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COMPUTATIONAL ASPECTS OF GROTHENDIECK LOCAL RESIDUES

by

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Dedicated to Professor Tatsuo Suwa on his sixtieth birthday

Abstract. — Grothendieck local residues are studied from a view point of algebraic analysis. The main idea in this approach is the use of regular holonomic $\mathcal{D}$-modules attached to a zero-dimensional algebraic local cohomology class. A new method for computing Grothendieck local residues is developed in the context of Weyl algebra. An effective computing algorithm that exploits first order annihilators is also described.

***Résumé* (Aspects effectifs des résidus locaux de Grothendieck).** — On étudie le résidu local de Grothendieck du point de vue de l'analyse algébrique. L'idée principale de cette approche est l'utilisation de $\mathcal{D}$-modules holonomes réguliers attachés à une classe algébrique de cohomologie locale en dimension zéro. On développe une méthode nouvelle pour calculer les résidus locaux de Grothendieck dans le cadre de l'algèbre de Weyl. Cette méthode permet de décrire un algorithme efficace, lequel utilise les annulateurs du premier ordre.

1. Introduction

In this paper, we consider Grothendieck local residues and its duality in the context of holonomic $\mathcal{D}$-modules. Upon using the regular holonomic system associated to a certain zero-dimensional algebraic local cohomology class, we derive a method for computing Grothendieck local residues. We also give an effective algorithm that serves exact computations.

In § 2, we study local residues from the viewpoint of the analytic $\mathcal{D}$-module theory. By using the local residue pairing, we associate to an algebraic local cohomology class attached to a given regular sequence an analytic linear functional acting on the space of germs of holomorphic functions. We apply Kashiwara-Kawai duality theorem on holonomic systems [**3**] to the residue pairing and show that the kernel of the above analytic functional can be described in terms of partial differential operators. This result ensures in particular the computability of the Grothendieck local residues.

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In § 3, we give a framework in the Weyl algebra, and develop there a method for computing Grothendieck local residues. The key ingredient of the present method is the annihilating ideal in the Weyl algebra of the given zero-dimensional algebraic local cohomology class. We show that the use of generators of the annihilating ideal in the Weyl algebra reduces the computation of the local residues to that of linear equations.

In § 4, we derive an algorithm for computing Grothendieck local residues that exploits only first order partial differential operators. The resulting algorithm (Algorithm R) is efficient and thus can be available in use for actual computations in many cases. We also present an criterion to the applicability of this algorithm.

In § 5, we give an example to illustrate an effectual way of using Algorithm R.

In Appendix, we present an algorithm that outputs the first order partial differential operators which annihilate a direct summand in question of the given algebraic local cohomology class.

2. Local duality theorem

Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X = \mathbb{C}^n$ and $\mathcal{F}$ a regular sequence given by n holomorphic functions $f_1, \dots, f_n$ on X. Denote by $\mathcal{I}$ the ideal of $\mathcal{O}_X$ generated by $f_1, \dots, f_n$ and Z the zero-dimensional variety

$$\boldsymbol{V}(\mathcal{I}) = \{z \in X \mid f_1(z) = \cdots = f_n(z) = 0\}$$

of the ideal $\mathcal{I}$ consisting of finitely many points.

There is a canonical mapping ι from the sheaf of n-th extension groups $\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \Omega^n_X)$ to the sheaf of n-th algebraic local cohomology groups $\mathcal{H}^n_{[Z]}(\Omega^n_X)$ with support on Z:

$$\iota : \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \Omega^n_X) \longrightarrow \mathcal{H}^n_{[Z]}(\Omega^n_X)$$

where Ω^n_X is the sheaf of holomorphic n-forms on X. We denote by $\omega_{\mathcal{F}} = \left[\dfrac{dz}{f_1 \cdots f_n}\right]$ the image by the mapping ι of the Grothendieck symbol

$$\begin{bmatrix} dz \\ f_1 \dots f_n \end{bmatrix} \in \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \Omega^n_X),$$

i.e.,

$$\omega_{\mathcal{F}} = \iota\left(\begin{bmatrix} dz \\ f_1 \dots f_n \end{bmatrix}\right) \in \mathcal{H}^n_{[Z]}(\Omega^n_X), \tag{1}$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$. Let $\omega_{\mathcal{F},\beta}$ denote the germ at $\beta \in Z$ of the algebraic local cohomology class $\omega_{\mathcal{F}}$:

$$\omega_{\mathcal{F},\beta} \in \mathcal{H}^n_{[\beta]}(\Omega^n_X),$$

where $\mathcal{H}^n_{[\beta]}(\Omega^n_X)$ stands for the algebraic local cohomology supported at β.

Let $\mathcal{H}^n_{\{\beta\}}(\Omega^n_X)$ be the sheaf of n-th local cohomology groups at $\beta \in Z$ and let $\mathrm{Res}_\beta : \mathcal{H}^n_{\{\beta\}}(\Omega^n_X) \to \mathbb{C}$ be the local residue map. Recall that the mapping

$$\mathcal{H}^n_{\{\beta\}}(\Omega^n_X) \times \mathcal{O}_{X,\beta} \longrightarrow \mathcal{H}^n_{\{\beta\}}(\Omega^n_X)$$

composed with the local residue map Res_β defines a natural pairing between two topological vector spaces $\mathcal{H}^n_{\{\beta\}}(\Omega^n_X)$ and $\mathcal{O}_{X,\beta}$. Thus, the algebraic local cohomology class $\omega_{\mathcal{F},\beta} \in \mathcal{H}^n_{[\beta]}(\Omega^n_X)$ which also belongs to $\mathcal{H}^n_{\{\beta\}}(\Omega^n_X)$ induces a linear functional $\mathrm{Res}_\beta(\omega_{\mathcal{F}})$ that acts on $\mathcal{O}_{X,\beta}$. Namely, $\mathrm{Res}_\beta(\omega_{\mathcal{F}})$ is defined to be

$$\mathrm{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z)) = \mathrm{Res}_\beta(\varphi(z)\omega_{\mathcal{F},\beta})$$

for $\varphi(z) \in \mathcal{O}_{X,\beta}$, $\beta \in Z$. We consider the kernel space Ker of the linear functional $\mathrm{Res}_\beta(\omega_{\mathcal{F}})$ defined to be

$$\mathrm{Ker} = \{\psi(z) \in \mathcal{O}_{X,\beta} \mid \mathrm{Res}_\beta(\omega_{\mathcal{F}})(\psi(z)) = 0\}.$$

Now we are going to give an alternative description of the kernel space Ker in terms of partial differential operators.

Let $\mathcal{D}_X$ be the sheaf on X of linear partial differential operators. Then the sheaves Ω^n_X, $\mathcal{H}^n_{[\beta]}(\Omega^n_X)$ and $\mathcal{H}^n_{[Z]}(\Omega^n_X)$ are *right* $\mathcal{D}_X$-modules. Note also that $\mathcal{O}_X$ and $\mathcal{H}^n_{[\beta]}(\mathcal{O}_X)$ have a structure of left $\mathcal{D}_X$-modules. We denote by $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ the *right* ideal of $\mathcal{D}_X$ consisting of linear partial differential operators which annihilate the cohomology class $\omega_{\mathcal{F}}$:

$$\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}}) = \{P \in \mathcal{D}_X \mid \omega_{\mathcal{F}} P = 0\}.$$

Note that, if we set $\omega_{\mathcal{F}} = \sigma_{\mathcal{F}} dz$ with $\sigma_{\mathcal{F}} \in \mathcal{H}^n_{[Z]}(\mathcal{O}_X)$, the right ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ can be rewritten as

$$\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}}) = \{P \in \mathcal{D}_X \mid P^* \sigma_{\mathcal{F}} = 0\},$$

where P^* stands for the formal adjoint operator of P.

The $\mathcal{D}_X$-module $\mathcal{D}_X / \mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ is isomorphic to $\mathcal{H}^n_{[Z]}(\Omega^n_X)$. We thus in particular have the following theorem (*cf.* [**2**], [**3**], [**7**]);

Theorem 2.1. — *Let $\mathcal{F}$ be a regular sequence given by n holomorphic functions and $\omega_{\mathcal{F}}$ an algebraic local cohomology class defined by* (1) *whose support contains a point β.*

(i) *$\mathcal{D}_X / \mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ is a regular singular holonomic system.*

(ii) *$\mathcal{D}_X / \mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ is simple at each point $\beta \in Z$.*

The theorem implies the folloiwng result on the local cohomology solution space of the holonomic system $\mathcal{D}_X / \mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$;

Corollary 2.2. — *Let $\beta \in Z$. Then*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X / \mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}}), \mathcal{H}^n_{\{\beta\}}(\Omega^n_X)) &= \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X / \mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}}), \mathcal{H}^n_{[\beta]}(\Omega^n_X)) \\ &= \mathbb{C}\omega_{\mathcal{F},\beta}\end{aligned}$$

holds.

The above result means that the holonomic system $\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ completely characterize the algebraic local cohomology class $\omega_{\mathcal{F}}$ as its solution.

***Example 2.3* (*cf.* [1]).** — Let $\mathcal{F} = \{f_1, f_2\}$ be a regular sequence and I be the ideal in $\mathbb{C}[x, y]$ generated by functions f_1 and f_2 given below. Let $j_{\mathcal{F}}(x, y) = \det\left(\dfrac{\partial(f_1, f_2)}{\partial(x, y)}\right)$ be the Jacobian of f_1 and f_2. We fix the lexicographical ordering $x \succ y$ and use the term ordering $\succ$ in computations of Gröbner basis of I.

