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VARIATION OF PARABOLIC COHOMOLOGY AND POINCARÉ DUALITY

by

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Abstract. — We continue our study of the variation of parabolic cohomology ([**DW**]) and derive an exact formula for the underlying Poincaré duality. As an illustration of our methods, we compute the monodromy of the Picard-Euler system and its invariant Hermitian form, reproving a classical theorem of Picard.

***Résumé* (Variation de la cohomologie parabolique et dualité de Poincaré).** — On continue l'étude de la variation de la cohomologie parabolique commencée dans [**DW**]. En particulier, on donne des formules pour l'accouplement de Poincaré sur la cohomologie parabolique, et on calcule la monodromie du système de Picard-Euler, confirmant un résultat classique de Picard.

Introduction

Let $x_1, \ldots, x_r$ be pairwise distinct points on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and set $U := \mathbb{P}^1(\mathbb{C}) - \{x_1, \ldots, x_r\}$. The Riemann–Hilbert correspondence [**Del70**] is an equivalence between the category of ordinary differential equations with polynomial coefficients and at most regular singularities at the points x_i and the category of local systems of $\mathbb{C}$-vectorspaces on U. The latter are essentially given by an r-tuple of matrices $g_1, \ldots, g_r \in \mathrm{GL}_n(\mathbb{C})$ satisfying the relation $\prod_i g_i = 1$. The Riemann–Hilbert correspondence associates to a differential equation the tuple (g_i), where g_i is the monodromy of a full set of solutions at the singular point x_i.

In [**DW**] the authors investigated the following situation. Suppose that the set of points $\{x_1, \ldots, x_r\} \subset \mathbb{P}^1(\mathbb{C})$ and a local system $\mathcal{V}$ with singularities at the x_i depend on a parameter s which varies over the points of a complex manifold S. More precisely, we consider a relative divisor $D \subset \mathbb{P}^1_S$ of degree r such that for all $s \in S$ the fibre $D_s \subset \mathbb{P}^1(\mathbb{C})$ consists of r distinct points. Let $U := \mathbb{P}^1_S - D$ denote the complement

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and let $\mathcal{V}$ be a local system on U. We call $\mathcal{V}$ a *variation of local systems* over the base space S. The *parabolic cohomology* of the variation $\mathcal{V}$ is the local system on S

$$\mathcal{W} := R^1\pi_*(j_*\mathcal{V}),$$

where $j : U \hookrightarrow \mathbb{P}^1_S$ denotes the natural injection and $\pi : \mathbb{P}^1_S \to S$ the natural projection. The fibre of $\mathcal{W}$ at a point $s_0 \in S$ is the parabolic cohomology of the local system $\mathcal{V}_0$, the restriction of $\mathcal{V}$ to the fibre $U_0 = U \cap \pi^{-1}(s_0)$.

A special case of this construction is the *middle convolution functor* defined by Katz [**Kat97**]. Here $S = U_0$ and so this functor transforms one local system $\mathcal{V}_0$ on S into another one, $\mathcal{W}$. Katz shows that all rigid local systems on S arise from one-dimensional systems by successive application of middle convolution. This was further investigated by Dettweiler and Reiter [**DR03**]. Another special case are the generalized hypergeometric systems studied by Lauricella [**Lau93**], Terada [**Ter73**] and Deligne–Mostow [**DM86**]. Here S is the set of ordered tuples of pairwise distinct points on $\mathbb{P}^1(\mathbb{C})$ of the form $s = (0, 1, \infty, x_4, \ldots, x_r)$ and $\mathcal{V}$ is a one-dimensional system on $\mathbb{P}^1_S$ with regular singularities at the (moving) points $0, 1, \infty, x_4, \ldots, x_r$. In [**DW**] we gave another example where S is a 17-punctured Riemann sphere and the local system $\mathcal{V}$ has finite monodromy. The resulting local system $\mathcal{W}$ on S does not have finite monodromy and is highly non-rigid. Still, by the comparison theorem between singular and étale cohomology, $\mathcal{W}$ gives rise to ℓ-adic Galois representations, with interesting applications to the regular inverse Galois problem.

In all these examples, it is a significant fact that the monodromy of the local system $\mathcal{W}$ (i.e. the action of $\pi_1(S)$ on a fibre of $\mathcal{W}$) can be computed explicitly, i.e. one can write down matrices $g_1, \ldots, g_r \in \mathrm{GL}_n$ which are the images of certain generators $\alpha_1, \ldots, \alpha_r$ of $\pi_1(S)$. In the case of the middle convolution this was discovered by Dettweiler–Reiter [**DR00**] and Völklein [**Vö1**]. In [**DW**] it is extended to the more general situation sketched above. In all earlier papers, the computation of the monodromy is either not explicit (like in [**Kat97**]) or uses ad hoc methods. In contrast, the method presented in [**DW**] is very general and can easily be implemented on a computer.

It is one matter to compute the monodromy of $\mathcal{W}$ explicitly (i.e. to compute the matrices g_i) and another matter to determine its image (i.e. the group generated by the g_i). In many cases the image of monodromy is contained in a proper algebraic subgroup of GL_n, because $\mathcal{W}$ carries an invariant bilinear form induced from Poincaré duality. To compute the image of monodromy, it is often helpful to know this form explictly. After a review of the relevant results of [**DW**] in Section 1, we give a formula for the Poincaré duality pairing on $\mathcal{W}$ in Section 2. Finally, in Section 3 we illustrate our method in a very classical example: the Picard–Euler system.

1. Variation of parabolic cohomology revisited

1.1. Let X be a compact Riemann surface of genus 0 and $D \subset X$ a subset of cardinality $r \geq 3$. We set $U := X - D$. There exists a homeomorphism $\kappa : X \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$ between X and the Riemann sphere which maps the set D to the real line $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. Such a homeomorphism is called a *marking* of (X, D).

Having chosen a marking κ, we may assume that $X = \mathbb{P}^1(\mathbb{C})$ and $D \subset \mathbb{P}^1(\mathbb{R})$. Choose a base point $x_0 \in U$ lying in the upper half plane. Write $D = \{x_1, \ldots, x_r\}$ with $x_1 < x_2 < \cdots < x_r \leq \infty$. For $i = 1, \ldots, r-1$ we let γ_i denote the open interval $(x_i, x_{i+1}) \subset U \cap \mathbb{P}^1(\mathbb{R})$; for $i = r$ we set $\gamma_0 = \gamma_r := (x_r, x_1)$ (which may include ∞). For $i = 1, \ldots, r$, we let $\alpha_i \in \pi_1(U)$ be the element represented by a closed loop based at x_0 which first intersects γ_{i-1} and then γ_i. We obtain the following well known presentation

$$\pi_1(U, x_0) = \Big\langle\ \alpha_1, \ldots, \alpha_r \ \mid\ \prod_i \alpha_i = 1\ \Big\rangle, \tag{1}$$

which only depends on the marking κ.

Let R be a (commutative) ring. A *local system of R-modules* on U is a locally constant sheaf $\mathcal{V}$ on U with values in the category of free R-modules of finite rank. Such a local system corresponds to a representation $\rho : \pi_1(U, x_0) \to \mathrm{GL}(V)$, where $V := \mathcal{V}_{x_0}$ is the stalk of $\mathcal{V}$ at x_0 (note that V is a free R-module of finite rank). For $i = 1, \ldots, r$, set $g_i := \rho(\alpha_i) \in \mathrm{GL}(V)$. Then we have

$$\prod_{i=1}^{r} g_i = 1,$$

and $\mathcal{V}$ can also be given by a tuple $\mathbf{g} = (g_1, \ldots, g_r) \in \mathrm{GL}(V)^r$ satisfying the above product-one-relation.

