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THE MAIN CONJECTURE OF MODULAR TOWERS AND ITS HIGHER RANK GENERALIZATION

by

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Abstract. — The genus of projective curves discretely separates decidedly different two variable algebraic relations. So, we can focus on the connected moduli $\mathcal{M}_g$ of genus g curves. Yet, modern applications require a data variable (function) on such curves. The resulting spaces are versions, depending on our need from this data variable, of *Hurwitz spaces*. A *Nielsen class* (§1) is a set defined by $r \geq 3$ conjugacy classes $\mathbf{C}$ in the data variable monodromy G. It gives a striking genus analog.

Using Frattini covers of G, every Nielsen class produces a projective system of related Nielsen classes for any prime p dividing $|G|$. A nonempty (infinite) projective system of braid orbits in these Nielsen classes is an infinite $(G, \mathbf{C})$ *component (tree) branch*. These correspond to projective systems of irreducible (dim $r-3$) components from $\{\mathcal{H}(G_{p,k}(G), \mathbf{C})\}_{k=0}^{\infty}$, the $(G, \mathbf{C}, p)$ Modular Tower (**MT**). The classical modular curve towers $\{Y_1(p^{k+1})\}_{k=0}^{\infty}$ (simplest case: G is dihedral, $r = 4$, $\mathbf{C}$ are involution classes) are an avatar.

The (weak) Main Conjecture 1.2 says, if G is *p-perfect*, there are no rational points at high levels of a component branch. When $r = 4$, **MT**s (minus their cusps) are systems of upper half plane quotients covering the j-line. Our topics.

- §3 and §4: Identifying component branches on a **MT** from *g-p', p and Weigel cusp branches* using the **MT** generalization of *spin structures*.
- §5: Listing cusp branch properties that imply the (weak) Main Conjecture and extracting the small list of towers that could possibly fail the conjecture.
- §6: Formulating a (strong) Main Conjecture for higher rank **MT**s (with examples): almost all primes produce a modular curve-like system.

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Support from NSF #DMS-99305590, #DMS-0202259 and #DMS-0455266. This contains many advances on my March 12, 2004, Luminy talk (subsumed by overheads in [**Fri05a**]). One of those centers on Weigel cusps and whether they exist. This interchange with Thomas Weigel occurred in Jerusalem and Milan during the long trip including Luminy. Prop. 3.12 is due to Darren Semmen, a constant modular representation consultant. Conversations with Anna Cadoret, Pierre Debes and Kinya Kimura influenced me to be more complete than otherwise I would have been.

***Résumé* (La conjecture principale sur les tours modulaires et sa généralisation en rang supérieur)**

Le genre des courbes projectives est un invariant discret qui permet une première classification des relations algébriques en deux variables. On peut ainsi se concentrer sur les espaces de modules connexes $\mathcal{M}_g$ des courbes de genre g donné. Pourtant de nombreux problèmes nécessitent la donnée supplémentaire d'une fonction sur la courbe. Les espaces de modules correspondants sont les espaces de Hurwitz, dont il existe plusieurs variantes, répondant à des besoins divers. Une classe de Nielsen (§1) est un ensemble, constitué à partir d'un groupe G et d'un ensemble $\mathbf{C}$ de $r \geq 3$ classes de conjugaison de G, qui décrit la monodromie de la fonction. C'est un analogue frappant du genre.

En utilisant les revêtements de Frattini de G, chaque classe de Nielsen fournit un système projectif de classes de Nielsen dérivées, pour tout premier p divisant $|G|$. Un système projectif non vide (infini) d'orbites d'actions de tresses dans ces classes de Nielsen est une branche infinie d'un arbre de composantes. Cela correspond à un système projectif de composantes irréductibles (de dimension $r-3$) de $\{\mathcal{H}(G_{p,k}(G), \mathbf{C})\}_{k=0}^{\infty}$, la tour modulaire. La tour classique des courbes modulaires $\{Y_1(p^{k+1})\}_{k=0}^{\infty}$ (le cas le plus simple où G est le groupe diédral D_{2p}, $r = 4$ et $\mathbf{C}$ la classe d'involution répétée 4 fois) en est un avatar.

La conjecture principale (faible) dit que, si G est p-parfait, il n'y a pas de points rationnels au delà d'un niveau suffisamment élevé d'une branche de composantes. Quand $r = 4$, les tours modulaires (privées des pointes) sont des systèmes de quotients du demi-plan supérieur au-dessus de la droite projective de paramètre j. Nos thèmes.

- §3 et §4 : Identification des branches de composantes sur une tour modulaire à partir des branches de pointes $g-p'$, p et Weigel, grâce à la généralisation des structures de spin.
- §5 : Énoncé d'un ensemble de propriétés des branches de pointes impliquant la conjecture principale (faible) et réduction à un nombre limité de cas de tours pouvant encore éventuellement la mettre en défaut.
- §6 : Formulation d'une conjecture principale forte pour des tours modulaires de rang supérieur (avec des exemples) : presque tous les premiers conduisent à un système semblable à celui des courbes modulaires.

Contents

Luminy in March 2004 gave me a chance to show the growing maturity of Modular Towers (**MT**s). Documenting its advances, however, uses two other sources: Papers from this conference; and a small selection from the author's work. §C.1 lists the former. While the first two papers in that list have their own agendas, they show the influence of **MT**s. The last two papers aim, respectively, at the arithmetic and group theory of **MT**s. This paper concentrates on (cusp) geometry. As [**Fri07**] is not yet complete, I've listed typos corrected from the print version of [**BF02**] — our basic reference — in the on-line version (§C.2). From it came the serious examples (see partial list of §6.2.3) that graphically demonstrate the theory.

A glance at the Table of Contents shows §4 is the longest and most theoretical in the paper. It will figure in planned later papers. We have done our best in §6 to get serious examples to illustrate everything in §4. (Constraints include assuring we had in print enough on the examples to have them work as we wanted.) So, we suggest referring to §4 after finding motivation from other sections.

Many items in this paper would seem to complicate looking at levels of a **MT**: types of cusps, Schur multipliers of varying groups, component orbits. It behooves us to have an organizing tool to focus, label and display crucial and difficult computations. Further, we find that arithmetic geometers with little group theory background just don't know where to start. What surely helps handle some of these problems is the **sh**-incidence matrix. I suggested to Kay Maagard that the braid package (for computing Nielsen class orbits) would gain greatly if it had a sub-routine for this. He said he would soon put such in [**MSV03**].

We use the **sh**-incidence Matrix on $\mathrm{Ni}(A_4,\mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ in §6.4.2 to show what we mean. More elaborate examples for level 1 of this **MT** and also for $\mathrm{Ni}(A_5,\mathbf{C}_{3^4})^{\mathrm{in,rd}}$ are in [**BF02**, Chaps. 8 and 9]. All these are done without [**Sch95**] or other computer calculation, and they figure in many places in this paper as nontrivial examples of the mathematical arguments that describe the structure of **MT** levels. Still, [**BF02**, §9.2.1 and 9.2.2] list what [**Sch95**] produced for all branch cycles (see §5.2.2 and §6.2.3) for both (j-line covering) components at level 1 in the $(A_5,\mathbf{C}_{3^4},p=2)$ **MT**.

1. Questions and topics

In this paper the branch point parameter $r \geq 3$ is usually 4 (or 3). Results (based on §3 and §4) on **MT**s with r arbitrary are in a companion paper [**Fri06a**] that contains proofs of several results from the author's long-ago preprints. For example: It describes all components of Hurwitz spaces attached to $(A_n,\mathbf{C}_{3^r})$, alternating groups with 3-cycle branch cycles running over all $n \geq 3$, $r \geq n-1$.

1.1. The case for investigating MTs.

— A group G and r conjugacy classes $\mathbf{C} = \mathrm{C}_1,\dots,\mathrm{C}_r$ from G define a *Nielsen class* (§2.4.1). The Hurwitz monodromy group H_r acts on (we say *braids*) elements in representing Nielsen classes. Components of **MT** levels correspond to H_r orbits. Other geometry, especially related to cusps, corresponds to statements about subgroups of H_r on Nielsen classes.

Sometimes we use the notation $r_{\mathbf{C}}$ for the number r of conjugacy classes. Mostly, however, we concentrate on **MT**s defined by reduced (inner) Nielsen classes $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in,rd}}$ where $r_{\mathbf{C}} = 4$ (sometimes one conjugacy class, repeated four times). Then, the sequence of reduced inner Hurwitz spaces $(\{\mathcal{H}(G_{p,k}(G),\mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty}$ below) defining their levels are curves. Here H_4, acting on a corresponding projective sequence of Nielsen classes, factors through a mapping class group we denote as $\bar{M}_4$. It is naturally isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$.

In this case, a projective sequence of finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ acting on the upper half-plane, indexed by powers of a prime p, do correspond to these levels. Yet, this sequence appears indirectly in MTs, unlike the classical approach to the special case of modular curve sequences. The closure $\bar{\mathcal{H}}(G_{p,k}(G),\mathbf{C})^{\mathrm{in,rd}}$ is a ramified cover of the j-line (§2.3) that includes *cusps* (lying over $j = \infty$). Each cusp identifies with a *Nielsen class cusp set* (as in (2.5a)).

Like modular curves towers, the usual cusp type is a p cusp. Also, like modular curve towers, special cusp sets correspond to actual cusps with special geometric properties. The technical theme of this paper: **MT**s with *g-p' cusps* (§3.2.1) have a special kinship to modular curves (a subcase). That is because g-p' cusps potentially generalize a classical meaning for those modular curve cusps akin to representing degenerating Tate elliptic curves. This relates to the topic of *tangential base points* (Princ. 4.10 and §6.2). The other kind of cusp type called o-p' has no modular curve analog. We give many examples of these occurring on **MT**s where $p = 2$ and G_0 is an alternating group.

Direct interpretation of cusps and other geometric properties of **MT** levels compensates for how they appear indirectly as upper half-plane quotients. This allows defining **MT**s for $r > 4$. These have many applications, and an indirect relation with Siegel upper half-spaces, though no direct analog with modular curves.

1.1.1. Why investigate **MT***s?*— We express **MT**s as a response to these topics.

T_1. They answer to commonly arising questions:
- T_1.a. Why has it taken so long to solve the Inverse Galois Problem?
- T_1.b. How does the Inverse Galois Problem relate to other deep or important problems?

T_2. Progress on **MT**s generates new applications:
- T_2.a. Proving the Main Conjecture shows **MT**s have some properties analogous to those for modular curves.
- T_2.b. Specific **MT** levels have many recognizable applications.