(i) Let $f_1 = x(x^2 - y^3 - y^4)$, $f_2 = x^2 - y^3$. We have $I = \langle x^2 - y^3, xy^4, y^7\rangle$ and $\boldsymbol{V}(I) = \{(0, 0)\}$ with the multiplicity 11. The algebraic local cohomology class $\omega_{\mathcal{F}} = \left[\frac{dx\wedge dy}{f_1 f_2}\right]$ is supported only at the origin $(0, 0)$. The annihilating ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ of $\omega_{\mathcal{F}}$ is generated by multiplication operators $x(x^2 - y^3 - y^4)$, $x^2 - y^3$ and a first order differential operator $P = 3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - 12$. By solving the system of differential equations $\omega_{\mathcal{F}} y^7 = \omega_{\mathcal{F}} xy^4 = \omega_{\mathcal{F}}(x^2 - y^3) = \omega_{\mathcal{F}} P = 0$ together with the formula $j_{\mathcal{F}}(x, y)\omega_{\mathcal{F}} = 11\delta_{(0,0)} dx \wedge dy$ where $\delta_{(0.0)} = \left[\frac{1}{xy}\right] \in \mathcal{H}^2_{[(0,0)]}(\mathcal{O}_X)$ is the delta function with support at the origin, we have the following representation of $\omega_{\mathcal{F}}$;

$$\omega_{\mathcal{F}} = \left[\left(\frac{1}{x^5 y} + \frac{1}{x^3 y^4} + \frac{1}{xy^7}\right) dx \wedge dy\right].$$

(ii) Let $f_1 = x$ and $f_2 = (x^2 - y^3)(x^2 - y^3 - y^4)$. We have $I = \langle x, y^7 + y^6\rangle$ and its primary decomposition $I = \langle x, y + 1\rangle \cap \langle x, y^6\rangle$. The annihilating ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ of the algebraic local cohomology class $\omega_{\mathcal{F}} = \left[\frac{dx\wedge dy}{f_1 f_2}\right]$ is generated by $x, y^7 + y^6$ and $P = (y^2 + y)\frac{\partial}{\partial y} - 5y - 5$. We have a representation

$$\left[\left(\frac{1}{xy} - \frac{1}{xy^2} + \frac{1}{xy^3} - \frac{1}{xy^4} + \frac{1}{xy^5} - \frac{1}{xy^6}\right) dx \wedge dy\right] + \left[\frac{dx \wedge dy}{x(y + 1)}\right]$$

of $\omega_{\mathcal{F}}$ by solving the system of differential equations $\omega_{\mathcal{F}} x = \omega_{\mathcal{F}}(y^6 + y^7) = \omega_{\mathcal{F}} P = 0$ together with the formula $j_{\mathcal{F}}(x, y)\omega_{\mathcal{F}} = (6\delta_{(0,0)} + \delta_{(0,-1)}) dx \wedge dy$ where $\delta_{(0,-1)} = \left[\frac{1}{x(y+1)}\right]$ is the delta function with support at $(0, -1)$.

(iii) Let $f_1 = x^2 - y^3 - y^4$ and $f_2 = x(x^2 - y^3)$. We have $I = \langle x^2 - y^4 - y^3, xy^4, y^8 + y^7\rangle$ and its primary decomposition $I = \langle x, y + 1\rangle \cap \langle x^2 - y^4 - y^3, xy^4, y^7\rangle$. The variety $\{(0, -1)\}$ is simple and $\{(0, 0)\}$ is of multiplicity 11. The annihilating ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ of the algebraic local cohomology class $\omega_{\mathcal{F}} = \left[\frac{dx\wedge dy}{f_1 f_2}\right]$ is generated by $x^2 - y^4 - y^3, xy^4, y^8 + y^7$ and $P = (4xy + 3x)\frac{\partial}{\partial x} + (2y^2 + 2y)\frac{\partial}{\partial y} - 12y - 12$. We have a representation of $\omega_{\mathcal{F}}$ as

$$\left[\left(\frac{1}{xy} + \frac{1}{x^5 y} - \frac{1}{xy^2} + \frac{1}{xy^3} - \frac{1}{xy^4} + \frac{1}{x^3 y^4} + \frac{1}{xy^5} - \frac{1}{xy^6} + \frac{1}{xy^7}\right) dx \wedge dy\right] + \left[-\frac{dx \wedge dy}{x(y + 1)}\right]$$

by solving $\omega_{\mathcal{F}}(x^2 - y^3 - y^4) = \omega_{\mathcal{F}} xy^4 = \omega_{\mathcal{F}}(y^8 + y^7) = \omega_{\mathcal{F}} P = 0$ together with the formula $j_{\mathcal{F}}(x,y)\omega_{\mathcal{F}} = (11\delta_{(0,0)} + \delta_{(0,-1)})dx \wedge dy$.

***Example 2.4* ([4])**. — Let $f = x^3+y^7+xy^5$. We consider the regular sequence given by partial derivarives $f_1 = 3x^2+y^5$ and $f_2 = 5xy^4+7y^6$ of f. The primary decomposition of the ideal $I = \langle f_1, f_2\rangle$ is given by $\langle 3125x + 151263, 25y + 147\rangle \cap I_0$ where $I_0 = \langle 3x^2 + y^5, 5xy^4 + 7y^6, y^8\rangle$.

For a direct summand ω_1 with support at $\{(-\frac{151263}{3125}, -\frac{147}{25})\}$of the algebraic local cohomology class $\omega_{\mathcal{F}} = \left[\frac{dx\wedge dy}{f_1 f_2}\right]$, the annihilating ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_1)$ is given by $\langle 25y+147, 3125x + 151263\rangle\mathcal{D}_X$.

For the other direct summand ω_0 with support at the origin $(0,0)$, its annihilating ideal $\mathrm{Ann}_{A_n}(\omega_0)$ is generated by the ideal I_0 and the second order differential operator

$$\begin{aligned}
&y\frac{\partial^2}{\partial y^2} + \Big(-\frac{43}{18}y^4 + \frac{84}{5}xy\Big)\frac{\partial^2}{\partial x^2} + \Big(\frac{50}{147}y + 9\Big)\frac{\partial}{\partial y}\\
&+ \bigg(\frac{6250}{1361367}y^4 + \frac{125}{9261}y^3 + \Big(-\frac{78125}{3176523}x - \frac{5}{63}\Big)y^2 + \Big(\frac{8125}{64827}x + \frac{252}{5}\Big)y - \frac{25}{441}x\bigg)\frac{\partial}{\partial x}\\
&\quad - \frac{762939453125}{218041257467152161}y^7 + \frac{6103515625}{494424620106921}y^6 - \frac{8300781250}{30270895108587}y^5\\
&\quad + \frac{156250000}{205924456521}y^4 + \Big(-\frac{37841796875}{211896265760109}x + \frac{781250}{1400846643}\Big)y^3\\
&\quad + \Big(\frac{927734375}{1441471195647}x - \frac{78125}{1361367}\Big)y^2 + \Big(-\frac{1953125}{1400846643}x + \frac{21250}{64827}\Big)y\\
&\quad - \frac{390625}{66706983}x + \frac{650}{441}.
\end{aligned}$$

Kashiwara-Kawai duality theory on holonomic systems ([**3**]) together with Theorem 2.1 implies the following result which gives a characterization of the space Ker.

Theorem 2.5. — *Let* Ker *be the kernel space of the residue mapping* $\mathrm{Res}_\beta(\omega_{\mathcal{F}})$. *Then*

$$\mathrm{Ker} = \{R\varphi(z) \mid \varphi(z) \in \mathcal{O}_{X,\beta}, R \in \mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})\}$$

holds.

Observe that the stalk at $\beta \in Z$ of $\mathcal{O}_X/\mathcal{I}$ is a finite dimensional vector space, the quotient space $\mathrm{Ker}/\mathcal{I} \subset \mathcal{O}_X/\mathcal{I}$ is a one codimensional vector subspace. Hence, if generators of the ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ are given, the determination of Ker can be reduced to a problem in the finite dimensional vector space.

Example 2.6. — Let $f_1 = x^3$ and $f_2 = y^2+2x^2+3x$. The variety $\boldsymbol{V}(I)$ of the ideal $I = \langle f_1, f_2\rangle$ is the origin $\{(0,0)\}$ with the multiplicity 6. Let $\omega_{\mathcal{F}} = \left[\frac{dx\wedge dy}{f_1 f_2}\right] \in \mathcal{H}^2_{[(0,0)]}(\Omega^n_X)$. Then the right ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ is generated by f_1, f_2 and the first order differential operator

$$P = 6x\frac{\partial}{\partial x} + (3y + 2xy)\frac{\partial}{\partial y} + (-2x - 15).$$

It is easy to verify that P enjoys the property $P(I) \subseteq I$. Under the identification $\mathcal{O}_X/I \cong \mathrm{Span}_{\mathbb{C}}\{1, y, x, xy, x^2, x^2y\}$,

$$\begin{aligned} P1 &= -2x - 15, \\ Px &= -2x^2 - 9x, \\ Px^2 &= -3x^2, \\ Py &= -9y, \\ Pxy &= -6xy, \\ Px^2y &= 0. \end{aligned}$$

Thus, by Theorem 2.5, we have $\mathrm{Ker}\,/I \cong \mathrm{Span}_{\mathbb{C}}\{1, y, x, xy, x^2\}$.

