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ADE SURFACE SINGULARITIES, CHAMBERS AND TORIC VARIETIES

by

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Abstract. — We study the link between the positive divisors supported on the exceptional divisor of the minimal resolution of a rational double point and the root systems of Dynkin diagrams. Then, we calculate the toric variety corresponding to the fundamental Weyl chamber.

***Résumé* (Singularités ADE des surfaces, chambres et variétés toriques).** — Nous étudions le lien entre les diviseurs positifs à support sur le diviseur exceptionnel de la résolution minimale d'un point double rationnel et les systèmes de racine des diagrammes de Dynkin. Puis, nous calculons la variété torique correspondant à la chambre fondamentale de Weyl.

1. Introduction

A singularity of a normal analytic surface is rational if the geometric genus of the surface doesn't change by a resolution of the singularity. These singularities are rather simple among surface singularities since they are absolutely isolated and their resolutions have some nice combinatoric properties. A classification of rational singularities is done by the dual graph of the minimal resolution according to their multiplicities (see [**11**] for details and related references).

First, DuVal observed that the dual graph of the minimal resolution of a rational singularity of multiplicity 2, called rational double point, with algebraically closed field is one of the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8, briefly ADE diagrams (see [**2**] or [**4**]). This means that the intersection matrix associated to the dual graph of the minimal resolution of a rational double point is the same as the Cartan matrix of the corresponding Dynkin diagram.

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The negative definiteness of the intersection matrix of the exceptional divisor of a resolution of a normal surface singularity permits us to study on a set of certain positive divisors supported on the exceptional divisor, which will be called the semigroup of Lipman. By using this set, we can associate a toric variety with a weighted graph whose intersection matrix is negative definite (see [**1**]).

In this work, motivated by a question appeared in [**9**], we give a geometric construction of the roots of an ADE diagram, listed in [**3**] (see Planche I,IV,V,VI,VII). Following [**14**], we observe that the semigroup of Lipman associated with an ADE diagram is the same as the fundamental Weyl chamber of the corresponding root system. In the last section, using [**1**], we describe the toric variety corresponding to the fundamental Weyl chamber of an ADE diagram (see [**1**], [**15**]).

2. Rational Singularities

Let S be a germ at ξ of a complex two dimensional normal space with a singularity at ξ. A *resolution* of S is a complex nonsingular surface with a proper map $\pi : X \to S$ such that its restriction to $X - \pi^{-1}(\xi)$ is an isomorphism and $X - \pi^{-1}(\xi)$ is dense in X. A resolution $\pi : X \to S$ is called *minimal resolution* if any other resolution $\pi' : X' \to S$ factorizes by π. It is well known that the exceptional divisor $E = \pi^{-1}(\xi)$ of π is connected and of dimension 1 (see [**7**], theorem V.5.2). Let us denote by $E_1, \ldots, E_n$ the irreducible components of E. The *intersection matrix* $M(E)$ associated with E is defined by the intersection $(E_i \cdot E_j)$ of the components E_i and E_j, which is the intersection number of E_i and E_j if $i \neq j$, and the first Chern class of the normal bundle to E_i if $i = j$. It is a negative definite matrix (see [**13**]).

Let $\mathcal{G}$ denote the free abelian group generated by the irreducible components of E:

$$\mathcal{G} = \left\{ \textstyle\sum_{i=1}^{n} m_i E_i,\ m_i \in \mathbb{Z} \right\}.$$

The elements of $\mathcal{G}$ are called the *divisors supported on* E. The support of a divisor $Y = \sum_i m_i E_i$ is the set of the components for which $m_i \neq 0$. In the free abelian group $\mathcal{G}$, the intersection matrix $M(E)$ defines a symmetric bilinear form. We shall denote $(Y \cdot Z)$ the value of this bilinear form on a pair (Y, Z) of elements in $\mathcal{G}$. An element of $\mathcal{G}$ in which all the coefficients are non-negative and at least one is positive, is called a *positive divisor*.

