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2-DIMENSIONAL VERSAL S_4-COVERS AND RATIONAL ELLIPTIC SURFACES

by

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Abstract. — We introduce the notion of a versal Galois cover, and study versal S_4-covers explicitly. Our goal of this article is to show that two S_4-covers arising from certain rational elliptic surfaces are versal.

Résumé **(S_4-revêtements galoisiens versels de dimension 2 et surfaces rationnelles elliptiques)**
On introduit la notion de revêtement galoisien versel et on étudie explicitement les S_4-revêtements galoisiens. Le but de cet article est de montrer que deux S_4-revêtements galoisiens obtenus à partir de certaines surfaces elliptiques rationnelles sont versels.

Introduction

Let G be a finite group. Let X and Y be normal projective varieties. X is called a G-cover of Y if there exists a finite surjective morphism $\pi : X \to Y$ such that the induced inclusion morphism $\pi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ gives a Galois extension with $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$, where $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ denote the rational function fields of X and Y, respectively.

G-covers have been used in various branches of algebraic geometry and topology, *e.g.*, to construct algebraic varieties having the prescribed invariants, to study the topology of the complement to a reduced plane algebraic curve, and so on. In this article, our main concern is not applications of G-covers, but G-covers themselves.

One of fundamental problems in the study of G-covers is to give an explicit "bottom-to-top" method in constructing G-covers from some geometric data of the base variety Y or intermediate covers, *i.e.*, covers corresponding to the intermediate field between $\mathbb{C}(X)$ and $\mathbb{C}(Y)$. This point of view resembles the constructive aspects of

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the inverse Galois problem: to construct a field extension of $\mathbb{Q}$ having a prescribed group as its Galois group over $\mathbb{Q}$.

In the investigation of the inverse Galois problem, many works have been done about "generic polynomials or versal polynomials" for these twenty years (see [**3**] for detail references, for example). The main purpose of this article is to make an analogous *geometric study* of them. Let us begin with the definition of a versal G-cover.

Definition 0.1. — A G-cover $\varpi : X \to M$ is said to be versal if it satisfies the following property:

For any G-cover $\pi : Y \to Z$, there exist a rational map $\nu : Z \cdots \to M$ and a Zariski open set U in Z such that

(i) $\nu|_U : U \to M$ is a morphism, and
(ii) $\pi^{-1}(U)$ is birational to $U \times_M X$ over U.

Note that we do not assume any uniqueness for ϖ and ν. Also we do not assume that ν is dominant. One could say that a versal G-cover is a geometric realization of the Galois closure of a versal G-polynomial introduced in [**1**].

Intuitively, any G-cover is obtained as *rational* pull-back of ϖ, if a versal G-cover exists. It is known that a versal G-cover exists for any G (see [**9**], [**10**]). Concretely, let $n = \#(G)$ and let $X = (\mathbb{P}^1)^n$ be the n-ple direct product of $\mathbb{P}^1$. By using the regular representation of G, one can regard G as a transitive subgroup of S_n (the symmetric group of n letters), and obtain a natural G-action on X by the permutation of the coordinates. Let $M := X/G$ be the quotient variety with respect to this action, and we denote the quotient morphism by $\varpi : X \to M$. Then we have

***Theorem 0.1* (Namba [9], [10])**. — *$\varpi : X \to M$ is a versal G-cover.*

By Theorem 0.1, the existence of a versal G-cover is assured for any G. Namba's model, however, has too large dimension to use it to consider concrete problems. Also his construction is "top-to-bottom," *i.e.*, the one to find a variety with a natural G-action first, and then to take the quotient with respect to this action. This approach is different from our viewpoint. This leads us to pose the following question:

Question 0.1. — Find a tractable versal G-cover (via a "bottom-to-top" construction if possible).

In order to obtain a tractable versal G-cover, it is natural to consider such cover of as small dimension as possible. To formulate our problem along this line, the notion of the essential dimension of G introduced by Buhler and Reichstein in [**1**] is at our disposal. The essential dimension of G gives the lower bound of dimensions of versal G-covers and it is denoted by $\mathrm{ed}_k(G)$, where k is the base field of variety ($k = \mathbb{C}$ in our case). We refer to [**1**] about details on $\mathrm{ed}_k(G)$, and put here some of the properties and results about $\mathrm{ed}_{\mathbb{C}}(G)$:

– $\mathrm{ed}_{\mathbb{C}}(G) = 1$ if and only if G is either a cyclic group $\mathbb{Z}/n\mathbb{Z}$ or a dihedral group D_{2r} (r: odd) of order $2r$. Versal G-covers of dimension 1 are classically well-known (see § 2 or [**1**]).

– $\mathrm{ed}_{\mathbb{C}}(G) = 2$ for $G = S_4, A_4, A_5, S_5$, where S_n and A_n denote the symmetric and alternating groups of n letters, respectively.

– $\mathrm{ed}_{\mathbb{C}}(G)$ is equal to the smallest dimension of a versal G-cover (Theorem 7.5 in [**1**]).

The purpose of this article is to study versal S_4-covers of dimension 2 as a first step of the study of versal G-covers. In § 1, we summarize for a method to deal with S_4-covers developed in [**15**]. In § 2, we give two examples of S_4-covers using this method. We denote them by $\pi_{431} : S_{431} \to \Sigma_{431}$ and $\pi_{9111} : S_{9111} \to \Sigma_{9111}$. Both of them are constructed from certain rational elliptic surfaces in a canonical way. Both of the actions of the Galois groups S_4 on S_{431} and S_{9111} are described by the language of the Mordell-Weil groups of the corresponding elliptic surfaces by the same idea. Our goal of this article is to prove the following:

Theorem 0.2. — *Both $\pi_{431} : S_{431} \to \Sigma_{431}$ and $\pi_{9111} : S_{9111} \to \Sigma_{9111}$ are versal S_4-covers.*

The rest of this article is devoted to proving this theorem. We first show that π_{9111} is versal by using Tsuchihashi's result in [**17**] in § 3. In § 4, we explain a method for a top-to-bottom method in constructing of a versal G-cover by using a linear representation of G. The method seems to be well-known to the specialists who are working on generic polynomials or versal polynomials. In fact, it is essentially used in [**1**]. Yet we put it here since we need it to prove the versality for π_{431}. We give several examples in § 5 by using this method. In § 6, we prove the versality for π_{431} by comparing S_{431} with an example in § 5.