Note that, the relative Čech representation

$$\omega_{\mathcal{F}} = \Big(\left[\frac{1}{x^3y^2}\right] - 2\left[\frac{1}{xy^4}\right] - 3\left[\frac{1}{x^2y^4}\right] + 9\left[\frac{1}{xy^6}\right] \Big) dx \wedge dy$$

of the cohomology class $\omega_{\mathcal{F}}$ implies the following formula;

$$\begin{aligned} &\mathrm{Res}_{(0,0)}\left(\frac{\varphi(x,y)dx \wedge dy}{x^3(y^2 + 2x^2 + 3x)}\right) \\ &\qquad = \frac{1}{2}\frac{\partial^3\varphi}{\partial x^2\partial y}(0,0) - \frac{1}{3}\frac{\partial^3\varphi}{\partial y^3}(0,0) - \frac{1}{2}\frac{\partial^4\varphi}{\partial x\partial y^3}(0,0) + \frac{3}{40}\frac{\partial^5\varphi}{\partial y^5}(0,0). \end{aligned}$$

3. A method for computing the local residues

Let K be the field $\mathbb{Q}$ of rational numbers. Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a regular sequence of n polynomials $f_i \in K[z] = K[z_1, \ldots, z_n]$, $i = 1, \ldots, n$ and I the ideal in $K[z]$ generated by these n polynomials. Let $I = I_1 \cap \cdots \cap I_\ell$ be the primary decomposition of the ideal I. Put $Z = \boldsymbol{V}(I)$, $Z_\lambda = \boldsymbol{V}(I_\lambda)$ and let $H^n_{[Z]}(\Omega^n_X) = \Gamma(X, \mathcal{H}^n_{[Z]}(\Omega^n_X))$, $H^n_{[Z_\lambda]}(\Omega^n_X) = \Gamma(X, \mathcal{H}^n_{[Z_\lambda]}(\Omega^n_X))$ for $\lambda = 1, \ldots, \ell$. We have the following direct sum decomposition;

$$H^n_{[Z]}(\Omega^n_X) = H^n_{[Z_1]}(\Omega^n_X) \oplus \cdots \oplus H^n_{[Z_\ell]}(\Omega^n_X).$$

Accordingly, the algebraic local cohomology class $\omega_{\mathcal{F}} = \left[\dfrac{dz}{f_1 \cdots f_n}\right]$ can be decomposed into

$$\omega_{\mathcal{F}} = \omega_1 + \cdots + \omega_\lambda + \cdots + \omega_\ell$$

with $\omega_\lambda \in H^n_{[Z_\lambda]}(\Omega^n_X)$, $\lambda = 1, \ldots, \ell$. Let $\varphi(z) \in K[z]$ and let $\beta \in Z_\lambda$. Since $\omega_{\mathcal{F}} = \omega_\lambda$ on Z_λ, we have

$$\mathrm{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z)) = \mathrm{Res}_\beta(\omega_\lambda)(\varphi(z)).$$

To compute the Grothendieck local residue $\mathrm{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z))$ at $\beta \in Z_\lambda$, it suffices to consider the linear functional $\mathrm{Res}_\beta(\omega_\lambda)$ associated to the direct summand ω_λ of $\omega_{\mathcal{F}}$. Since $\omega_\lambda I_\lambda = 0$ holds, $\mathrm{Res}_\beta(\omega_\lambda)$ defines a linear functional acting on the space $K[z]/I_\lambda$.

Taking these facts in account, we introduce vector spaces $E_{I_\lambda} = K[z]/I_\lambda$ and $E_{\sqrt{I_\lambda}} = K[z]/\sqrt{I_\lambda}$. Let $j_{\mathcal{F}}(z) = \det\left(\dfrac{\partial(f_1,\dots,f_n)}{\partial(z_1,\dots,z_n)}\right)$ be the Jacobian of $f_1,\dots,f_n$. Let us consider the correspondence γ which assignes $j_{\mathcal{F}}(z)g(z) \bmod I_\lambda$ to $g(z)$.

Lemma 3.1. — *Let* $\gamma(g) = j_{\mathcal{F}}(z)g(z) \bmod I_\lambda$. *Then*

(i) $\gamma : E_{\sqrt{I_\lambda}} \to E_{I_\lambda}$ *is a well-defined linear map.*
(ii) $\gamma : E_{\sqrt{I_\lambda}} \to E_{I_\lambda}$ *is injective.*

Proof. — Let $J_{\mathcal{F}} = \langle j_{\mathcal{F}}(z)\rangle \subset K[z]$ be the ideal generated by $j_{\mathcal{F}}(z)$. Then the ideal quotient $I_\lambda : J_{\mathcal{F}}$ is equal to the radical $\sqrt{I_\lambda}$.

(i) Let $g \in \sqrt{I_\lambda}$. Then $j_{\mathcal{F}}(z)g(z)$ is in I_λ which means the well-definedness of the map $\gamma : E_{\sqrt{I_\lambda}} \to E_{I_\lambda}$.

(ii) Let $g \in E_{\sqrt{I_\lambda}}$ and assume $\gamma(g) = 0$ in E_{I_λ}. Then, $j_{\mathcal{F}}(z)g(z) \in I_\lambda$ and thus $g(z) \in \sqrt{I_\lambda}$, *i.e.*, $g = 0$ in $E_{\sqrt{I_\lambda}}$. □

Let $j_{\mathcal{F},\lambda}(z) = j_{\mathcal{F}}(z) \bmod I_\lambda \in E_{I_\lambda}$. Then, $\gamma(g) = j_{\mathcal{F},\lambda}(z)g(z) \bmod I_\lambda$. We introduce $E_{J,\lambda}$ to be

$$E_{J,\lambda} = \mathrm{Im}\gamma = \{j_{\mathcal{F},\lambda}(z)g(z) \bmod I_\lambda \mid g \in E_{\sqrt{I_\lambda}}\}.$$

Let $E_{K,\lambda}$ denote the subspace of E_{I_λ} defined to be

$$E_{K,\lambda} = \{h(z) \in E_{I_\lambda} \mid \mathrm{Res}_\beta(\omega_{\mathcal{F}})(h(z)) = 0, \beta \in Z_\lambda\}.$$

Proposition 3.2. — $E_{I_\lambda} = E_{J,\lambda} \oplus E_{K,\lambda}$.

Proof. — It follows from $\dim E_{I_\lambda} = \#Z_\lambda + \dim E_{K,\lambda}$ and $\#Z_\lambda = \dim E_{\sqrt{I_\lambda}}$ that

$$\dim E_{I_\lambda} = \dim E_{\sqrt{I_\lambda}} + \dim E_{K,\lambda}.$$

Thus, Lemma 3.1 implies $\dim E_{\sqrt{I_\lambda}} = \dim E_{J,\lambda}$, which gives

$$\dim E_{I_\lambda} = \dim E_{J,\lambda} + \dim E_{K,\lambda}.$$

The proof of Lemma 3.1 also yields $E_{J,\lambda} \cap E_{K,\lambda} = \{0\}$, which completes the proof. □

By Proposition 3.2, we see that, for any polynomial $\varphi(z) \in K[z]$, there exist polynomials $g_\lambda(z) \in E_{\sqrt{I_\lambda}}$ and $h_\lambda(z) \in E_{K,\lambda}$ such that

$$\varphi(z) = j_{\mathcal{F},\lambda}(z)g_\lambda(z) + h_\lambda(z) \mod I_\lambda. \tag{2}$$

Let $\mu_\lambda = \dim E_{I_\lambda} / \dim E_{\sqrt{I_\lambda}}$.

Lemma 3.3. — *Let* $\varphi(z) \in K[z]$ *and* $\varphi_\lambda(z) = \varphi(z) \bmod I_\lambda \in E_{I_\lambda}$. *Assume* $\varphi_\lambda(z) = \gamma(g(z)) + h(z)$. *Then,* $\mathrm{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z)) = \mu_\lambda g(\beta)$.

Proof. — Since $\beta \in Z_\lambda$,

$$\begin{aligned}\operatorname{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z)) &= \operatorname{Res}_\beta(\omega_\lambda)(\varphi(z))\\ &= \operatorname{Res}_\beta(\omega_\lambda)(\varphi_\lambda(z))\\ &= \operatorname{Res}_\beta(\omega_\lambda)(\gamma(g(z)) + h(z))\\ &= \operatorname{Res}_\beta(\omega_\lambda)(j_{\mathcal{F},\lambda}(z)g_\lambda(z)).\end{aligned}$$

Let $\delta_{Z_\lambda} \in H^n_{[Z_\lambda]}(\Omega^n_X)$ be the delta function supported on Z_λ. Since the multiplicity at $\beta \in Z_\lambda$ of the ideal I_λ is equal to μ_λ, we have $j_{\mathcal{F},\lambda}(z)\omega_\lambda = \mu_\lambda \delta_{Z_\lambda}$. Thus we have

$$\begin{aligned}\operatorname{Res}_\beta(\omega_\lambda)(j_{\mathcal{F},\lambda}(z)g(z)) &= \operatorname{Res}_\beta(j_{\mathcal{F},\lambda}(z)\omega_\lambda)(g(z))\\ &= \mu_\lambda \operatorname{Res}_\beta(\delta_{Z_\lambda})(g(z))\\ &= \mu_\lambda g(\beta),\end{aligned}$$

which implies

$$\operatorname{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z)) = \mu_\lambda g(\beta). \qquad \square$$

Let $A_n := K[z_1, \dots, z_n]\langle \partial/\partial z_1, \dots, \partial/\partial z_n\rangle$ be the Weyl algebra on n variables $z = (z_1, \dots, z_n) \in X$. Let $\operatorname{Ann}_{A_n}(\omega_\lambda)$ be the right ideal of A_n given by annihilators of the cohomology class ω_λ. The right module $A_n/\operatorname{Ann}_{A_n}(\omega_\lambda)$ is a simple holonomic system at each point $\beta \in Z_\lambda$. And thus the dimension of the solution space $\operatorname{Hom}_{A_n}(A_n/\operatorname{Ann}_{A_n}(\omega_\lambda), \omega_\lambda A_n)$ is equal to $\# Z_\lambda = \dim E_{\sqrt{I_\lambda}}$. Reasoning on the duality for the holonomic system $A_n/\operatorname{Ann}_{A_n}(\omega_\lambda)$ yields the following result, which is the counterpart in the Weyl algebra of the Theorem 1.