Convention 1.1. — Let α, β be two elements of $\pi_1(U, x_0)$, represented by closed path based at x_0. The composition $\alpha\beta$ is (the homotopy class of) the closed path obtained by first walking along α and then along β. Moreover, we let $\mathrm{GL}(V)$ act on V *from the right.*

1.2. Fix a local system of R-modules $\mathcal{V}$ on U as above. Let $j : U \hookrightarrow X$ denote the inclusion. The *parabolic cohomology* of $\mathcal{V}$ is defined as the sheaf cohomology of $j_*\mathcal{V}$, and is written as $H^n_p(U, \mathcal{V}) := H^n(X, j_*\mathcal{V})$. We have natural morphisms $H^n_c(U, \mathcal{V}) \to H^n_p(U, \mathcal{V})$ and $H^n_p(U, \mathcal{V}) \to H^n(U, \mathcal{V})$ (H_c denotes cohomology with compact support). Moreover, the group $H^n(U, \mathcal{V})$ is canonically isomorphic to the group cohomology $H^n(\pi_1(U, x_0), V)$ and $H^1_p(U, \mathcal{V})$ is the image of the cohomology with compact support in $H^1(U, \mathcal{V})$, see [**DW**, Prop. 1.1]. Thus, there is a natural inclusion

$$H^1_p(U, \mathcal{V}) \hookrightarrow H^1(\pi_1(U, x_0), V).$$

Let $\delta : \pi_1(U) \to V$ be a cocycle, i.e. we have $\delta(\alpha\beta) = \delta(\alpha) \cdot \rho(\beta) + \delta(\beta)$ (see Convention 1.1). Set $v_i := \delta(\alpha_i)$. It is clear that the tuple (v_i) is subject to the relation

$$v_1 \cdot g_2 \cdots g_r + v_2 \cdot g_3 \cdots g_r + \cdots + v_r = 0. \tag{2}$$

By definition, δ gives rise to an element in $H^1(\pi_1(U, x_0), V)$. We say that δ is a *parabolic* cocycle if the class of δ in $H^1(\pi_1(U), V)$ lies in $H^1_p(U, \mathcal{V})$. By [**DW**, Lemma 1.2], the cocycle δ is parabolic if and only if v_i lies in the image of $g_i - 1$, for all i. Thus, the assignment $\delta \mapsto (\delta(\alpha_1), \ldots, \delta(\alpha_r))$ yields an isomorphism

$$H^1_p(U, \mathcal{V}) \cong W_{\mathbf{g}} := H_{\mathbf{g}}/E_{\mathbf{g}}, \tag{3}$$

where

$$H_{\mathbf{g}} := \{ (v_1, \ldots, v_r) \mid v_i \in \mathrm{Im}(g_i - 1), \text{relation (2) holds}\} \tag{4}$$

and

$$E_{\mathbf{g}} := \{ (v \cdot (g_1 - 1), \ldots, v \cdot (g_r - 1)) \mid v \in V\}. \tag{5}$$

1.3. Let S be a connected complex manifold, and $r \geq 3$. An r-*configuration* over S consists of a smooth and proper morphism $\bar{\pi} : X \to S$ of complex manifolds together with a smooth relative divisor $D \subset X$ such that the following holds. For all $s \in S$ the fiber $X_s := \bar{\pi}^{-1}(s)$ is a compact Riemann surface of genus 0. Moreover, the natural map $D \to S$ is an unramified covering of degree r. Then for all $s \in S$ the divisor $D \cap X_s$ consists of r pairwise distinct points $x_1, \ldots, x_r \in X_s$.

Let us fix an r-configuration (X, D) over S. We set $U := X - D$ and denote by $j : U \hookrightarrow X$ the natural inclusion. Also, we write $\pi : U \to S$ for the natural projection. Choose a base point $s_0 \in S$ and set $X_0 := \bar{\pi}^{-1}(s_0)$ and $D_0 := X_0 \cap D$. Set $U_0 := X_0 - D_0 = \pi^{-1}(s_0)$ and choose a base point $x_0 \in U_0$. The projection $\pi : U \to S$ is a topological fibration and yields a short exact sequence

$$1 \longrightarrow \pi_1(U_0, x_0) \longrightarrow \pi_1(U, x_0) \longrightarrow \pi_1(S, s_0) \longrightarrow 1. \tag{6}$$

Let $\mathcal{V}_0$ be a local system of R-modules on U_0. A *variation* of $\mathcal{V}_0$ over S is a local system $\mathcal{V}$ of R-modules on U whose restriction to U_0 is identified with $\mathcal{V}_0$. The *parabolic cohomology* of a variation $\mathcal{V}$ is the higher direct image sheaf

$$\mathcal{W} := R^1\bar{\pi}_*(j_*\mathcal{V}).$$

By construction, $\mathcal{W}$ is a local system with fibre

$$W := H^1_p(U_0, \mathcal{V}_0).$$

(Since an r-configuration is locally trivial relative to S, it follows that the formation of $\mathcal{W}$ commutes with arbitrary basechange $S' \to S$.) Thus $\mathcal{W}$ corresponds to a representation $\eta : \pi_1(S, s_0) \to \mathrm{GL}(W)$. We call ρ the *monodromy representation* on the parabolic cohomology of $\mathcal{V}_0$ (with respect to the variation $\mathcal{V}$).

1.4. Under a mild assumption, the monodromy representation η has a very explicit description in terms of the *Artin braid group*. We first have to introduce some more notation. Define

$$\mathscr{O}_{r-1} := \{\, D' \subset \mathbb{C} \mid |D'| = r-1 \,\} = \{\, D \subset \mathbb{P}^1(\mathbb{C}) \mid |D| = r,\ \infty \in D \,\}.$$

The fundamental group $A_{r-1} := \pi_1(\mathscr{O}_{r-1}, D_0)$ is the *Artin braid group* on $r-1$ strands. Let $\beta_1, \dots, \beta_{r-2}$ be the standard generators, see e.g. [**DW**, § 2.2.] (The element β_i switches the position of the two points x_i and x_{i+1}; the point x_i walks through the lower half plane and x_{i+1} through the upper half plane.) The generators β_i satisfy the following well known relations:

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}, \qquad \beta_i\beta_j = \beta_j\beta_i \quad (\text{for } |i-j| > 1). \tag{7}$$

Let R be a commutative ring and V a free R-module of finite rank. Set

$$\mathscr{E}_r(V) := \{\, \mathbf{g} = (g_1, \dots, g_r) \mid g_i \in \mathrm{GL}(V),\ \prod_i g_i = 1 \,\}.$$

We define a right action of the Artin braid group A_{r-1} on the set $\mathscr{E}_r(V)$ by the following formula:

$$\mathbf{g}^{\beta_i} := (g_1, \dots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \dots, g_r). \tag{8}$$

One easily checks that this definition is compatible with the relations (7). For $\mathbf{g} \in \mathscr{E}_r(V)$, let $H_{\mathbf{g}}$ be as in (4). For all $\beta \in A_{r-1}$, we define an R-linear isomorphism

$$\Phi(\mathbf{g}, \beta) :\ H_{\mathbf{g}} \xrightarrow{\ \sim\ } H_{\mathbf{g}^\beta},$$

as follows. For the generators β_i we set

$$(v_1, \dots, v_r)^{\Phi(\mathbf{g},\beta_i)} := (v_1, \dots, v_{i+1}, \underbrace{v_{i+1}(1 - g_{i+1}^{-1} g_i g_{i+1}) + v_i g_{i+1}}_{(i+1)\text{th entry}}, \dots, v_r). \tag{9}$$

For an arbitrary word β in the generators β_i, we define $\Phi(\mathbf{g}, \beta)$ using (9) and the 'cocycle rule'

$$\Phi(\mathbf{g}, \beta) \cdot \Phi(\mathbf{g}^\beta, \beta') = \Phi(\mathbf{g}, \beta\beta'). \tag{10}$$

(Our convention is to let linear maps act from the right; therefore, the left hand side of (9) is the linear map obtained from first applying $\Phi(\mathbf{g}, \beta)$ and then $\Phi(\mathbf{g}^\beta, \beta')$.) It is easy to see that $\Phi(\mathbf{g}, \beta)$ is well defined and respects the submodule $E_{\mathbf{g}} \subset H_{\mathbf{g}}$ defined by (5). Let

$$\bar\Phi(\mathbf{g}, \beta) :\ W_{\mathbf{g}} \xrightarrow{\ \sim\ } W_{\mathbf{g}^\beta}$$

denote the induced map on the quotient $W_{\mathbf{g}} = H_{\mathbf{g}}/E_{\mathbf{g}}$.

Given $\mathbf{g} \in \mathscr{E}_r(V)$ and $h \in \mathrm{GL}(V)$, we define the isomorphism

$$\Psi(\mathbf{g}, h) :\ \begin{cases} H_{\mathbf{g}^h} & \xrightarrow{\ \sim\ } \quad H_{\mathbf{g}} \\ (v_1, \dots, v_r) & \longmapsto \quad (v_1 \cdot h, \dots, v_r \cdot h). \end{cases},$$

where $\mathbf{g}^h := (h^{-1}g_1h, \dots, h^{-1}g_rh)$. It is clear that $\Psi(\mathbf{g}, h)$ maps $E_{\mathbf{g}^h}$ to $E_{\mathbf{g}}$ and therefore induces an isomorphism $\bar{\Psi}(\mathbf{g}, h) : W_{\mathbf{g}^h} \xrightarrow{\sim} W_{\mathbf{g}}$.

