

Affine geometries have long been studied from various points of view and are intimately related to the concept of a linear connection. They are modelled on a presentation of a flat vector space $\mathfrak{u}$ as a homogeneous space $P\ltimes\mathfrak{u}/P$ for some subgroup $P\subset\mathbf{GL}\,\mathfrak{u}$. Curved analogues of this flat model structure are reductions $\mathcal{G}\subset\mathbf{GL}^1M$ of the bundle of 1–frames of M to the structure group $P\subset\mathbf{GL}^1\mathfrak{u}$ possibly satisfying additional conditions. The strongest condition we may impose is called integrability and allows only the flat model space as local solution. Integrability excludes the presence of curvature and is thus too strong a condition to provide a rich local geometry, although of course the global geometry may be interesting in its own right.

In general it is more useful to ask for a torsion–free connection tangent to the reduction $\mathcal{G}\subset\mathbf{GL}^1M$. Historically this differential condition has provided some of the most fruitful concepts in differential geometry. Say in Riemannian geometry modelled on $\mathbf{O}_n\mathbb{R}\ltimes\mathbb{R}^n/\mathbf{O}_n\mathbb{R}$ it is automatically satisfied for a unique connection and the subgroups of $\mathbf{O}_n\mathbb{R}$ which are in suitable sense minimal among those allowing non–trivial examples have been classified and studied in detail [**Be**], [**Br**]. Similarly this differential condition characterizes symplectic and complex manifolds among almost symplectic and almost complex manifolds respectively.

In this section we consider a straightforward generalization of this differential condition essentially based on the modified definition of torsion given in the last section. In this way we get around the difficulties and inconsistencies which are almost inevitable if we cling to the concept of torsion–freeness in the form it arises in affine geometries. Reductions $\mathcal{G} \subset \mathbf{GL}^{\,d}M$, $d \geq 1$, of the holonomic frame bundle of a manifold M satisfying this new differential condition are called holonomic geometries of order $d \geq 1$ on M modelled on the homogeneous space G/P. Similarly we will call reductions $\mathcal{G} \subset \overline{\mathbf{GL}}^{\,d}M$, $d \geq 1$, of the semi-holonomic frame bundle of a manifold M satisfying a suitable variant of the differential condition semi-holonomic geometries.

Resorting to semi-holonomic frame bundles we allow for torsion and at first sight it seems that we have eventually eliminated any dependence on the choice of translations altogether. However they still intervene through the homomorphism of the structure group $P \subset \overline{\mathbf{GL}}^{\,d}\mathfrak{u}$ via the diagonal $\Delta : \ \mathbf{GL}^{\,d}\mathfrak{u} \longrightarrow \overline{\mathbf{GL}}^{\,d}\mathfrak{u}$. Although this fact looks almost negligible it makes a crucial difference in the main result of this note. Modulo a slightly technical construction in the presence of isospin we will identify the category of Cartan geometries with the category of semi-holonomic geometries of suitable order $d \geq 1$ for all homogeneous model spaces G/P satisfying the regularity assumption of Definition 2.2. In particular this result implies that in the absence of isospin all Cartan geometries possess a classifying geometric object of order $d \geq 1$ satisfying an explicitly known partial differential equation.

Holonomic and semi-holonomic geometries are modelled on a homogeneous space G/P called the flat model space. According to Definition 2.2 we will suppose that G/P is admissible, i. e. the image of P under the group homomorphism

$$\Phi_{\mathfrak{u}} : \ P \longrightarrow \mathbf{GL}^{\,k}\mathfrak{u}, \quad p \longmapsto \mathrm{jet}_0^k[\upsilon \longmapsto \exp^{-1}(pe^{\upsilon}P)] \ = \ \mathrm{jet}_0^k\Phi_{\mathfrak{u}}(p,\cdot)$$

is a closed subgroup of $\mathbf{GL}^{\,k}\mathfrak{u}$ for all $k \geq 1$. The kernel of $\Phi_{\mathfrak{u}}$ is a closed normal subgroup P_d of P. We will denote the analytic quotient of $\mathbf{GL}^{\,k}\mathfrak{u}$ by the image P/P_d of P by $P\backslash\mathbf{GL}^{\,k}\mathfrak{u}$ for short as no confusion is likely to occur.

In essence holonomic or semi-holonomic geometries of order $d \geq 1$ on a manifold M modelled on G/P will be reductions $\mathcal{G}$ of the holonomic or semi-holonomic frame bundle $\mathbf{GL}^{\,d}M$ or $\overline{\mathbf{GL}}^{\,d}M$ respectively. This definition is absolutely classical [**K**], but we will impose a first order partial differential equation on the classifying section $\Omega_{\mathcal{G}}$ of this reduction

$$\Omega_{\mathcal{G}} \quad \in \quad \Gamma(\,\mathbf{GL}^{\,d}M \times_{\mathbf{GL}^{\,d}\mathfrak{u}} (\,{}_P\backslash\mathbf{GL}^{\,d}\mathfrak{u})\,) \ = \ C^{\infty}(\,\mathbf{GL}^{\,d}M,\ {}_P\backslash\mathbf{GL}^{\,d}\mathfrak{u}\,)^{\mathbf{GL}^{\,d}\mathfrak{u}}$$

defined by the condition $\mathrm{jet}_0^d m \star \Omega_{\mathcal{G}}(\mathrm{jet}_0^d m)^{-1} \in \mathcal{G}$ for all $\mathrm{jet}_0^d m \in \mathbf{GL}^{\,d}M$, such that