Here is the answer to T_1.a. in a nutshell. **MT**s shows a significant part of the Inverse Galois Problem includes precise generalizations of many renown statements from modular curves. Like those statements, **MT** results say you can't find very many of certain specific structures over $\mathbb{Q}$.

For example, §6.1.2 cites [**Cad05b**] to say the weak (but not the strong) Main Conjecture of **MT**s follows from the Strong Torsion Conjecture (STC) on abelian varieties. Still, there is more to say: Progress on our Main Conjecture implies specific insight and results on the STC (subtle distinctions on the type of torsion points in question), and relations of it to the Inverse Galois Problem.

1.1.2. Frattini extensions of a finite group G lie behind **MT***s.* — Use the notation $\mathbb{Z}/n$ for congruences $\mod n$ and $\mathbb{Z}_p$ for the p-adic integers. Denote the profinite completion of $\mathbb{Z}$ by $\tilde{\mathbb{Z}}$ and its automorphisms (invertible profinite integers) by $\tilde{\mathbb{Z}}^*$.

Suppose p is a prime dividing $|G|$. Group theorists interpret p' as an *adjective* applying to sets related to G: A set is p' if p does *not* divide orders of its elements.

We say G is *p-perfect* if it has no $\mathbb{Z}/p$ quotient. For $H \leq G$, denote the subgroup of G generated by commutators ($hgh^{-1}g^{-1}$, $h \in H, g \in G$) by (H, G). Then, G is *perfect* if and only it is p-perfect for each p dividing $|G|$ (equivalent to $(G, G) = G$). §2.1 explains the point of the p-perfect condition.

A covering homomorphism $\varphi : H \to G$ of pro-finite groups is *Frattini* if for any proper subgroup $H^* < H$, the image $\varphi(H^*)$ is a proper subgroup of G. Alternatively, the kernel $\ker(\varphi)$ of φ lies in the Frattini subgroup (intersection of all proper maximal subgroups of G) of G. For P a pro-p group, the closure of the group containing pth powers and commutators is its *Frattini* subgroup $\Phi(P)$. Iterate this k times for

$$\Phi^k(P) < \Phi^{k-1}(P) < \cdots < P.$$

Consider a *reduced* Nielsen class (§2.4.2) defined by r (p') conjugacy classes

$$\mathbf{C} = (\mathrm{C}_1, \ldots, \mathrm{C}_r) \text{ in a finite group } G = G_0.$$

Defining the characteristic (projective) series of Nielsen classes from this requires the characteristic (projective) sequence $\{G_k\}_{k=0}^{\infty}$ of p-Frattini covers of G_0. Each G_k covers G and is a factor of the universal p-Frattini cover $\psi : {}_p\tilde{G} \to G$, versal for all extensions of G by p-groups ([**Dèb06**, §1.2], [**FJ86**, Chap. 20]):

$$\{G_k = G_{p,k}(G) \overset{\text{def}}{=} {}_p\tilde{G}/\Phi^k(\tilde{P}_p)\}_{k=0}^{\infty} \text{ with } \tilde{P}_p = \ker(\psi : {}_p\tilde{G} \to G).$$

Then, $G_{k+1} \to G_k$ is the maximal Frattini cover of G_k with elementary abelian p-group as kernel. Further, $\ker(G_{k+1} \to G_k)$ is a G_k module whose composition factors consist of irreducible G_0 modules. The most important of these is $\mathbf{1}_{G_k} = \mathbf{1}_{G_0}$, the trivial 1-dimensional G_k module.

[**Fri02**, §2.2] shows how to find the rank of the pro-p, pro-free group $\tilde{P}_p$. Its subquotients figure in the geometry of the attached **MT** levels.

Consider any cover $H \to G$ of profinite groups with kernel ($\ker(H \to G)$ a (pro-)p group. If C is a p' conjugacy class in G, then above it in H there is a unique p' conjugacy class. This is the most elementary case of the Schur-Zassenhaus Lemma. When we have this situation it is natural to retain the notation C for the conjugacy class in H, so long as we are clear on which group contains the class. Conversely, if C is a p' conjugacy class of H it has a unique image p' conjugacy class in G.

This setup applies whenever we refer to **MT**s, as in this. The **MT** attached to $(G, \mathbf{C}, p)$ is a projective sequence of spaces $\{\mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}} \overset{\text{def}}{=} \mathcal{H}_k\}_{k=0}^{\infty}$. We also use this lifting principle even when $H \to G$ is not a Frattini cover (as in §4.3).

1.1.3. **MT***s and the Regular Inverse Galois Problem.* — Use the acronym RIGP for the Regular Inverse Galois Problem. For any field K, K^{cyc} is K with all roots of 1 adjoined. Let F ($\leq \mathbb{C}$ for simplicity) be a field and $G = G_0$ any finite p-perfect group. An F regular realization of G^* is a Galois cover $\varphi^* : X^* \to \mathbb{P}^1_z$ over F with group G^* (with automorphisms also defined over F). Then, the branch point set $\boldsymbol{z}$ of φ^* is an F set, with corresponding conjugacy classes $\mathbf{C}^*$ in G^*.

We use the *Branch Cycle Lemma* (BCL, §3.1.1; [**Dèb06**, Thm. 1.5] has example uses when $\mathbb{Q} = F$). It says the branch points and respective conjugacy classes satisfy a compatibility condition: For each $\tau \in \mathrm{Aut}(\mathbb{C}/F)$, $z_i^\tau = z_j$ implies

$$(\mathrm{C}_i^*)^{n_\tau} = \mathrm{C}_j^* \text{ with } \tau \mapsto n_\tau \in G(\mathbb{Q}^{\mathrm{cyc}}/F \cap \mathbb{Q}^{\mathrm{cyc}}) \le \tilde{\mathbb{Z}}^*(\S 1.1.2). \tag{1.1}$$

We say the conjugacy classes are *F-rational* if (1.1) holds without our having to know anything more about the branch points than they are an F set. That is, if (as a set with multiplicity) $(\mathbf{C}^*)^n = \mathbf{C}^*$ for each $n \in G(\mathbb{Q}^{\mathrm{cyc}}/F \cap \mathbb{Q}^{\mathrm{cyc}})$.

A significant conclusion is that if G^* is centerless, and $\mathbf{C}^*$ is F-rational, then such φ^* s correspond one-one with F points on the space $\mathcal{H}(G^*, \mathbf{C}^*)^{\mathrm{in}}$ ([**FV91**, Thm. 1]; each then gives an F point in $\mathcal{H}(G^*, \mathbf{C})^{\mathrm{in,rd}}$). The quotients of ${}_p\tilde{G}$ differ in a style akin to the difference between D_p and $D_{p^{k+1}}$; in some ways not a big difference at all. So, we ask if they are all regular realizations from one rubric?

(1.2a) Minimum: Can all be realized with some bound on the number of branch points (dependent on G_0 and p)?

(1.2b) Maximum: Can all be realized with the same branch point set $\boldsymbol{z}$?

For many fields F, including number fields (Rem. 1.3), the hypothesis of Prop. 1.1 implies its conclusion ([**Dèb06**, Thm. 2.6] outlines the proof). That is, if (1.2a), then there is a specific **MT** with F points at each level.

Proposition 1.1. — *Assume there is r_0 so each G_k has an F regular realization, with $\le r_0$ branch points. Then, there is a* **MT** *from $(G, \mathbf{C})$ with $r_{\mathbf{C}} \le r_0$ and each $\mathcal{H}(G_k, \mathbf{C})^{\mathrm{in}}$ (and therefore $\mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}}$), $k \ge 0$, has an F point.*

The last half answer to Quest. T_1.a is the conjecture that the *conclusion* (and therefore the hypothesis) of Prop. 1.1 doesn't hold for number fields.

***Conjecture 1.2* (Weak Main Conjecture)**. — Suppose G_0 is p-perfect and K is a number field. Then, there cannot be K points at every level of a **MT**. So, regular realizations of all the G_k s over K requires an unbounded number of branch points.

A modular curve case of this is that $Y_1(p^{k+1})$ (modular curve $X_1(p^{k+1})$ minus its cusps) has no K points for $k >> 0$. Thm. 5.1 says the Main Conj. holds for $(G_0, \mathbf{C}, p)$ unless there is a K projective sequence of components $\{\mathcal{H}'_k \subset \mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^\infty$ and either none of the $\mathcal{H}'_k$ has a p cusp; or $\mathcal{H}'_{k+1}/\mathcal{H}'_k$ is equivalent to a degree p rational function $f_k : \mathbb{P}_z^1 \to \mathbb{P}_z^1$ with f_k either a polynomial, or totally ramified over two places.

***Remark 1.3* (F for which Prop. 1.1 holds)**. — Recall, compatible with (1.1), an element g in a profinite group is F-rational if g^n is conjugate to g for all $n \in G(\mathbb{Q}^{\mathrm{cyc}}/F \cap \mathbb{Q}^{\mathrm{cyc}}) \le \tilde{\mathbb{Z}}^*$. Denote the the field generated by roots of 1 of p' order by $\mathbb{Q}^{\mathrm{cyc},p'}$ and let $F_{p'} = F \cap \mathbb{Q}^{\mathrm{cyc},p'}$. [**FK97**, Thm. 4.4] shows that if *no* p-power element $g \in {}_p\tilde{G}$ is F-rational, then F satisfies Prop. 1.1. Further, this holds if $[F_{p'} : \mathbb{Q}] < \infty$.

1.1.4. Limit groups. — Finding F regular realizations, and their relation to Conj. 1.2, breaks into three considerations for the collection of p-Frattini covers $G^* \to G$.

(1.3a) For what G^*s is $\mathcal{H}(G^*, \mathbf{C})^{\mathrm{in,rd}}$ nonempty (so it can have F points)?
(1.3b) Which of those nonempty $\mathcal{H}(G^*, \mathbf{C})^{\mathrm{in,rd}}$s have some absolutely irreducible F component $\mathcal{H}'(G^*, \mathbf{C})^{\mathrm{in,rd}}$?
(1.3c) Which of the $\mathcal{H}'(G^*, \mathbf{C})^{\mathrm{in,rd}}$s have F points.

Limit groups (a braid orbit invariant) are a profinite summary of what (1.3a) is about (§4.1): A positive answer for G^* holds in (1.3a) if and only if G^* is a quotient of a limit group for some braid orbit on $\mathrm{Ni}(G_0, \mathbf{C})$. Note: There may be several limit groups for a given level 0 braid orbit (as in App. B.1). Braid orbits in $\mathrm{Ni}(G_0, \mathbf{C})$ containing g-p' cusps have the whole of ${}_p\tilde{G}$ as one limit group (Princ. 3.6). §4.5 documents much evidence this is also necessary.