Note that the computation of the maps $\bar{\Phi}(\mathbf{g}, \beta)$ and $\bar{\Psi}(\mathbf{g}, h)$ can easily be implemented on a computer.

1.5. Let S be a connected complex manifold, $s_0 \in S$ a base point and (X, D) an r-configuration over S. As before we set $U := X - D$, $D_0 := D \cap X_{s_0}$ and $U_0 := U \cap X_{s_0}$. Let $\mathcal{V}_0$ be a local system of R-modules on U_0 and $\mathcal{V}$ a variation of $\mathcal{V}_0$ over S. Let $\mathcal{W}$ be the parabolic cohomology of the variation $\mathcal{V}$ and let $\eta : \pi_1(S, s_0) \to \mathrm{GL}(W)$ be the corresponding monodromy representation. In order to describe η explicitly, we find it convenient to make the following assumption on (X, D):

Assumption 1.2

1. $X = \mathbb{P}^1_S$ is the relative projective line over S.
2. The divisor D contains the section $\infty \times S \subset \mathbb{P}^1_S$.
3. There exists a point $s_0 \in S$ such that $D_0 := D \cap \bar{\pi}^{-1}(s_0)$ is contained in the real line $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C}) = \bar{\pi}^{-1}(s_0)$.

In practise, this assumption is not a big restriction. See [**DW**] for a more general setup.

By Assumption 1.2, we can consider D_0 as an element of $\mathcal{O}_{r-1}$. Moreover, the divisor $D \subset \mathbb{P}^1_S$ gives rise to an analytic map $S \to \mathcal{O}_{r-1}$ which sends $s_0 \in S$ to $D_0 \in \mathcal{O}_{r-1}$. We let $\varphi : \pi_1(S, s_0) \to A_{r-1}$ denote the induced group homomorphism and call it the *braiding map* induced by (X, D).

For $t \in \mathbb{R}^+$ let $\Omega_t := \{ z \in \mathbb{C} \mid |z| > t,\ z \notin (-\infty, 0) \}$. Since Ω_t is contractible, the fundamental group $\pi_1(U_0, \Omega_t)$ is well defined for $t \gg 0$ and independent of t, up to canonical isomorphism. We write $\pi_1(U_0, \infty) := \pi_1(U_0, \Omega_t)$. We can define $\pi_1(U, \infty)$ in a similar fashion, and obtain a short exact sequence

$$1 \to \pi_1(U_0, \infty) \longrightarrow \pi_1(U, \infty) \longrightarrow \pi_1(S, s_0) \to 1. \tag{11}$$

It is easy to see that the projection $\pi : U \to S$ has a continuous section $\zeta : S \to U$ with the following property. For all $s \in S$ there exists $t \gg 0$ such that the region Ω_t is contained in the fibre $U_s := \pi^{-1}(s) \subset \mathbb{P}^1(\mathbb{C})$ and such that $\zeta(s) \in \Omega_t$. The section ζ induces a splitting of the sequence (11), which is actually independent of ζ. We will use this splitting to consider $\pi_1(S, s_0)$ as a subgroup of $\pi_1(U, \infty)$.

The variation $\mathcal{V}$ corresponds to a group homomorphism $\rho : \pi_1(U, \infty) \to \mathrm{GL}(V)$, where V is a free R-module. Let ρ_0 denote the restriction of ρ to $\pi_1(U_0, \infty)$ and χ the restriction to $\pi_1(S, s_0)$. By Part (iii) of Assumption 1.2 and the discussion in § 1.1 we have a natural ordering $x_1 < \dots < x_r = \infty$ of the points in D_0, and a natural choice of a presentation $\pi_1(U_0, \infty) \cong \langle \alpha_1, \dots, \alpha_r \mid \prod_i \alpha_i = 1 \rangle$. Therefore, the local system $\mathcal{V}_0$ corresponds to a tuple $\mathbf{g} = (g_1, \dots, g_r) \in \mathcal{E}_r(V)$, with $g_i := \rho_0(\alpha_i)$. One checks that the homomorphism $\chi : \pi_1(S, s_0) \to \mathrm{GL}(V)$ satisfies the condition

$$\mathbf{g}^{\varphi(\gamma)} = \mathbf{g}^{\chi(\gamma)^{-1}}, \tag{12}$$

for all $\gamma \in \pi_1(S, s_0)$. Conversely, given $\mathbf{g} \in \mathcal{E}_r(V)$ and a homomorphism $\chi : \pi_1(S, s_0)$ such that (12) holds then there exists a unique variation $\mathcal{V}$ which induces the pair $(\mathbf{g}, \chi)$.

With these notations one has the following result (see [**DW**, Thm. 2.5]):

Theorem 1.3. — *Let $\mathcal{W}$ be the parabolic cohomology of $\mathcal{V}$ and $\eta : \pi_1(S, s_0) \to \mathrm{GL}(W_{\mathbf{g}})$ the corresponding monodromy representation. For all $\gamma \in \pi_1(S, s_0)$ we have*

$$\eta(\gamma) = \bar{\Phi}(\mathbf{g}, \varphi(\gamma)) \cdot \bar{\Psi}(\mathbf{g}, \chi(\gamma)).$$

Thus, in order to compute the monodromy action on the parabolic cohomology of a local system $\mathcal{V}_0$ corresponding to a tuple $\mathbf{g} \in \mathcal{E}_r(V)$, we need to know the braiding map $\varphi : \pi_1(S, s_0) \to A_{r-1}$ and the homomorphism $\chi : \pi_1(S, s_0) \to \mathrm{GL}(V)$.

Remark 1.4. — Suppose that R is a field and that the local system $\mathcal{V}_0$ is irreducible, i.e. the subgroup of $\mathrm{GL}(V)$ generated by the elements g_i acts irreducibly on V. Then the homomorphism χ is determined, modulo the scalar action of $R^\times$ on V, by $\mathbf{g}$ and φ (via (12)). It follows from Theorem 1.3 that the projective representation $\pi_1(S, s_0) \to \mathrm{PGL}(V)$ associated to the monodromy representation η is already determined by (and can be computed from) $\mathbf{g}$ and the braiding map φ.

The above result is crucial for recent work of the first author [**Det05**] on the middle convolution, where the above methods are used to realize special linear groups as Galois groups over $\mathbb{Q}(t)$.

2. Poincaré duality

Let $\mathcal{V}$ be a local system of R-modules on the punctured Riemann sphere U. If $\mathcal{V}$ carries a non-degenerate symmetric (resp. alternating) form, then Poincaré duality induces on the parabolic cohomology group $H^1_p(U, \mathcal{V})$ a non-degenerate alternating (resp. symmetric) form. Similarly, if $R = \mathbb{C}$ and $\mathcal{V}$ carries a Hermitian form, then we get a Hermitian form on $H^1_p(U, \mathcal{V})$. In this section we derive an explicit expression for this induced form.

2.1. Let us briefly recall the definition of *singular (co)homology with coefficients in a local system*. See e.g. [**Spa93**] for more details. For $q \geq 0$ let $\Delta^q = |y_0, \ldots, y_q|$ denote the standard q-simplex with vertices $y_0, \ldots, y_q$. We will sometimes identify Δ^1 with the closed unit interval $[0, 1]$. Let X be a connected and locally contractible topological space and $\mathcal{V}$ a local system of R-modules on X. For a continuous map $f : Y \to X$ we denote by $\mathcal{V}_f$ the group of global sections of $f^*\mathcal{V}$.