$$\mathcal{G} \xrightarrow{\ \subset\ } \mathbf{GL}^{\,d}M \xrightarrow{\ \Omega_{\mathcal{G}}\ } {}_P\backslash\mathbf{GL}^{\,d}\mathfrak{u}$$

is exact in the middle with a pointed set on the right. We want to impose a first order partial differential equation on $\Omega_{\mathcal{G}}$, so let's consider $\mathrm{jet}^1\Omega_{\mathcal{G}}$ as a function on $\mathbf{GL}^{d+1}M$

$$\begin{aligned}\mathrm{jet}^1\Omega_{\mathcal{G}} \quad\in\quad &\Gamma(\,\mathbf{GL}^{d+1}M \times_{\mathbf{GL}^{d+1}\mathfrak{u}} \mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d}\mathfrak{u})\,)\\ &\qquad = C^\infty(\,\mathbf{GL}^{d+1}M,\, \mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d}\mathfrak{u})\,)^{\mathbf{GL}^{d+1}\mathfrak{u}}\end{aligned}$$

where we employed the diagonal $\Delta:\ \mathbf{GL}^{d+1}M \longrightarrow \mathbf{GL}^1(\mathbf{GL}^dM, M)$ to pull back $\mathrm{jet}^1\Omega_{\mathcal{G}}$ to a function on $\mathbf{GL}^{d+1}M$. Now the crucial difference made by choosing the translations adapted to the geometry of G/P is the presence of the map $\mathcal{J}$ constructed in Corollary 2.4:

$$\mathcal{J}:\quad {}_P\backslash\mathbf{GL}^{d}\mathfrak{u} \longrightarrow \mathrm{Jet}^1_0(\,{}_P\backslash\mathbf{GL}^{d-1}\mathfrak{u}\,)$$

Definition 3.1. — A holonomic geometry of order d on a manifold M is a reduction $\mathcal{G}$ of the bundle $\mathbf{GL}^dM$ of holonomic d–frames of M to the structure group $P/P_d \subset \mathbf{GL}^d\mathfrak{u}$, such that the first order jet $\mathrm{jet}^1\Omega_{\mathcal{G}}$ of the classifying section considered as a function on $\mathbf{GL}^{d+1}M$ takes values in the double kernel

$$\mathbf{GL}^{d+1}M \ \xrightarrow{\mathrm{jet}^1\Omega_{\mathcal{G}}}\ \mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d}\mathfrak{u}) \ \rightrightarrows\ \mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d-1}\mathfrak{u})$$

of the two maps $\mathcal{J}\circ\mathrm{ev}$ and $\mathrm{Jet}^1_0\mathrm{pr}$. As the action of $\mathbf{GL}^{d+1}\mathfrak{u}$ on $\mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d}\mathfrak{u})$ respects this double kernel it suffices to check this condition at an arbitrary point of $\mathbf{GL}^{d+1}M$.

The partial differential equation imposed on holonomic geometries is modelled on the naive Spencer operator and is far less restrictive than the integrability of the subbundle $\mathcal{G}$. E. g. an affine holonomic geometry of order $d = 2$ modelled on a homogeneous space $P \ltimes \mathfrak{u}/P$ with $P \subset \mathbf{GL}^1\mathfrak{u}$ is the same as a torsion–free but not necessarily flat connection tangent to the reduction $\mathrm{pr}\,\mathcal{G} \subset \mathbf{GL}^1M$ of $\mathbf{GL}^1M$ compare Lemma 2.5. Moreover in Riemannian and conformal geometry we have a natural bijection

$$\mathcal{J}:\ \ {}_P\backslash\mathbf{GL}^{2}\mathfrak{u} \ \xrightarrow{\cong}\ \mathrm{Jet}^1_0(\,{}_P\backslash\mathbf{GL}^{1}\mathfrak{u}\,)$$

and so the holonomy constraint on the geometry of order $d = 2$ is the holonomy constraint on the first order jet of the Riemannian or conformal structure in disguise.

Although the notion of holonomic geometries is intuitively linked to the vanishing of torsion the straightforward generalization to reductions of the semi-holonomic frame bundles $\overline{\mathbf{GL}}^dM$ is equally interesting. Namely the homomorphism $P \longrightarrow \mathbf{GL}^d\mathfrak{u}$ with the inclusion $\Delta:\ \mathbf{GL}^d\mathfrak{u} \longrightarrow \overline{\mathbf{GL}}^d\mathfrak{u}$ realizes P/P_d as a closed subgroup of $\overline{\mathbf{GL}}^d\mathfrak{u}$. We only have to replace the map $\mathcal{J}$ from above by the map

$$\mathcal{J}:\quad {}_P\backslash\overline{\mathbf{GL}}^{d}\mathfrak{u} \longrightarrow \mathrm{Jet}^1_0(\,{}_P\backslash\overline{\mathbf{GL}}^{d-1}\mathfrak{u}\,)$$

coming from $\overline{\mathbf{GL}}^d\mathfrak{u} \subset \mathbf{GL}^1(\overline{\mathbf{GL}}^{d-1}\mathfrak{u}, \mathfrak{u}) \longrightarrow \mathrm{Jet}^1_0\overline{\mathbf{GL}}^{d-1}\mathfrak{u}$. This is compatible with the inclusions $\mathbf{GL}^d\mathfrak{u} \longrightarrow \overline{\mathbf{GL}}^d\mathfrak{u}$ and $\mathrm{Jet}^1_0\mathbf{GL}^{d-1} \longrightarrow \mathrm{Jet}^1_0(\overline{\mathbf{GL}}^{d-1}\mathfrak{u})$ and hence descends to quotients, too. This way of fixing $\mathcal{J}$ has evidently the merit that holonomic reductions of $\mathbf{GL}^dM$ are automatically semi-holonomic reductions of $\overline{\mathbf{GL}}^dM$. Further details are left to the reader as we will give another definition below, which is perhaps easier to handle and certainly closer in spirit to the concept of Cartan connections.