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1. S_4-covers

In [**15**], the author has developed a method in studying Galois covers having S_4 as their Galois groups. We here explain it briefly (see [**15**] for a proof). For a finite surjective morphism $\pi : X \to Y$, the branch locus of π is the subset of Y given by

$$\{y \in Y \mid \pi \text{ is not locally isomorphic over } y\}.$$

We denote it by $\Delta(X/Y)$ or Δ_π.

Let $\pi : X \to Y$ be an S_4-cover. Let $V_4 (\cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2})$ be the subgroup given by

$$\{1, (12)(34), (13)(24), (14)(23)\},$$

and let $\mathbb{C}(X)^{V_4}$ be the V_4-invariant subfield of $\mathbb{C}(X)$. We denote the $\mathbb{C}(X)^{V_4}$-normalization of Y by $D(X/Y, V_4)$. There are canonical morphisms:

$$\beta_1(\pi, V_4) : D(X/Y, V_4) \longrightarrow Y, \quad \beta_2(\pi, V_4) : X \longrightarrow D(X/Y, V_4).$$

Note that $\beta_2(\pi, V_4)$ is a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover, while $\beta_1(\pi, V_4)$ is an S_3-cover, where S_3 denotes the symmetric group of 3 letters.

Proposition 1.1. — *Let $f : Z \to Y$ be an S_3-cover of Y. Suppose that Z is smooth and there exist three different reduced divisors, D_1, D_2 and D_3 on Z satisfying the following conditions:*

(i) *There is no common component among D_1, D_2 and D_3. Put* $\mathrm{Gal}(Z/Y) = S_3 = \langle \sigma, \tau \mid \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1\rangle$*, then $(i-a)$ $D_1^\sigma = D_2$ and $D_3^\sigma = D_3$, and $(i-b)$ $D_1^\tau = D_2$, $D_2^\tau = D_3$, $D_3^\tau = D_1$. (D^σ and D^τ denote the pull-back of D by σ and τ, respectively).*

(ii) *There exists a line bundle, $\mathbb{L}$, such that D_1 is linearly equivalent to $2\mathbb{L}$.*

Then there exists an S_4-cover $\pi : X \to Y$ satisfying (i) $D(X/Y, V_4) = Z$ and (ii) $\Delta(X/Z) = \mathrm{Supp}(D_1 + D_2 + D_3)$.

2. S_4-covers arising from certain rational elliptic surfaces

In this section, we make use of various results in the theory of elliptic surfaces freely in order to construct two example which play main roles in this article. See for [**4**], [**6**], [**7**] and [**13**] for the details about the theory of elliptic surfaces. Note that our method in this section can be generalized to any elliptic surface $\varphi : S \to \mathbb{P}^1$ with 3-torsion

2.1. The surface S_{431}. — Let $\varphi : X_{431} \to \mathbb{P}^1$ be a rational elliptic surface obtained by blowing up base points $q : X_{431} \to \mathbb{P}^2$ of the pencil of cubic curves

$$\Lambda : \{\lambda_0(X_0X_1X_2) + \lambda_1(X_0 + X_1 + X_2)^3 = 0\}_{[\lambda_0, \lambda_1] \in \mathbb{P}^1},$$

where X_0, X_1, X_2 are homogeneous coordinates of $\mathbb{P}^2$. The notation X_{431} is due to [**7**]. It is known that $\varphi : X_{431} \to \mathbb{P}^1$ satisfies the following properties(see [**7**]):

– The Mordell-Weil group, $\mathrm{MW}(X_{431})$, is isomorphic to $\mathbb{Z}/3\mathbb{Z}$; and we denote its elements by O, s_1 and s_2.

– φ has three singular fibers and their types are of I_1, I_3 and IV^*.

We may assume that the three singular fibers , s_1 and s_2 sit in X_{431} as in Figure 1 below. The curves O, s_1, s_2, $C_{2,i}$ $(i = 0, 1, 2, 4, 5, 6)$ are the exceptional curves of q.

Let

– σ_φ = the inversion morphism with respect to the group law

– τ_{s_i} = the translation by s_i.

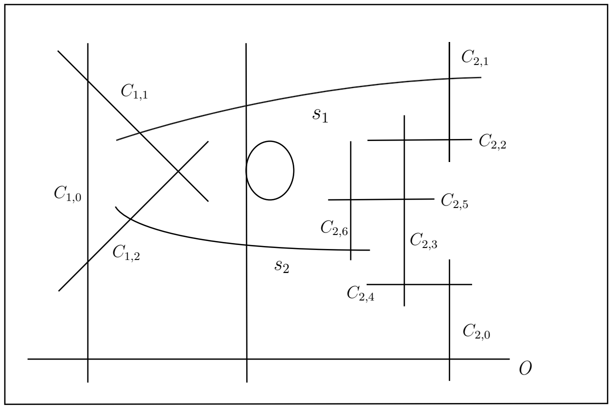

FIGURE 1

Both σ_φ and τ_{s_1} are fiber preserving automorphisms on X_{431} such that $\sigma_\varphi^2 = \tau_{s_1}^3 = (\sigma_\varphi\tau_{s_1})^2 = 1$. Hence σ_φ and τ_{s_1} define an S_3-action on X_{431}. We put $\Sigma_{431} = X_{431}/S_3$, and denote its quotient morphism by $f_{431} : X_{431} \to \Sigma_{431}$. On a smooth fiber of φ, this S_3-action is a natural one: the S_3-action induced by the inversion and the translation by a 3-torsion on an elliptic curve.

Lemma 2.1. — *The S_3-action on the singular fibers are described as follows:*

I_1-fiber: σ_φ and τ_{s_1} give non-trivial automorphisms. By taking a suitable local coordinate (z_1, z_2) around the node P, they are described as follows:

$$\begin{aligned}\sigma_\varphi &: (z_1, z_2) \longmapsto (z_2, z_1),\\ \tau_{s_1} &: (z_1, z_2) \longmapsto (\omega z_1, \omega^2 z_2),\end{aligned}$$

where $P := (0,0)$ and $\omega = \exp(2\pi\sqrt{-1}/3)$.