Fields F that are ℓ-adic completions of a number field are examples for which the maximum condition (1.2b) holds (see [**Dèb06**, §2.4]; though $[F_{p'} : \mathbb{Q}] = \infty$ in Rem. 1.3). That means there is an F component branch (§1.2.1 — all levels defined over F) on some **MT** with a projective system of F points $\{\boldsymbol{p}_k \in \mathcal{H}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$. By contrast, though (1.2a) (with Prop. 1.1) postulates F points at all levels of some **MT**, over a number field we know they cannot form a projective system [**BF02**, Thm. 6.1].

Denote the completion of a field K at a valuation ν of K by K_ν. Evidence from the case of shifts of Harbater-Mumford representatives (H-M reps.) suggests an affirmative answer for the following. §1.2.1 explains the hypotheses opening Quest. 1.4.

Question 1.4. — Let K be a number field with $\{\mathcal{H}'(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$ a K component branch defined by a g-p' cusp branch. Does it have a projective system of K_ν points for each ν over any prime ℓ not dividing $|G_0|$?

App. A and App. B give cases of Nielsen classes with limit groups other than ${}_p\tilde{G}$. App. A is a different angle on modular curves, where a universal Heisenberg group obstruction explains the unique limit group.

App. B includes applying Thm. 4.12 (and Ex. 4.13). Here, each layer of an H-M cusp branch has above it at least two components, one not an H-M component. Something similar happens for the main example **MT** of [**BF02**] (for $G = A_5$; Ex. B.2). So, each level of these examples has at least two components, one with ${}_p\tilde{G}$ in its limit group, and the other with ${}_p\tilde{G}$ not in its limit groups.

A rephrase of (1.3b) would be to decide which limit groups produce a $\mathbb{Q}$ component branch. When the limit group is ${}_p\tilde{G}$ and the component branch is from an H-M cusp branch it is sufficient that all H-M reps. fall in one braid orbit (see §1.4). We expect this to generalize to g-p' reps. The criterion of [**Fri95**, Thm. 3.21] for H-M reps. to fall in one braid orbit holds at all levels of a **MT**, if it holds at level 0. Still, that criterion never holds when $r = 4$, the main case of this paper.

Finally, given that we know the answers for a particular Nielsen class to (1.3a) and (1.3b), (1.3c) gets to the nub of our Main Conjecture: High **MT** levels should have no rational points over a number field K (at least when the limit group is ${}_p\tilde{G}$).

§6.3 gives a solid example of how to use the cusp rubric to compute. It shows the nature of the two components, $\mathcal{H}_0^+ \cup \mathcal{H}_0^-$ at level 0 of a significant **MT**. Both have genus 0, and $\mathcal{H}_0^+$ is an H-M component: Indeed, it contains *all* H-M cusps (Ex. 3.7, shifts of special reps. in g-p' cusps).. The other has nontrivial lifting invariant (§4.2) and so nothing above it at level 1. Both are parameter spaces of genus 1 curves, and both are upper half plane quotients. Yet, neither is a modular curve.

1.2. Five parts on a MT structure. — From this point $r = 4$. So, **MT** levels are j-line covers [**BF02**, Prop. 2.3 and §2.3.1]. We list this paper's six main topics.

(1.4a) §2.4.2: Tools for computing cusp widths (ramification orders) and elliptic ramification of levels.

(1.4b) §3.1.1, §3.2.1 and §4.1: Relating infinite branches on the cusp and component trees, a classification of cusp types and limit Nielsen classes.

(1.4c) §4.3 and §4.4: Describing infinite component branches.

(1.4d) §5: Outlining for $r = 4$ how to prove the (weak) Main Conjecture.

(1.4e) §6.1: Formulating the Strong Main Conjecture and comparing its expectations with that for modular curve towers.

(1.4f) §4.1, §6.2 and §6.3: Showing specific **MT** components apply to significant Inverse Galois and modular curve topics.

These contribute to T_1.b ((1.4a), (1.4c) and (1.4e)) and T_2 ((1.4b), (1.4d) and (1.4f)).

1.2.1. Results on cusps. — Conj. 2.2 interprets the Main Conjecture as a statement on computing genera of components. That starts the proof outline that (1.4c) alludes to. §2.4 turns that computation into group theory and combinatorics.

Our main results relate cusps at a **MT** level to the components on which they lie. The language uses a cusp (resp. component) tree $\mathcal{C}_{G,\mathbf{C},p}$ (resp. $\mathcal{T}_{G,\mathbf{C},p}$) on a **MT** (§3.1). The natural map $\mathcal{C}_{G,\mathbf{C},p} \to \mathcal{T}_{G,\mathbf{C},p}$ is from containment of cusps in components. This interprets from a cusp set being in a braid orbit (2.5).

An infinite (geometric) component branch (§3.1) is a maximal projective sequence

$$B' = \{\bar{\mathcal{H}}_k' \subset \bar{\mathcal{H}}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty} \text{ of (geometric) Hurwitz space components.}$$

With F a field, call B' an F *component branch* if all levels have definition field F. An infinite cusp branch is a maximal projective sequence

$$B = \{\bar{\boldsymbol{p}}_k \in \bar{\mathcal{H}}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty} \text{ of (geometric) points over } j = \infty.$$

There also exist finite branches, where the last component $\mathcal{H}_k'$ has nothing above it on $\mathcal{H}(G_{k+1}, \mathbf{C})^{\mathrm{in,rd}}$. Our Main Conjecture only applies to infinite K component branches where K is a number field. Still, describing the infinite component branches forces dealing with the finite branches. From §2.4.2, B corresponds to a sequence of cusp sets defined by a projective system $\{{}_k\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$ of Nielsen class elements. Characterizations of such a B come from definitions of p, g-p' and o-p' cusps (§3.2.1). Three *Frattini Principles* 3.5, 3.6 and 4.24 imply one of these three happens.

(1.5a) For k large, $\bar{\boldsymbol{p}}_k$ is a p cusp (p branch).
(1.5b) For all k, $\bar{\boldsymbol{p}}_k$ is a g-p' cusp (g-p' branch).
(1.5c) For k large, $\bar{\boldsymbol{p}}_k$ is an o-p' cusp (Weigel branch).

In case (1.5a) there could be a string consisting of g-p' and/or o-p' cusps before the p cusp part of the sequence. For many g-p' cusps there are no o-p' cusps above them (for cusps of shifts of H-M reps., for example as prior to Prop. 3.12). So, if at level 0 you only have such g-p' cusps, no projective sequence will include both g-p' and o-p' cusps.

Still, Prop. 3.12 produces **MT**s where an o-p' cusp lies over some g-p' cusps at each *high* level. When finite exceptional strings don't occur at the start of cusp branches, we call them *pure*. Any **MT** level can be the start of the tower by applying a fixed shift of the indices. Then these names would apply to cusps at that level.

1.2.2. g-p' (cusp) versus Weigel cusp branches. — Any cusp branch B determines a component branch B'. This allows naming an infinite component branch B' of $\mathcal{T}_{G,\mathbf{C},p}$ by the name of the cusp branch.

For example, a g-p' branch (as in Princ. 3.6) on the cusp tree produces a g-p' branch on the component tree. A succinct phrasing of Princ. 3.6:

(1.6) Any g-p' cusp starts at least one (infinite) g-p' branch.

A succinct converse of this would help so much to decide which **MT**s most resemble modular curve towers. Here is our best guess for such a converse.

***Conjecture 1.5* (g-p' Conjecture).** — Show for K is a number field, each K component branch (§1.2.1) on a **MT** is defined by some g-p' cusp branch.

Many papers consider H(arbater)-M(umford) cusp (Ex. 3.7) and component branches ([**Cad05b**], [**DD04**], [**DE06**], [**Wew02**]; not using the term branch).

By contrast Weigel cusp branches are an enigma. Identifying g-p' cusps and a corresponding branch of $\mathcal{C}_{G,\mathbf{C},p}$ has given the successes for finding infinite branches of $\mathcal{T}_{G,\mathbf{C},p}$. The gist of Conj. 1.6 is they are necessary for a component branch. §4.6 lists evidence for it. Examples in §4.6.2 show the main issues.

Conjecture 1.6. — With K a number field, there are no Weigel cusp branches on any infinite K component branch of a **MT**.

If Conj. 1.6 is true, then for any (infinite) K component branch either a g-p' branch defines it or it has only p cusp branches (see §1.3.3). We also suspect the latter cannot hold, for such component branches would lack classical aspects.

1.2.3. Setup for proving the (weak) Main Conjecture. — The group H_4 acts (through $\bar{M}_4$) compatibly on all Nielsen class levels of a **MT**. So any $q \in H_4$ acts on a projective system $\{{}_k\boldsymbol{g}\}_{k=0}^{\infty}$ defining a cusp branch B, with $\{({}_k\boldsymbol{g})q\}_{k=0}^{\infty}$ defining a new sequence of cusps. (A different projective system of representatives for B likely gives a different projective system of cusps from the q action.)

From this, many cusp branches may define the same component branch. So any component branch could simultaneously be a g-p', p and Weigel component branch.

Thm. 5.1 says, the (weak) Main Conjecture 1.2 essentially follows if there must be more than one p cusp branch on a component branch. Since modular curve towers, and all presently analyzed **MT**s have ∞-ly many p-cusp branches, this seems a sure bet. An affirmative result like [**BF02**] paved the way if $\bar{\boldsymbol{p}}_k$ is a p cusp or the cusp of a shifted H-M rep. So, here is the hardest remainder (modulo Conj. 1.6) for [**Fri06b**]:

(1.7) For k large, a g-p' cusp braids to a p cusp.

We abstract the framework from [**BF02**, §8] for H-M cusps and $p = 2$ in §5.3 to show both its likelihood and nontriviality.

1.3. MTs of arbitrary rank and full component branches.

— For both applications and technical analysis we expand in two ways on what spaces come attached to a definition of a **MT**.

1.3.1. Intermediate spaces and groups acting on free groups. — Our applications use spaces intermediate to $\mathcal{H}_k \to U_\infty$ (notation of §2.3), just as modular curves use $Y_0(p^{k+1})$ as a space intermediate to $Y_1(p^{k+1}) \to U_\infty$. This gives the notions of *full* cusp and component graphs (§1.3.2; these are rarely trees).

Also, starting with a finite group H acting faithfully on a free group F_u (or a lattice $\mathbb{Z}^u$) replacing a finite group G, gives the concept of a **MT** of rank u. This allows running over all primes, not explicitly excluded by our usual assumptions: G is p-perfect and $\mathbf{C}$ consists of p' classes.

We have two immediate reasons for doing this.

(1.8a) §1.4: For a version of Serre's *O(pen)I(mage)T(heorem)* (OIT) [**Ser98**].