***Theorem 2.1* (see [2])**. — *The singularity ξ of S is a rational singularity if and only if the arithmetic genus $\frac{1}{2}(Y \cdot Y + \sum_{i=1}^{n} m_i(w_i - 2)) + 1$ of each positive divisor $Y = \sum_{i=1}^{n} m_i E_i$ in $\mathcal{G}$ is $\leqslant 0$ where $w_i = -(E_i \cdot E_i)$.*

Assume that $\pi : X \to S$ is a resolution of a normal surface singularity which is not necessarily rational. Let f be an element of the maximal ideal $\mathcal{M}$ of $\mathcal{O}_{S,\xi}$. Then the divisor $(\pi^* f)$ of f on X is written as $(\pi^* f) = Y + T_f$ where Y is a positive divisor supported on the exceptional divisor E of π and T_f, called the strict transform of f

by π, intersects E in finitely many point at most. Since $((\pi^* f)\cdot E_i) = 0$ for all i, we obtain $(Y\cdot E_i) \leqslant 0$ for all i. The inverse is true when the singularity ξ is rational. We mean that, if Y is a positive divisor on X such that $(Y\cdot E_i) = -(T\cdot E_i)$ for all i, then there exists a function f in $\mathcal{M}$ such that $(\pi^* f) = Y + T$ (see [**2**]). Now, as in [**12**] (see section 18), let us consider the set

$$\mathcal{E}^+(E) = \{Y \in \mathcal{G} \mid (Y\cdot E_i) \leqslant 0 \text{ for all } i\}$$

By [**18**], this set is not empty. It is an additive semigroup: For $Y_1, Y_2 \in \mathcal{E}^+(E)$, we have $Y_1 + Y_2 \in \mathcal{E}^+(E)$.

Definition 2.2. — The set $\mathcal{E}^+(E)$ is called the semigroup of Lipman.

Since E is connected, for all $Y = \sum m_i E_i$ in $\mathcal{E}^+(E)$, we have $m_i \geqslant 1$ for all i. A partial order on $\mathcal{E}^+(E)$ is defined as follows: For two elements $Y_1 = \sum_{i=1}^n a_i E_i$ and $Y_2 = \sum_{i=1}^n b_i E_i$ of $\mathcal{E}^+(E)$, we say $Y_1 \leqslant Y_2$ if $a_i \leqslant b_i$ for all i. The smallest element of this set is called *the fundamental cycle* of the resolution π. The proposition 4.1 in [**10**], gives the following algorithm to construct the fundamental cycle of a given E:

Let us denote by Z the fundamental cycle of π. Consider $Z_1 = \sum_{i=1}^n E_i$. If $(Z_1\cdot E_i) \leqslant 0$ for all i, then $Z_1 = Z$. If else, there exists an E_{i_1} such that $(Z_1\cdot E_{i_1}) > 0$; in this case, we put $Z_2 = Z_1 + E_{i_1}$ and we see whether $(Z_2\cdot E_i) \leqslant 0$ for all i. The term Z_j, $(j \geqslant 1)$, of the sequence satisfies, either $(Z_j\cdot E_i) \leqslant 0$ for all i, then we put $Z = Z_j$, or there is an irreducible component E_{i_j} such that $(Z_j\cdot E_{i_j}) > 0$, then we put $Z_{j+1} = Z_j + E_{i_j}$. Thus the fundamental cycle of π is the first cycle Z_k of this sequence such that $(Z_k\cdot E_i) \leqslant 0$ for all i. By the same method, we can construct all other elements of $\mathcal{E}^+(E)$ (see [**14**] or [**17**]).

The following result of Artin characterize what an exceptional divisor of a resolution of a rational singularity looks like:

***Theorem 2.3* (see [2])**. — *A singularity of a normal analytic surface in $\mathbb{C}^N$ is rational if and only if the arithmetic genus of the fundamental cycle of the exceptional divisor of a resolution of the singularity vanishes.*

This gives:

***Corollary 2.4* (see [2])**. — *The exceptional divisor of any resolution of a rational singularity is normal crossing, with each E_i nonsingular and of genus zero, and any two distinct components intersect transversally at most in one point.*

A proof of this corollary can be found also in [**17**].