Theorem 3.4. — *Let $R_1, \dots, R_s$ be generators of $\operatorname{Ann}_{A_n}(\omega_\lambda)$. For $h(z) \in E_{I,\lambda}$, the following two conditions are equivalent;*

(i) *$h(z) \in E_{K,\lambda}$, i.e., $\operatorname{Res}_\beta(\omega_{\mathcal{F}})(h(z)) = 0$ holds for $^\forall \beta \in Z_\lambda$.*

(ii) *There exist $u_1(z), \dots, u_s(z) \in K[z]$ such that $h(z) = \sum_{k=1}^s R_k u_k(z) \bmod I_\lambda$.*

Example 3.5. — Let $f_1 = 144y^4 + (288x^2 + 2304x + 952)y^2 + 144x^4 - 768x^3 + 952x^2 - 343$ and $f_2 = 36y^2 + 36x^2 - 49$.

The primary decomposition of the ideal $I = \langle f_1, f_2\rangle$ is given by $I = I_1 \cap I_2$ where $I_1 = \langle 144x^2 + 168x + 49, 144y^2 - 168x - 245\rangle$ and $I_2 = \langle 6x - 7, y^2\rangle$. Let $\omega_{\mathcal{F}} = \left[\frac{dx\wedge dy}{f_1 f_2}\right] = \omega_1 + \omega_2$ where $\omega_1 \in H^2_{[Z_1]}(\Omega^2_X)$ and $\omega_2 \in H^2_{[Z_2]}(\Omega^2_X)$. A monomial basis of $K[x, y]/I_1$ is $\{1, x, y, xy\}$. The annihilating ideal of the algebraic local cohomology classes ω_1 is generated by I_1 and $P_1 = (84x + 49)\frac{\partial}{\partial x} + (48xy + 28y)\frac{\partial}{\partial y} - 84$. By the computation

$$\begin{aligned}P_1 1 &= -84,\\ P_1 x &= 49,\\ P_1 y &= 48xy - 56y,\\ P_1 xy &= -28xy + \tfrac{98}{3}y\end{aligned}$$

in $E_{I,1}$, we find $\operatorname{Span}_K\{P_1 1, P_1 x, P_1 y, P_1 xy\} = \operatorname{Span}_K\{1, 6xy - 7y\}$.

We find $K[x,y]/I_2 \cong \mathrm{Span}_K\{1,y\}$. The annihilating ideal of ω_2 is generated by I_2 and $y\frac{\partial}{\partial y} - 1$. We have $\mathrm{Span}_K\{P_1 1, P_1 y\} = \mathrm{Span}_K\{1,y\}$.

Remark 3.6. — There is an algorithm, due to T.Oaku ([**6**], see also [**10**]), for computing the Gröbner basis of the left ideal $\mathrm{Ann}_{A_n}(\sigma_{\mathcal{F}})$ of the algebraic local cohomology class $\sigma_{\mathcal{F}} = \left[\dfrac{1}{f_1 \dots f_n}\right] \in H^n_{[Z]}(\mathcal{O}_X)$. The annihilating ideal $\mathrm{Ann}_{A_n}(\sigma_\lambda)$ of the direct summand $\sigma_\lambda \in H^n_{[Z_\lambda]}(\mathcal{O}_X)$ of $\sigma_{\mathcal{F}}$ is equal to the left ideal in A_n generated by $\mathrm{Ann}_{A_n}(\sigma_{\mathcal{F}}) \cup I_\lambda$, *i.e.*,

$$\mathrm{Ann}_{A_n}(\sigma_\lambda) = A_n\langle \mathrm{Ann}_{A_n}(\sigma_{\mathcal{F}}), I_\lambda\rangle.$$

Thus generators of the right ideal $\mathrm{Ann}_{A_n}(\omega_\lambda) = \{R \in A_n \mid R^* \in \mathrm{Ann}_{A_n}(\sigma_\lambda)\}$ can be explicitly constructed by using Gröbner basis computation in A_n. For an alternative approach, we refer the reader to [**5**].

Theorem 3 ensures that the decomposition (2) of $\varphi(z)$ can be rewritten in the form

$$\varphi(z) = j_{\mathcal{F},\lambda}(z)g_\lambda(z) + \sum_{k=1}^{s} R_k u_k(z) \mod I_\lambda$$

where $g_\lambda(z) \in E_{\sqrt{I_\lambda}}$, $u_k(z) \in K[z]$, $k = 1,\dots,s$ and $R_k \in \mathrm{Ann}_{A_n}(\omega_\lambda)$, $k = 1,\dots,s$ are generators of $\mathrm{Ann}_{A_n}(\omega_\lambda)$. Then, the formula $\rho - \mu_\lambda g_\lambda(\beta) = 0$ represents the relation between residues and the variety. The final step of the computation of residues is achieved in the following manner; Find a generator $r_\lambda(\rho)$ of the intersection of $K[\rho]$ and the ideal in $K[\rho,z]$ given by $\sqrt{I_\lambda}$ and $\rho - \mu_\lambda g_\lambda(z)$, *i.e.*, $\langle r_\lambda(\rho)\rangle = K[\rho] \cap \langle \sqrt{I_\lambda}, \rho - \mu_\lambda g_\lambda(z)\rangle$. Then the roots of $r_\lambda(\rho) = 0$ are exactly the residue $\mathrm{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z))$, $\beta \in Z_\lambda$. This computation can be done, for instance, through Gröbner basis computations in the polynomial ring $K[\rho,z]$.

4. Algorithm for computing residues with first order differential operators

We use the notation as in the preceding section and we recall properties of the first order partial differential operator that annihilates the algebraic local cohomology class ω_λ. Upon using first order annihilators, we derive an algorithm for computing Grothendieck local residues which works for almost every case.

4.1. Use of first order annihilators. — Let W_λ be the vector space defined by

$$W_\lambda = \{h(z)\omega_{\mathcal{F}} \mid h(z) \in K[z],\ \mathrm{supp}(h(z)\omega_{\mathcal{F}}) \subseteq Z_\lambda\},$$

the image by ι of the extension group $Ext^n_{K[z]}(K[z]/I_\lambda, K[z]dz)$. Since $W_\lambda = K[z]\omega_\lambda$ and $\mathrm{Ann}_{K[z]}(\omega_\lambda) = I_\lambda$ hold, we have the following proposition;

***Proposition 4.1* ([8])**. — *Let $P \in A_n$ be a first order differential operator which annihilates ω_λ. Then*

(i) W_λ *is closed under the right action of* P*, i.e.,* $\omega P \in W_\lambda$, $^\forall \omega \in W_\lambda$.
(ii) I_λ *is closed under the left action of* P*, i.e.,* $P(f) \in I_\lambda$, $^\forall f \in I_\lambda$.

Let $\mathrm{Ann}^{(1)}_{A_n}(\omega_\lambda)$ be the right ideal in A_n generated by differential operators of order at most one that annihilate the algebraic local cohomology class ω_λ. Let $E_{L,\lambda}$ be a subspace of $E_{K,\lambda}$ defined to be

$$E_{L,\lambda} = \{Rh \bmod I_\lambda \mid R \in \mathrm{Ann}^{(1)}_{A_n}(\omega_\lambda), h \in K[z]\},$$

which is equal to

$$\{Rh \bmod I_\lambda \mid \omega_\lambda R = 0,\, \mathrm{ord}(R) = 1,\, h \in K[z]\}. \tag{3}$$

Proposition 4.1 yields

$$E_{L,\lambda} = \{Rh \bmod I_\lambda \mid \omega_\lambda R = 0,\, \mathrm{ord}(R) = 1,\, h \in E_{I_\lambda}\}.$$

Let M_{I_λ} be a finite set of monomials of $K[z]$ satisfying the condition that the space $B_{I_\lambda} = \mathrm{Span}_K\{m(z) \in M_{I_\lambda}\}$ generated by these monomials is isomorphic to E_λ as a vector space. Such monomials M_{I_λ} can be obtained by Gröbner basis computations with respect to a term order in $K[z]$.

Definition 4.2. — $L_\lambda = \{P = p_1(z)\dfrac{\partial}{\partial z_1} + \cdots + p_n(z)\dfrac{\partial}{\partial z_n} + q(z) \in A_n \mid p_i(z), q(z) \in B_{I_\lambda},\, i = 1, \dots, n,\, \omega_\lambda P = 0\}$.

We have the following lemma;

Lemma 4.3. — *Let* $R \in A_n$ *be a linear partial differential operator of order one which annihilates* ω_λ*. Then there exists a linear partial differential operator* P *in* L_λ *such that* $R - P \in I_\lambda A_n$.