I_3-fiber: No irreducible component is pointwise fixed. σ_φ^* and $\tau_{s_1}^*$ permute the irreducible components as follows:

$$\sigma_\varphi^* : \begin{array}{l} C_{1,0} \mapsto C_{1,0},\\ C_{1,1} \mapsto C_{1,2},\\ C_{1,2} \mapsto C_{1,1},\end{array} \quad \tau_{s_1}^* : \begin{array}{l} C_{1,0} \mapsto C_{1,2},\\ C_{1,1} \mapsto C_{1,0},\\ C_{1,2} \mapsto C_{1,1}.\end{array}$$

IV^*-fiber: $C_{2,4}$ is the unique component which is pointwise fixed by σ_φ and no irreducible component is pointwise fixed by τ_{s_1}. σ_φ^* and $\tau_{s_1}^*$ permute irreducible components as follows:

$$\sigma_\varphi^* : \begin{array}{l} C_{2,1} \mapsto C_{2,6},\, C_{2,2} \mapsto C_{2,5},\, C_{2,3} \mapsto C_{2,3},\\ C_{2,4} \mapsto C_{2,4},\, C_{2,0} \mapsto C_{2,0},\end{array}$$

$$\tau_{s_1}^* : \begin{array}{l} C_{2,0} \mapsto C_{2,6},\, C_{2,1} \mapsto C_{2,0},\, C_{2,2} \mapsto C_{2,4} \\ C_{2,3} \mapsto C_{2,3},\, C_{2,4} \mapsto C_{2,5},\, C_{2,5} \mapsto C_{2,2} \\ C_{2,6} \mapsto C_{2,1}. \end{array}$$

Proof. — We only prove the second half of (i). It is easy to see that the stabilizer group at P is S_3. Hence the representation at the tangent space at P in X_{431} give a 2-dimensional faithful representation of S_3, and the statement follows. For other statements, see § 9, [**4**], § 5, [**8**], and [**11**], for example. □

As for the action on s_1, s_2, note that

$$\begin{array}{l} s_1^{\sigma_\varphi} = s_2, \quad O^{\sigma_\varphi} = O, \\ s_1^{\tau_{s_1}} = O, \quad s_2^{\tau_{s_1}} = s_1, \quad O^{\tau_{s_1}} = s_2. \end{array}$$

By Lemmas 8.1 and 8.2 in [**13**], we have

$$\begin{aligned} s_1 \approx_{\mathbb{Q}} O + F &- \frac{1}{3}\left(2C_{1,1} + C_{1,2}\right) \\ &- \frac{1}{3}\left(4C_{2,1} + 5C_{2,2} + 6C_{2,3} + 3C_{2,4} + 4C_{2,5} + 2C_{2,6}\right), \\ s_2 \approx_{\mathbb{Q}} O + F &- \frac{1}{3}\left(C_{1,1} + 2C_{1,2}\right) \\ &- \frac{1}{3}\left(4C_{2,6} + 5C_{2,5} + 3C_{2,4} + 6C_{2,3} + 4C_{2,2} + 2C_{2,1}\right), \end{aligned}$$

where F denotes a fiber of φ, and $\approx_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-algebraic equivalence of divisors. Since X_{431} is simply connected, one can replace the algebraic equivalence by the linear equivalence. Hence we have

$$\begin{aligned} &s_1 + s_2 + C_{1,1} + C_{1,2} + C_{2,2} + C_{2,5} \\ &\qquad \sim 2(O + F - C_{2,1} - C_{2,2} - 2C_{2,3} - C_{2,4} - C_{2,5} - C_{2,6}), \end{aligned}$$

where $\sim$ denotes the linear equivalence of divisors. Put

$$D = s_1 + s_2 + C_{1,1} + C_{1,2} + C_{2,2} + C_{2,5},$$

and define

$$\begin{aligned} D_1 &= D^{\tau_{s_1}} \\ D_2 &= D^{\tau_{s_1}^2} \\ D_3 &= D. \end{aligned}$$

Then, by Proposition 1.1, there exists $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover $g_{431} : S_{431} \to X_{431}$ such that

(i) $\pi_{431} = f_{431} \circ g_{431} : S_{431} \to \Sigma_{431}$ is an S_4-cover,
(ii) $D(S_{431}/\Sigma_{431}, V_4) = X_{431}$, and
(iii) $\Delta_{g_{431}} = \mathrm{Supp}(D_1 + D_2 + D_3)$.

2.2. The surface S_{9111}. — Let $([s_0, s_1], [t_0, t_1])$ be a (bi-) homogeneous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the pencil

$$\Lambda_1 : \{\lambda_0(s_0 s_1 t_0^2 + s_0^2 t_0 t_1 + s_1^2 t_1^2) + \lambda_1(s_0 s_1 t_0 t_1) = 0\}_{[\lambda_0:\lambda_1]\in\mathbb{P}^1}.$$

By blowing up the base points of Λ_1, we obtain a rational elliptic surface. Following to [**7**], we denote this elliptic surface by $\varphi_1 : X_{9111} \to \mathbb{P}^1$ and denote the blowing-ups $X_{9111} \to \mathbb{P}^1 \times \mathbb{P}^1$ by q_1. It is known that $\varphi_1 : X_{9111} \to \mathbb{P}^1$ satisfies the following properties(see [**7**]):

– $\mathrm{MW}(X_{9111}) \cong \mathbb{Z}/3\mathbb{Z}$; and we denote its elements by O, s_1 and s_2.

– φ has four singular fibers and their types are of I_9, I_1, I_1, I_1.

We may assume that the four singular fibers , s_1 and s_2 sit in X_{9111} as in Figure 2 below. The curves $O, s_1, s_2, C_0, C_2, C_3, C_6, C_7$ are the exceptional curves for q_1.