Then the *dual graph* associated with the exceptional divisor of a resolution of a rational singularity, in which each E_i is represented by a vertex and each intersection point is represented by an edge between the vertices corresponding to the intersecting components, is a tree. Each vertex in the dual graph is weighted by $-(E_i\cdot E_i)$. Conversely, with a given weighted graph, by plumbing, we can associate a *configuration* *of curves* embedded in a nonsingular surface and, if such a configuration of curves satisfies theorem 2.3, its contraction gives a rational singularity of a normal analytic surface (see [**6**], [**11**]).

Example 2.5. — A configuration of curves associated with an ADE diagram is contracted to a rational singularity of a normal analytic surface.

Moreover, we have:

***Proposition 2.6* (see [2])**. — *Let $\pi : X \to S$ be the minimal resolution of the rational singularity ξ of S. Then the multiplicity of S at ξ equals $-(Z \cdot Z)$ where Z is the fundamental cycle of π.*

Recall that the minimal resolution is characterized by $(E_i \cdot E_i) \leqslant -2$ for all irreducible components E_i of the exceptional divisor. A rational double point is a rational singularity for which the fundamental cycle of the minimal resolution satisfies $(Z \cdot Z) = -2$. We know that a rational double point of a surface is defined by the power series with the form $f(x, y) + z^2 = 0$. By using the results given above, we deduce:

***Proposition 2.7* (see [2] or [4])**. — *A normal analytic surface singularity is a rational double point if and only if the exceptional divisor of the minimal resolution of the singularity is a configuration of curves associated with one of the ADE diagrams.*

3. Root systems of rational double points

There is a well known construction of ADE diagrams starting from a semisimple Lie algebra. In this section, we are interested in the inverse of that construction, as suggested in [**9**]. We will see that, using the geometry of a Dynkin diagram, we can obtain the roots of the corresponding semisimple Lie algebra. This gives a partial answer to the question of Ito and Nakamura (see [**9**], p. 194).

Let V be an euclidean space endowed with a positive definite symmetric bilinear form $(,)$. A reflection s on V is an orthogonal transformation $s : V \to V$ such that, for $v \in V$, $s(v) = -v$ and it fixes pointwise the hyperplane $H_v = \{u \in V \mid (u, v) = 0\}$ of V. We can describe the reflection by the formula $s_v(u) = u - \frac{2(u,v)}{(v,v)} v$.

Definition 3.1. — A subset R of V is called a root system if

(i) it is finite, generates V and doesn't contain 0,
(ii) for every $v \in R$, there exists a unique reflection s_v such that $s_v(R) = R$,
(iii) for every $v \in R$, the only multiples of v in R are $\pm v$,
(iv) for $u, v \in R$, we have $\frac{2(u,v)}{(v,v)} \in \mathbb{Z}$.

The finite group generated by the reflections is called the Weyl group.
See [**8**] for more details.

In what follows, E will denote a configuration of curves associated with an ADE diagram, called ADE configuration. Now, following [**14**], (see p. 158), we want to establish the relation between the root systems and the semigroup of Lipman of E. Denote by $E_1, \dots, E_n$ the irreducible components of E. We know that $(E_i \cdot E_j)$ equals -2 if $i = j$ and equals 0 or 1 if $i \neq j$ (see [**4**] or [**12**]). Now, consider the following subset of $\mathcal{G}$:

$$R(E) = \{Y \in \mathcal{G} \mid (Y \cdot Y) = -2\}.$$

***Proposition 3.2* (see [14])**. — *The set $R(E)$ is a root system.*

Replacing the inner product in the definition above by the symmetric bilinear form defined by the intersection matrix $M(E)$, we can see that $R(E)$ satisfies the conditions of the definition above.

We will call *root divisors* the elements of $R(E)$. By definition, $E_1, \dots, E_n$ and $-E_1, \cdots, -E_n$ are root divisors but $E_i - E_j$ is not a root divisor since $(E_i - E_j \cdot E_i - E_j) \neq -2$ for any $i \neq j$. Let us denote $B = \{E_1, \dots, E_n\}$. We can see that B is a vector space basis of $R(E)$ in $\mathcal{G} \otimes_{\mathbb{Z}} \mathbb{R}$ and every element Y in $R(E)$ can be written as the sum of E_i's with coefficients all nonnegative or all nonpositive (compare with [**8**], pp. 47-48). If we denote by $R^+(E)$ the set of the elements of $R(E)$ with coefficients all nonnegative, then we have $R(E) = R^+(E) \cup (-R^+(E))$.