We thus have the following results;

***Proposition 4.4* ([5])**. — I_λ *and* L_λ *generate the ideal* $\mathrm{Ann}^{(1)}_{A_n}(\omega_\lambda)$ *over* A_n*, i.e.,* $\mathrm{Ann}^{(1)}_{A_n}(\omega_\lambda) = (I_\lambda \cup L_\lambda)A_n$ *holds.*

Proposition 4.5

$$E_{L,\lambda} \cong \{q(z) \in B_{I_\lambda} \mid P = p_1(z)\frac{\partial}{\partial z_1} + \cdots + p_n(z)\frac{\partial}{\partial z_n} + q(z) \in L_\lambda\}.$$

Proof. — By (3),

$$E_{L,\lambda} = \{Rh \bmod I_\lambda \mid \omega_\lambda R = 0,\, \mathrm{ord}(R) = 1,\, h \in K[z]\}.$$

Since the first order partial differential operator $R \circ h$, a composition of R and the multiplication operator h, annihilates ω_λ, it follows from the above lemma that there exists a $P \in L_\lambda$ such that $Rh - P \in I_\lambda A_n$. We thus have

$$Rh = (R \circ h)1 = (R \circ h - P)1 + P1 = P1 \mod I_\lambda,$$

which completes the proof. □

In Appendix of the present paper, we give an algorithm that computes the vector space L_λ.

We arrive at the following result;

Theorem 4.6. — *The following conditions are equivalent;*

(i) $E_{K,\lambda} = E_{L,\lambda}$.

(ii) $\mathrm{Ann}_{A_n}(\omega_\lambda) = \mathrm{Ann}^{(1)}_{A_n}(\omega_\lambda)$.

Corollary 4.7. — *If* $\dim E_{L,\lambda} = \dim E_{I_\lambda} - \dim E_{\sqrt{I_\lambda}}$, *then* $E_{K,\lambda} = E_{L,\lambda}$.

4.2. Algorithm for computing residues. — When conditions in Theorem 4.6 are satisfied, or equivalently $\dim E_{L,\lambda} = \dim E_{I_\lambda} - \dim E_{\sqrt{I_\lambda}}$ holds, one can compute the residues $\mathrm{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z))$ for $\beta \in Z_\lambda$ in the following manner; Let $g_1(z), \ldots, g_{d_\lambda}(z)$ be a canonical monomial basis of the vector space $E_{\sqrt{I_\lambda}}$, where $d_\lambda = \dim E_{\sqrt{I_\lambda}}$. Let $q_1(z), \ldots, q_{\kappa_\lambda}(z)$ ($\kappa_\lambda = \dim E_{I_\lambda} - d_\lambda$) be a basis of $E_{L,\lambda}$. A polynomial $\varphi(z) \in K[z]$ can be represented in $K[z]/I_\lambda$ as

$$\varphi(z) = j_{\mathcal{F},\lambda}(z)g(z) + (b_1 q_1(z) + \cdots + b_{\kappa_\lambda} q_{\kappa_\lambda}(z)) \mod I_\lambda$$

with $g(z) = \sum_{j=1}^{d_\lambda} c_j g_j(z)$, $c_j \in K$, $j = 1, \ldots, d_\lambda$ and $b_k \in K$, $k = 1, \ldots, \kappa_\lambda$. Thus, by Lemma 3.3, we have

$$\mathrm{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z)) = \mu_\lambda g(\beta).$$

The output of the following algorithm is the desired univariate polynomial $r_\lambda(\rho)$ such that

$$\{\rho \in \mathbb{C} \mid r_\lambda(\rho) = 0\} = \{\rho \in \mathbb{C} \mid \rho = \mathrm{Res}_\beta([\frac{\varphi(z)dz}{f_1 \cdots f_n}]),\ \beta \in Z_\lambda\}.$$

Algorithm R (Computation of the Grothendieck local residue)

Input : a regular sequence $f_1(z), \ldots, f_n(z)$, a holomorphic n-form $\varphi(z)dz$, the Gröbner bases of primary ideals I_λ, $\sqrt{I_\lambda}$.

(i) Choose a basis $q_1(z), \ldots, q_{\kappa_\lambda}(z)$ of $E_{L,\lambda}$ from the output of Algorithm A (in Appendix).

(ii) Choose the basis $g_1(z), \ldots, g_{d_\lambda}(z) \in E_{\sqrt{I_\lambda}}$ and compute $e_j := j_{\mathcal{F},\lambda}(z) g_j(z) \mod I_\lambda$, $j = 1, \ldots, d_\lambda$, where $j_{\mathcal{F},\lambda}(z) = j_{\mathcal{F}}(z) \mod I_\lambda$ with $j_{\mathcal{F}}(z) = \det\left(\frac{\partial(f_1,\ldots,f_n)}{\partial(z_1,\ldots,z_n)}\right)$.

(iii) Compute $\varphi_\lambda(z) = \varphi(z) \mod I_\lambda \in E_{I_\lambda}$.

(iv) Determine the coefficients c_j, $j = 1, \ldots, d_\lambda$ by solving the following linear equation;

$$\varphi_\lambda = c_1 e_1 + \cdots + c_{d_\lambda} e_{d_\lambda} + b_1 q_1 + \cdots + b_t q_t.$$

(v) Put $g(z) = \sum_{j=1}^{d_\lambda} c_j g_j(z)$.

(vi) Compute a generator $r_\lambda(\rho)$ of the ideal $K[\rho] \cap \langle \sqrt{I_\lambda}, \rho - \mu_\lambda g(z) \rangle$ where $\mu_\lambda = \dim E_{I_\lambda} / \dim E_{\sqrt{I_\lambda}}$ is the multiplicity of the point β.

Output : the polynomial $r_\lambda(\rho)$.

The above algorithm may admit several extension. One of the most natural generalizations is probably the use of higher order annihilators. Such a generalization, which involves construction of higher order annihilators, will be treated in elsewhere ([**9**]). In the rest of this section, we give an example for illustration.

Example 4.8. — Let $f_1 = (x^2+y^2-1)^2$ and $f_2 = (x^2+y^2)^2 + 3x^2y - y^3$.

The primary decomposition of the ideal $I = \langle f_1, f_2\rangle$ is given by $I_1 \cap I_2$ where

$$I_1 = \langle 16y^4 + 32y^3 + 24y^2 + 8y + 1, 80y^3 + 107y^2 + 48y - x^2 + 8\rangle$$

and

$$I_2 = \langle y^2 - 2y + 1, y + 5x^2 - 1\rangle$$

with the radical $\sqrt{I_1} = \langle 4x^2 - 3, 2y+1\rangle$ and $\sqrt{I_2} = \langle x, y-1\rangle$. The varieties are $Z_1 = \boldsymbol{V}(I_1) = \{(\frac{\sqrt{3}}{2}, -\frac{1}{2}), (\frac{-\sqrt{3}}{2}, -\frac{1}{2})\}$ and $Z_2 = \boldsymbol{V}(I_2) = \{(0,1)\}$. Let $\omega_{\mathcal{F}} = \omega_1 + \omega_2$ with $\omega_1 \in H^2_{[Z_1]}(\Omega^2_X)$ and $\omega_2 \in H^2_{[Z_2]}(\Omega^2_X)$. The Jacobian $j_{\mathcal{F}}(x,y)$ is $-36xy^4 + (-24x^3+36x)y^2 + 12x^5 - 12x^3$.

Let us compute residues $\mathrm{Res}_\beta\left(\left[\frac{\varphi(x,y)}{f_1 f_2}dx\wedge dy\right]\right)$ on Z_1 and at Z_2.

(a) Computation on $Z_1 = \{(\frac{\sqrt{3}}{2}, -\frac{1}{2}), (-\frac{\sqrt{3}}{2}, -\frac{1}{2})\}$. — We identify

$$K[x,y]/I_1 \cong \mathrm{Span}_K\{1, x, xy, xy^2, xy^3, y, y^2, y^3\}$$

and $K[x,y]/\sqrt{I_1} \cong \mathrm{Span}_K\{1, x\}$. Algorithm A outputs following six first order annihilators that form a basis of the vector space L_1;

$$\begin{aligned}
&\Big(-\frac{197}{62}y^3 - \frac{927}{248}y^2 - \frac{699}{496}y - \frac{83}{496}\Big)\frac{\partial}{\partial x}\\
&\qquad + \Big(-\frac{11}{31}xy^3 - \frac{81}{62}xy^2 - \frac{285}{248}xy - \frac{145}{496}x\Big)\frac{\partial}{\partial y} + xy^3 + \frac{14}{31}x,\\
&\Big(-\frac{941}{93}y^3 - \frac{3501}{248}y^2 - \frac{407}{62}y - \frac{3029}{2976}\Big)\frac{\partial}{\partial x}\\
&\qquad + \Big(\frac{607}{186}xy^3 + \frac{159}{31}xy^2 + \frac{647}{248}xy + \frac{40}{93}x\Big)\frac{\partial}{\partial y} + xy^2 - \frac{1}{31}x,\\
&\Big(\frac{861}{31}y^3 + \frac{2333}{62}y^2 + \frac{526}{31}y + \frac{79}{31}\Big)\frac{\partial}{\partial x}\\
&\qquad + \Big(-\frac{200}{31}xy^3 - \frac{277}{31}xy^2 - \frac{233}{62}xy - \frac{14}{31}x\Big)\frac{\partial}{\partial y} + xy - \frac{16}{31}x,\\
&\Big(\frac{5045}{558}xy^3 + \frac{401}{31}xy^2 + \frac{13775}{2232}xy + \frac{274}{279}x\Big)\frac{\partial}{\partial x}\\
&\qquad + \Big(-\frac{151}{279}y^3 - \frac{2351}{2232}y^2 - \frac{347}{558}y - \frac{343}{2976}\Big)\frac{\partial}{\partial y} + y^3 - \frac{3}{31}y,\\
&\Big(-\frac{1012}{279}xy^3 - \frac{165}{31}xy^2 - \frac{1451}{558}xy - \frac{118}{279}x\Big)\frac{\partial}{\partial x}\\
&\qquad + \Big(\frac{175}{279}y^3 + \frac{665}{558}y^2 + \frac{202}{279}y + \frac{53}{372}\Big)\frac{\partial}{\partial y} + y^2 + \frac{16}{31}y,
\end{aligned}$$

$$\Big(-\frac{416}{9}xy^3-64xy^2-\frac{266}{9}xy-\frac{41}{9}x\Big)\frac{\partial}{\partial x} + \Big(-\frac{16}{9}y^3-\frac{34}{9}y^2-\frac{25}{9}y-\frac{2}{3}\Big)\frac{\partial}{\partial y}+1.$$