The partial differential equation characterizing holonomic geometries is modelled on the naive Spencer operator and so we expect that these equations are not stable, i. e. the first order jet of a solution has to lie in a strictly smaller subset of the double kernel, although the additional first order conditions become manifest only upon the first prolongation. In the following argument we will use a naive prolongation procedure, which is justified by the coassociativity of the diagonal $\Delta : \mathbf{GL}^{k+d}M \longrightarrow \mathbf{GL}^{k}(\mathbf{GL}^{d}M, M)$. The failure of coassociativity would thus be disastrous for the whole approach and the meaning of the partial differential equation itself would remain dubious. In order to derive the stable version of the equation we rewrite the definition of the double kernel in the following way

$$\mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d}\mathfrak{u}) \stackrel{\mathrm{Jet}^1_0\mathcal{J}}{\longrightarrow} \mathrm{Jet}^1_0\mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d-1}\mathfrak{u}) \Longrightarrow \mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d-1}\mathfrak{u})$$

where the two maps on the right are now simply the two possible evaluation maps from $\mathrm{Jet}^1_0\mathrm{Jet}^1_0\mathcal{F}$ to $\mathrm{Jet}^1_0\mathcal{F}$ with $\mathcal{F} = P\backslash\mathbf{GL}^{d-1}\mathfrak{u}$. In other words the double kernel appearing in the definition of holonomic reductions is just the preimage of the space $\overline{\mathrm{Jet}}^2_0\mathcal{F}$ of semi-holonomic 2–jets. Consider now the intersection $\mathrm{Jet}^k_0\overline{\mathrm{Jet}}^2_0\mathcal{F} \cap \mathrm{Jet}^{k+1}_0\mathrm{Jet}^1_0\mathcal{F}$ in $\mathrm{Jet}^k_0\mathrm{Jet}^1_0\mathrm{Jet}^1_0\mathcal{F}$. It is easily proved by writing out the definitions that this intersection is mapped to $\mathrm{Jet}^{k+1}_0\mathcal{F} \subset \mathrm{Jet}^{k-1}_0\mathrm{Jet}^1_0\mathrm{Jet}^1_0\mathcal{F}$ under the projection $\mathrm{pr} : \mathrm{Jet}^k_0\mathrm{Jet}^1_0\mathrm{Jet}^1_0\mathcal{F} \longrightarrow \mathrm{Jet}^{k-1}_0\mathrm{Jet}^1_0\mathrm{Jet}^1_0\mathcal{F}$. Hence the stable version of the partial differential equation characterizing holonomic reductions reads:

Remark 3.2. — A reduction $\mathcal{G} \subset \mathbf{GL}^{d}M$ is a holonomic reduction if and only if the value of the k–jet $\mathrm{jet}^k\Omega_{\mathcal{G}}(\mathrm{jet}^{k+d}_0 m), k \geq 1$, of its classifying section at one and hence every point $\mathrm{jet}^{k+d}_0 m \in \mathbf{GL}^{k+d}M$ is mapped to $\mathrm{Jet}^{k+1}_0(P\backslash\mathbf{GL}^{d-1}\mathfrak{u})$ under the prolongation of $\mathcal{J}$:

$$\mathrm{Jet}^k_0({}_P\backslash\mathbf{GL}^{d}\mathfrak{u}) \stackrel{\Delta\mathcal{J}}{\longrightarrow} \mathrm{Jet}^{k-1}_0\mathrm{Jet}^1_0\mathrm{Jet}^1_0({}_P\backslash\mathbf{GL}^{d-1}\mathfrak{u})$$

A similar result holds true for semi-holonomic geometries, but its formulation is somewhat confusing, because the semi-holonomic reduction $\mathcal{G}$ has to be prolonged as a holonomic object. Forgetting about the interpretation of the classifying section $\Omega_{\mathcal{G}}$ as a reduction of the semi-holonomic frame bundle we have to think of $\Omega_{\mathcal{G}}$ simply as a section of the fibre bundle $\mathbf{GL}^{d}M \times_{\mathbf{GL}^{d}\mathfrak{u}} (P\backslash\overline{\mathbf{GL}}^{d}\mathfrak{u})$ to get the result in its strongest possible form.