***Proposition 3.3* (see [14])**. — *Let $Z = \sum_{i=1}^n a_i E_i$ be the fundamental cycle of E. Then, for each root divisor $Y = \sum_{i=1}^n m_i E_i$ in $R(E)$, we have $m_1 \leqslant a_1, \dots, m_n \leqslant a_n$.*

The fundamental cycle is called the highest (or biggest) root divisor in $R(E)$.

Proof. — Since E is the exceptional divisor of the minimal resolution of a rational double point, we have $(Z \cdot Z) = -2$. So $Z \in R(E)$. Assume that there is a positive divisor Y in $R(E)$ such that $Y > Z$ and $(Y \cdot Y) = -2$. So we have $Y = Z + D$ where D is a positive divisor. This gives $(Y \cdot Y) = (Z \cdot Z) + 2(Z \cdot D) + (D \cdot D)$. Thus $2(Z \cdot D) = -(D \cdot D)$. Since Z is the fundamental cycle, we have $(Z \cdot E_i) \leqslant 0$ for all i, so $(Z \cdot D) \leqslant 0$. This implies $D = 0$. □

Hence, we can calculate the highest root divisor by the algorithm of Laufer given in the preceding section. The following proposition gives an algorithm to construct all elements of $R(E)$ from Z by using B:

Theorem 3.4. — *Let $R^+(E) = \{Y_0, \dots, Y_k\}$ with $Y_k = Z$. Then, for each $j = 0, \dots, k-1$, there exists an element Y_t in $R^+(E)$ such that $(Y_t \cdot E_i) = k_i < 0$ and $Y_j = Y_t + k_i E_i$ for some i. Inversely, for each E_i in B such that $(Y_t \cdot E_i) = k_i < 0$, $Y_t + k_i E_i$ is a root divisor in $R(E)$.*

Proof. — The existence of at least one irreducible component E_i in each Y_j such that $(Y_j \cdot E_i) < 0$ is due to negative definiteness of the intersection matrix. Then, theorem follows from the fact that $(Y_t + k_i E_i) \cdot (Y_t + k_i E_i) = -2$. □

(Compare the root divisors obtained by the theorem with the roots given in [**3**] (see Planche I,IV,V,VI,VII).)

In particular, we have:

Corollary 3.5. — *The divisor $Y = \sum_{i=1}^{n} E_i$ (i.e. $m_i = 1$ for all i) is an element of $R^+(E)$.*

Proof. — It follows from theorem 3.4. □

Now, for each $E_i \in B$, consider the hyperplane $H_i = \{P \in \mathbb{R}^n \mid (P \cdot E_i) = 0\}$. It divides $\mathbb{R}^n$ into two parts such that:

$$H_i^+ := \{D \in \mathbb{R}^n \mid (D \cdot E_i) > 0\} \quad \text{and} \quad H_i^- := \{D \in \mathbb{R}^n \mid (D \cdot E_i) < 0\}.$$

We have $H_i^+ = -H_i^-$. A connected component of $\mathbb{R}^n - \bigcup_{i=1}^{n} H_i$ is called a (Weyl) *chamber* and the chamber defined by $C(E) := \bigcap_{E_i \in B}(H_i^-)$ is called the fundamental (Weyl) chamber (see [**8**]). Thus the closure of $C(E)$,

$$\overline{C}(E) := \{D \in \mathbb{R}^n \mid (D \cdot E_i) \leqslant 0\},$$

is a closed convex cone. Then:

Remark 3.6. — Let E be an ADE configuration. The semigroup $\mathcal{E}^+(E)$ of Lipman is the fundamental chamber $\overline{C}(E)$. In particular, the highest root of $R(E)$ belongs to $\overline{C}(E)$.

4. Toric varieties

The fundamental chamber, or equivalently the semigroup of Lipman, of an ADE configuration defines a polyhedral cone in $\mathbb{R}^n$. In this section, by using [**1**], we will construct the toric variety corresponding to that cone.