(1.8b) Res. 1.7: To compare **MT**s with the most compelling arithmetic statement we know on modular curve towers.

Result 1.7 **(Mazur-Merel)**. — *For each number field K, there is a constant A_K (dependent only on K) so there are no rational points on $Y_1(p^{k+1})$ (modular curve $X_1(p^{k+1})$ minus its cusps) if $p^{k+1} > A_K$.*

Our (strong) Main Conjecture (Conj. 6.2) formulates this to **MT**s of arbitrary rank. [**Fri06a**] has applications to statements independent of **MT**s. Though **MT** levels are rarely modular curves (quotients of *congruence* subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ acting on the upper half plane), modular curve thinking guides their use.

1.3.2. Expanding on cusp and component branches. — Using groups intermediate to the G_k s produces *(p)-limit* Nielsen classes $\mathrm{Ni}(G^*, \mathbf{C})$ with G^* a maximal quotient (limit group) of ${}_p\tilde{G}$ having $\mathrm{Ni}(G^*, \mathbf{C})$ nonempty. Limit groups are braid invariants on projective systems of Nielsen class elements. Unless G^* is ${}_p\tilde{G}$, these give *full* **MT**s whose infinite branches don't have components of $\{\mathcal{H}_k\}_{k=0}^{\infty}$ cofinal among them. This generalization has three motivations.

(1.9a) To include *all* modular curves for p odd in this rubric (not just those closely related to $Y_1(p^{k+1})$ s) requires a rank 2 **MT**.
(1.9b) Higher rank **MT**s for p can have special *F-quotients* (§6.2.2) — still based on the universal p-Frattini cover — from low-level quotients.
(1.9c) Using [**Wei05**] gives some precise limit group properties.

App. A gives a full comparison of **MT**s with all modular curves. It shows the unique limit group for (1.9a) is $(\mathbb{Z}_p)^2 \times^s \{\pm 1\}$ (for $p \neq 2$). Cor. 4.20 explains how each limit group is defined by a unique obstruction. Here that obstruction is universal across all primes, coming from a Heisenberg group. §1.3.3 shows how (1.9c) helps decide when the limits groups are ${}_p\tilde{G}$, the case of our Main Conjecture. §1.4 is on how F-quotients in (1.9b) point to generalizations of Serre's OIT.

1.3.3. Component branches and Schur multipliers. — [**BF02**, §8] gave a procedure for figuring components on a **MT** level. Making the computations at level 0 requires detailed handling of conjugacy classes **C** for the group G_0. Level 0 components in the case of simple groups have contributed much of the success of the braid approach to the Inverse Galois Problem. Though predicting how components and cusps work at level 0 is still an art, various families of groups (simple and otherwise) do exhibit similar patterns when using related conjugacy classes (witness A_n and 3-cycles [**Fri06a**]). Given the level 0 work, we organize for higher levels in three steps.

(1.10a) Inductive setup from level k to $k+1$: List cusps at level k within each braid orbit, and choose one representative ${}_k\boldsymbol{g}$ for each braid orbit ${}_kO_b$.
(1.10b) List all preimages in $\mathrm{Ni}(G_{k+1}, \mathbf{C})$ lying over ${}_k\boldsymbol{g}$ and use this to list all cusps ${}_{k+1}O_c$ at level $k+1$ lying over cusps ${}_kO_c$ in ${}_kO_b$.
(1.10c) Then, partition cusps lying over ${}_k\boldsymbol{g}$ according to their braid orbits.

The G_k module $M_k = \ker(G_{k+1} \to G_k)$ controls going from G_k to G_{k+1}. A characteristic sequence of M_k subquotients (called Loewy layers; example §A.2.1 will help the reader) are semi-simple G_0 modules. Since [**Fri95**] we've known it is the $\mathbf{1}_{G_0}$ s in the Loewy layers that are critical to properties of higher **MT** levels.

The cardinality of the fiber in (1.10b) is a braid invariant. The first business is a version of (1.3a): Decide effectively when the fiber is nonempty. Cor. 4.19 shows it is the $\mathbf{1}_{G_0}$ s in the first Loewy layer of M_k — the maximal elementary p quotient of the Schur multiplier of G_k (§2.1) — that controls this.

Suppose O is a braid orbit in $\mathrm{Ni}(G = G_0, \mathbf{C})$. Then, O defines a profinite cover $\psi_O : M_O \to G$ with this versal property (Lem. 4.14). For any quotient G' of ${}_p\tilde{G}$, each braid orbit $O' \le \mathrm{Ni}(G', \mathbf{C})$ over O corresponds to $\psi' : M_O \to G'$ factoring through ψ_O. Weigel's Th. 4.15 says M_O is an *oriented p-Poincaré duality group*.

One consequence: Cor. 4.19 says that if the fiber over the orbit ${}_kO_b$ is empty (as in (1.10b)), then some $\mathbb{Z}/p$ quotient in the *first* Loewy layer of $\ker(G_{k+1} \to G_k)$ obstructs it. To wit, if $R \to G$ is the central extension with $\ker(R \to G_k)$ giving this $\mathbb{Z}/p$ quotient, then $M_O \to G_k$ for ${}_kO_b$ does not extend to $M_O \to R$.

Further, Cor. 4.20 says that if G^* is a limit group in a Nielsen class and it is different from ${}_p\tilde{G}$, then the following hold.

(1.11a) G^* has exactly one nonsplit extension by a $\mathbb{Z}/p[G^*]$ module M'.
(1.11b) M' is the trivial (one-dimensional) $\mathbb{Z}/p[G^*]$ module.

App. A and B give explicit examples identifying M'.

The example of §6.3 combines the **sh**-*incidence matrix* with the natural division into cusp types from §3.2.1 to show how we often manage figuring (1.10c). Princ. 4.24 frames in pure group theory how to deal with o-p' cusps. So, it sets a module approach for, say, Conj. 1.6. Here's how this refined tool relates cusps with their components.

Suppose $\boldsymbol{g} \in O \le \mathrm{Ni}(G_k, \mathbf{C})$ defines an o-p' cusp. Then, having an o-p' cusp $\boldsymbol{g}' \in O' \in \mathrm{Ni}(G_{k+1}, \mathbf{C})$ over $\boldsymbol{g}$ restates as a versal property for two profinite groups extensions that induce $\psi_{O'} : M_{O'} \to G_{k+1}$. This characterizes with group theory whether there are Weigel cusp branches through O. These formulas will generalize to **MT**s of arbitrary rank and any value of r.

1.4. Generalizing complex multiplication and Serre's OIT. —

App. B gives a significant example when there are several limit groups G^* (one, at least, $\ne {}_p\tilde{G}$) and — as we show — the spaces are not modular curves. So, it is nontrivial that we can here be explicit in formulating an OIT and a **MT** version of complex multiplication.

1.4.1. Decomposition groups. — Suppose $j' \in U_\infty(F)$ (§2.3; with F a number field) is a j value. Then, there is a decomposition group $D_{j'}$ from G_F acting on projective systems of points $\mathbf{Fib}_{j'}(G^*, \mathbf{C})$ on the full **MT** over j' defined by $(G^*, \mathbf{C})$. [**BF02**, Thm. 6.1] (when $G^* = {}_p\tilde{G}$) says no orbit has length one. It is far stronger than the Main Conjecture to have $D_{j'}$ with *large* orbits on $\mathbf{Fib}_{j'}(G^*, \mathbf{C})$, for all j'.

To go, however, beyond naiveté requires estimating how large $D_{j'}$ is. Lem. 3.1 explains how to use cusp branch types: Practical knowledge of how G_F acts on systems of components comes from knowing how G_F acts on specific types of cusps.

The historical example is where we know all H-M cusps fall in one braid orbit. Then, [**Fri95**, Thm. 3.21] says a component containing the H-M cusps has definition field given by the BCL (§1.1.3; this is $\mathbb{Q}$ if $\mathbf{C}$ is $\mathbb{Q}$-rational). [**Cad05a**] exploits this for arbitrary r to produce many Nielsen classes where the corresponding reduced Hurwitz space contains absolutely irreducible curves over $\mathbb{Q}$ (the first result of its kind).

We expect g-p' cusp *types* to be the main tool for many results. For example, [**Fri95**, Thm. 3.21] should generalize to describe component branches with all levels defined over some fixed number field. We guess this is exactly when all g-p' cusps of a fixed *type* fall in a bounded (independent of the level) number of orbits.

Here is another example. §6.2.4 notes that a g-p' cusp branch B provides a *tangential base-point* in the sense of Nakamura. Related cusps would allow following the proof of Serre's OIT for "large" j-invariant, by considering the arithmetic of these cusps over all rank 1 complete fields.

1.4.2. Seeking OIT examples. — App. A has a (rank 2, as in §1.3.1) **MT** attached to $F_2 \times^s \mathbb{Z}/2$. It describes the full **MT** whose levels identify with standard modular curves. Here, for all (odd) p, there is a unique limit **MT**, and a unique (proper) F-quotient of it. For each there is a (full) component graph, which we respectively denote by $\mathcal{T}_{\mathrm{GL}_2}$ and $\mathcal{T}_{\mathrm{CM}}$.

So, in this language, we expect j' values that produce decomposition groups that correspond to $\mathcal{T}_{\mathrm{GL}_2}$ (or to GL_2) and to $\mathcal{T}_{\mathrm{CM}}$ (or to CM). That this is so is Serre's OIT, in our language. Our next example shows how to extend this to general higher rank **MT**s. Seeking an OIT type result uses analog properties from Serre's example. It is crucial that we expect there to be Frattini properties for monodromy groups of **MT** component branches, as in (6.2).

App. B has a rank 2 **MT** attached to $G = F_2 \times^s \mathbb{Z}/3$ that shows possibilities for general results like Serre's OIT. We see the g-p' cusp criterion (Princ. 3.6) for identifying infinite component branches in a **MT**. For both $p = 2$ and $p \equiv -1 \mod 3$, one limit group is ${}_p\tilde{G} = \tilde{F}_{2,p} \times^s \mathbb{Z}/3$, and its **MT** has no F-quotient. At least for $p = 2$, there are other limit groups, explicitly showing Cor. 4.19. We conjecture $D_{j'}$ in these cases always has a type we call F_2.

For $p \equiv +1 \mod 3$, $\tilde{F}_{2,p} \times^s \mathbb{Z}/3$ is also a limit group, but its **MT** has a unique F-quotient. In this case we expect $D_{j'}$ has either type F_2 or a type we call CM (and both types occur).