Thus, by Proposition 4.5, we have

$$E_{L,1}=\mathrm{Span}_K\left\{1, y^2+\frac{16}{31}y, y^3-\frac{3}{31}y, xy-\frac{16}{31}x, xy^2-\frac{1}{31}x, xy^3+\frac{14}{31}x\right\}.$$

Put $q_1(x,y)=1$, $q_2(x,y)=y^2+\frac{16}{31}y$, $q_3(x,y)=y^3-\frac{3}{31}y$, $q_4(x,y)=xy-\frac{16}{31}x$, $q_5(x,y)=xy^2-\frac{1}{31}x$, $q_6(x,y)=xy^3+\frac{14}{31}x$ and $g_1(x,y)=1$, $g_2(x,y)=x$. We find that $e_1(x,y)=-576xy^3-864xy^2-432xy-72x$ and $e_2(x,y)=-432y^3-648y^2-324y-54$. For $\varphi_1(x,y)=-159y^3-216y^2+(-4x-96)y-14$, we have

$$\varphi_1=\frac{8}{243}e_1-\frac{1}{324}e_2-\frac{85}{6}q_1-218q_2-\frac{481}{3}q_3+\frac{92}{9}q_4+\frac{256}{9}q_5+\frac{512}{27}q_6,$$

which implies

$$\varphi(x,y)=\Big(\frac{8}{243}+\Big(-\frac{1}{324}x\Big)\Big)j_{\mathcal{F}}(x,y)-\frac{85}{6}-218\Big(y^2+\frac{16}{31}y\Big)-\frac{481}{3}\Big(y^3-\frac{3}{31}y\Big) + \Big(\frac{92}{9}\Big)\Big(xy-\frac{16}{31}x\Big)+\Big(\frac{256}{9}\Big)\Big(xy^2-\frac{1}{31}x\Big)+\Big(\frac{512}{27}\Big)\Big(xy^3+\frac{14}{31}x\Big) \mod I_1.$$

Thus we find $g(x,y)=\frac{8}{243}-\frac{1}{324}x$. The Gröbner basis of $\langle 4x^2-3, 2y+1, \rho-2g(x,y)\rangle$ with respect to the lexicographical ordering $x\succ y\succ\rho$ is $\{236196\rho^2-62208\rho+4069, 2y+1, 3x+243\rho-32\}$. $r_\lambda(\rho)=236196\rho^2-62208\rho+4069$ is the desired polynomial.

(b) Computation at $Z_2=\{(0,1)\}$. — We identify $K[x,y]/I_2\cong\mathrm{Span}_K\{1,x,y,xy\}$ and $K[x,y]/\sqrt{I_2}\cong\mathrm{Span}_K\{1\}$. The following three first order annihilators are the output of the Algorithm A ;

$$\Big(\frac{3}{62}y-\frac{3}{62}\Big)\frac{\partial}{\partial x}+\Big(-\frac{15}{31}xy+\frac{15}{31}x\Big)\frac{\partial}{\partial y}+xy-\frac{16}{31}x,$$
$$\Big(\frac{17}{45}xy-\frac{32}{45}x\Big)\frac{\partial}{\partial x}+\Big(-\frac{2}{3}y+\frac{2}{3}\Big)\frac{\partial}{\partial y}+y,$$
$$\Big(\frac{62}{45}xy-\frac{77}{45}x\Big)\frac{\partial}{\partial x}+\Big(-\frac{2}{3}y+\frac{2}{3}\Big)\frac{\partial}{\partial y}+1.$$

Thus, we have

$$E_{L,2}=\mathrm{Span}_K\left\{1, y, xy-\frac{16}{31}x\right\}.$$

Put $q_1(x,y)=1$, $q_2(x,y)=y$, $q_3(x,y)=xy-\frac{16}{31}x$. We find

$$e_1(x,y)=j_{\mathcal{F}}(x,y) \mod I_2=-\frac{324}{5}xy+\frac{324}{5}x.$$

Then, $\varphi_2(x,y) = \varphi(x,y) \bmod I_2$ is rewritten in the following form;

$$\begin{aligned}\varphi_2(x,y) &= \Big(-4x - \frac{3}{5}\Big)y + \frac{8}{5}\\ &= -\frac{16}{243}e_1(x,y) + \frac{8}{5}q_1(x,y) - \frac{3}{5}q_2(x,y) - \frac{124}{15}q_3(x,y).\end{aligned}$$

We find $g(x,y) = -\dfrac{16}{243}$. Thus we have $\langle x, y-1, \rho - 4g(x,y)\rangle = \langle 243\rho + 64, x, y-1\rangle$. We arrive at $\mathrm{Res}_{(0,0)}\Big(\Big[\frac{\varphi(x,y)dx\wedge dy}{f_1 f_2}\Big]\Big) = -64/243$.

5. Example

Let $\varphi_\lambda(z)$ be a polynomial in $K[z]$ and let $\varphi_\lambda(z) = \varphi(z) \bmod I_\lambda \in E_{I_\lambda}$. It is obvious that if the condition

$$\varphi_\lambda(z) \in E_{J,\lambda} \oplus E_{L,\lambda} \tag{4}$$

holds, one can apply Algorithm R to compute the Grothendieck local residue $\mathrm{Res}_\beta\Big(\Big[\frac{\varphi(z)dz}{f_1\cdots f_n}\Big]\Big)$, $\beta \in Z_\lambda$. This fact does not imply that the range of application of Algorithm R is the condition (4). In this section, we present an example to show the usage of the Algorithm R and illustrate a method to extend the range of application of Algorithm R.

Example 5.1. — Let I be an ideal generated by $f_1 = 3x^4 - 6x^3 + 3x^2 + y^5$, $f_2 = 5y^4x + 7y^6$. The Gröbner basis of the ideal $I = \langle f_1, f_2\rangle$ with respect to the graded total lexicographical ordering (with $x \succ y$) is given by

$$\{-3087x^8 + 12348x^7 - 18522x^6 + 12348x^5 - 3087x^4 - 125x^3y^4,$$
$$-21x^4y + 42x^3y - 21x^2y + 5xy^4, 3x^4 - 6x^3 + 3x^2 + y^5\}.$$

Its primary decomposition is $I = I_1 \cap I_2 \cap I_3$ where

$$\begin{aligned}&I_1 = \langle 147x^2 - 294x + 25y + 147, 5x + 7y^2\rangle,\\ &\sqrt{I_1} = \langle 147x^2 - 294x + 25y + 147, 5x + 7y^2\rangle,\\ &I_2 = \langle y^4, x^2 - 2x + 1\rangle, \sqrt{I_2} = \langle x-1, y\rangle,\\ &I_3 = \langle 21x^2y - 5xy^4, 6x^3 - 3x^2 - y^5, x^4, x^3y, 5x^3 + 7x^2y^2\rangle, \sqrt{I_3} = \langle x, y\rangle.\end{aligned}$$

We find $Z_1 = V(I_1)$ consists of four simple points, $Z_2 = V(I_2) = \{(0,1)\}$ with the multiplicity $\mu_2 = 8$ and $Z_3 = V(I_3) = \{(0,0)\}$ with the multiplicity $\mu_3 = 12$. Let $\omega_{\mathcal F} = \omega_1 + \omega_2 + \omega_3$ with $\omega_\lambda \in H^2_{[Z_\lambda]}(\Omega^2_X)$, $\lambda = 1,2,3$.

(a) Since $\mu_1 = 1$, $E_{I_1} = E_{J,1}$ and thus $E_{K,1} = E_{L,1} = \{0\}$, one can apply Algorithm R to compute $\mathrm{Res}_\beta\left(\left[\dfrac{\varphi(x,y)dx\wedge dy}{f_1 f_2}\right]\right)$ for any $\varphi(x,y) \in K[x,y]$ without computing $E_{L,1}$