We start by recalling what a toric variety is. Let N be a lattice which is isomorphic to $\mathbb{Z}^n$. Let σ be a rational polyhedral cone in the real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ which contains no line through the origin. Denote by $M = \operatorname{Hom}(N, \mathbb{Z})$ the dual lattice of N. The dual cone $\check{\sigma}$ is the set of vectors in $M_{\mathbb{R}}$ which are nonnegative on σ. The semigroup $S_\sigma := \check{\sigma} \cap M = \{u \in M \mid (u, v) \geqslant 0, \text{ for all } i\}$ is finitely generated. We denote by χ^u the element in the algebra $\mathbb{C}[S_\sigma]$ corresponding to the element u of S_σ. Each element of $\mathbb{C}[S_\sigma]$ is in the form of a finite sum $\sum a_i \chi^{u_i}$ for $a_i \in \mathbb{C}$ and $u_i \in S_\sigma$. The variety $\operatorname{Spec} \mathbb{C}[S_\sigma]$ is an affine toric variety (see [**5**] for more details).

Here we want to find the toric variety $\operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$ when σ is defined by $\overline{C}(E)$ where E is an ADE configuration. Notice that $\overline{C}(E)$ satisfies the conditions on the cone by which we construct an affine toric variety above. In order to construct the toric variety corresponding to $\overline{C}(E)$, we first need to find the generators of $\overline{C}(E)$, which are the generators of $\mathcal{E}^+(E)$: Consider F_i' such that $(F_i' \cdot E_j) = -\delta_{ij}$. We obtain $F_i' = \sum_{i=1}^{n} m_{ij} E_j$ with $m_{ij} \in \mathbb{Q}^+$. The divisor F_i such that $F_i' = k_i \cdot F_i$ is a positive divisor where k_i denotes the least common factor of the denominators of the coefficients m_{ij}, $(j = 1, \ldots, n)$.

***Theorem 4.1* (see [1])**. — *With the preceding notation, $F_1, \ldots, F_n$ belong to $\mathcal{E}^+(E)$ and they generate the cone $\mathcal{E}^+(E)$ over $\mathbb{Q}^+$.*

Proof. — By construction, $F_1, \ldots, F_n$ belong to $\mathcal{E}^+(E)$. We will show that each element in $\mathcal{E}^+(E)$ can be written as a linear combination of the elements F_i's with coefficients in $\mathbb{Q}^+$.

Let G be the semigroup generated by $F_1, \ldots, F_n$ with coefficients in $\mathbb{Q}^+$ and $\mathcal{G}$ be as defined before. We need to show that $\mathcal{E}^+(E) = \mathcal{G} \cap G$: Let $Y = \sum_{i=1}^n m_i E_i$ be an element of $\mathcal{E}^+(E)$. Consider $M(E) \cdot (m_1, \ldots, m_n)^t = (y_1, \ldots, y_n)^t$ where $M(E)$ is the intersection matrix (same as the Cartan matrix multiplied by -1) of E. Notice that $(Y \cdot E_i) = y_i$, so $y_i \leqslant 0$ for all i. Let $D = \sum_{i=1}^n d_i E_i$ be an element of G. So, $d_i \in \mathbb{Q}^+$ for all i. Assume $M(E) \cdot (d_1, \ldots, d_n)^t = (0, \ldots, 0, -1, 0, \ldots 0)^t$ where the entry -1 is in the i-th row. The fact that $M(E)$ is an invertible matrix gives $Y = -\sum_{i=1}^n d_i y_i E_i$. So, the coefficient $-d_i y_i$ is in $\mathbb{Q}^+$ for all i. This says $Y \in \mathcal{G} \cap G$.