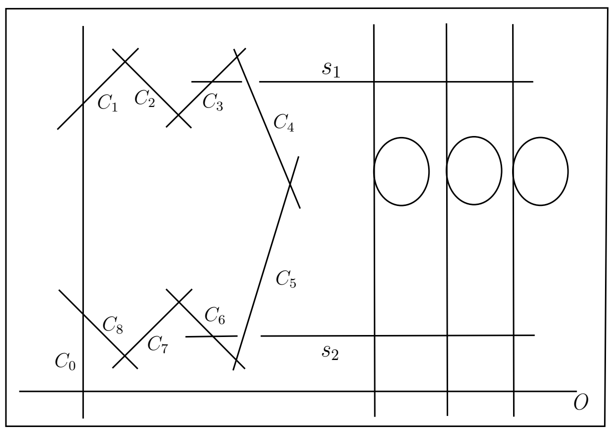

FIGURE 2

Likewise the case of X_{431}, we canonically obtain an S_3-action given by σ_{φ_1} (the inversion with respect to the group law) and τ_{s_1} (the translation by s_1). Let $\Sigma_{9111} := X_{9111}/S_3$ and we denote the quotient morphism by $f_{9111} : X_{9111} \to \Sigma_{9111}$.

Lemma 2.2. — *The S_3-action on the I_9 fiber and I_1 fibers are described as follows:*

(i) *I_1 fiber: the same as that in Lemma 2.1.*

(ii) *I_9 fiber:*

$$\sigma_{\varphi_1}^* : C_i \to C_{9-i(\bmod 9)},$$
$$\tau_{s_1}^* : C_i \to C_{i-3(\bmod 9)}.$$

Also note that

$$s_1^{\sigma_{\varphi_1}} = s_2, \quad O^{\sigma_{\varphi_1}} = O,$$
$$s_1^{\tau_{s_1}} = O, \quad s_2^{\tau_{s_1}} = s_1, \quad O^{\tau_{s_1}} = s_2.$$

For a proof, see § 9, [**4**], § 5, [**8**], and [**11**], for example.

By Lemmas 8.1 and 8.2 in [**13**], we have

$$s_1 \approx_{\mathbb{Q}} O + F - \frac{1}{3}(2C_1 + 4C_2 + 6C_3 + 5C_4 + 4C_5 + 3C_6 + 2C_7 + C_8),$$

and

$$s_2 \approx_{\mathbb{Q}} O + F - \frac{1}{3}(C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5 + 6C_6 + 4C_7 + 2C_8),$$

where$F$ denotes a fiber of φ_1, and $\approx_{\mathbb{Q}}$ denotes $\mathbb{Q}$-algebraic equivalence. Since X_{9111} is simply connected, we can replace the algebraic equivalence by the linear equivalence. Hence

$$\begin{aligned} s_1 + s_2 + C_1 + C_3 + C_4 + C_5 + C_6 + C_8 & \\ \sim 2(O + F - C_2 - C_3 - C_4 - C_5 - C_6 - C_7). & \end{aligned}$$

Now we put

$$D = s_1 + s_2 + C_1 + C_3 + C_4 + C_5 + C_6 + C_8,$$

and define three effective divisors D_1, D_2 and D_3 on X_{9111} as follows:

$$D_1 = D^{\tau_{s_1}}, \quad D_2 = D^{\tau_{s_1}^2}, \quad D_3 = D.$$

Then, by Proposition 1.1, we have a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover $g_{9111} : S_{9111} \to X_{9111}$ such that

(i) $\pi_{9111} = f_{9111} \circ g_{9111} : S_{9111} \to \Sigma_{9111}$ is an S_4-cover,
(ii) $D(S_{9111}/\Sigma_{9111}, V_4) = X_{9111}$, and
(iii) $\Delta g_{9111} = \mathrm{Supp}(D_1 + D_2 + D_3)$.

3. Versality for $\pi_{9111} : S_{9111} \to \Sigma_{9111}$

Let us make a quick review for Tsuchihashi's versal S_4-cover $\varpi_{ts} : X_{ts} \to M_{ts}$ in [**17**]. Let Y be a surface obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up two points: $(s,t) = (0,\infty), (\infty, 0)$, where (s,t) denotes an inhomonogenous coordinate. Y admits an S_3-action induced by birational automorphisms on $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\sigma : (s,t) \longmapsto (t,s) \qquad \tau : (s,t) \longmapsto \left(\frac{1}{st}, s\right).$$

We write

$$\begin{aligned} D_1 &= \text{the strict transform of } \mathbb{P}^1 \times \{0\} \\ D_2 &= \text{the strict transform of } \{0\} \times \mathbb{P}^1 \\ D_3 &= \text{the strict transform of } \{\infty\} \times \mathbb{P}^1 \\ D_4 &= \text{the strict transform of } \mathbb{P}^1 \times \{\infty\} \\ E_1 &= \text{the exceptional divisor from } (\infty, 0) \\ E_2 &= \text{the exceptional divisor from } (0, \infty). \end{aligned}$$

With these notations, we have

$$\mathrm{Pic}(Y) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2,$$

and

$$D_4 \sim D_1 + E_1 - E_2, \quad \text{and} \quad D_3 \sim D_2 - E_1 + E_2.$$

As for the S_3-action on D_i $(i = 1, 2, 3, 4)$ and E_j $(j = 1, 2)$, we have

$$\begin{gathered} D_1^\sigma = D_2,\ D_3^\sigma = D_4,\ E_1^\sigma = E_2, \\ D_1^\tau = E_2,\ D_2^\tau = D_4,\ D_3^\tau = D_1, \\ D_4^\tau = E_1 \quad E_1^\tau = D_2 \quad E_2^\tau = D_3 \end{gathered}$$

Let $M_{ts} = Y/S_3$, and we denote the quotient morphism by $f_{ts} : Y \to M_{ts}$. By [**17**], M_{ts} is $\mathbb{P}^2$ and the branch locus $\Delta_{f_{ts}}$ is a 3 cuspidal quartic curve. We now construct a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover, $g_{ts} : X \to Y$ so that the composition $f_{ts} \circ g_{ts}$ gives an S_4-cover. Let R_1, R_2, and R_3 be effective divisors on Y given by

$$\begin{aligned} R_1 &= D_1 + D_4 + E_1 + E_2, \\ R_2 &= D_2 + D_3 + E_1 + E_2, \text{ and} \\ R_3 &= D_1 + D_2 + D_3 + D_4. \end{aligned}$$