We want to close this section with an alternative characterization of holonomic geometries in terms of connections. If $\mathcal{G}$ is a reduction of the holonomic frame bundle $\mathbf{GL}^{d}M$ to the structure group P/P_d then its image $\mathrm{pr}\,\mathcal{G}$ under the projection $\mathrm{pr} : \mathbf{GL}^{d}M \longrightarrow \mathbf{GL}^{d-1}M$ is still a reduction, namely to the structure group P/P_{d-1}. Considering $\mathrm{pr}\,\Omega_{\mathcal{G}}$ as a function on $\mathbf{GL}^{d}M$ it is constant along the orbits of the kernel $\mathrm{pr} : \mathbf{GL}^{d}\mathfrak{u} \longrightarrow \mathbf{GL}^{d-1}\mathfrak{u}$ and hence descends to a well defined function on $\mathbf{GL}^{d-1}M$, which is just the classifying section $\Omega_{\mathrm{pr}\,\mathcal{G}}$ of $\mathrm{pr}\,\mathcal{G}$ considered as a function on $\mathbf{GL}^{d-1}M$. With this in mind we state the following lemma for holonomic reductions, leaving the formulation of the corresponding statement for semi-holonomic reductions to the reader:

Lemma 3.3. — *A reduction $\mathcal{G}$ of the bundle GL^dM of holonomic frames of order d on M is a holonomic reduction if and only if there is exists a map $\mathcal{G} \longrightarrow \mathbf{GL}^1(\mathrm{pr}\,\mathcal{G},\, M)$, which makes the following diagram commute:*

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathbf{GL}^1(\mathrm{pr}\,\mathcal{G},\, M) \\ \subset\big\downarrow & & \subset\big\downarrow \\ \mathbf{GL}^dM & \xrightarrow{\Delta} & \mathbf{GL}^1(\mathbf{GL}^{d-1}M,\, M) \end{array}$$

In particular if such a map exists it is just the restriction of Δ to $\mathcal{G}$.

Note that similarly to Lemma 2.5 every point of $\mathbf{GL}^1(\mathrm{pr}\,\mathcal{G}, M)$ can be thought of as defining a principal connection on $\mathrm{pr}\,\mathcal{G}$. Hence a holonomic reduction $\mathcal{G}$ of $\mathbf{GL}^dM$ can be thought of as a fibre bundle of principal connections on $\mathrm{pr}\,\mathcal{G}$. It is certainly a leitmotif in the formal theory of partial differential equations that an object of order d gives rise to a connection on objects of order $d-1$.

Proof. — We only need to check the condition on $\mathrm{jet}^1\Omega_{\mathcal{G}}$ at some point of $\mathbf{GL}^{d+1}M$ and we choose a point $\overline{g} \in \mathbf{GL}^{d+1}M$ projecting to $g \in \mathcal{G} \subset \mathbf{GL}^dM$. In this way we have for certain that $\mathrm{ev}\,(\mathrm{jet}^1\Omega_{\mathcal{G}}(\overline{g})) = \Omega_{\mathcal{G}}(g)$ is the base point of $P\backslash\mathbf{GL}^d\mathfrak{u}$. So then is its image under $\mathcal{J}$:

$$(\mathcal{J} \circ \mathrm{ev}\,)(\,\mathrm{jet}^1\Omega_{\mathcal{G}}(\overline{g})\,) \;=\; \mathrm{Jet}^1_0 P \cdot \mathrm{jet}^1_0[\upsilon \longmapsto (\,\upsilon,\, \mathrm{jet}^{d-1}_0[\tilde{\upsilon} \longmapsto \tilde{\upsilon}]\,)]$$

On the other hand we observe $\mathrm{Jet}^1_0\mathrm{pr}\,(\mathrm{jet}^1\Omega_{\mathcal{G}}(\overline{g})) = \mathrm{jet}^1(\mathrm{pr}\,\Omega_{\mathcal{G}})(\overline{g})$ by definition. However $\mathrm{jet}^1(\mathrm{pr}\,\Omega_{\mathcal{G}})$ is constant along the orbits of the kernel of $\mathbf{GL}^{d+1}\mathfrak{u} \longrightarrow \mathbf{GL}^d\mathfrak{u}$ and descends from $\mathbf{GL}^{d+1}M$ to the function $\mathrm{jet}^1\Omega_{\mathrm{pr}\,\mathcal{G}}$ on $\mathbf{GL}^dM$, so that we conclude:

$$\mathrm{Jet}^1_0\mathrm{pr}\,(\,\mathrm{jet}^1\Omega_{\mathcal{G}}(\overline{g})\,) \;=\; \mathrm{jet}^1\Omega_{\mathrm{pr}\,\mathcal{G}}(g)$$

Consequently the value of $\mathrm{jet}^1\Omega_{\mathcal{G}}$ at $\overline{g} \in \mathbf{GL}^{d+1}M$ will lie in the double kernel of $\mathcal{J} \circ \mathrm{ev}$ and $\mathrm{Jet}^1_0\mathrm{pr}$ if and only if the value of $\mathrm{jet}^1\Omega_{\mathrm{pr}\,\mathcal{G}}$ at $g \in \mathbf{GL}^dM$ will be the 1–jet of the constant map to the base point in $P\backslash\mathbf{GL}^{d-1}\mathfrak{u}$. However $\Omega_{\mathrm{pr}\,\mathcal{G}}$ is the classifying section of $\mathrm{pr}\,\mathcal{G}$ and hence $\mathrm{jet}^1\Omega_{\mathrm{pr}\,\mathcal{G}}$ is the classifying section of $\mathbf{GL}^1(\mathcal{G}, M) \subset \mathbf{GL}^1(\mathbf{GL}^{d-1}M, M)$. □

4. Classification of Cartan Geometries

In 1935 Cartan [**C**] introduced Cartan geometries modelled on a homogeneous space G/P to make the idea of a curved analogue of G/P precise. Certainly one hope connected with this definition was that analysis on these curved analogues should behave quite similar to analysis on the flat model G/P. In fact on a homogeneous space G/P all questions of analysis on homogeneous vector bundles can be cast into the language of representation theory, say the determination of the spectra of Laplace or Dirac operators as a popular sport. The impact of this concept has been tremendous and many beautiful results have fulfilled this hope since.