Now we will see the inclusion $\mathcal{G} \cap G \subset \mathcal{E}^+(E)$: Let $D \in \mathcal{G} \cap G$. This means $D = \sum_{i=1}^n b_j F_j$ with $b_j \in \mathbb{Q}^+$. Consider $(D \cdot E_i) = \sum_{i=1}^n b_j (F_j \cdot E_i)$. Since F_j, $(j = 1, \ldots, n)$, is an element of $\mathcal{E}^+(E)$ and $b_j \in \mathbb{Q}^+$ for all j, we have $(D \cdot E_i) \leqslant 0$ for all i. Hence $D \in \mathcal{E}^+(E)$. Then $\mathcal{E}^+(E) = \mathcal{G} \cap G$. □

Definition 4.2. — The elements $F_1, \ldots, F_n$ are called the generators of $\mathcal{E}^+(E)$ (or $\overline{C}(E)$).

Now, let N be a lattice generated by $E_1, \ldots, E_n$ and M be its dual lattice generated by $E_1^*, \ldots, E_n^*$ such that $(E_i^* \cdot E_j) = \delta_{ij}$. Let N' be the lattice generated by $F_1, \ldots, F_n$ and M' be its dual lattice generated by $F_1^*, \ldots, F_n^*$ such that $(F_i^* \cdot F_j) = \delta_{ij}$. Since N' is a subgroup of N of finite index, we have:

***Theorem 4.3* (see [5])**. — *With preceding notation, we have $\mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}[M']^{N/N'}$.*

This means that the affine toric variety $\operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$ is the quotient $\mathbb{C}^n/G$ where G is the finite group N/N'.

Now, let us see the construction method of the affine toric variety corresponding to $\overline{C}(E)$ when E is associated with the diagram A_2. For this, it is enough to describe the finite group N/N' and to see the action of this group on $\mathbb{C}[M']$: It is well known that the intersection matrix $M(E)$ associated with A_2 is $\left(\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right)$. From the formula $(F_i' \cdot E_j) = -\delta_{ij}$ given above, we find the generators of $\mathcal{E}^+(E)$ as $F_1 = 2E_1 + E_2$ and $F_2 = E_1 + 2E_2$.

Consider the lattice $N = \langle E_1, E_2 \rangle$ and its sublattice $N' = \langle F_1, F_2 \rangle$ and, denote by $M = \langle E_1^*, E_2^* \rangle$ and $M' = \langle F_1^*, F_2^* \rangle$ the dual lattices of N and N' respectively. It is easy to see that $F_1^* = \frac{-1}{3}(-2E_1^* + E_2^*)$ and $F_2^* = \frac{-1}{3}(E_1^* - 2E_2^*)$. Notice that

$\det M(E) = 3$ and M' is generated by the rows of the intersection matrix multiplied by $-1/\det M(E)$.

Now, let us describe the finite group N/N':

Proposition 4.4. — *The group N/N' is generated by $\overline{E}_1$ and $\overline{E}_2$ over $\mathbb{Z}$ with* $\mathrm{ord}(\overline{E}_i) = 3$ *for $i = 1, 2$ where $\overline{E}_i = E_i + N'$ and* $\mathrm{ord}(\overline{E}_i)$ *is the order of $\overline{E}_i$ in N/N'.*

Proof. — Let $\overline{F} = F + N' \in N/N'$. This says $F \in N'$ if and only if there exist $a_i \in \mathbb{Z}$ such that $F = a_1F_1 + a_2F_2$. Hence there exist the $b_i \in \mathbb{Z}$ such that $F = b_1E_1 + b_2E_2$. Using the generators F_i obtained above, we find $(2a_1 + a_2)E_1 + (a_1 + 2a_2)E_2 = b_1E_1 + b_2E_2$. For $b_2 = 0$, we find $\mathrm{ord}(\overline{E}_1)$ as the smallest $b_1 \in \mathbb{Z}$ satisfying this equation, so $b_1 = 3$. For $b_1 = 0$, we find $b_2 = \mathrm{ord}(\overline{E}_2) = 3$. Therefore the proposition follows. $\square$