By the S_3-action described as above, we have

$$R_1^\sigma = R_2, \quad R_3^\sigma = R_3, \quad \text{and} \quad R_1^\tau = R_2, \quad R_2^\tau = R_3.$$

Also,

$$R_1 \sim 2(D_1 + E_1).$$

Since there is no common irreducible component among R_1, R_2 and R_3, by Proposition 1.1, there exists a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover $g_{ts} : X_{ts} \to Y$ branched at $\mathrm{Supp}(R_1 + R_2 + R_3)$ so that $\varpi_{ts} = f_{ts} \circ g_{ts}$ gives an S_4-cover. In [**17**], Tsuchihashi showed

Theorem 3.1. — *The S_4-cover $\varpi_{ts} : X_{ts} \to M_{ts}$ is versal.*

Since S_3 acts six divisors D_i $(i = 1, 2, 3, 4)$ and E_i $(i = 1, 2)$ transitively, they are mapped to one plane curve. By [**17**], it is the unique bitangent line of Δ_{ts}. Hence $\Delta_{\varpi_{ts}}$ consists of a 3-cuspidal quartic and its unique bitangent line. Note that such configuration is unique up to projective transformations (see [**2**] Ch. 4, § 4.) Concretely, it is given by the equation

$$(X + Y + Z)\{(XY + YZ + ZX)^2 - 4XYZ(X + Y + Z)\} = 0,$$

where $[X, Y, Z]$ denotes a homogeneous coordinate of $\mathbb{P}^2$.

Lemma 3.1. — *Let $\pi : X \to \mathbb{P}^2$ be an S_4-cover such that*

(i) *Δ_π consists of two irreducible components: a 3-cuspidal quartic curve Q and its unique bitangent line L, and*

(ii) *$\beta_1(\pi, V_4)$ is branched at Q, and $\beta_2(\pi, V_4)$ is branched at $(\beta_1(\pi, V_4))^*(L)$.*

Then there exists a covering isomorphism $\phi : X \to X_{ts}$ such that $\pi = \varpi_{ts} \circ \phi$ and it induces a covering isomorphism $\overline{\phi} : D(X/\mathbb{P}^2, V_4) \to Y$.

Proof. — One may assume that both covers is branched at the same quintic curve after a suitable covering transformation. Since $\beta_1(\pi, V_4) : D(X/\mathbb{P}^2, V_4) \to \mathbb{P}^2$ is branched at Q, it is determined by a normal subgroup, N, of $\pi_1(\mathbb{P}^2 \smallsetminus Q)$ such that $\pi_1(\mathbb{P}^2 \smallsetminus Q) \cong S_3$. As $\pi_1(\mathbb{P}^2 \smallsetminus Q)$ is isomorphic to the binary dihedral group of order 12, such a normal subgroup is unique. Hence, up to covering isomorphisms, we can consider that $D(X/\mathbb{P}^2, V_4) = Y$, $f_{ts} = \beta_1(\pi, V_4)$, and $(\beta_1(\pi, V_4))^*(L) = \sum_i D_i + \sum_j E_j$. Since $\pi_1(D(X/\mathbb{P}^2, V_4) \smallsetminus (\beta_1(\pi, V_4))^*(L)) = \pi_1(Y \smallsetminus \sum_i D_i + \sum_j E_j) \cong \mathbb{Z} \oplus \mathbb{Z}$, there exists a unique $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover branched at $\beta_1(\pi, V_4))^*(L)$. Thus, up to covering isomorphisms, we have $X = X_{ts}$, and $\beta_2(\pi, V_4) = g_{ts}$. □

Lemma 3.2. — *Let $f_{9111} : X_{9111} \to \Sigma_{9111}$ be the S_3-cover given in § 2. Then Σ_{9111} is described as follows:*

Let $B = Q + L$ be a quintic curve consisting of two irreducible components Q and L, where Q is a 3-cuspidal quartic curve, and L is the bitangent line to Q. Choose $x \in Q \cap L$. (Note that by using the above explicit equation, one can easily check that the there exists a linear transformation $\mathbb{P}^2 \to \mathbb{P}^2$ such that (i) it preserves $Q + L$ and (ii) it exchanges the two points in $Q \cap L$). Let $q_x : (\mathbb{P}^2)_x \to \mathbb{P}^2$ be the blowing up at x. We denote the strict transform of Q and L by Q' and L', respectively and the exceptional divisor by E. Let y be the unique point in $Q' \cap L' \cap E$. Blow up $(\mathbb{P}^2)_x$ at y. Then the resulting surface Σ_{9111}, and we denote the blowing up by $q_y : \Sigma_{9111} \to (\mathbb{P}^2)_x$. Moreover, (i) $\Delta_{f_{9111}} = Q'' + E'$, where Q'' and E' are strict transforms of Q' and E, respectively, and (ii) the preimage of L consists of some of the irreducible components of the I_9-fiber.

Proof. — By Lemma 2.2, the quotient surface $X_{9111}/\langle \tau_{s_1} \rangle$ by $\langle \tau_{s_1} \rangle$ has three A_2 singularities corresponding to the nodes of I_1. The minimal resolution of $X_{9111}/\langle \tau_{s_1} \rangle$ is again an elliptic surface having 4 I_3-fibers. Following to [**7**], we denote it by X_{3333}. The image of O is a section $\overline{O}$ and the induced action by σ_{φ_1} again gives the inversion morphism over the generic fiber, where O is the neutral element. By the results in [**7**], § 6 (Table 6.8), our lemma follows. □

Proof of Theorem 0.2 for S_{9111}. — Consider the composition $q \circ q_1 \circ \pi_{9111} : S_{9111} \to \mathbb{P}^2$, and let $\overline{\pi}_{9111} : \overline{S}_{9111} \to \mathbb{P}^2$ be the Stein factorization. Then $\overline{S}_{9111}$ is an S_4-cover of $\mathbb{P}^2$ satisfying the conditions in Lemma 3.1. Hence it coincides with $\varpi_{ts} : X_{ts} \to \mathbb{P}^2$. By Theorem 3.1, the versality for S_{9111} follows. □