1.4.3. Low **MT** *levels apply to the RIGP and to Andre's Theorem.* — (1.4f) alluded to the specific applications of its level 0 and 1 components for $p = 2$. None of its levels are modular curves. Also, unlike modular curve levels, these levels have several components. §6.3 labels the two level 0 components as $\mathcal{H}_0^+$ and $\mathcal{H}_0^-$. Level 1 has six, labeled $\mathcal{H}_1^x$ with the x decoration signifying some special property. Here appear generalizations of spin invariants (as in §1.3.3) that produce varying types of component branches.

For $p = 2$, and level 0, $\mathcal{H}_0^{\pm}$ (parametrizing families of genus 3 curves) map to their absolute (reduced) Hurwitz space versions $\mathcal{H}_0^{\pm,\mathrm{abs}}$. Each, like a modular curve, parametrizes genus 1 curves with extra structure and embeds naturally in $\mathbb{P}^1_j \times \mathbb{P}^1_j$.

Suppose in this embedding the components have infinitely many coordinates in complex quadratic extensions of $\mathbb{Q}$. Then, we might be suspicious when $p = 2$ that this **MT** would have some complex multiplication property. A theorem, however, of André's (Prop. 6.15) says they don't. This further corroborates our guess that for $p = 2$, almost all $D_{j'}$ have type F_2.

Two level 1 components (§6.4.5) contain H-M reps. We show what a serious challenge is deciding whether their defining field is $\mathbb{Q}$, with its effect on the RIGP (applied to the exponent 2 Frattini cover of A_5).

2. Ingredients for a MT level

We start with some notation and an explanation of how Schur multipliers appear here. Then we briefly try to comfort a reader about Hurwitz spaces as families of covers of the Riemann sphere: $\mathbb{P}^1_z = \mathbb{C}_z \cup \{\infty\}$.

2.1. p-perfectness and Schur multipliers. — Consider r conjugacy classes, $\mathbf{C}$, in G and $\boldsymbol{g} = (g_1, \dots, g_r) \in G^r$. Then, $\boldsymbol{g} \in \mathbf{C}$ means $g_{(i)\pi}$ is in C_i, for some π permuting $\{1, \dots, r\}$. Also, $\Pi(\boldsymbol{g}) \overset{\text{def}}{=} \prod_{i=1}^r g_i$ (order matters). Lem. 2.1 shows how p-perfect enters.

Lemma 2.1. — *If p is a prime with G not p-perfect and $\mathbf{C}$ are p' classes of G, then elements in $\mathbf{C}$ are in the kernel of G to the corresponding $\mathbb{Z}/p$ quotient. So, if $\boldsymbol{g} \in \mathbf{C}$ then $\langle \boldsymbol{g} \rangle = G$ is impossible:* $\mathrm{Ni}(G, \mathbf{C})$ *(and the Hurwitz space) is empty.*

Here is another technical plus from the p-perfect condition. There is a Frattini cover $R_p \to G$ with $\ker(R_p \to G)$ in the center of R_p and equal to the p part of the Schur multiplier of G. Further, $R_p \to G$ is *universal* for central p extensions of G (for example, [**BF02**, §3.6.1]; call it the representation cover for (G, p)). We use the notation SM_G (resp. $\mathrm{SM}_{G,p}$) for the Schur multiplier (resp. p-part of the Schur multiplier) of G. If G is p-perfect for all $p || \mathrm{SM}_G|$, then the fiber product over G of all such R_p is truly a universal Frattini central extension of G. §2.5 lists properties we use of Schur multipliers.

Identifying components of **MT** levels is a recurring theme. Whether a component at level k has some component above it at level $k+1$ — the level k component is *unobstructed* — is controlled by Schur multipliers. Lem. 4.9 and Cor. 4.19 are our main tools. Applying them is the heart of describing the type of infinite branches in a **MT**. We conclude with comments on the literature.

The definition of homology groups of G (with coefficients in $\mathbb{Z}$) came from topology. These were the homology groups of a space with fundamental group G whose simply connected cover is contractible. [**Bro82**, p. 2] discusses how Hopf used it to describe $H_2(G, \mathbb{Z})$. Write $G = F/R$ with F free. Then, $H_1(G, \mathbb{Z}) = G/(G, G)$ and $H_2(G, \mathbb{Z}) = R \cap (F, F)/(F, R)$ (the Schur multiplier of G).

The expression for H_1 is from general principles. For H_2 it is not obvious. It is usual to compute H_2 using tricks to identify E that suits (2.1). If G is perfect, then there is a universal (short exact) sequence

$$0 \to H_2(G,\mathbb{Z}) \to E \to G \to 1. \tag{2.1}$$

The group E factors through all central extensions of G [**Bro82**, p. 97, Ex. 7] (by a unique map through p group extensions if G is p-perfect). By contrast, the universal Frattini cover $\tilde{G} \to G$ of G is versal: It factors through all extensions of G including E, but the factoring map isn't unique. Then, $R_p \to G$ is the extension of G from modding E out by the p' part of $\ker(E \to G)$. It is easy that p-perfectness is the same as $R_p \to G$ being a universal p-central extension of G.

Also, if G is p-perfect and centerless, then all the characteristic Frattini quotients (§1.1.2) $G_{p,k}$ are too. That implies $\mathcal{H}(G_{p,k},\mathbf{C})^{\mathrm{in}}$ (see below) has *fine* moduli [**BF02**, Prop. 3.21]. Take $R_{p,k}$ as the representation cover of $(G_{p,k},p)$. Then, $\mathcal{H}(R_{p,k},\mathbf{C})^{\mathrm{in}}$ does not have fine moduli. Both statements produce many Hurwitz space applications.

2.2. One cover defines a family of covers.

— An analytic cover, $\varphi : X \to \mathbb{P}^1_z$ of compact Riemann surfaces, ramifies over a finite set of points

$$\boldsymbol{z} = z_1,\dots,z_r \subset \mathbb{P}^1_z :\ \mathbb{P}^1_z \setminus \{\boldsymbol{z}\} = U_{\boldsymbol{z}}.$$

Such a φ defines a system of covers by applying Riemann's existence theorem and deforming the branch points (keeping them distinct). We explain.

Represent projective r space $\mathbb{P}^r$ as nonzero polynomials of degree at most r modulo scalar multiples. Then, polynomials (r unordered points) with at least two equal zeros form its *discriminant* locus D_r. Denote $\mathbb{P}^r \setminus D_r$ by U_r. By moving branch points $\boldsymbol{z}$, you can form along any path in U_r a unique continuation of the cover φ.

Given $\boldsymbol{z}$ and classical generators at $\boldsymbol{z}^0$ ([**BF02**, §2.1-2.2] or §4.3), this interprets homotopy classes of paths in $\pi_1(U_r,\boldsymbol{z})$ as Hurwitz monodromy H_r (§2.4.1). Its action on Nielsen classes then reproduces this deformation of covers.

Suppose given $(G_0,\mathbf{C},p)$ with p' classes $\mathbf{C} = (\mathrm{C}_1,\dots,\mathrm{C}_r)$. [**Dèb06**, §1.2] reminds how this produces a projective sequence $\{\mathcal{H}^{\mathrm{in}}_k\}_{k=0}^{\infty}$, of inner Hurwitz spaces. Assuming it is nonempty, the level k space has dimension r and is an affine variety étale over U_r. These levels correspond to inner Nielsen classes as in §2.4.

Any $\boldsymbol{p} \in \mathcal{H}^{\mathrm{in}}_k$ corresponds to an equivalence class of Galois covers $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$, with group denoted $\mathrm{Aut}(X_{\boldsymbol{p}}/\mathbb{P}^1_z)$. The representative includes a specific isomorphism $\mu : \mathrm{Aut}(X_{\boldsymbol{p}}/\mathbb{P}^1_z) \to G_k(G)$. Another cover $\varphi' : X' \to \mathbb{P}^1_z$ is in the same *inner* class if the following holds. There is a continuous $\psi : X' \to X_{\boldsymbol{p}}$, commuting with the maps to $\mathbb{P}^1_z$, inducing conjugation by some $g \in G_k(G)$ between identifications of $\mathrm{Aut}(X_{\boldsymbol{p}}/\mathbb{P}^1_z)$ and $\mathrm{Aut}(X'/\mathbb{P}^1_z)$ with $G_k(G)$. We say the cover is in the Nielsen class $\mathrm{Ni}(G_k(G) = G_{p,k}(G),\mathbf{C})^{\mathrm{in}}$.

More detail is in [**BF02**, §2], [**Fri07**, Chap. 4], [**Völ96**, Chap. 10]. The first two especially discuss the motivation and basic definitions for **MT**s.

2.3. Reduced inner spaces. — We use *reduced* inner Nielsen classes. This references triples (ψ, μ, β) (not just (ψ, μ) as in §2.2): $\beta \in \mathrm{PGL}_2(\mathbb{C})$, and $\varphi_{\boldsymbol{p}} \circ \psi = \beta \circ \varphi'$.

2.3.1. The j-invariant. — To an unordered 4-tuple $\boldsymbol{z} \in U_4$ we associate the *j-invariant* $j_{\boldsymbol{z}}$ of $\boldsymbol{z}$, a point of $U_\infty \overset{\mathrm{def}}{=} \mathbb{P}^1_j \setminus \{\infty\}$. To simplify, normalize so $j = 0$ and 1 are the usual elliptic points corresponding to $j_{\boldsymbol{z}}$ having non-trivial (more than a Klein 4-group; §2.4 and [**Fri07**, Chap. 4, §4.2]) stabilizer in $\mathrm{PGL}_2(\mathbb{C})$.

Given $j' \in U_\infty \setminus \{0, 1\}$, there is an uncanonical one-one association: covers with j-invariant j' in the reduced Nielsen class $\Leftrightarrow$ elements of the reduced Nielsen classes (§2.4.2). So, reduced Nielsen classes produce $\{\mathcal{H}_k = \mathcal{H}(G_k(G), \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^\infty$: a projective sequence of inner reduced Hurwitz spaces.

The map $\mathcal{H}_{k+1} \to \mathcal{H}_k$ is a cover over every unobstructed component (§2.1) of $\mathcal{H}_k$. By cover we include that it is possibly ramified for k at points over $j = 0$ or 1. Each *nonempty* component of $\mathcal{H}_k$ is an upper half-plane quotient and U_∞ cover (ramified only over $j = 0$ and 1) [**BF02**, §2].

Since the components of $\{\mathcal{H}_k\}_{k=0}^\infty$ are curves, they have natural nonsingular projective closures $\{\bar{\mathcal{H}}_k\}_{k=0}^\infty$, with each $\bar{\mathcal{H}}_k$ extending to give a finite map to $\mathbb{P}^1_j$. As expected, we call the (geometric) points of $\bar{\mathcal{H}}_k \setminus \mathcal{H}_k$ the level k *cusps*.