In the following discussion, a *q-chain* will be a function φ which assigns to each singular q-simplex $\sigma : \Delta^q \to X$ a section $\varphi(\sigma) \in \mathcal{V}_\sigma$. Let $\Delta^q(X, \mathcal{V})$ denote the set of all q-chains, which is made into an R-module in the obvious way. A q-chain φ is said to have *compact support* if there exists a compact subset $A \subset X$ such that $\varphi_\sigma = 0$ whenever $\mathrm{supp}(\sigma) \subset X - A$. The corresponding R-module is denoted by $\Delta^q_c(X, \mathcal{V})$. We define coboundary operators $d : \Delta^q(X, \mathcal{V}) \to \Delta^{q+1}(X, \mathcal{V})$ and $d : \Delta^q_c(X, \mathcal{V}) \to \Delta^{q+1}_c(X, \mathcal{V})$ through the formula

$$(d\,\varphi)(\sigma) := \sum_{0 \le i \le q} (-1)^i \cdot \overline{\varphi(\sigma^{(i)})}.$$

Here $\sigma^{(i)}$ is the ith face of σ (see [**Spa66**]) and $\overline{\varphi(\sigma^{(i)})}$ denotes the unique extension of $\varphi(\sigma^{(i)})$ to an element of $\mathcal{V}_\sigma$. It is proved in [**Spa93**] that we have canonical isomorphisms

$$H^n(X, \mathcal{V}) \cong H^n(\Delta^\bullet(X, \mathcal{V}), d), \qquad H^n_c(X, \mathcal{V}) \cong H^n(\Delta^\bullet_c(X, \mathcal{V}), d), \tag{13}$$

i.e. singular cohomology agrees with sheaf cohomology. Let $x_0 \in X$ be a base point and V the fibre of $\mathcal{V}$ at x_0. Then we also have an isomorphism

$$H^1(X, \mathcal{V}) \cong H^1(\pi_1(X, x_0), V). \tag{14}$$

Let φ be a 1-chain with $d\varphi = 0$. Let $\alpha : [0, 1] \to X$ be a closed path with base point x_0. By definition, $\varphi(\alpha)$ is a global section of $\alpha^*\mathcal{V}$. Then $\alpha \mapsto \delta(\alpha) := \varphi(\alpha)(1)$ defines a cocycle $\delta : \pi_1(X, x_0) \to V$, and this cocycle represents the image of φ in $H^1(X, \mathcal{V})$.

A q-chain φ is called *finite* if $\varphi(\sigma) = 0$ for all but finitely many simplexes σ. It is called *locally finite* if every point in X has a neighborhood $U \subset X$ such that $\varphi(\sigma) = 0$ for all but finitely many simplexes σ contained in U. We denote by $\Delta_q(X, \mathcal{V})$ (resp. by $\Delta^{lf}_q(X, \mathcal{V})$) the R-module of all finite (resp. locally finite) q-chains. For a fixed q-simplex σ and a section $v \in \mathcal{V}_\sigma$, the symbol $v \otimes \sigma$ will denote the q-chain which assigns v to σ and 0 to all $\sigma' \neq \sigma$. Obviously, every finite (resp. locally finite) q-chain can be written as a finite (resp. possibly infinite) sum $\sum_\mu v_\mu \otimes \sigma_\mu$. We define boundary operators $\partial : \Delta_q(X, \mathcal{V}) \to \Delta_{q-1}(X, \mathcal{V})$ and $\partial : \Delta^{lf}_q(X, \mathcal{V}) \to \Delta^{lf}_{q-1}(X, \mathcal{V})$ through the formula

$$\partial(v \otimes \sigma) := \sum_{0 \le i \le q} (-1)^i \cdot v|_{\sigma^{(i)}} \otimes \sigma^{(i)}.$$

We define homology (resp. locally finite homology) with coefficients in $\mathcal{V}$ as follows:

$$H_q(X, \mathcal{V}) := H_q(\Delta_\bullet(X, \mathcal{V})), \qquad H^{lf}_q(X, \mathcal{V}) := H_q(\Delta^{lf}_\bullet(X, \mathcal{V})).$$

2.2. Let $X := \mathbb{P}^1(\mathbb{C})$ be the Riemann sphere and $D = \{x_1, \ldots, x_r\} \subset \mathbb{P}^1(\mathbb{R})$ a subset of $r \geq 3$ points lying on the real line, with $x_1 < \cdots < x_r \leq \infty$. Let $\mathcal{V}$ be a local system of R-modules on $U = X - D$. Choose a base point x_0 lying in the upper half plane. Then $\mathcal{V}$ corresponds to a tuple $\mathbf{g} = (g_1, \ldots, g_r)$ in $\mathrm{GL}(V)$ with $\prod_i g_i = 1$, where $V := \mathcal{V}_{x_0}$. See § 1.1. Let $\mathcal{V}^* := \underline{\mathrm{Hom}}(\mathcal{V}, R)$ denote the local system dual to $\mathcal{V}$.

It corresponds to the tuple $\mathbf{g}^* = (g_1^*, \dots, g_r^*)$ in $\mathrm{GL}(V^*)$, where V^* is the dual of V and for each $g \in \mathrm{GL}(V)$ we let $g^* \in \mathrm{GL}(V^*)$ be the unique element such that

$$\langle w \cdot g^*, v \cdot g \rangle = \langle w, v \rangle$$

for all $w \in V^*$ and $v \in V$. Note that $V^{**} = V$ because V is free of finite rank over R.

Let φ be a 1-chain with compact support and with coefficients in $\mathcal{V}^*$. Let $a = \sum_\mu v_\mu \otimes \alpha_\mu$ be a locally finite 1-chain with coefficients in $\mathcal{V}$. By abuse of notation, we will also write φ (resp. a) for its class in $H_c^1(U, \mathcal{V}^*)$ (resp. in $H_1^{lf}(U, \mathcal{V})$). The *cap product*

$$\varphi \cap a := \sum_\mu \langle \varphi(\alpha_\mu), v_\mu \rangle$$

induces a bilinear pairing

$$\cap \,:\, H_c^1(U, \mathcal{V}^*) \otimes H_1^{lf}(U, \mathcal{V}) \;\longrightarrow\; R. \tag{15}$$

It is easy to see from the definition that $H_0^{lf}(U, \mathcal{V}) = 0$. Therefore, it follows from the Universal Coefficient Theorem for cohomology (see e.g. [**Spa66**, Thm. 5.5.3]) that the pairing (15) is nonsingular on the left, i.e. identifies $H_c^1(U, \mathcal{V}^*)$ with $\mathrm{Hom}\,(H_1^{lf}(U, \mathcal{V}), R)$. The cap product also induces a pairing

$$\cap \,:\, H^1(U, \mathcal{V}^*) \otimes H_1(U, \mathcal{V}) \;\longrightarrow\; R. \tag{16}$$

(This last pairing may not be non-singular on the left. The reason is that

$$H_0(U, \mathcal{V}) \cong V / \langle\, \mathrm{Im}(g_i - 1) \mid i = 1, \dots, r \,\rangle$$

may not be a free R-module, and so $\mathrm{Ext}^1(H_0(U, \mathcal{V}), R)$ may be nontrivial.) Let $f^1 : H_c^1(U, \mathcal{V}^*) \to H^1(U, \mathcal{V}^*)$ and $f_1 : H_1(U, \mathcal{V}) \to H_1^{lf}(U, \mathcal{V})$ denote the canonical maps. Going back to the definition, one can easily verify the rule

$$f^1(\varphi) \cap a = \varphi \cap f_1(a). \tag{17}$$

Let $\varphi \in H_c^1(U, \mathcal{V}^*)$ and $\psi \in H^1(U, \mathcal{V})$. The *cup product* $\varphi \cup \psi$ is defined as an element of $H_c^2(U, R)$, see [**Ste43**] or [**Spa93**]. The standard orientation of U yields an isomorphism $H_c^2(U, R) \cong R$. Using this isomorphism, we shall view the cup product as a bilinear pairing

$$\cup \,:\, H_c^1(U, \mathcal{V}^*) \otimes H^1(U, \mathcal{V}) \;\longrightarrow\; R.$$

Similarly, one can define the cup product $\varphi \cup \psi$, where $\varphi \in H^1(U, \mathcal{V}^*)$ and $\psi \in H_c^1(U, \mathcal{V})$. Given $\varphi \in H_c^1(U, \mathcal{V}^*)$ and $\psi \in H_c^1(U, \mathcal{V})$, one checks that

$$f^1(\varphi) \cup \psi = \varphi \cup f^1(\psi). \tag{18}$$

***Proposition 2.1* (Poincaré duality).** — *There exist unique isomorphisms of R-modules*

$$p :\, H_1(U, \mathcal{V}) \;\xrightarrow{\sim}\; H_c^1(U, \mathcal{V}), \qquad p :\, H_1^{lf}(U, \mathcal{V}) \;\xrightarrow{\sim}\; H^1(U, \mathcal{V})$$

such that the following holds. If $\varphi \in H^1_c(U,\mathcal{V}^*)$ and $a \in H^{lf}_1(U,\mathcal{V})$ or if $\varphi \in H^1(U,\mathcal{V}^*)$ and $a \in H_1(U,\mathcal{V})$ then we have

$$\varphi \cap a = \varphi \cup p(a).$$

These isomorphisms are compatible with the canonical maps f_1 and f^1, i.e. we have $p\circ f_1 = f^1 \circ p$.