4. Versal G-covers and linear representations of G

In this section, we explain a method to construct versal G-covers. Our argument here is a geometric analog to that in Proposition 1.1.4, Chapter 1, [**3**]. Although we treat simpler cases than those in [**3**], it is enough for our purpose. Let G be a finite group as before. Let $\rho_1 (= \mathbf{1}_G), \ldots, \rho_s$ be the set of all non-equivalent irreducible representations of G, where $\mathbf{1}_G$ is the trivial representation. We denote the degree of ρ_i by $\deg \rho_i$. Let $\rho : G \to \mathrm{GL}(r, \mathbb{C})$ be a faithful representation which is equivalent to $\oplus_{i \geqslant 2} \rho_i^{\oplus m_i}$, $m_i \leqslant \deg \rho_i$. Put $\widetilde{\rho} = \mathbf{1}_G \oplus \rho$. By using $\widetilde{\rho}$ and ρ, we define the G-actions on $\mathbb{P}^r$ and $\mathbb{P}^{r-1}$ in the following way:

$$g\left([z_0 : \ldots : z_r]\right) := [z_0 : \ldots : z_r]\left(\widetilde{\rho}(g)\right)^{-1},$$

for $\mathbb{P}^r$ and

$$g\left([z_0 : \ldots : z_{r-1}]\right) := [z_0 : \ldots : z_{r-1}]\left(\rho(g)\right)^{-1},$$

for $\mathbb{P}^{r-1}$. Let $M_{G,\widetilde{\rho}} = \mathbb{P}^r/G$ and $M_{G,\rho} = \mathbb{P}^{r-1}/G$ be the quotient space with respect to these actions. We denote the quotient morphisms by $\varpi_{G,\widetilde{\rho}}$ and $\varpi_{G,\rho}$, respectively. Our goal of this section is to prove the following proposition.

Proposition 4.1

(i) $\varpi_{G,\widetilde{\rho}} : \mathbb{P}^r \to M_{G,\widetilde{\rho}}$ *is a versal G-cover.*

(ii) *If the action on $\mathbb{P}^{r-1}$ is faithful (i. e., $G \subset \mathrm{Aut}(\mathbb{P}^{r-1})$), then $\varpi_{G,\rho} : \mathbb{P}^{r-1} \to M_{G,\rho}$ is a versal G-cover.*

Corollary 4.1. — *If $\rho(g) \neq$ a scalar matrix for any $g \in G, g \neq 1$, then $\varpi_{G,\rho}$ is versal. In particular, if the center of G is trivial, then $\varpi_{G,\rho}$ is versal.*

In order to prove Proposition 4.1, let us first recall the normal basis theorem from Galois theory.

Theorem 4.1. — *Let k be a field and let K/k be a finite Galois extension with $[K : k] = n$. Put $G = \mathrm{Gal}(K/k)$ and let $g_1(= 1), \ldots, g_n$ be the element of G. Then there exists an element, $\omega \in K$ such that $g_1(\omega), \ldots, g_n(\omega)$ form a basis of K over k.*

For a proof, see [**5**], p. 229.

Since K is considered as an n-dimensional k-vector space, the action of the Galois group gives rise to an n dimensional representation of G, and we denote it by $\rho_{\mathrm{Gal}} : G \to \mathrm{GL}(n, k)$. By Theorem 4.1, we have the following:

Corollary 4.2. — *ρ_{Gal} is equivalent to the regular representation R_G.*

Proposition 4.2. — *Let G be a finite group and let $\rho_1, \ldots, \rho_r$ denote all irreducible distinct representations of G over $\mathbb{C}$. Then R_G is equivalent to $\oplus_i \rho_i^{\deg \rho_i}$.*

See [**12**] for a proof.

Let X be a G-variety, *i.e.*, a variety which admits a G-action. We define a subset, $\mathrm{Fix}(G)$, of X as follows:

$$\mathrm{Fix}(G) = \{x \in X \mid G_x \neq \{1\}\},$$

where G_x denotes the stabilizer at x. Also we call X a faithful G-variety if $G \subset \mathrm{Aut}(X)$.

Lemma 4.1. — *Let X be a faithful G-variety. Let $\pi : Y \to Z$ be a G-cover and let $\mu : Y \to X$ be a G-equivalent rational map such that $\mu(Y) \not\subset \mathrm{Fix}(G)$. We denote the induced rational map from Z to X/G by ν. Choose a Zariski open subset U on Z in such a way that*

(i) *ν is a morphism on U, and*
(ii) *μ is a morphism on $\pi^{-1}(U)$.*

Then the normalization of $U \times_{X/G} X$ is birational to $\pi^{-1}(U)$ over U.

Proof. — Let Φ be the canonical morphism $\pi^{-1}(U) \to U \times_{X/G} X$.

Claim. — *Φ is surjective and generically one to one.*

Proof of Claim. — Since μ is G-equivalent, Φ is surjective. Let $p_1 : U \times_{X/G} X \to U$ be the projection to the first factor. Since $\mu(Y) \not\subset \mathrm{Fix}(G)$, $\#p_1^{-1} = \#(G)$ for a general point $u \in U$. Hence Φ is generically one to one. □

By Claim, Lemma 4.1 follows. □

Proof of Proposition 4.1. — We prove (ii) only, since our proof of (i) is similar. Let $\pi : Y \to Z$ be an arbitrary G-cover. By the definition of a G-cover, $\mathbb{C}(Y)$ is a G-extension of $\mathbb{C}(Z)$ and the G-action is given by $\xi \in \mathbb{C}(Y) \mapsto g(\xi) = \xi \circ g^{-1}$. We apply Theorem 4.1 to the case when $k = \mathbb{C}(Z)$ and $K = \mathbb{C}(Z)$. Then there exists $\omega \in \mathbb{C}(Y)$ such that $g_1(\omega), \dots, g_n(\omega)$ form a basis over $\mathbb{C}(Z)$. Let $\rho_{\mathrm{Gal}} : G \to \mathrm{GL}(n, \mathbb{C}(Z))$ be the representation of G induced by the Galois action with respect to the basis $g_1(\omega), \dots, g_n(\omega)$. By Corollary 4.2, $\rho_{\mathrm{Gal}}(g) \in \mathrm{GL}(n, \mathbb{C})$ for any $g \in G$. Hence, by Proposition 4.2, there exist $\xi_i = \sum_{j=1}^n c_{ij} g_j(\omega)$, $c_{ij} \in \mathbb{C}$, $i = 1, \dots, r$, such that

(i) $\xi_1, \dots, \xi_r$ generate r-dimensional $\mathbb{C}$ vector subspace W of $\mathbb{C}(Y)$,
(ii) W is G-invariant and the representation $G \to \mathrm{GL}(W)$ induced by ρ_{Gal} coincides with ρ.