To see why we use reduced spaces consider the following statement (encapsulating (6.11b)) where ∞-*ly many* means no two are reduced equivalent.

(2.2) For there to be ∞-ly many 4 branch point, reduced inequivalent $\mathbb{Q}$ regular realizations of $G_1(A_5)$, the H-M components of $\mathcal{H}(G_1(A_5), \mathbf{C}_{\pm 5^2})^{\mathrm{in,rd}}$ must have infinitely many $\mathbb{Q}$ points.

The (two) H-M components in question have genus 1. We ask if they have infinitely many $\mathbb{Q}$ points. Even one $\mathbb{Q}$ point $\boldsymbol{p}$ (not a cusp) on one of these components would give a geometric cover $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ over $\mathbb{Q}$ with group $G_1(A_k)$. Further, running over $\beta \in \mathrm{PGL}_2(\mathbb{Q})$ the covers $\beta \circ \varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ give ∞-ly many inner inequivalent covers with the same group also over $\mathbb{Q}$. These, however, are all reduced equivalent. It is more significant to consider the outcome of (2.2).

The following statement implies Conj. 1.2 (special case of [**BF02**, Thm. 6.1]; outline in [**Dèb06**, Thm. 2.6]).

Conjecture 2.2. — For large k, all components of $\bar{\mathcal{H}}_k$ have genus exceeding 1.

2.3.2. Definition fields. — All **MT** levels, with their moduli space structure, have minimal definition field the same common cyclotomic field (§1.1.3). If $\mathbf{C}$ is $\mathbb{Q}$-rational, then this definition field is $\mathbb{Q}$. Still, it is the absolutely irreducible components of levels that require attention. For example, if our base field is $\mathbb{Q}$, and some **MT** level has no $\mathbb{Q}$ components, then this (or any higher) level can have no $\mathbb{Q}$ points. This case of the weak Main Conjecture is then trivial (for $\mathbb{Q}$).

§6.2.4 reminds of methods to find **MT**s with component branches over $\mathbb{Q}$. They don't, however, apply when $r_{\mathbf{C}} = 4$. So, some component branch of a MT might have no number field definition: No matter what is K with $[K : \mathbb{Q}] < \infty$, there may be a value of k so the level k component has definition field outside K. Lem. 3.1 uses cusp branches to limit, though not yet eliminate, this possibility. Thus, our approach to the Main Conjecture aims at deciding it based only on the **MT** (cusp) geometry.

2.4. Nielsen classes, Hurwitz monodromy and computing genera.

— We can compute the genera of the components of $\bar{\mathcal{H}}_k$ using the Riemann-Hurwitz formula by answering the following questions.

(2.3a) What are the $\bar{\mathcal{H}}_k$ components.

(2.3b) What are the cusp widths (ramification orders over ∞) in each component.

(2.3c) What points ramify in each component over elliptic points ($j = 0$ or 1).

2.4.1. A Nielsen class dictionary. — Use notation of §2.1. *Reduced* Nielsen classes let us calculate components, cusp and elliptic ramification. We'll see how the Frattini property controls growth of cusp widths (ramification) with k.

Here are definitions of Nielsen classes, and their absolute (requires adding a transitive permutation representation $T : G \to S_n$) and inner quotients. In the absolute case we equivalence Nielsen class elements $\boldsymbol{g}$ and $h\boldsymbol{g}h^{-1}$ with h in the normalizer $N_{S_n}(G)$ of G in S_n.

$$\begin{aligned} \text{Nielsen classes: } & \operatorname{Ni}(G,\mathbf{C}) = \{\boldsymbol{g} \in \mathbf{C} \mid \langle \boldsymbol{g}\rangle = G; \Pi(\boldsymbol{g}) = 1\} \\ \text{Absolute classes: } & \operatorname{Ni}(G,\mathbf{C})/N_{S_n}(G,\mathbf{C}) \stackrel{\text{def}}{=} \operatorname{Ni}(G,\mathbf{C},T)^{\text{abs}};\ \text{and} \\ \text{Inner classes: } & \operatorname{Ni}(G,\mathbf{C})/G \stackrel{\text{def}}{=} \operatorname{Ni}(G,\mathbf{C})^{\text{in}}. \end{aligned}$$

Elements q_i, $i = 1, 2, 3$ (braids), generate the degree 4 *Hurwitz monodromy group* H_4. Each acts on any Nielsen classes by a twisting on its 4-tuples. Example:

$$q_2 : \boldsymbol{g} \mapsto (\boldsymbol{g})q_2 = (g_1, g_2g_3g_2^{-1}, g_2, g_4).$$

For $\beta \in \mathrm{PGL}_2(\mathbb{C})$, reduced equivalence of covers (as in §2.3) works as follows:

$$\varphi : X \to \mathbb{P}^1_z \iff \beta \circ \varphi : X \to \mathbb{P}^1_z.$$

This equivalence preserves the $j = j_{\boldsymbol{z}}$-invariant of the branch point set $\boldsymbol{z} = \boldsymbol{z}_\varphi$.

Reduced equivalence on Nielsen classes results from each set $\boldsymbol{z}_\varphi$ having some Klein 4-group subgroup of $\mathrm{PGL}_2(\mathbb{C})$ fixing it. This corresponds to modding out the Nielsen class by $\mathcal{Q}'' = \langle (q_1q_2q_3)^2, q_1q_3^{-1}\rangle \le H_4$ [**BF02**, Prop. 4.4].

So, the action of H_4 on reduced Nielsen classes factors through the *mapping class group*: $\bar{M}_4 \stackrel{\text{def}}{=} H_4/\mathcal{Q}'' \equiv \mathrm{PSL}_2(\mathbb{Z})$. [**BF02**, §2.7] has normalized this identification with $\mathrm{PSL}_2(\mathbb{Z})$ (see §2.4.2). It uses generators

$$\begin{aligned} &\langle \gamma_0, \gamma_1, \gamma_\infty\rangle, \gamma_0 = q_1q_2, \gamma_1 = \mathbf{sh} = q_1q_2q_3, \gamma_\infty = q_2, \\ &\text{satisfying the product-one relation: } \gamma_0\gamma_1\gamma_\infty = 1. \end{aligned} \tag{2.4}$$

2.4.2. Reduced Nielsen classes and cusps. — Regard the words $\gamma_0, \gamma_1, \gamma_\infty$ in the q_i s of (2.4) as in H_4. Usually the γ notation expresses them as acting in the quotient group $\bar{M}_4$, on reduced Nielsen classes.

Here is the notation for absolute (resp. inner) reduced representatives:

$$\mathrm{Ni}(G, \mathbf{C})/\langle N_{S_n}(G, \mathbf{C}), \mathcal{Q}''\rangle \overset{\mathrm{def}}{=} \ \mathrm{Ni}^{\mathrm{abs,rd}} \text{ and}$$
$$\mathrm{Ni}(G, \mathbf{C})/\langle G, \mathcal{Q}''\rangle \overset{\mathrm{def}}{=} \ \mathrm{Ni}^{\mathrm{in,rd}}.$$

The element **sh** acts like the *shift*. It sends a reduced rep. $\boldsymbol{g} = (g_1, \dots, g_4)$ to the reduced class of (g_2, g_3, g_4, g_1). On reduced Nielsen classes, **sh** has order 2 (not 4 as it does on Nielsen classes). Similarly, γ_0 has order 3 on reduced Nielsen classes (absolute or inner). Yes, these identify with the generating elements in $\mathrm{PSL}_2(\mathbb{Z})$ having orders 2 and 3 corresponding respectively to $j = 1$ and $j = 0$! The action of $\gamma_\infty = q_2$ then gives a combinatorial interpretation of cusps.

Definition 2.3. — The *cusp group* (a subgroup of H_4) is $\mathrm{Cu}_4 = \langle q_2, \mathcal{Q}''\rangle$.

Orbits of Cu_4 (resp. $\bar{M}_4$) on Nielsen classes correspond to cusps (resp. components) of the corresponding Hurwitz spaces [**BF02**, Prop. 2.3]. In computational notation, running over $\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in,rd}}$:

(2.5a) Cusps on $\bar{\mathcal{H}}_k \Leftrightarrow (\boldsymbol{g})\mathrm{Cu}_4$, a *cusp set* in the Nielsen classes.
(2.5b) Components on $\bar{\mathcal{H}}_k \Leftrightarrow (\boldsymbol{g})\bar{M}_4$, a braid orbit on Nielsen classes.

We often refer to $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ as a cusp, shortening reference to its cusp set.

2.4.3. Riemann-Hurwitz on components. — Now we interpret Riemann-Hurwitz: $(\gamma_0, \gamma_1, \gamma_\infty)$ act on a $\bar{M}_4$ orbit $\Leftrightarrow$ branch cycles for a component of $\bar{\mathcal{H}}(G, \mathbf{C})^{\mathrm{rd}} \to \mathbb{P}^1_j$.

(2.6a) Ramified points over 0 $\Leftrightarrow$ orbits of γ_0.
(2.6b) Ramified points over 1 $\Leftrightarrow$ orbits of γ_1.
(2.6c) The index contribution $\mathrm{ind}(\gamma_\infty)$ from a cusp with rep. $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{in,rd}}$ is $|(\boldsymbol{g})\mathrm{Cu}_4/\mathcal{Q}''| - 1$.

Reminder: The *index* of $g \in S_n$ with t orbits is $\mathrm{ind}(g) \overset{\mathrm{def}}{=} n - t$. App. B does one example computation of (2.6). [**BF02**, §2.8] computes modular curve genera from this viewpoint, while [**BF02**, §2.10] and [**BF02**, Cor. 8.3] show how the **sh**-incidence matrix works effectively to do much harder genus computations where the group is respectively A_5 and $G_1(A_5)$.

2.5. More on Schur multipliers and Frattini covers of a subgroup. — We list results on Schur multipliers and Frattini covers used, say, in examples like Ex. B.2 and Ex. B.3. One thing they say is that a $\mathbb{Z}/p$ quotient at the head of $M_k = \ker(G_{k+1} \to G_k)$ makes a special contribution to the $\mathbb{Z}/p$ quotients at the head of all M_t s, $t \geq k$. So, the appearance of a Schur multiplier of a simple group at level 0 affects all levels of a **MT**.

2.5.1. Two Schur multiplier topics. — Use notation of §2.1. A $\mathbb{Z}/p$ quotient of SM_G has *height* the largest u with $\mathrm{SM}_{G,p} \to \mathbb{Z}/p$ factoring through $\mathbb{Z}/p^u$.

(2.7a) Given a $\mathbb{Z}/p$ quotient of SM_G, what is its height?