(b) We use the following identification; $E_{I_2} \cong \mathrm{Span}_K\{1, y, x, y^2, xy, y^3, xy^2, xy^3\}$. Algorithm A outputs the following 10 operators which form a basis of the vector space L_2;

$$(\tfrac{5}{14}xy - \tfrac{5}{14}y)\tfrac{\partial}{\partial x} + (-\tfrac{15}{28}xy^2 + \tfrac{5}{28}y^2)\tfrac{\partial}{\partial y} + xy^3 + \tfrac{5}{14}y,$$
$$(-xy^3)\tfrac{\partial}{\partial y} + xy^2,$$
$$(-\tfrac{1}{4}xy + \tfrac{1}{4}y)\tfrac{\partial}{\partial x} + (-\tfrac{1}{8}xy^2 + \tfrac{1}{8}y^2)\tfrac{\partial}{\partial y} + xy - \tfrac{3}{4}y,$$
$$(-\tfrac{1}{3}xy + \tfrac{14}{15}y^3)\tfrac{\partial}{\partial y} + x,$$
$$(\tfrac{5}{28}xy - \tfrac{5}{28}y)\tfrac{\partial}{\partial x} + (-\tfrac{15}{56}xy^2 - \tfrac{5}{56}y^2)\tfrac{\partial}{\partial y} + y^3 + \tfrac{15}{28}y,$$
$$-y^3\tfrac{\partial}{\partial y} + y^2,$$
$$(-\tfrac{14}{15}xy^3 + \tfrac{28}{15}y^3 - \tfrac{1}{3}y)\tfrac{\partial}{\partial y} + 1,$$
$$(-xy^3 - \tfrac{5}{7}xy + y^3 + \tfrac{5}{7}y)\tfrac{\partial}{\partial x} + (\tfrac{15}{14}xy^2 - \tfrac{5}{7}y^2)\tfrac{\partial}{\partial y},$$
$$(-xy^2 + y^2)\tfrac{\partial}{\partial x} + (3xy^3 - 2y^3)\tfrac{\partial}{\partial y},$$
$$(-x+1)\tfrac{\partial}{\partial x} + (-\tfrac{49}{15}xy^3 + xy + \tfrac{7}{3}y^3 - \tfrac{2}{3}y)\tfrac{\partial}{\partial y}.$$

Taking the zeroth order parts of these operators, we have

$$E_{L,2} = \mathrm{Span}_K\{1, y^2, y^3 + \frac{15}{28}y, x, xy - \frac{3}{4}y, xy^2, xy^3 + \frac{5}{14}y\}.$$

Since $\dim E_{L,2} = 7$ which is equal to $\kappa_2 = 8 - 1$, we have$E_{K,2} = E_{L,2}$. Thus, one can compute local residues $\mathrm{Res}_\beta\left(\left[\frac{\varphi(x,y)}{f_1 f_2}dx \wedge dy\right]\right)$ at $\beta = (1, 0)$ by using the Algorithm R.

(c) We identify $E_{I_3} \cong \mathrm{Span}_K\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, xy^3, y^4\}$. Algorithm A outputs the following 14 operators which form a basis of the vector space L_3;

$$\begin{aligned}
&(-\tfrac{51584680665}{475773195424}x^2y - \tfrac{27238684725}{1903092781696}xy - \tfrac{661775625}{1903092781696}x^2)\tfrac{\partial}{\partial x}\\
&+ (-\tfrac{94539375}{1903092781696}y^4 + \tfrac{2924301625}{237886597712}xy^3 + \tfrac{308828625}{1903092781696}y^3 - \tfrac{27238684725}{3806185563392}xy^2\\
&- \tfrac{38134158615}{7612371126784}y^2 - \tfrac{10014110299529}{4757731954240}x^2y - \tfrac{44118375}{1903092781696}xy + \tfrac{44628681825}{475773195424}x^3\\
&- \tfrac{159058523925}{3806185563392}x^2 + \tfrac{3891240675}{7612371126784}x)\tfrac{\partial}{\partial y}\\
&+ xy^3 + \tfrac{16343210835}{475773195424}y,
\end{aligned}$$

$$\begin{aligned}
&(-\tfrac{791242092165}{475773195424}x^2y + \tfrac{637344986607}{1903092781696}xy + \tfrac{15484572075}{1903092781696}x^2)\tfrac{\partial}{\partial x}\\
&+ (\tfrac{2212081725}{1903092781696}y^4 - \tfrac{11784672235}{237886597712}xy^3 - \tfrac{7226133635}{1903092781696}y^3 + \tfrac{637344986607}{3806185563392}xy^2\\
&+ \tfrac{4461414906249}{38061855633920}y^2 + \tfrac{26125989295}{951546390848}x^2y + \tfrac{1032304805}{1903092781696}xy - \tfrac{1044245230899}{475773195424}x^3\\
&+ \tfrac{3721734504591}{3806185563392}x^2 - \tfrac{91049283801}{7612371126784}x)\tfrac{\partial}{\partial y}\\
&+ y^4 - \tfrac{1912034959821}{2378865977120}y.
\end{aligned}$$

$$\begin{aligned}
&(\tfrac{743128890965999}{59471649428000}x^2y + \tfrac{7848009842967}{47577319542400}xy + \tfrac{7626831723}{1903092781696}x^2)\tfrac{\partial}{\partial x}\\
&+ (\tfrac{1089547389}{1903092781696}y^4 - \tfrac{168509956839}{1189432988560}xy^3 - \tfrac{17795940687}{9515463908480}y^3 + \tfrac{7848009842967}{95154639084800}xy^2
\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{54936068900769}{951546390848000}y^2 + \tfrac{263609004939}{4757731954240}x^2y + \tfrac{2542277241}{9515463908480}xy - \tfrac{47827745671083}{11894329885600}x^3\\
&+ \tfrac{45827941913271}{95154639084800}x^2 - \tfrac{1121144263281}{190309278169600}x)\tfrac{\partial}{\partial y}\\
&+ x^3 - \tfrac{23544029528901}{59471649428000}y,
\end{aligned}
$$

$$
\begin{aligned}
&\quad(\tfrac{72218552931}{475773195424}x^2y + \tfrac{38134158615}{1903092781696}xy + \tfrac{926485875}{1903092781696}x^2)\tfrac{\partial}{\partial x}\\
&+ (\tfrac{132355125}{1903092781696}y^4 - \tfrac{4094022275}{237886597712}xy^3 - \tfrac{432360075}{1903092781696}y^3 + \tfrac{38134158615}{3806185563392}xy^2\\
&+ \tfrac{53387822061}{7612371126784}y^2 + \tfrac{6404494775}{951546390848}x^2y + \tfrac{61765725}{1903092781696}xy + \tfrac{752071906675}{3330412367968}x^3\\
&+ \tfrac{222681933495}{3806185563392}x^2 - \tfrac{5447736945}{7612371126784}x)\tfrac{\partial}{\partial y}\\
&+ x^2y - \tfrac{22880495169}{475773195424}y,
\end{aligned}
$$

$$
\begin{aligned}
&\quad(\tfrac{1107738929031}{475773195424}x^2y - \tfrac{4461414906249}{9515463908480}xy - \tfrac{21678400905}{1903092781696}x^2)\tfrac{\partial}{\partial x}\\
&+ (-\tfrac{3096914415}{1903092781696}y^4 - \tfrac{221388056583}{237886597712}xy^3 + \tfrac{10116587089}{1903092781696}y^3 - \tfrac{4461414906249}{19030927816960}xy^2\\
&- \tfrac{31229904343743}{190309278169600}y^2 - \tfrac{36576385013}{951546390848}x^2y - \tfrac{1445226727}{1903092781696}xy - \tfrac{2534551518303}{475773195424}x^3\\
&+ \tfrac{10775551059819}{3806185563392}x^2 + \tfrac{637344986607}{38061855633920}x)\tfrac{\partial}{\partial y}\\
&+ xy^2 + \tfrac{13384244718747}{11894329885600}y,
\end{aligned}
$$

$$
\begin{aligned}
&\quad(\tfrac{219769342315}{3330412367968}x^2y - \tfrac{162869373975}{1903092781696}xy - \tfrac{3956981875}{1903092781696}x^2)\tfrac{\partial}{\partial x}\\
&+ (-\tfrac{952111673973}{1903092781696}y^4 + \tfrac{23558397625}{713659793136}xy^3 + \tfrac{5539774625}{5709278345088}y^3 - \tfrac{5897817572065}{26643298943744}xy^2\\
&- \tfrac{228017123565}{7612371126784}y^2 - \tfrac{40777421125}{2854639172544}x^2y - \tfrac{791396375}{5709278345088}xy + \tfrac{1181324524075}{23312886575776}x^3\\
&- \tfrac{465582240375}{3806185563392}x^2 + \tfrac{23267053425}{7612371126784}x)\tfrac{\partial}{\partial y}\\
&+ y^3 + \tfrac{97721624385}{475773195424}y,
\end{aligned}
$$

$$
\begin{aligned}
&\quad(-\tfrac{7754172503217}{2378865977120}x^2y + \tfrac{31229904343743}{47577319542400}xy + \tfrac{30349761267}{1903092781696}x^2)\tfrac{\partial}{\partial x}\\
&+ (\tfrac{4335680181}{1903092781696}y^4 + \tfrac{2983943004259}{3568298965680}xy^3 - \tfrac{70816109623}{9515463908480}y^3 + \tfrac{31229904343743}{95154639084800}xy^2\\
&+ \tfrac{218609330406201}{951546390848000}y^2 - \tfrac{3989627868967}{14273195862720}x^2y + \tfrac{10116587089}{9515463908480}xy + \tfrac{17741860628121}{2378865977120}x^3\\
&- \tfrac{75428857418733}{19030927816960}x^2 - \tfrac{4461414906249}{190309278169600}x)\tfrac{\partial}{\partial y}\\
&+ x^2 - \tfrac{93689713031229}{59471649428000}y,
\end{aligned}
$$

$$
\begin{aligned}
&\quad(-\tfrac{43953868463}{475773195424}x^2y + \tfrac{228017123565}{1903092781696}xy + \tfrac{5539774625}{1903092781696}x^2)\tfrac{\partial}{\partial x}\\
&+ (\tfrac{791396375}{1903092781696}y^4 - \tfrac{32981756675}{713659793136}xy^3 - \tfrac{7755684475}{5709278345088}y^3 - \tfrac{723529267283}{3806185563392}xy^2\\
&+ \tfrac{319223972991}{7612371126784}y^2 + \tfrac{57088389575}{2854639172544}x^2y + \tfrac{1107954925}{5709278345088}xy - \tfrac{236264904815}{3330412367968}x^3\\
&+ \tfrac{651815136525}{3806185563392}x^2 - \tfrac{32573874795}{7612371126784}x)\tfrac{\partial}{\partial y}\\
&+ xy - \tfrac{136810274139}{475773195424}y,
\end{aligned}
$$