Using $\xi_1, \dots, \xi_r$, we define a rational map $\mu : Y \to \mathbb{P}^{r-1} = \mathbb{P}(W^\vee)$, where $W^\vee$ denote the dual vector space of W, by

$$p \in Y \longmapsto [\xi_1(p) : \dots : \xi_r(p)] \in \mathbb{P}^{r-1}.$$

Since

$$\begin{aligned} [g(\xi_1)(p) : \dots : g(\xi_r)(p)] &= [\xi_1(g^{-1}(p)) : \dots : \xi_r(g^{-1}(p))] \\ &= [\xi_1(p) : \dots : \xi_r(p)]\rho(g) \\ &= [\xi_1(p) : \dots : \xi_r(p)](\rho(g^{-1}))^{-1}, \end{aligned}$$

μ is G-equivalent. By our assumption, $\mathrm{Fix}(G)$ with respect to the G-action on $\mathbb{P}^{r-1}$ is a union of proper linear subspace. Since $\xi_1, \dots, \xi_r$ are linear independent over $\mathbb{C}(Y)$, $\mu(Y) \not\subset \mathrm{Fix}(G)$. Hence Proposition 4.1 follows from Lemma 4.1. □

5. Examples

In this section, we give some examples of versal Galois covers of dimension $\mathrm{ed}_{\mathbb{C}}(G)$. By Theorem 6.2, [**1**], there exists one dimensional versal G-cover, if and only if G is isomorphic to either a cyclic group, $\mathbb{Z}/n\mathbb{Z}$, or a dihedral group, D_{2n}, of order $2n$ (n: odd). For these groups, one can construct versal G-covers of dimension 1 in the following manner.

Example 5.1. — $G = \mathbb{Z}/n\mathbb{Z}$. Let $\rho : G \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^\times$ be an arbitrary faithful representation, and put $\widetilde{\rho} = \mathbf{1}_G \oplus \rho$. Then, by Proposition 4.1 (i), $\varpi_{\mathbb{Z}/n\mathbb{Z},\widetilde{\rho}} : \mathbb{P}^1 \to M_{\mathbb{Z}/n\mathbb{Z},\widetilde{\rho}} (= \mathbb{P}^1)$ gives a one dimensional versal $\mathbb{Z}/n\mathbb{Z}$-cover.

Example 5.2. — $G = D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1 \rangle$ (n: odd). Let $\rho : D_{2n} \to \mathrm{GL}(2,\mathbb{C})$ be the irreducible representation given by

$$\sigma \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau \longmapsto \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix},$$

where $\zeta_n = \exp(2\pi\sqrt{-1}/n)$. Then, by Proposition 4.1 (ii), $\varpi_{D_{2n},\rho} : \mathbb{P}^1 \to M_{D_{2n},\rho} (= \mathbb{P}^1)$ gives a one dimensional versal D_{2n}-cover. In [**14**], we essentially use this cover in constructing D_{2n}-covers. The versal D_{2n}-cover given by Tsuchihashi in [**17**] is the same as this example.

We now go on to examples of versal G-covers of dimension 2. By Theorem 6.2, [**1**] and Proposition 4.1, we have the following theorem.

Theorem 5.1. — *Let G be a finite group isomorphic to neither $\mathbb{Z}/n\mathbb{Z}$ nor D_{2n} (n: odd).*

(i) *If G has a two dimensional faithful representation ρ, then $\varpi_{G,\widetilde{\rho}} : \mathbb{P}^2 \to M_{G,\widetilde{\rho}}$ gives a versal G-cover.*

(ii) *If G has a three dimensional faithful representation ρ and $\rho(g)$ is not a scalar matrix for any g, then $\varpi_{G,\rho} : \mathbb{P}^2 \to M_{G,\rho}$ gives a versal G-cover.*

By Theorem 5.1, we have 2-dimensional versal G-covers for D_{2n} (n: even), A_4, S_4 and A_5. Note that the essential dimensions for these groups are 2

Example 5.3. — Let $\rho_{S_4} : S_4 \to \mathrm{GL}(3,\mathbb{C})$ be the three dimensional irreducible representation given by

$$(12)(34) \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (123) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (12) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

An irreducible representation $\rho_{A_4} : A_4 \to \mathrm{GL}(3,\mathbb{C})$ is also given by considering the first two matrices. Put $X_{S_4} = X_{A_4} = \mathbb{P}^2$. We denote the versal S_4- and A_4-covers coming from these representations by $\varpi_{S_4} : X_{S_4} \to M_{S_4}$ and $\varpi_{A_4} : X_{A_4} \to M_{A_4}$, respectively.

6. Versality for $\pi_{431} : S_{431} \to \Sigma_{431}$

Let $\varpi_{S_4} : X_{S_4} \to M_{S_4}$ be the versal S_4-cover in Example 5.3. For this S_4-cover, $D(X_{S_4}/M_{S_4}, V_4) = \mathbb{P}^2$. The morphism $\beta_2(\varpi_{S_4}, V_4)$ is given by

$$\beta_2(\varpi_{S_4}, V_4) : [Z_0 : Z_1 : Z_2] \longmapsto [X_0 : X_1 : X_2] = [Z_0^2 : Z_1^2 : Z_2^2],$$

where $[Z_0 : Z_1 : Z_2]$ and $[X_0 : X_1 : X_2]$ are homogeneous coordinates of X_{S_4} and $D(X_{S_4}/M_{S_4}, V_4)$, respectively. The induced S_3-action on $D(X_{S_4}/M_{S_4}, V_4)$ is given by $(12) : [X_0, X_1, X_2] \mapsto [X_1, X_0, X_2]$ and $(123) : [X_0, X_1, X_2] \mapsto [X_1, X_2, X_0]$. The goal of this section is to show that $\varpi_{S_4} : X_{S_4} \to M_{S_4}$ essentially coincide with $\pi_{431} : S_{431} \to \Sigma_{431}$ in §2.