Proof. — See [**Ste43**] or [**Spa93**]. □

Corollary 2.2. — *The cup product induces a non-degenerate bilinear pairing*

$$\cup : H^1_p(U,\mathcal{V}^*) \otimes H^1_p(U,\mathcal{V}) \longrightarrow R.$$

Proof. — Let $\varphi \in H^1_p(U,\mathcal{V}^*)$ and $\psi \in H^1_p(U,\mathcal{V})$. Choose $\varphi' \in H^1_c(U,\mathcal{V}^*)$ and $\psi' \in H^1_c(U,\mathcal{V})$ with $\varphi = f^1(\varphi')$ and $\psi = f^1(\psi')$. By (18) we have $\varphi'\cup\psi = \varphi\cup\psi'$. Therefore, the expression $\varphi\cup\psi := \varphi'\cup\psi$ does not depend on the choice of the lift φ' and defines a bilinear pairing between $H^1_p(U,\mathcal{V}^*)$ and $H^1_p(U,\mathcal{V})$. By Proposition 2.1 and since the cap product (15) is non-degenerate on the left, this pairing is also non-degenerate on the left. But the cup product is alternating (i.e. we have $\varphi\cup\psi = -\psi\cup\varphi$, where the right hand side is defined using the identification $\mathcal{V}^{**} = \mathcal{V}$), so our pairing is also non-degenerate on the right. □

For $a \in H^{lf}_1(U,\mathcal{V}^*)$ and $b \in H_1(U,\mathcal{V})$, the expression

$$(a,b) := p(a)\cup p(b)$$

defines another bilinear pairing $H^{lf}_1(U,\mathcal{V}^*) \otimes H_1(U,\mathcal{V}) \to R$. It is shown in [**Ste43**] that this pairing can be computed as an 'intersection product of loaded cycles', generalizing the usual intersection product for constant coefficients, as follows. We may assume that a is represented by a locally finite chain $\sum_\mu v^*_\mu \otimes \alpha_\mu$ and that b is represented by a finite chain $\sum_\nu v_\nu \otimes \beta_\nu$ such that for all μ,ν the 1-simplexes α_μ and β_ν are smooth and intersect each other transversally, in at most finitely many points. Suppose x is a point where α_μ intersects β_ν. Then there exists $t_0 \in [0,1]$ such that $x = \alpha(t_0) = \beta(t_0)$ and $(\frac{\partial\alpha}{\partial t}|_{t_0}, \frac{\partial\beta}{\partial t}|_{t_0})$ is a basis of the tangent space of U at x. We set $\imath(\alpha,\beta,x) := 1$ (resp. $\imath(\alpha,\beta,x) := -1$) if this basis is positively (resp. negatively) oriented. Furthermore, we let $\alpha_{\mu,x}$ (resp. $\beta_{\nu,x}$) be the restriction of α (resp. of β) to the interval $[0,t_0]$. Then we have

$$(a,b) = \sum_{\mu,\nu,x} \imath(\alpha_\mu,\beta_\nu,x)\cdot\langle\, (v^*)^{\alpha_{\mu,x}}, v^{\beta_{\nu,x}}\,\rangle. \tag{19}$$

2.3. Let $\mathcal{V}\otimes\mathcal{V} \to \underline{R}$ be a non-degenerate symmetric (resp. alternating) bilinear form, corresponding to an injective homomorphism $\kappa : \mathcal{V} \hookrightarrow \mathcal{V}^*$ with $\kappa^* = \kappa$ (resp. $\kappa^* = -\kappa$). We denote the induced map $H^1_p(U,\mathcal{V}) \to H^1_p(U,\mathcal{V}^*)$ by κ as well. Then

$$\langle\varphi,\psi\rangle := \kappa(\varphi)\cup\psi$$

defines a non-degenerate alternating (resp. symmetric) form on $H^1_p(U,\mathcal{V})$.

Similarly, suppose that $R=\mathbb{C}$ and let $\mathcal{V}$ be equipped with a non-degenerate Hermitian form, corresponding to an isomorphism $\kappa:\bar{\mathcal{V}}\xrightarrow{\sim}\mathcal{V}^*$. Then the pairing

$$(\varphi,\psi) := -i\cdot(\kappa(\bar\varphi)\cup\psi) \tag{20}$$

is a nondegenerate Hermitian form on $H^1_p(U,\mathcal{V})$ (we identify $H^1_p(U,\bar{\mathcal{V}})$ with the complex conjugate of the vector space $H^1_p(U,\mathcal{V})$ in the obvious way).

Suppose that the Hermitian form on $\mathcal{V}$ is positive definite. Then we can express the signature of the form (20) in terms of the tuple $\mathbf{g}$, as follows. For $i=1,\ldots,r$, let

$$g_i \sim \begin{pmatrix} \alpha_{i,1} & & \\ & \ddots & \\ & & \alpha_{i,n}\end{pmatrix} \tag{21}$$

be a diagonalization of $g_i\in\mathrm{GL}(V)$. Since the g_i are unitary, the eigenvalues $\alpha_{i,j}$ have absolute value one and can be uniquely written in the form $\alpha_{i,j}=\exp(2\pi i\mu_{i,j})$, with $0\le\mu_{i,j}<1$. Set $\bar\mu_{i,j}:=1-\mu_{i,j}$ if $\mu_{i,j}>0$ and $\bar\mu_{i,j}:=0$ otherwise.

Theorem 2.3. — *Suppose that $\mathcal{V}$ is equipped with a positive definite Hermitian form and that $H^0(U,\mathcal{V})=0$. Then the Hermitian form* (20) *on $H^1_p(U,\mathcal{V})$ has signature*

$$\big(\,(\sum_{i,j}\mu_{i,j})-\dim_{\mathbb{C}}V,\ (\sum_{i,j}\bar\mu_{i,j})-\dim_{\mathbb{C}}V\,\big).$$

Proof. — If $\dim_{\mathbb{C}}V=1$, this formula is proved in [**DM86**, § 2]. The general case is proved in a similar manner. We will therefore only sketch the argument.

Let $\Omega^\bullet(\mathcal{V}):\mathcal{O}(\mathcal{V})\to\Omega^1(\mathcal{V})$ be the holomorphic $\mathcal{V}$-valued de Rham complex on U ([**DM86**, § 2.7]). Let $j^m_*\Omega^\bullet(\mathcal{V})$ denote the subcomplex of $j_*\Omega^\bullet(\mathcal{V})$ consisting of sections which are meromorphic at all the singular points. Then we have

$$H^1(U,\mathcal{V}) \;=\; \mathbb{H}^1(X,j^m_*\Omega^\bullet(\mathcal{V})) \;=\; H^1\Gamma(X,j^m_*\Omega^\bullet(\mathcal{V})).$$

We define a subbundle $\mathcal{E}$ of $j^m_*\mathcal{O}(\mathcal{V})$ as follows. Fix an index i and let $U_i\subset X$ be a disk-like neighborhood of x_i which does not contain any other singular point. Set $U_i^*:=U_i-\{x_i\}$. We obtain a decomposition

$$\mathcal{V}|_{U_i^*} \;=\; \oplus_j L_j$$

into local systems of rank one, corresponding to the diagonalization (21) of the monodromy matrix g_i. In the notation of [**DM86**, § 2.11], we set

$$\mathcal{E}|_{U_i} \;:=\; \oplus_j\mathcal{O}(\mu_{i,j}\cdot x_i)(L_j).$$

In other words: a holomorphic section of $\mathcal{E}$ on U_i can be written as $\sum_j z^{-\mu_{i,j}}f_jv_j$, where z is a local parameter on U_i vanishing at x_i, f_j is a holomorphic function and v_j is a (multivalued) section of L_j on (the universal cover of) U_i^*. It is clear that

$\mathcal{E}$ is a vectorbundle of rank $\dim_{\mathbb{C}} V$. Moreover, it is easy to see (compare [**DM86**], Proposition 2.11.1) that

$$\deg \mathcal{E} \;=\; \sum_{i,j} \mu_{i,j}. \tag{22}$$

In the same manner we define a subbundle $\mathcal{E}'$ of $j_*^m \Omega^1(\bar{\mathcal{V}})$. It is clear that

$$\deg \mathcal{E}' \;=\; \sum_{i,j} \bar{\mu}_{i,j}, \tag{23}$$

where $\bar{\mu}_{i,j}$ is defined as above.