$$
\begin{aligned}
&\quad(\tfrac{34743651830461}{11894329885600}x^2y - \tfrac{2205324794687}{1903092781696}xy + \tfrac{13148030151145}{39964948415616}x^2)\tfrac{\partial}{\partial x}\\
&+ (\tfrac{8528573235}{1903092781696}y^4 - \tfrac{2136373705261}{2140979379408}xy^3 - \tfrac{5484245202773}{17127835035264}y^3 - \tfrac{2205324794687}{3806185563392}xy^2\\
&- \tfrac{15437273562809}{38061855633920}y^2 + \tfrac{23620397530591}{59947422623424}x^2y - \tfrac{11643589832491}{119894845246848}xy - \tfrac{11804151944417}{2378865977120}x^3\\
&+ \tfrac{95057716174309}{19030927816960}x^2 + \tfrac{315046399241}{7612371126784}x)\tfrac{\partial}{\partial y}\\
&+ y^2 + \tfrac{6615974384061}{2378865977120}y,
\end{aligned}
$$

$$
\begin{aligned}
&\quad(-\tfrac{243205562813227}{59471649428000}x^2 + \tfrac{15437273562809}{9515463908480}xy - \tfrac{2629606030229}{5709278345088}x^2)\tfrac{\partial}{\partial x}\\
&+ (-\tfrac{11940002529}{1903092781696}y^4 + \tfrac{14954615936827}{10704896897040}xy^3 - \tfrac{315046399241}{17127835035264}y^3 + \tfrac{15437273562809}{19030927816960}xy^2\\
&+ \tfrac{108060914939663}{190309278169600}y^2 - \tfrac{23620397530591}{42819587588160}x^2y - \tfrac{16902801892949}{85639175176320}xy + \tfrac{82629063610919}{11894329885600}x^3\\
&- \tfrac{665404013220163}{95154639084800}x^2 - \tfrac{2205324794687}{38061855633920x})\tfrac{\partial}{\partial y}\\
&+ x - \tfrac{46311820688427}{11894329885600}y,
\end{aligned}
$$

$$(-xy^3 - \tfrac{5}{7}x^2y)\tfrac{\partial}{\partial x} + (\tfrac{1}{2}y^4 + \tfrac{5}{14}xy^2 - \tfrac{21}{5}x^2y + \tfrac{25}{49}x^3)\tfrac{\partial}{\partial y},$$
$$(-y^4 - \tfrac{42}{5}x^2y + \tfrac{21}{5}xy)\tfrac{\partial}{\partial x} + (\tfrac{21}{10}xy^2 - \tfrac{21}{20}y^2 + \tfrac{3}{2}x^2 - \tfrac{3}{4}x)\tfrac{\partial}{\partial y},$$
$$(\tfrac{1029}{125}x^2y - x^3)\tfrac{\partial}{\partial x} + (-\tfrac{7}{5}xy^3 - \tfrac{588}{25}x^3 + \tfrac{147}{25}x^2)\tfrac{\partial}{\partial y},$$
$$(-xy^2 - \tfrac{5}{7}x^2)\tfrac{\partial}{\partial x} + (-\tfrac{2}{3}xy^3 + \tfrac{1}{3}y^3 - \tfrac{10}{21}x^2y + \tfrac{5}{21}xy)\tfrac{\partial}{\partial y}.$$

Taking zeroth order parts of these operators, we obtain a basis of $E_{L,3}$:

$$\Big\{x - \frac{46311820688427}{11894329885600}y,\ y^2 + \frac{6615974384061}{2378865977120}y,\ xy - \frac{136810274139}{475773195424}y,$$
$$x^2 - \frac{93689713031229}{59471649428000}y,\ y^3 + \frac{97721624385}{475773195424}y,\ xy^2 + \frac{13384244718747}{11894329885600}y,$$
$$x^2y - \frac{22880495169}{475773195424}y,\ x^3 - \frac{23544029528901}{59471649428000}y,\ y^4 - \frac{1912034959821}{2378865977120}y,$$
$$xy^3 + \frac{16343210835}{475773195424}y\Big\}.$$

Since $\dim E_{L,3} = 10$ and $\kappa_3 = \dim E_{K,3} = 11$, we have $E_{K,3} \supsetneq E_{L,3}$.

If $\varphi_3(x,y) = \varphi(x,y) \bmod I_3$ happens to be in $E_{J,3} \oplus E_{L,3}$, one can directly apply Algorithm R to compute $\mathrm{Res}_\beta\left(\left[\dfrac{\varphi(z)dx\wedge dy}{f_1f_2}\right]\right)$. But for the general case where $\varphi_3(x,y) \notin E_{J,3} \oplus E_{L,3}$, one can not *directly* use Algorithm R for computing local residues. Still, for such a case, one can compute local residues by applying Algorithm R in the following way;

Let $\mathcal{F}'$ be the regular sequence given by f_1^2, f_2, and let $\omega_{\mathcal{F}'} = \left[\dfrac{dx\wedge dy}{f_1^2f_2}\right]$. Let $I' = \langle f_1^2, f_2\rangle$ and its primary decomposition $I' = I_1' \cap I_2' \cap I_3'$ with $\sqrt{I_1'} = \sqrt{I_1}$, $\sqrt{I_2'} = \sqrt{I_2}$ and $\sqrt{I_3'} = \sqrt{I_3}$. We apply Algorithm A and compute the space $E_{L',3}$. One finds $E_{K',3} \neq E_{L',3}$ while one can verify

$$f_1\varphi_3 \in E_{J',3} \oplus E_{L',3} \quad \text{for any} \quad \varphi_3 \in E_{I_3}.$$

Thus by the use of the relation

$$\mathrm{Res}_\beta\left[\frac{\varphi_3 dx\wedge dy}{f_1f_2}\right] = \mathrm{Res}_\beta\left[\frac{f_1\varphi_3 dx\wedge dy}{f_1^2f_2}\right],\quad \beta \in Z_3 = \{(0,0)\},$$

one can apply Algorithm R to compute the Grothendieck local residue.

6. Appendix

In this section, as an appendix, we introduce a method for constructing first order annihilators of a direct summand of the algebraic local cohomology class $\omega_{\mathcal{F}}$.

Let $f_1, \dots, f_n$ be a regular sequence of polynomials in $K[z]$. Let I be the ideal generated by $f_1, \dots, f_n$ and $I_1 \cap \dots \cap I_\ell$ its primary decomposition. Let $\omega_\lambda \in H^n_{[Z_\lambda]}(\Omega^n_X)$ be the direct summand of $\omega_{\mathcal{F}}$.

Definition 6.1

$$L_\lambda = \{P = p_1(z)\frac{\partial}{\partial z_1} + \dots + p_n(z)\frac{\partial}{\partial z_n} + q(z) \mid \omega_\lambda P = 0, p_i(z) \in B_{I_\lambda}, i = 1, \dots, n, q(z) \in B_{I_\lambda}\}$$

Let $V_\lambda = \{v = \sum_{j=1}^n p_j(z)\frac{\partial}{\partial z_j} \mid vh(z) \in I_\lambda \quad \text{for} \quad \forall\, h \in I_\lambda, p_j(z) \in B_{I_\lambda}\}$.

The following result which is an immediate consequence of Proposition 4.1 is the base of the Algorithm A;

***Lemma 6.2* ([8]).** — *The mapping from L_λ to V_λ which associates the first order part v_P to $P \in L_\lambda$ is a surjective mapping.*

Let $I_\lambda^{(2)}$ be the primary component of the ideal

$$I^{(2)} = \langle f_1^2, \dots, f_n^2 \rangle$$

with the radical $\sqrt{I_\lambda^{(2)}} = \sqrt{I_\lambda}$, $\lambda = 1, \dots, \ell$. The following algorithm gives a basis of the vector space L_λ;

Algorithm A (A construction of first order annihilators)
Input : a regular sequence $f_1(z), \dots, f_n(z)$, primary ideals I_λ, $I_\lambda^{(2)}$.

(i) Determine coefficients $p_i(z) \in B_{I_\lambda}$, $i = 1, \dots, n$ so that the operator

$$v = p_1(z)\frac{\partial}{\partial z_1} + \dots + p_n(z)\frac{\partial}{\partial z_n}$$

satisfies $vh(z) = 0 \bmod I_\lambda$ for any $h(z) \in I_\lambda$.

(ii) Determine a zeroth order part $q(z) \in B_{I_\lambda}$ so that

$$(-\sum_{j=i}^n \frac{\partial p_i(z)}{\partial z_i} + q(z)) f_1 \dots f_n + \sum_{j=i}^n p_j(z)\frac{\partial (f_1 \dots f_n)}{\partial z_i} \in I_\lambda^{(2)}.$$

(iii) Put

$$P = p_1(z)\frac{\partial}{\partial z_1} + \dots + p_n(z)\frac{\partial}{\partial z_n} + q(z)$$

with $n+1$ tuples $(p_1(z), \dots, p_n(z), q(z))$ determined by the above step.

Output : a basis of L_λ.

If we drop all subscripts λ, we obtain analogous results and a corresponding algorithm that computes a vector space of annihilators of the cohomology class $\omega_{\mathcal{F}}$.

Note that the same idea is applicable to the construction of higher order annihilators to the cohomology class ω_λ (see [**5**], [**4**]).

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