Nevertheless the definition relies on the introduction of an auxiliary principal P–bundle and it is not at all clear how we can possibly arrange the construction of this bundle. Taking the definition at face value there is no classifying geometric object on M associated to a Cartan geometry, because the auxiliary principal bundle is a totally new geometric entity to be dealt with. There is a saying that "a Cartan geometry should be the result of a theorem and not of a definition" reflecting if anything else the need for a classifying geometric object.

For many important model spaces G/P however this problem is not that bad because the classifying geometric object is known from the very beginning or easy to guess. The geometries associated to other model spaces have been discovered via Cartan's method of equivalence in the process of classifying structures arising independently in differential geometry, e. g. normal parabolic geometries were introduced by Tanaka [**T**] in order to classify regular differential systems with simple automorphism groups. Recently Čap & Schichl [**CS**] have given a more geometric but essentially equivalent construction of normal parabolic Cartan geometries along these lines.

Taking into account the remarks on the delicate problems caused by the use of principal bundles instead of groupoids in describing geometric structures we will propose a completely different point of view in this section. If only we abandon the idea of "canonical" translations and play the groupoid card, the homogeneous space G/P will take care of this part of the geometry, too. In this spirit we will review Cartan geometries and prove in this section that the categories of Cartan geometries modelled on G/P and of semi-holonomic geometries of suitable order $d+1$ are equivalent in the absence of isospin. In particular this result entails the existence of a classifying object $\Omega_{\mathcal{G}}$ of order $d+1$ for any Cartan geometry modelled on G/P and forms a fundamental existence result for local covariants of Cartan geometries.

Definition 4.1. — A Cartan geometry on a manifold M modelled on a homogeneous space G/P is a principal P–bundle $\mathcal{G}$ over M endowed with a $\mathfrak{g}$–valued 1–form $\theta : T\mathcal{G} \longrightarrow \mathfrak{g}$, the Cartan connection, which induces an isomorphism of vector spaces at every point $g \in \mathcal{G}$ and satisfies the equivariance condition:

$$\theta(\frac{d}{dt}\Big|_0 g_t \star p_t) \quad = \quad \mathrm{Ad}_{p_0^{-1}}\,\theta(\frac{d}{dt}\Big|_0 g_t) \;+\; \frac{d}{dt}\Big|_0 p_0^{-1}p_t$$

Its curvature is then by definition the $\mathfrak{g}$–valued 2–form $\kappa \;:=\; d\theta \,+\, \frac{1}{2}[\theta\wedge\theta]$ on $\mathcal{G}$.

On the principal P–bundle G over G/P the Maurer–Cartan form provides us with a Cartan connection and the Maurer–Cartan equation tells us that the Cartan geometry defined this way has vanishing curvature. In this sense the model space G/P is always the flat model space in the category of manifolds with Cartan geometries modelled on G/P. In order to reformulate this definition we recall from the previous section that there is a homomorphism $\Phi :\; P \longrightarrow \mathbf{GL}^{\,k}(P,\mathfrak{u})$ completing the commutative square (1)

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d, "\Phi_{\mathfrak{u}}"'] & \mathbf{GL}^{\,k}(P,\mathfrak{u}) \arrow[d, "\mathbf{GL}^{\,k}\Phi_{\mathfrak{u}}"'] \\ \mathbf{GL}^{\,k+l}\mathfrak{u} \arrow[r, "\Delta"] & \mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}\mathfrak{u},\mathfrak{u}) \end{tikzcd}$$

which has to be distinguished carefully from the composition $P \longrightarrow \mathrm{Jet}_0^k P \longrightarrow \mathbf{GL}^k(P,\mathfrak{u})$. Using this homomorphism we can replace the Cartan connection θ featuring in the definition of Cartan geometries by a morphism of principal bundles, which is more convenient to use in our current language:

Definition 4.2. — A Cartan geometry on a manifold M modelled on a homogeneous space G/P is a principal P–bundle $\mathcal{G}$ over M together with a morphism

$$\theta^{-1}: \quad \mathcal{G} \quad\longrightarrow\quad \mathbf{GL}^1(\mathcal{G}, M)$$

of principal bundles equivariant over $\Phi:\ P \longrightarrow \mathbf{GL}^1(P,\mathfrak{u})$ and satisfying $\mathrm{ev}\circ\theta^{-1} = \mathrm{id}_{\mathcal{G}}$.