We define the subspace $H^{1,0}(U,\mathcal{V})$ of $H^1(U,\mathcal{V})$ as the image of the map

$$H^0(X, \mathcal{E}\otimes\Omega^1_X) \;\to\; \mathbb{H}^1(X, j_*^m\Omega^\bullet(\mathcal{V})) = H^1(U,\mathcal{V}).$$

A local computation shows that $H^{1,0}(U,\mathcal{V})$ is actually contained in $H^1_p(U,\mathcal{V}) = H^1(X, j_*\mathcal{V})$. Let ω be a global section of $\mathcal{E}\otimes\Omega^1_X$ and let $[\omega]$ denote the corresponding class in $H^{1,0}(U,\mathcal{V})$. The pairing (20) applied to $[\omega]$ is then given by the following integral

$$([\omega],[\omega]) \;=\; -i\cdot\int_U \omega\wedge\bar{\omega},$$

see [**DM86**, § 2.18]. Here the integrand is defined as follows: if we write locally $\omega = v\alpha$, where v is a section of $\mathcal{V}$ and α is a holomorphic one-form, then $\omega\wedge\bar{\omega} := ||v||^2\alpha\wedge\bar{\alpha}$. The definition of $\mathcal{E}$ ensures that the above integral converges. It follows that the pairing (20) is positive definite on $H^{1,0}(U,\mathcal{V})$ and that $H^{1,0}(U,\mathcal{V}) = H^0(X,\mathcal{E}\otimes\Omega^1_X)$. By Riemann–Roch and (22) we have

$$\begin{aligned}\dim H^{1,0}(U,\mathcal{V}) \;&\geq\; \deg(\mathcal{E}\otimes\Omega^1_X) + \operatorname{rank}(\mathcal{E}\otimes\Omega^1_X)\\ &\geq\; \sum_{i,j}\mu_{i,j} - \dim V.\end{aligned} \tag{24}$$

We define $H^{0,1}(U,\mathcal{V})$ as the complex conjugate of $H^{1,0}(U,\bar{\mathcal{V}})$, considered as a subspace of $H^1_p(U,\mathcal{V})$. Note that the latter space is the image of $H^0(X,\mathcal{E}'\otimes\Omega^1_X)$, and we can represent an element in $H^{0,1}(U,\mathcal{V})$ as an antiholomorphic form with values in $\mathcal{E}'$. The same reasoning as above shows that the pairing (20) is negative definite on $H^{0,1}(U,\mathcal{V})$ and that $H^{0,1}(U,\mathcal{V})$ is equal to the complex conjugate of $H^0(X,\mathcal{E}'\otimes\Omega^1_X)$. Furthermore, we have

$$\dim H^{0,1}(U,\mathcal{V}) = \deg(\mathcal{E}'\otimes\Omega^1_X) + \operatorname{rank}(\mathcal{E}'\otimes\Omega^1_X) \;\geq\; \sum_{i,j}\bar{\mu}_{i,j} - \dim V. \tag{25}$$

Together with (24) we get the inequality

$$\begin{aligned}\dim H^1_p(U,\mathcal{V}) &\geq \dim H^{1,0}(U,\mathcal{V}) + \dim H^{0,1}(U,\mathcal{V})\\ &\geq \sum_{i,j}(\mu_{i,j}+\bar{\mu}_{i,j}) - 2\dim V\\ &= (r-2)\dim V - \sum_i \dim\operatorname{Ker}(g_i - 1).\end{aligned}$$

But according to [**DW**, Remark 1.3], this inequality is an equality. It follows that (24) and (25) are equalities as well. The theorem is now a consequence of the fact pointed out before that the pairing (20) is positive definite on $H^{1,0}(U,\mathcal{V})$ and negative definite on $H^{0,1}(U,\mathcal{V})$. □

Remark 2.4. — The authors expect several applications of the above results, such as the construction of totally real Galois representations of classical groups (in combination with the results of [**Det05**]). Another possible application would be to find new examples of differential equations with a full set of algebraic solutions, in the spirit of the work of Beukers and Heckman [**BH89**].

2.4. We are interested in an explicit expression for the pairing of Corollary 2.2. We use the notation introduced at the beginning of § 2.2, with the following modification. By γ_i we now denote a homeomorphism between the open unit interval $(0,1)$ and the open interval (x_i,x_{i+1}). We assume that γ_i extends to a path $\bar\gamma_i:[0,1]\to\mathbb{P}^1(\mathbb{R})$ from x_i to x_{i+1}. We denote by $U^+\subset\mathbb{P}^1(\mathbb{C})$ (resp. U^-) the upper (resp. the lower) half plane and by $\bar U^+$ (resp. $\bar U^-$) its closure inside $U=\mathbb{P}^1(\mathbb{C})-\{x_1,\dots,x_r\}$. Since $\bar U^+$ is simply connected and contains the base point x_0, an element of V extends uniquely to a section of $\mathcal{V}$ over $\bar U^+$. We may therefore identify V with $\mathcal{V}(\bar U^+)$ and with the stalk of $\mathcal{V}$ at any point $x\in\bar U^+$.

Choose a sequence of numbers ϵ_n, $n\in\mathbb{Z}$, with $0<e_n<e_{n+1}<1$ such that $\epsilon_n\to 0$ for $n\to-\infty$ and $\epsilon_n\to 1$ for $n\to\infty$. Let $\gamma_i^{(n)}:[0,1]\to U$ be the path $\gamma_i^{(n)}(t):=\gamma_i(\epsilon_n t+\epsilon_{n-1}(1-t))$. Let $w_1,\dots,w_r\in V$. Since $\operatorname{supp}(\gamma_i)\subset\bar U^+$, it makes sense to define

$$w_i\otimes\gamma_i:=\sum_n w_i\otimes\gamma_i^{(n)}.$$

This is a locally finite 1-chain. Set

$$c:=\sum_{i=1}^r w_i\otimes\gamma_i.$$

Note that $\partial(c)=0$, so c represents a class in $H_1^{lf}(U,\mathcal{V})$.

Lemma 2.5

1. *The image of c under the Poincaré isomorphism $H_1^{lf}(U,\mathcal{V})\cong H^1(U,\mathcal{V})$ is represented by the unique cocycle $\delta:\pi_1(U,x_0)\to V$ with*

$$\delta(\alpha_i)=w_i-w_{i-1}\cdot g_i.$$

2. *The cocycle δ in (i) is parabolic if and only if there exist elements $u_i\in V$ with $w_i-w_{i-1}=u_i\cdot(g_i-1)$, for all i.*

Proof. — For a path $\alpha:[0,1]\to U$ in U, consider the following conditions:

(a) The support of α is contained either in U^+ or in U^-.

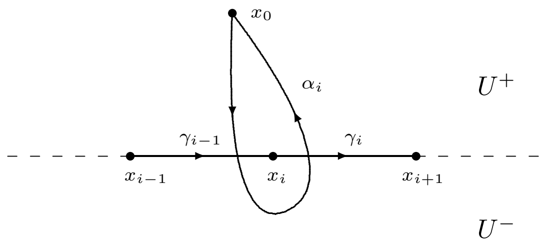

FIGURE 1.

(b) We have $\alpha(0) \in U^+$, $\alpha(1) \in U^-$ and α intersects γ_i transversally in a unique point.
(c) We have $\alpha(0) \in U^-$, $\alpha(1) \in U^+$ and α intersects γ_i transversally in a unique point.