Theorem 6.1. — *Let $q : X_{431} \to D(X_{S_4}/M_{S_4}, V_4)(= \mathbb{P}^2)$ be the blowing-up the nine base points of the pencil Λ in §2. Then:*

(i) *The S_3-action on $D(X_{S_4}/M_{S_4}, V_4)$ induced by ρ_{S_4} also defines the one on X_{431}. It coincides with the S_3-action given by σ_φ and τ_{s_1}.*

(ii) *Let X be the $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover of X_{431} obtained as the $\mathbb{C}(X_{S_4})$-normalization of X_{431}. Then $X = S_{431}$.*

The versality for π_{431} immediately follows from Theorem 6.1

Proof of Theorem 6.1

(i) Let us introduce a new coordinate $[U : V : W]$ of $\mathbb{P}^2$ by

$$U := X_0 + X_1$$
$$V := X_1$$
$$W := X_2$$

Let $x = X_0/X_2$, $y = X_1/X_2$, $u = U/W$ and $v = V/W$. Then we have $u = x+y$, $v = y$. With respect to the (u, v)-coordinate, the S_3-action on $D(X_{S_4}/M_{S_4}, V_4)$ induced by ρ_{S_4} is expressed in the following way:

$$(12) : (u, v) \longmapsto (u, u-v), \quad (123) : (u, v) \longmapsto \left(\frac{v+1}{u-v}, \frac{1}{u-v}\right)$$

Also with the (u, v) coordinate, the pencil of cubic curves is expressed by

$$\{v(v-u) = \mu(u+1)^3\}_{\mu=\frac{\lambda_1}{\lambda_0}}.$$

Note that this equation gives a "Weierstrass equation" of X_{431}, which is an elliptic curve over $\mathbb{C}(\mu)$. Using this equation, we can write down the operation σ_φ and τ_{s_2}, explicitly. For this Weierstrass form, we may assume that O, s_1 and s_2 correspond to the point at infinity, $P_1 = (-1, 0)$ and $P_2 = (-1, -1)$. Let (u_0, v_0) be an arbitrary $\mathbb{C}(\mu)$-rational point on X_{431}. By the definition of the addition law with O as the neutral element, we can write down the action of σ_φ and τ_{s_2} for the rational functions (u, v) explicitly as follows:

For σ_φ, we have

$$\sigma_\varphi^* : (u, v) \longmapsto (u, u - v).$$

Hence $\sigma_\varphi(u) = u \circ \sigma_\varphi^{-1} = u$ and $\sigma_\varphi(v) = v \circ \sigma_\varphi^{-1} = u - v$.

For an element $\xi \in \mathbb{C}(X_{431})$, $\tau_{s_2}(\xi) = \xi \circ \tau_{s_2}^{-1} = \xi \circ \tau_{s_1}$. Hence the action of τ_{s_2} on rational functions is nothing but adding the point P_1 on X_{431}. Let (u_1, v_1) be the third intersection point with the line connecting (u, v) and P_1. Then we have

$$u_1 = -1 - \frac{1}{\mu}\frac{v}{(u+1)^2} = -1 - \frac{1}{\mu}\frac{v(u+1)}{(u+1)^3} = -1 - \frac{u+1}{v-u} = \frac{v+1}{u-v},$$

and

$$v_1 = \frac{v}{u-v}.$$

Hence the point corresponding to $(u, v) + P_1$, where $+$ denotes the addition on X_{431}, is

$$\left(\frac{v+1}{u-v}, \frac{1}{u-v}\right)$$

Therefore the birational action induced by the S_3-action on $\mathbb{P}^2$ coincides with that of σ_φ and τ_{s_1} on the generic fiber $(X_{431})_\eta$ of $\varphi : X_{431} \to \mathbb{P}^1$. Since the latter is the restriction of fiber preserving automorphisms, the statement (i) follows.

(ii) The $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover $\beta_2(\varpi_{S_4}, V_4) : X_{S_4}(= \mathbb{P}^2) \to D(X_{S_4}/M_{S_4}, V_4)(= \mathbb{P}^2)$ is given by

$$[Z_0 : Z_1 : Z_2] \longmapsto [Z_0^2 : Z_1^2 : Z_2^2].$$

Hence the three lines $X = 0$, $Y = 0$ and $Z = 0$ are the branch locus. This implies that $\pi_1(\mathbb{P}^2 \smallsetminus \Delta_{\beta_2(\varpi_{S_4}, V_4)}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since the subgroup H of $\mathbb{Z} \oplus \mathbb{Z}$ with $\mathbb{Z} \oplus \mathbb{Z}/H \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is unique, $\beta_2(\varpi_S, V_4) : X_{S_4} \to \mathbb{P}^2$ is the unique $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover.

The nine exceptional curves for $X_{431} \to \mathbb{P}^2$ are

$$\begin{array}{lll} O, & C_{2,0} & C_{2,4} \\ s_1 & C_{2,1} & C_{2,2} \\ s_2 & C_{2,6} & C_{2,5}, \end{array}$$

and we may assume that the strict transform of the lines $X = 0$, $Y = 0$ and $Z = 0$ are $C_{1,0}$, $C_{1,1}$ and $C_{1,2}$, respectively. On the other hand, the V_4-cover $S_{431} \to X_{431}$ is branched at

$$\operatorname{Supp}(O + s_1 + s_2 + C_{1,0} + C_{1,1} + C_{1,2} + C_{2,2} + C_{2,4} + C_{2,5}).$$

This means that the Stein factorization S'_{431} of $S_{431} \to X_{431} \to \mathbb{P}^2$ is a V_4-cover branched at three lines $X = 0$, $Y = 0$ and $Z = 0$. This implies that $S'_{431} \cong X_{S_4}$ over $\mathbb{P}^2$. Hence $\mathbb{C}(X) \cong \mathbb{C}(X_{S_4}) \cong \mathbb{C}(S'_{431}) \cong \mathbb{C}(S_{431})$ over $\mathbb{C}(X_{431})$, and our assertion follows. □

Remark 6.1. — Since both π_{431} and π_{9111} are versal, one is obtained from another by a rational pull-back. An explicit study for these rational maps will be done in [**16**].

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