In fact for every $g \in G$ the inverse vector space isomorphism $\theta^{-1}:\ \mathfrak{g} \longrightarrow T_g\mathcal{G}$ restricted to $\mathfrak{u} \subset \mathfrak{g}$ is the differential of all smooth maps $\mathfrak{u} \longrightarrow \mathcal{G}$ in a well–defined equivalence class in $\mathbf{GL}^1(\mathcal{G}, M)$ projecting back to $g \in \mathcal{G}$. It is more tedious to check that the equivariance condition for the Cartan connection θ is equivalent to the equivariance condition of θ^{-1}. In this alternative formulation of Cartan geometries curvature is characterized by the classifying section Ω_{curv} of the reduction $\mathbf{GL}^2(\mathcal{G}, M)$ of the principal bundle $\overline{\mathbf{GL}}^2(\mathcal{G}, M)$ to the structure group $\mathbf{GL}^2(P,\mathfrak{u})$

(2)

$$\mathcal{G} \overset{\theta^{-1}}{\longrightarrow} \overline{\mathbf{GL}}^1(\mathcal{G}, M) \overset{\overline{\mathbf{GL}}^1\theta^{-1}}{\longrightarrow} \overline{\mathbf{GL}}^2(\mathcal{G}, M) \overset{\Omega_{\mathrm{curv}}}{\longrightarrow} {}_{\mathbf{GL}^2(P,\mathfrak{u})}\backslash\overline{\mathbf{GL}}^2(P,\mathfrak{u}) \;\cong\; \Lambda^2\mathfrak{u}^*\otimes\mathfrak{g}$$

in the spirit of the description of torsion in Lemma 2.5:

$$\kappa(\,\theta^{-1}(X), \theta^{-1}(Y)\,) \quad=\quad -\,2\,\Omega_{\mathrm{curv}}(X,Y) \qquad\qquad X, Y \in \mathfrak{u}$$

Any closed subgroup P of a finite dimensional Lie group G has a unique maximal closed subgroup $P_\infty \subset P$ which is normal in G, namely the kernel of the representation of G in the diffeomorphisms $G \longrightarrow \mathrm{Diff}\, G/P$ of the manifold G/P. We will call P_∞ the isospin subgroup of P in G. Since the quotient G/P is connected and carries a natural analytic structure we may alternatively characterize P_∞ as the kernel of the homomorphism

$$P_\infty \;\overset{\subset}{\longrightarrow}\; P \;\longrightarrow\; \mathbf{GL}^\infty\mathfrak{u} \;:=\; \varprojlim_d \mathbf{GL}^d\mathfrak{u}$$

into the inverse limit $\mathbf{GL}^\infty\mathfrak{u}$ of the groups $\mathbf{GL}^d\mathfrak{u}$, $d \geq 1$. In this way we get a descending filtration of P by a sequence of normal subgroups

$$G \;:=\; P_{-1} \;\supset\; P \;:= P_0 \;\supset\; P_1 \;\supset\; P_2 \;\supset\; \ldots \;\supset\; P_\infty$$

and a corresponding filtration of the Lie algebra $\mathfrak{p}$ of P by ideals:

$$\mathfrak{g} \;:=\; \mathfrak{p}_{-1} \;\supset\; \mathfrak{p} \;:= \mathfrak{p}_0 \;\supset\; \mathfrak{p}_1 \;\supset\; \mathfrak{p}_2 \;\supset\; \ldots \;\supset\; \mathfrak{p}_\infty$$

The latter filtration can be described in purely algebraic terms using only the Lie algebra structure of $\mathfrak{g}$ and the subalgebra $\mathfrak{p}$. By construction this algebraic version of the filtration is strictly falling in the sense that $\mathfrak{p}_d = \mathfrak{p}_{d+1}$ implies $\mathfrak{p}_d = \mathfrak{p}_\infty$. Moreover the quotient P_k/P_{k+1} embeds as a closed subgroup into the kernel of $\mathbf{GL}^{k+1}\mathfrak{u} \longrightarrow \mathbf{GL}^k\mathfrak{u}$, which is a vector group for $k \geq 1$, and thus P_k/P_{k+1} is a product of $\mathfrak{p}_k/\mathfrak{p}_{k+1}$

by some lattice. Because G/P is assumed to be connected and each P_{k+1} is normal in P we need only verify that $\mathfrak{p}_{d+1}$ is an ideal in $\mathfrak{g}$ to assert that P_{d+1} is normal in G:

Remark 4.3. — If $\mathfrak{p}_d = \mathfrak{p}_{d+1}$ or equivalently $\mathfrak{p}_d = \mathfrak{p}_\infty$ for some $d \geq 1$, then $P_{d+1} \subset P$ is normal in G and a fortiori equal to $P_\infty = P_{d+1}$.

In particular both filtrations get stationary very quickly and this is the main reason why virtually all geometries studied in differential geometry with finite dimensional isometry groups are first or second order. The total lack of any natural example of a higher order geometry with finite dimensional isometry groups is unsettling indeed, because infinite order geometries abound in differential geometry, certainly symplectic geometry is the most prominent example.

In any case there are degenerate examples of finite dimensional geometries of arbitrarily high order, though these examples are not maximally prolonged. As far as we know the race for maximally prolonged finite dimensional geometries of higher order is still open and a modest negative hint is given by the following lemma, which is easy to prove and certainly has appeared in the literature before:

Lemma 4.4. — *For a reductive group G and a closed subgroup P the subalgebra $\mathfrak{p}_2$ is an ideal in $\mathfrak{g}$. In particular $P_3 = P_\infty$ is normal in G and all geometries modelled on homogeneous quotients of G are first, second or at most third order.*

The reader is invited to decide about his or her favorite assumption to exclude the third order case. With these preliminary remarks about the filtration of P and $\mathfrak{p}$ by jet order we turn to the classification of Cartan geometries modelled on admissible homogeneous spaces G/P as semi-holonomic geometries:

Theorem 4.5. — *Consider a homogeneous quotient G/P of a finite dimensional Lie group G by a closed subgroup P such that the image of P in $\mathbf{GL}^k\mathfrak{u}$ is closed for all $k \geq 1$. Let $d \geq 1$ be the smallest integer with $\mathfrak{p}_d = \mathfrak{p}_\infty$. If the isospin group $P_\infty = \{1\}$ is trivial then there is a natural correspondence between Cartan geometries and semi-holonomic geometries $\mathcal{G}$ of order $d{+}1$ modelled on G/P. In general a Cartan geometry $\mathcal{G}$ on M is an isospin P_∞–bundle over a semi-holonomic geometry $\mathcal{G}/P_\infty \subset \overline{GL}^{d+1}M$ of order $d+1$ modelled on G/P. The semi-holonomic geometry fixes the Cartan connection on $\mathcal{G}$ up to an affine subspace of isospin connections modelled on:*

$$\Gamma(\,M,\ T^*M \otimes (\mathcal{G} \times_P \mathfrak{p}_\infty)\,) \quad \subset \quad \Gamma(\,\mathcal{G}/P_\infty,\ T^*(\mathcal{G}/P_\infty) \otimes (\mathcal{G} \times_{P_\infty} \mathfrak{p}_\infty)\,)$$

Note that $\mathcal{G} \times_{P_\infty} \mathfrak{p}_\infty$ can be identified with the pullback of $\mathcal{G} \times_P \mathfrak{p}_\infty$ from M to $\mathcal{G}/P_\infty$.

Physically speaking a Cartan geometry $\mathcal{G}$ is an isospin gauge theory coupling to gravity by the choice of an affine subspace of isospin connections. In particular only the subgroup $\Gamma(M, \mathcal{G} \times_P P_\infty)$ of the full gauge group $\Gamma(M, \mathcal{G} \times_P P)$ of $\mathcal{G}$ survives in the semi-holonomic gauge. It may be a delicate problem involving the topology of M however to characterize the principal P_∞–bundles over a principal P/P_∞–bundle over M, which are principal P–bundles over M in a way compatible with the actions of P/P_∞ and P_∞:

Example 4.6. — The isospin group of the homogeneous space $\mathbf{Spin}_n\mathbb{R}\ltimes\mathbb{R}^n/\mathbf{Spin}_n\mathbb{R}$ is the discrete central subgroup $P_1 = P_\infty \cong \mathbb{Z}_2$ of $\mathbf{Spin}_n\mathbb{R}$. Semi-holonomic reductions of the semi-holonomic frame bundle $\overline{\mathbf{GL}}^2 M$ of a manifold M to $\mathbf{Spin}_n\mathbb{R}/\mathbb{Z}_2 = \mathbf{SO}_n\mathbb{R}$ correspond bijectively to the choice of an orientation, a Riemannian metric and a metric but not necessarily torsion–free connection on M. According to Lemma 4.5 Cartan geometries modelled on $\mathbf{Spin}_n\mathbb{R}\ltimes\mathbb{R}^n/\mathbf{Spin}_n\mathbb{R}$ over M are spin structures thought of as special $\mathbb{Z}_2$–principal bundles over the orthonormal frame bundle $\mathbf{SO}(M)$.

In this example of an affine geometry the topological obstructions against the existence of Cartan geometries are well known, compare [**LM**], as well as the parametrization of spin structures by suitable cohomology groups of M. The same argument applies more or less verbatim in the more general case of Cartan geometries modelled on G/P with P connected and P_∞ discrete, hence central in G.

Proof. — Consider the projection $\mathrm{pr} : \overline{\mathbf{GL}}^{d+1}M \longrightarrow \overline{\mathbf{GL}}^{d}M$ which maps a given reduction $\mathcal{G}\subset\overline{\mathbf{GL}}^{d+1}M$ to the structure group P/P_{d+1} to a reduction $\mathrm{pr}\,\mathcal{G}\subset\overline{\mathbf{GL}}^{d}M$ to the structure group P/P_d. According to Lemma 3.3 the reduction $\mathcal{G}$ is semi-holonomic if and only if the diagonal $\Delta : \overline{\mathbf{GL}}^{d+1}M \longrightarrow \mathbf{GL}^1(\overline{\mathbf{GL}}^{d}M, M)$ induces a map $\theta^{-1} : \mathcal{G} \longrightarrow \mathbf{GL}^1(\mathrm{pr}\,\mathcal{G},\, M)$. The short exact sequence of groups

$$P_d/P_{d+1} \quad\longrightarrow\quad P/P_{d+1} \quad\longrightarrow\quad P/P_d$$

is in fact a covering by the choice of d with $\mathfrak{p}_d = \mathfrak{p}_{d+1}$ and so is the projection $\mathrm{pr} : \mathcal{G} \longrightarrow \mathrm{pr}\,\mathcal{G}$. Hence θ^{-1} has a unique lift to a map $\theta^{-1} : \mathcal{G} \longrightarrow \mathbf{GL}^1(\mathcal{G},\, M)$ satisfying $\mathrm{ev}\circ\theta^{-1} = \mathrm{id}_{\mathcal{G}}$. Thus all semi-holonomic reductions $\mathcal{G}\subset\overline{\mathbf{GL}}^{d+1}M$ of order $d+1$ come along with a distinguished Cartan connection.