In Case (b) (resp. in Case (c)) we identify $\mathcal{V}_\alpha$ with V via the stalk $\mathcal{V}_{\alpha(0)}$ (resp. via $\mathcal{V}_{\alpha(1)}$. Let $\varphi \in C^1(U, \mathcal{V})$ be the unique cocycle such that

$$\varphi(\alpha) = \begin{cases} 0, & \text{if } \alpha \text{ is as in Case (a)} \\ -w_i, & \text{if } \alpha \text{ is as in Case (b)} \\ w_i^{\alpha^{-1}}, & \text{if } \alpha \text{ is as in Case (c).} \end{cases}$$

(To show the existence and uniqueness of φ, choose a triangulation of U in which all edges satisfy Condition (a), (b) or (c). Then use simplicial approximation.) We claim that φ represents the image of the cycle c under the Poincaré isomorphism. Indeed, this follows from the definition of the Poincaré isomorphism, as it is given in [**Ste43**]. Write $\alpha_i = \alpha_i'\alpha_i''$, with $\alpha_i'(1) = \alpha_i''(0) \in U^-$. Using the fact that φ is a cocycle we get

$$\varphi(\alpha_i) = \varphi(\alpha_i') + \varphi(\alpha_i'')^{\alpha_i'^{-1}} = -w_{i-1} + w_i \cdot g_i^{-1}.$$

Therefore we have $\delta(\alpha_i) = \varphi(\alpha_i) \cdot g_i = w_i - w_{i-1} \cdot g_i$. See Figure 1. This proves (i).

By Section 1.1, the cocycle δ is parabolic if and only if v_i lies in the image of $g_i - 1$. So (ii) follows from (i) by a simple manipulation. □

Theorem 2.6. — *Let $\varphi \in H_p^1(U, \mathcal{V}^*)$ and $\psi \in H_p^1(U, \mathcal{V})$, represented by cocycles $\delta^* : \pi_1(U, x_0) \to V^*$ and $\delta : \pi_1(U, x_0) \to V$. Set $v_i := \delta(\alpha_i)$ and $v_i^* = \delta^*(\alpha_i)$. If we choose $v_i' \in V$ such that $v_i' \cdot (g_i - 1) = v_i$ (see Lemma 2.5), then we have*

$$\varphi \cup \psi = \sum_{i=1}^{r} \Big(\langle v_i^*, v_i' \rangle + \sum_{j=1}^{i-1} \langle v_j^* g_{j+1}^* \cdots g_{i-1}^* (g_i^* - 1), v_i' \rangle \Big).$$

Proof. — Let $w_1 := v_1$, $w_1^* := v_1^*$ and

$$w_i := v_i + w_{i-1} \cdot g_i, \qquad w_i^* := v_i^* + w_{i-1}^* \cdot g_i^*$$

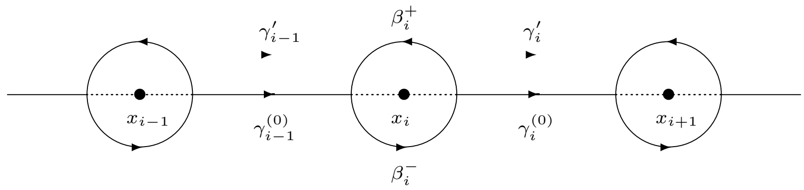

FIGURE 2.

for $i = 2, \ldots, r$. By Lemma 2.5, we can choose $u_i \in V$ with $w_i - w_{i-1} = u_i \cdot (g_i - 1)$, for $i = 1, \ldots, r$. The claim will follow from the following formula:

$$\varphi \cup \psi = \sum_{i=1}^{r} \langle w_i^* - w_{i-1}^*, u_i - w_{i-1} \rangle. \tag{26}$$

To prove Equation (26), suppose δ is parabolic, and choose $u_i \in V$ such that $w_i - w_{i-1} = u_i \cdot (g_i - 1)$. Let $D_i \subset X$ be a closed disk containing x_i but none of the other points x_j, $j \neq i$. We may assume that the boundary of D_i intersects γ_{i-1} in the point $\gamma_{i-1}^{(0)}(1)$ but nowhere else, and that D_i intersects γ_i in the point $\gamma_i^{(0)}(0)$ but nowhere else. Set $D_i^+ := D_i \cap \bar{U}^+$ and $D_i^- := D_i \cap \bar{U}^-$. Let $u_i^+ := u_i - w_{i-1}$, considered as a section of $\mathcal{V}$ over D_i^+ via extension over the whole upper half plane U^+. It makes sense to define the locally finite chain

$$u_i^+ \otimes D_i^+ := \sum_{\sigma} u_i^+ \otimes \sigma,$$

where σ runs over all 2-simplexes of a triangulation of D_i^+. (Note that $x_i \notin D_i^+$, so this triangulation cannot be finite.) Similarly, let $u_i^- \in \mathcal{V}_{D_i^-}$ denote the section of $\mathcal{V}$ over D_i^- obtained from $u_i \in V$ by continuation along a path which enters U^- from U^+ by crossing the path γ_{i-1}; define $u_i^- \otimes D_i^-$ as before. Let

$$c' := c + \partial\,(u_i^+ \otimes D_i^+ + u_i^- \otimes D_i^-).$$

It is easy to check that c' is homologous to the cocycle

$$c'' := \sum_i \big(\, w_i \otimes \gamma_i^{(0)} + u_i^+ \otimes \beta_i^+ + u_i^- \otimes \beta_i^- \,\big),$$

where β_i^+ (resp. β_i^-) is the path from $\gamma_i^{(0)}(0)$ to $\gamma_{i-1}^{(0)}(1)$ (resp. from $\gamma_{i-1}^{(0)}(1)$ to $\gamma_i^{(0)}(0)$) running along the upper (resp. lower) part of the boundary of D_i. See Figure 2. Note that c'' is finite and that, by construction, the image of c'' under the canonical map $f_1 : H_1(U, \mathcal{V}) \to H_1^{lf}(U, \mathcal{V})$ is equal to the class of c. Let $\psi' \in H_c^1(U, \mathcal{V})$ denote the image of c'' under the Poincaré isomorphism $H_1(U, \mathcal{V}) \cong H_c^1(U, \mathcal{V})$. The last statement of Proposition 2.1 shows that ψ' is a lift of $\psi \in H_p^1(U, \mathcal{V})$.

Let $c^* := \sum_i w_i^* \otimes \gamma_i \in C_1(U, \mathcal{V}^*)$. By (i) and the choice of w_i^*, the image of c^* under the Poincaré isomorphism $H_1^{lf}(U, \mathcal{V}^*) \cong H^1(U, \mathcal{V}^*)$ is equal to φ. By definition, we have $\varphi \cup \psi = (c^*, c'')$. To compute this intersection number, we have to replace c^* by a homologous cycle which intersects the support of c'' at most transversally. For instance, we can deform the open paths γ_i into open paths γ_i' which lie entirely in the upper half plane. See Figure 2. It follows from (19) that

$$(c^*, c'') = \sum_i \langle w_{i-1}^*, u_i^+\rangle - \langle w_i^*, u_i^+\rangle = \sum_i \langle w_i^* - w_{i-1}^*, u_i - w_{i-1}\rangle.$$

This finishes the proof of (26). The formula in (iv) follows from (26) from a straightforward computation, expressing w_i and u_i in terms of v_i and v_i'. □

Remark 2.7. — In the somewhat different setup, a similar formula as in Theorem 2.6 can be found in [**VÖ1**, § 1.2.3].

3. The monodromy of the Picard–Euler system

Let

$$S := \{\, (s,t) \in \mathbb{C}^2 \ \mid\ s,t \neq 0,1,\ s \neq t \,\},$$

and let $X := \mathbb{P}^1_S$ denote the relative projective line over S. The equation

$$y^3 = x(x-1)(x-s)(x-t) \tag{27}$$

defines a finite Galois cover $f : Y \to X$ of smooth projective curves over S, tamely ramified along the divisor $D := \{0,1,s,t,\infty\} \subset X$. The curve Y is called the *Picard curve*. Let G denote the Galois group of f, which is cyclic of order 3. The equation $\sigma^* y = \chi(\sigma)\cdot y$ for $\sigma \in G$ defines an injective character $\chi : G \hookrightarrow \mathbb{C}^\times$. As we will see below, the χ-eigenspace of the cohomology of Y gives rise to a local system on S whose associated system of differential equations is known as the *Picard–Euler system*.