Let us denote $\overline{E}_i$ by η_i for $i = 1, 2$. We have $\eta_i = (exp 2\pi i)^{1/3}$ such that $\eta_i^3 = 1$. Denote $\mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}[x_1, x_2]$ and $\mathbb{C}[M'] = \mathbb{C}[u_1, u_2]$ with $u_1 = x_1^{2/3}x_2^{-1/3}$ and $u_2 = x_1^{-1/3}x_2^{2/3}$. The action of N/N' on the coordinates of $\mathbb{C}[M']$ (see p. 34 in [**5**]) gives:

$$\eta_1(u_1, u_2) = (\eta_1^2u_1, \eta_1^{-1}u_2) \quad \text{and} \quad \eta_2(u_1, u_2) = (\eta_2^{-1}u_1, \eta_2^2u_2).$$

Using proposition 4.4, we find:

Theorem 4.5. — *With the preceding notation, the ring of invariants $\mathbb{C}[M']^{N/N'}$ is generated by u_1^3, $u_1^2u_2$, $u_1u_2^2$, u_2^3.*

Proof. — Let $u = u_1^{k_1}u_2^{k_2}$. By the action of the finite group N/N' on u, we have:

$$\eta_1(u) = \eta_1^{2k_1-k_2}u \quad \text{and} \quad \eta_2(u) = \eta_2^{-k_1+2k_2}u$$

Since the ring of invariants $\mathbb{C}[M']^{N/N'}$ is determined by the smallest k_1 and k_2 satisfying $\eta_i(u) = u$ for $i = 1, 2$, we obtain the following system of equations:

$$2k_1 - k_2 = 3l_1 \quad \text{and} \quad -k_1 + 2k_2 = 3l_2.$$

for some l_1 and l_2. Hence $k_i = 0, 1, 2 \pmod 3$ for $i = 1, 2$: When $k_1 = 3$ (resp. $k_2 = 3$) we have $k_2 = 0$ (resp. $k_1 = 0$); so, u_1^3 and u_2^3 are in $\mathbb{C}[M']^{N/N'}$. When $k_1 = 2$, we obtain $k_2 = 1$; so, $u_1^2u_2 \in \mathbb{C}[M']^{N/N'}$. When $k_1 = 1$, we obtain $k_2 = 2$; so, $u_1u_2^2 \in \mathbb{C}[M']^{N/N'}$. $\square$

Now, we need to find the ideal, called *toric ideal*, whose zero set is the affine toric variety $\operatorname{Spec}\mathbb{C}[\check{\sigma} \cap M]$. For this, we use [**16**]. The idea is to identify a 2×4 matrix $\mathcal{A} = (m_1 \dots m_4)$ with $m_i = (m_{1i}, m_{2i})^t$ to the generators $u^{m_i} = u_1^{m_{1i}}u_2^{m_{2i}}$ given in theorem 4.5. By lemma 1.1 in [**16**], the toric ideal $\mathcal{I}_{\mathcal{A}}$ is generated by $z^{v_+} - z^{v_-}$ for all integer vectors $v = v_+ - v_-$ in the kernel of $\mathcal{A}$. Hence, we conclude:

Corollary 4.6. — *Let E be the configuration of curves associated with the diagram A_2. The affine toric variety* $\operatorname{Spec}\mathbb{C}[\check{\sigma}\cap M]$ *corresponding to* $\overline{C}(E)$ *is defined as the zero set of the toric ideal*

$$\mathcal{I}_{\mathcal{A}} = \langle z_1z_3 - z_2^2, z_1z_4 - z_2z_3, z_2z_4 - z_3^2\rangle$$

where $z_i = u^{m_i}$.

Proof. — The matrix $\mathcal{A}$ corresponding to the generators given in theorem 4.5 is $\begin{pmatrix} 3 & 2 & 1 & 0\\ 0 & 1 & 2 & 3\end{pmatrix}$. So, the vectors

$$\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\2\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\0\\2\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

of the kernel of $\mathcal{A}$ generate our toric ideal $\mathcal{I}_{\mathcal{A}}$ (see lemma 1.1 and example 1.2.(a) in [**16**]). ☐

Applying the same method to any ADE configuration, we can obtain the corresponding toric variety. The reader can find each of these types in detail in [**1**]. We remark that our interest for the construction method of the affine toric variety corresponding to $\overline{C}(E)$ is coming from [**15**]. One of the natural continuations is to explore the possibility of a relation between the invariants of the affine toric variety and those of the corresponding normal surface singularity.

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