Conversely let us suppose that $\mathcal{G}$ is a Cartan geometry on M modelled on the homogeneous space G/P. Iterating the Cartan connection θ^{-1} as we did before to define curvature (2)

$$\begin{array}{ccccccccc}
\mathcal{G} & \xrightarrow{\theta^{-1}} & \overline{\mathbf{GL}}^1(\mathcal{G},\, M) & \xrightarrow{\overline{\mathbf{GL}}^1\theta^{-1}} & \overline{\mathbf{GL}}^2(\mathcal{G},\, M) & \xrightarrow{\overline{\mathbf{GL}}^2\theta^{-1}} & \cdots & \xrightarrow{\overline{\mathbf{GL}}^{k-1}\theta^{-1}} & \overline{\mathbf{GL}}^k(\mathcal{G},\, M) \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \overline{\mathbf{GL}}^1 M & & \overline{\mathbf{GL}}^2 M & & & & \overline{\mathbf{GL}}^k M
\end{array}$$

we get a sequence of P–equivariant maps $\mathcal{G} \longrightarrow \overline{\mathbf{GL}}^k M$. By equivariance the image of $\mathcal{G}$ is a reduction of the semi-holonomic frame bundle $\overline{\mathbf{GL}}^k M$ to the group P/P_k. The main point in the proof is now the assertion that all these reductions naturally associated to the Cartan geometry $\mathcal{G}$ are semi-holonomic. In fact the projection $\mathrm{pr} : \overline{\mathbf{GL}}^k M \longrightarrow \overline{\mathbf{GL}}^{k-1}M$ maps the image $\mathcal{G}/P_k$ of $\mathcal{G}$ in $\overline{\mathbf{GL}}^k M$ to $\mathrm{pr}\,(\mathcal{G}/P_k) = \mathcal{G}/P_{k-1}$ in $\overline{\mathbf{GL}}^{k-1}M$. Moreover the iterated Cartan connection $\theta^{-k} : \mathcal{G} \longrightarrow \overline{\mathbf{GL}}^k(\mathcal{G}, M)$ written in the way

$$\mathcal{G} \;\xrightarrow{\theta^{-1}}\; \mathbf{GL}^1(\mathcal{G},\, M) \;\xrightarrow{\mathbf{GL}^1\theta^{-k+1}}\; \overline{\mathbf{GL}}^k(\mathcal{G},\, M) \;\subset\; \mathbf{GL}^1(\overline{\mathbf{GL}}^{k-1}(\mathcal{G},\, M),\, M)$$

makes the left square of the diagram
(3)

$$\begin{array}{ccccc} \mathcal{G} & \xrightarrow{\theta^{-k}} & \overline{\mathbf{GL}}^{k}(\mathcal{G},\, M) & \longrightarrow & \overline{\mathbf{GL}}^{k} M \\ {\scriptstyle \theta^{-1}}\big\downarrow & & {\scriptstyle \Delta}\big\downarrow & & {\scriptstyle \Delta}\big\downarrow \\ \mathbf{GL}^{1}(\mathcal{G},\, M) & \xrightarrow{\mathbf{GL}^{1}\theta^{-k+1}} & \mathbf{GL}^{1}(\overline{\mathbf{GL}}^{k-1}(\mathcal{G},\, M),\, M) & \longrightarrow & \mathbf{GL}^{1}(\overline{\mathbf{GL}}^{k-1} M,\, M) \end{array}$$

commute, whereas the right one commutes trivially. In particular the reduction $\mathcal{G}/P_k$ of $\overline{\mathbf{GL}}^{k} M$ is semi-holonomic according to Lemma 3.3. Applying this argument for $k = d+1$ we see that the semi-holonomic reduction $\mathcal{G}/P_{d+1} \subset \overline{\mathbf{GL}}^{d+1} M$ of order $d+1$ comes along with a Cartan connection $\overline{\theta}^{-1} : \mathcal{G}/P_{d+1} \longrightarrow \mathbf{GL}^{1}(\mathcal{G}/P_{d+1},\, M)$. Moreover by Remark 4.3 we have equality $P_{d+1} = P_\infty$, hence the principal bundle $\mathcal{G}$ is a principal P_∞–bundle over the semi-holonomic reduction $\mathcal{G}/P_{d+1} = \mathcal{G}/P_\infty \subset \overline{\mathbf{GL}}^{d+1} M$. Finally the commutativity of the diagram (3) ensures that the Cartan connection θ^{-1} on $\mathcal{G}$ projects to the Cartan connection $\overline{\theta}^{-1}$ of the semi-holonomic reduction $\mathcal{G}/P_\infty$. □

The classification of Cartan geometries as semi-holonomic reductions is a generalization of a result of Morimoto [**M**], who constructed a P–equivariant embedding of the principal bundle $\mathcal{G}$ of a Cartan geometry modelled on G/P with trivial isospin $P_\infty = \{1\}$ into the infinitely prolonged semi-holonomic frame bundle $\overline{\mathbf{GL}}^{\infty} M$. Although far from obvious the homomorphism $P \longrightarrow \overline{\mathbf{GL}}^{\infty}\mathfrak{u}$ underlying Morimoto's construction agrees with the composition $P \longrightarrow \mathbf{GL}^{\infty}\mathfrak{u} \longrightarrow \overline{\mathbf{GL}}^{\infty}\mathfrak{u}$ used above. No doubt the rest of the extremely intricate construction results in the same embedding $\mathcal{G} \longrightarrow \overline{\mathbf{GL}}^{\infty} M$, in particular Morimoto's construction stabilizes at order $d+1$, too.

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