(2.7b) When do $\mathbb{Z}/p$ quotients of $\mathrm{SM}_{G,p}$ arise from pullback of Schur multipliers of classical groups?

[**Fri02**] and [**FS06**] have a general classification of Schur multipliers by how they append to $M_k = \ker(G_{p,k+1} \to G_{p,k})$. Also, a Schur multiplier appearing at level k *replicates* to higher levels in a form called *antecedent* (§4.2.2).

The archetype is the sequence of groups $\{G_{2,k}(A_n)\}_{k=0}^{\infty}$, $n \geq 4$. For each k, there is a $\mathbb{Z}/2$ quotient of the Schur multiplier of $G_{2,k}(A_n)$ that is the antecedent of the 2-Frattini central Spin cover $\mathrm{Spin}_n \to A_n$. Often antecedents inherit properties from the original Schur multiplier. Here are two examples.

(2.8a) If u is the height of a $\mathbb{Z}/p$ quotient of SM_G, then it is also the height of its antecedent in $\mathrm{SM}_{G_{p,k}}$ [**FS06**, §4.4].

(2.8b) For $p = 2$, if a $\mathbb{Z}/2$ quotient of SM_G is the pullback to Spin_N of an embedding $G \leq A_N$, some N, then an effective test decides if the antecedent of $\mathrm{SM}_{G_{p,k}}$ is from an embedding $G_{p,k} \leq A_{N'}$, some N'.

[**BF02**, §9.4] shows by example how (2.8b) contributes. It separates the two braid orbits of $\mathrm{Ni}(G_1(A_5), \mathbf{C}_{3^4})$ (as at the top of §1) by the lifting invariant (§4.2) from the pullback of $G_1(A_5) \leq A_{N'}$ with various values of N' (40, 60 and 120). This isn't so effective as to decide in one fell swoop the story of braid orbits for $\{\mathrm{Ni}(G_k(A_5), \mathbf{C}_{3^4})\}_{k=0}^{\infty}$. Still, that is our heading.

Finally, Prop. 2.4 shows, even for $p = 2$, Schur multipliers relating to spin covers of groups don't exhaust all Schur multipliers that conceivably affect computations on **MT** levels. [**BF02**, §5.7] explains its dependence on [**GS78**]: That the condition that M_0 (and so M_k) being 1-dimensional is equivalent to G_0 being a slight generalization (supersolvable) of dihedral groups. As a special case, if M_0 is not 1-dimensional, then $\mathbf{1}_{G_k}$ (see §1.1.2) appears with an explicit positive density in M_k for k large. Though effective, for small k it is subtle to predict the appearance of $\mathbf{1}_{G_k}$ and, for all k, where in the Loewy display the $\mathbf{1}_{G_k}$ s appear.

Recall: Over an algebraically closed field the set of simple G_0 modules has the same cardinality as the set of p' conjugacy classes. Let S be any simple G_0 module. Let K be algebraically closed and retain the notation M_k after tensoring with K. We use $\langle S, M_k\rangle$, and related compatible notation, for the total multiplicity of S in all Loewy layers of the G_k module M_k. Let $O_{p'}(G)$ be the maximal normal p'subgroup of finite group G (it is the same for each G_k).

Proposition 2.4 **(**[**Sem2**, Thm. 4.1]**).** — *If* $\dim_K(M_0) \neq 1$*, then*

$$\lim_{n \mapsto \infty} \frac{\langle S, M_k\rangle}{\dim_K(M_k)} = \frac{\langle S, K[G/O_{p'}(G)]\rangle}{\dim_K(K[G/O_{p'}(G)])}.$$

2.5.2. Frattini covers of a subgroup of G. — I can't find the following useful lemma (applied in Rem. 2.6, Lem. 4.23 and Princ. 4.24) in my previous publications.

Lemma 2.5. — *Let $H \le G$. Then, for each k there is an embedding (not unique) $\beta_k : G_{p,k}(H) \to G_{p,k}(G)$ lying over the embedding of H in G.*

Proof. — The lemma follows from Schur-Zassenhaus if H is a p' group where we use $G_{p,k}(H)$ to be H itself. Now assume H is not p'. The pullback $\mathrm{inj}_k^{-1}(H)$ of H in $G_{p,k}(G)$ is an extension with p group kernel having exponent p^k. From the versal property of $G_{p,k}(H)$ that produces $\beta_k : G_{p,k}(H) \to \mathrm{inj}_k^{-1}(H) \le G_{p,k}(G)$ lying over the embedding of H in G.

Denote pullback of H in ${}_p\tilde{G}$ by $\mathrm{inj}^{-1}(H)$. Since ${}_p\tilde{H} \to H$ is the *minimal* cover of H with kernel pro-free p-Sylow [**FJ86**, Prop. 20.33], there is a homomorphism $\mathrm{inj}^{-1}(H) \to {}_p\tilde{H}$. This induces $\psi_k : \mathrm{inj}_k^{-1}(H) \to G_{p,k}(H)$ in the other direction. The compositions $\psi_k \circ \beta_k : G_{p,k}(H) \to G_{p,k}(H)$ are onto: They lie over the identity on H and $G_{p,k}(H) \to H$ is a Frattini cover. So, acting on a finite group, they must be one-one. In particular, β_k is one-one. □

Remark 2.6. — The proof that gives β_k in Lem. 2.5 extends it inductively to some β_{k+1}. So, we may choose $\{\beta_k\}_{k=0}^{\infty}$ compatibly, coming from an injection $\beta : {}_p\tilde{H} \to {}_p\tilde{G}$. Also, if $G_k \to G$ factors through any $\mu : G' \to G$, then we may compose β_k with μ. When notation allows, continue to denote the resulting map $G_{p,k}(H) \to G'$ by β_k.

3. Projective systems of braid orbits

We consider two natural trees attached to the levels of a **MT**.

3.1. Projective systems of components. — Restrict the maps $\bar{\mathcal{H}}_{k+1} \to \bar{\mathcal{H}}_k$ to cusps and components to respectively define a *cusp–tree* $\mathcal{C}_{G,\mathbf{C},p}$ and a *component-tree* $\mathcal{T}_{G,\mathbf{C},p}$ directed by increasing levels. A *branch* on one of these trees is a maximal (directed upward) path; so it starts at level 0. Containment of cusps in their components induces a map from $\mathcal{C}_{G,\mathbf{C},p}$ to $\mathcal{T}_{G,\mathbf{C},p}$.

3.1.1. Cusp branches. — The Nielsen class view of this regards the vertices of $\mathcal{C}_{G,\mathbf{C},p}$ (resp. $\mathcal{T}_{G,\mathbf{C},p}$) as Cu_4 (resp. $\bar{M}_4$) orbits on the collections $\{\mathrm{Ni}(G_k,\mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty}$. Yet, we need the spaces to consider absolute Galois groups acting on these trees.

Let $F_{\mathbf{C}}$ be the subfield in the cyclotomic numbers fixed by $\{n \in \tilde{\mathbb{Z}}^* \mid \mathbf{C}^n = \mathbf{C}\}$, where equality is of sets with multiplicities. [**FV91**, Prop. 1] says (in general) the spaces $\mathcal{H}(G,\mathbf{C})^{\mathrm{in}}$ (with their maps to U_r interpreted as moduli spaces) have minimal definition field $F_{\mathbf{C}}$. This implies $F_{\mathbf{C}}$ is a definition field for $\mathcal{H}(G,\mathbf{C})^{\mathrm{in,rd}}$ (with its similar moduli properties), and so for the system of spaces $\{\bar{\mathcal{H}}(G_k,\mathbf{C})\}^{\mathrm{in,rd}}\}_{k=0}^{\infty}$ (with their compatible maps to $\mathbb{P}^1_j$).

Lemma 3.1. — *The absolute Galois group $G_{F_{\mathbf{C}}}$ acts compatibly on the vertices of $\mathcal{C}_{G,\mathbf{C},p}$ and $\mathcal{T}_{G,\mathbf{C},p}$. So, $G_{F_{\mathbf{C}}}$ acts compatibly as permutations on (finite or infinite) branches of $\mathcal{C}_{G,\mathbf{C},p}$ and $\mathcal{T}_{G,\mathbf{C},p}$.*

Assume a cusp branch B defines component branch B'. If, modulo braiding, $G_{F_{\mathbf{C}}}$ has a finite orbit on (resp. fixes) B, then it has a finite orbit on (resp. fixes) B'.

§6.2.4 notes we know many places where the "finite orbit on B" hypothesis of Lem. 3.1 holds, with B an H-M cusp branch (Ex. 3.7). The modular curve tower $\{X_1(p^{k+1})\}_{k=0}^{\infty}$ has just one component-branch. We understand its cusp-branches well. Manin-Demjanenko ([**Ser97b**, Chap. 5] or [**Fri02**, §5.3]) gave this case of Conj. 1.2 long before Faltings' Theorem. (We apply Faltings to treat general **MT**s.) It is typical to define a branch of $\mathcal{T}_{G,\mathbf{C},p}$ by labeling it from the image of a branch of $\mathcal{C}_{G,\mathbf{C},p}$. See Princ. 3.6 and Ex. 3.7.

There is nothing to prove in Conj. 1.2 if the $\mathcal{H}_k$ (or $\mathrm{Ni}(G_k,\mathbf{C})$) are empty for large k. This happens in one of the two components of the **MT** for $(A_n,\mathbf{C}_{3^r},p=2)$ with $r\ge n\ge 4$ (or if $r=n-1$ and n is even) [**Fri06a**, Main Result]. For $n=4=r$ see App. B.1. This gives a necessary situation for a number field K for considering Conj. 1.2: There is an infinite component branch

$$B' \overset{\mathrm{def}}{=} \{\bar{\mathcal{H}}'_k \Leftrightarrow \bar{M}_4 \text{ orbit } \mathrm{Ni}'_k\}_{k=0}^{\infty} \text{ fixed by } G_K \text{ (as in Lem. 3.1).} \tag{3.1}$$

§3.2.1 divides cusps into three types. It is easier to describe the cusps than to place them in components. §5.1 describes how projective systems of p cusps contribute to indices in the Riemann-Hurwitz formula.

3.1.2. Sequences of component genera. — Restrict the γs of (2.4) to Ni'_k in (3.1). This gives $(\gamma'_{0,k},\gamma'_{1,k},\gamma'_{\infty,k})$ defining the genus $g_{\bar{\mathcal{H}}'_k}$ of $\bar{\mathcal{H}}'_k$:

$$2(\deg(\bar{\mathcal{H}}'_k/\mathbb{P}^1_j)+g_{\bar{\mathcal{H}}'_k}-1)=\mathrm{ind}(\gamma'_{0,k})+\mathrm{ind}(\gamma'_{1,k})+\mathrm{ind}(\gamma'_{\infty,k}). \tag{3.2}$$

Below we denote the genera sequence for the branch B' by $\mathrm{Ge}_{B'}\overset{\mathrm{def}}{=}\{g_{\bar{\mathcal{H}}'_k}\}_{k=0}^{\infty}$. The strongest results toward the Main Conjectures require two contributions:

(3.3a) Deciphering the infinite branches from the finite branches.