We fix a generator σ of G and set $\omega := \chi(\sigma)$. Let $K := \mathbb{Q}(\omega)$ be the quadratic extension of $\mathbb{Q}$ generated by ω and $\mathcal{O}_K = \mathbb{Z}[\omega]$ its ring of integers. The family of G-covers $f : Y \to X$ together with the character χ of G corresponds to a local system of $\mathcal{O}_K$-modules on $U := X - D$. Set $s_0 := (2,3) \in S$ and let $\mathcal{V}_0$ denote the restriction of $\mathcal{V}$ to the fibre $U_0 = \mathbb{A}^1_{\mathbb{C}} - \{0,1,2,3\}$ of $U \to S$ over s_0. We consider $\mathcal{V}$ as a variation of $\mathcal{V}_0$ over S. Let $\mathcal{W}$ denote the parabolic cohomology of this variation; it is a local system of $\mathcal{O}_K$-modules of rank three, see [**DW**, Rem. 1.4]. Let $\chi' : G \hookrightarrow \mathbb{C}^\times$ denote the conjugate character to χ and $\mathcal{W}'$ the parabolic cohomology of the variation of local systems $\mathcal{V}'$ corresponding to the G-cover f and the character χ'. We write $\mathcal{W}_{\mathbb{C}}$ for the local system of $\mathbb{C}$-vectorspaces $\mathcal{W}\otimes\mathbb{C}$. The maps $\pi_Y : Y \to S$ and $\pi_X : X \to S$ denote the natural projections.

Proposition 3.1. — *We have a canonical isomorphism of local systems*

$$R^1\pi_{Y,*}\underline{\mathbb{C}} \cong \mathcal{W}_{\mathbb{C}} \oplus \mathcal{W}'_{\mathbb{C}}.$$

This isomorphism identifies the fibres of $\mathcal{W}_{\mathbb{C}}$ with the χ-eigenspace of the singular cohomology of the Picard curves of the family f.

Proof. — The group G has a natural left action on the sheaf $f_*\underline{\mathbb{C}}$. We have a canonical isomorphism of sheaves on X

$$f_*\underline{\mathbb{C}} \cong \underline{\mathbb{C}} \oplus j_*\mathcal{V}_{\mathbb{C}} \oplus j_*\mathcal{V}',$$

which identifies $j_*\mathcal{V}_{\mathbb{C}}$, fibre by fibre, with the χ-eigenspace of $f_*\underline{\mathbb{C}}$. Now the Leray spectral sequence for the composition $\pi_Y = \pi_X \circ f$ gives isomorphisms of sheaves on S

$$R^1\pi_{Y,*}\underline{\mathbb{C}} \cong R^1\pi_{X,*}(f_*\underline{\mathbb{C}}) \cong \mathcal{W}_{\mathbb{C}} \oplus \mathcal{W}'_{\mathbb{C}}.$$

Note that $R^1\pi_{X,*}\underline{\mathbb{C}} = 0$ because the genus of X is zero. Since the formation of $R^1\pi_{Y,*}$ commutes with the G-action, the proposition follows. □

The comparison theorem between singular and deRham cohomology identifies $R^1\pi_{Y,*}\underline{\mathbb{C}}$ with the local system of horizontal sections of the relative deRham cohomology module $R^1_{\mathrm{dR}}\pi_{Y,*}\mathcal{O}_Y$, with respect to the Gauss-Manin connection. The χ-eigenspace of $R^1_{\mathrm{dR}}\pi_{Y,*}\mathcal{O}_Y$ gives rise to a Fuchsian system known as the Picard–Euler system. In more classical terms, the Picard–Euler system is a set of three explicit partial differential equations in s and t of which the period integrals

$$I(s,t;a,b) := \int_a^b \frac{\mathrm{d}\,x}{\sqrt[3]{x(x-1)(x-s)(x-t)}}$$

(with $a, b \in \{0, 1, s, t, \infty\}$) are a solution. See [**Pic83**], [**Hol86**], [**Hol95**]. It follows from Proposition 3.1 that the monodromy of the Picard–Euler system can be identified with the representation $\eta : \pi_1(S) \to \mathrm{GL}_3(\mathcal{O}_K)$ corresponding to the local system $\mathcal{W}$.

***Theorem 3.2* (Picard).** — *For suitable generators $\gamma_1, \dots, \gamma_5$ of the fundamental group $\pi_1(S)$, the matrices $\eta(\gamma_1), \dots, \eta(\gamma_5)$ are equal to*

$$\begin{pmatrix} \omega^2 & 0 & 1-\omega \\ \omega-\omega^2 & 1 & \omega^2-1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 & 1-\omega^2 \\ 1-\omega^2 & 1 & \omega^2-1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & \omega^2-1 \\ 0 & \omega^2-1 & -2\omega \end{pmatrix},$$
$$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega^2 & \omega-\omega^2 & 0 \\ 0 & 1 & 0 \\ 1-\omega & \omega^2-1 & 1 \end{pmatrix}.$$

The invariant Hermitian form (induced by Poincaré duality, see Corollary 2.2) is given by the matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix},$$

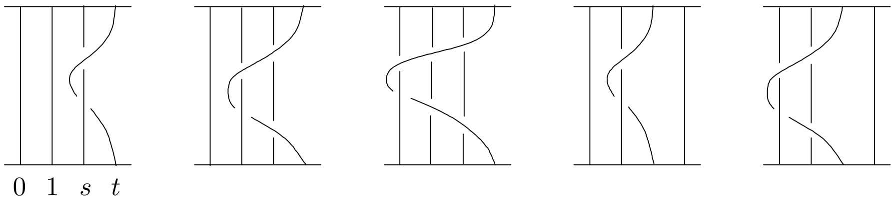

FIGURE 3. The braids $\gamma_1, \ldots, \gamma_5$

where $a = \frac{i}{3}(\omega^2 - \omega)$.

Proof. — The divisor $D \subset \mathbb{P}^1_S$ satisfies Assumption 1.2. Let $\varphi : \pi_1(S, s_0) \to A_4$ be the associated braiding map. Using standard methods (see e.g. [**Vö1**] and [**DR00**]), or by staring at Figure 3, one can show that the image of φ is generated by the five braids

$$\beta_3^2, \ \beta_3\beta_2^2\beta_3^{-1}, \ \beta_3\beta_2\beta_1^2\beta_2^{-1}\beta_3^{-1}, \ \beta_2^2, \ \beta_2\beta_1^2\beta_2^{-1}.$$

It is clear that these five braids can be realized as the image under the map φ of generators $\gamma_1, \ldots, \gamma_5 \in \pi_1(S, s_0)$.

Considering the ∞-section as a 'tangential base point' for the fibration $U \to S$ as in § 1.5, we obtain a section $\pi_1(S) \to \pi_1(U)$. We use this section to identify $\pi_1(S)$ with a subgroup of $\pi_1(U)$. Let $\alpha_1, \ldots, \alpha_5$ be the standard generators of $\pi_1(U_0)$. Let $\rho : \pi_1(U) \to K^\times$ denote the representation corresponding to the G-cover $f : Y \to X$ and the character $\chi : G \to K^\times$, and $\rho_0 : \pi_1(U_0) \to G$ its restriction to the fibre above s_0. Using (27) one checks that ρ_0 corresponds to the tuple $\mathbf{g} = (\omega, \omega, \omega, \omega, \omega^2)$, i.e. that $\rho_0(\alpha_i) = g_i$. Also, since the leading coefficient of the right hand side of (27) is one, the restriction of ρ to $\pi_1(S)$ is trivial. Hence, by Theorem 1.3, we have

$$\eta(\gamma_i) = \bar{\Phi}(\mathbf{g}, \varphi(\gamma_i)).$$

A straightforward computation, using (9) and the cocycle rule (10), gives the value of $\eta(\gamma_i)$ (in form of a three-by-three matrix depending on the choice of a basis of $W_{\mathbf{g}}$). For this computation, it is convenient to take the classes of $(1, 0, 0, 0, -\omega^2)$, $(0, 1, 0, 0, -\omega)$ and $(0, 0, 1, 0, -1)$ as a basis. In order to obtain the 5 matrices stated in the theorem, one has to use a different basis, i.e. conjugate with the matrix

$$B = \begin{pmatrix} 0 & -\omega - 1 & -\omega \\ \omega + 1 & \omega + 1 & \omega + 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The claim on the Hermitian form follows from Theorem 2.6 by another straightforward computation. □

Remark 3.3. — Theorem 3.2 is due to Picard, see [**Pic83**, p. 125] and [**Pic84**, p. 181]. He obtains exactly the matrices given above, but he does not list all of the corresponding braids. A similar list as above is obtained in [**Hol86**] using different methods.

Remark 3.4. — It is obvious from Theorem 3.2 that the Hermitian form on $\mathcal{W}$ has signature $(1,2)$ or $(2,1)$, depending on the choice of the character χ. This confirms Theorem 2.3 in this special case.

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