(3.3b) Separating cusp branches into types that indicate their contributions to Riemann-Hurwitz.

[**Fri05a**, Lect. 1] starts by computing modular curve genera from a **MT** viewpoint. §3.2.1 describes those cusp types, including the significant special cusps called g-p', and the corresponding g-p' cusp branches. The following is a prototype modular curve property, and [**FS06**] uses it as an explicit target.

Question 3.2. — Suppose K is a number field and B' is an (infinite) K component branch with B' the image of a g-p' cusp branch $B\in\mathcal{C}_{G,\mathbf{C},p}$. Is it possible to give a closed expression for the elements of $\mathrm{Ge}_{B'}$?

3.1.3. Reduction to $G_0 = G$ has no p-part to its center. — One part of Princ. 3.5 says that p cusps contribute highly to cusp ramification. That result is a subtle use of Prop. 3.3. This reduces considering **MT**s (or at least the Main Conjecture) to the case where for all k, the p-part of the center is trivial. Denote the center of a group G by $Z(G)$, and the p-part of the center by $Z_p(G)$.

Proposition 3.3 **(p-Center Reduction)**. — *Suppose $G = G_0$ is a p-perfect group with $Z_p(G) \neq \{1\}$. Then, there is a p-Frattini cover $\psi^c : G \to G^c$ with $Z_p(G^c)$ trivial (and G^c is p-perfect). Any p' conjugacy class* C *of G has a unique image class in G^c which we also donate by* C *(§1.1.2). In particular, Main Conj. 1.2 holds for $(G^c, \mathbf{C}^c, p)$, if and only if it holds for $(G, \mathbf{C}, p)$.*

Proof. — Let U_p be the maximal normal p-Sylow of G, and let $\Phi(U_p)$ be the Frattini subgroup of U_p. Then, $G \to G/\Phi(U_p)$ is a p-Frattini cover.

First consider the case G is p-split: $G = U_p \times^s G/U_p$. From G being p-perfect, G/U_p has no fixed points on $U_p/\Phi(U_p)$. So $Z_p(G/\Phi(U_p)) = \{1\}$. General case: Form $G/\Phi(U_p)$. We're done if $Z_p(G/\Phi(U_p))$ is trivial. Otherwise iterate this to achieve G^c.

Now consider the last sentence of the proposition. Since $G \to G^c$ is a p-Frattini cover, the universal p-Frattini cover of G^c is the same as that of G. Denote the kth characteristic Frattini extension of G^c by G^c_k. From the construction, there is a k_0 so that $G^c_{k_0} \to G^c$ factors through $G \to G^c$. Conclude easily for each k there is a corresponding k' so that $G^c_{k'} \to G^c$ factors through $G_k \to G$. Also, the map $\psi_k : G_k \to G$ composed with ψ^c factors through $G^c_k \to G$.

In particular, this means for $k >> 0$ there is a k' so that $\mathcal{H}(G^c_{k'}, \mathbf{C})^{\mathrm{in,rd}}$ naturally maps (surjectively, over any field containing their simultaneous definition fields) to $\mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}}$. So: if $\mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}}(K) = \varnothing$, then $\mathcal{H}(G^c_{k'}, \mathbf{C})^{\mathrm{in,rd}}(K) = \varnothing$; if $\mathcal{H}(G^c_{k'}, \mathbf{C})^{\mathrm{in,rd}}(K) \neq \varnothing$, then $\mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}}(K) \neq \varnothing$; etc. Conclude $(G^c, \mathbf{C}, p)$ and $(G, \mathbf{C}, p)$ simultaneously pass or fail the conclusion of the Main Conjecture. □

Remark 3.4 **(Center considerations)**. — Do not conclude from Prop. 3.3 that **MT**s can't handle groups with centers. All our sections and also of [**BF02**] must consider that ${}_p\tilde{G}$ is full of subquotient sequences of the form $\psi' : R' \to G'$, a central extension of G', with $\ker(\psi')$ a quotient of G's Schur multiplier. As in §4.4, it is the maximal elementary p-quotient of G_k 's Schur multiplier that controls major properties of **MT** levels.

Use the notation of Prop. 3.3. Denote the p' part of $Z(G)$ by $Z_{p'}(G)$. Then, for all k, $Z_{p'}(G_k) = Z_{p'}(G^c_k) = Z_{p'}(G)$. (See this by identifying ${}_p\tilde{G}$ with the universal p-Frattini of $G/Z_{p'}$ fiber product with G over $G/Z_{p'}$.) We could have continued the map $\psi^c : G \to G^c$ through $G_c \to G_c/Z_{p'}(G)$. That would, however, complicate the final conclusion of Prop. 3.3. No longer could we canonically identify the image conjugacy classes with $\mathbf{C}$. So, while **MT**s already deals seriously with the p-part of centers, K. Kimura's master's thesis [**Kim05**] has a point in considering phenomena that arise from the p'-part.

3.2. g-p' and o-p' cusps, and Frattini Principles 1 and 2.

— §3.2.1 defines the three cusp types using a representative $\boldsymbol{g} = (g_1, \dots, g_4)$ of the cusp orbit. We expect g-p' cusp branches to give outcomes like that of Quest. 3.2. Modulo Conj. 1.6, we expect some g-p' cusp branch defines any component branch *with all levels having a fixed number field as definition field*. §3.2.3 considers cases when we can use g-p' cusps to get a handle on o-p' cusps.

3.2.1. The cusp types. — Use these notations:

$$H_{2,3}(\boldsymbol{g}) \overset{\text{def}}{=} \langle g_2, g_3\rangle \text{ and } H_{1,4}(\boldsymbol{g}) = \langle g_1, g_4\rangle;$$

and $(\boldsymbol{g})\mathbf{mp} \overset{\text{def}}{=} \text{ord}(g_2g_3)$, the order of the *middle product*. Primary contributions after level 0 to (3.2) come from *p cusps*: $p|(\boldsymbol{g})\mathbf{mp}$. Here are the other types.

(3.4a) *g(roup)-p'*: $H_{2,3}(\boldsymbol{g})$ and $H_{1,4}(\boldsymbol{g})$ are p' groups.

(3.4b) *o(nly)-p'*: $p \not| (\boldsymbol{g})\mathbf{mp}$, but the cusp is not g-$p'$.

Let $\{{}_k\boldsymbol{g} = ({}_kg_1, {}_kg_2, {}_kg_3, {}_kg_4) \in \text{Ni}'_k\}_{k=0}^{\infty}$ be a projective system of cusp representatives. Then ${}_k\boldsymbol{g}$ corresponds to a braid orbit $\text{Ni}'_k \subset \text{Ni}(G_k, \mathbf{C})$, and therefore to a component $\mathcal{H}'_k \subset \mathcal{H}(G_k, \mathbf{C})^{\text{in,rd}}$. Denote the corresponding projective system of cusps by $\{\boldsymbol{p}_k \in \mathcal{H}'_k\}_{k=0}^{\infty}$. When a point $\boldsymbol{p}'$ on some space lies over another point $\boldsymbol{p}$, denote the ramification order (*index, or width*) of $\boldsymbol{p}'/\boldsymbol{p}$ by $e(\boldsymbol{p}'/\boldsymbol{p})$. Crucial to our Main Conjecture is the phenomenon that p cusp widths grow automatically as we go up **MT** levels. The formal statement, coming mostly from [**BF02**, §8.1], is our first use of the Frattini property. Recall: $Z_p(G)$ is the p-part of the center of G (§3.1.3).

***Principle 3.5* (Frattini Princ. 1).** — *If $p^u|({}_k\boldsymbol{g})\mathbf{mp}$, $u \geq 1$, then $p^{u+1}|({}_{k+1}\boldsymbol{g})\mathbf{mp}$.*

Assume $Z_p(G)$ is trivial. Then, for p odd (resp. $p = 2$) and $k \geq 0$ (resp. $k >> 0$) $e(\boldsymbol{p}_{k+1}/\boldsymbol{p}_k)$ is p.

Comments on explicitness. — The first part is a consequence of [**FK97**, Lift Lem. 4.1] (for example). It comes from this simple statement: All lifts to G_{k+1} of an element of order p in G_k have order p^2. That concludes the first part.

Denote the operator that takes any $(a, b) \in G^2$ to (aba^{-1}, a) by γ. Then, [**BF02**, Prop. 2.17] —§C.2 has a typo free statement with (g_1, g_2) replacing (a, b) —tells how to compute the length of the orbit (using no equivalence between pairs) of γ generated by (a, b). The length of the γ^2 orbit is

$$o(a, b) \overset{\text{def}}{=} o = \text{ord}(a \cdot b)/|\langle a \cdot b\rangle \cap Z(a, b)|.$$

Then, one of the following holds for the length $o'(a, b) = o'$ of the γ orbit on (a, b). Either: $a = b$ and $o' = 1$, or;

(3.5) if o is odd and $b(a \cdot b)^{\frac{o-1}{2}}$ has order 2, then $o' = o$; or else $o' = 2 \cdot o$.

[**BF02**, Lem. 8.2] of necessity was intricate, for it's goal was to nail $e(\boldsymbol{p}_{k+1}/\boldsymbol{p}_k)$ from data on the group theoretic cusp from ${}_k\boldsymbol{g}$ and ${}_{k+1}\boldsymbol{g}$. This was to precisely list genera of examples. We now say this result in a more relaxed way.

Assume $Z_p(G_0)$ is trivial. From [**BF02**, Prop. 3.21] the same therefore holds for $Z_p(G_k)$ for all $k \geq 0$. All we care about in our conclusion is the p part of $e(\boldsymbol{p}_{k+1}/\boldsymbol{p}_k)$. We divide the contribution to the p part $e_p(\boldsymbol{p}_k/\infty)$ into two cases: p odd, and $p = 2$.

When p is odd, [**BF02**, Lem. 8.2] gives $e_p(\boldsymbol{p}_k/\infty)$ as the p-part of $o({}_kg_2, {}_kg_3)$. If the p-part of $|\langle {}_kg_2 \cdot {}_kg_3\rangle \cap Z({}_kg_2, {}_kg_3)| \overset{\text{def}}{=} k_p(2,3)$ is trivial, then the result is the p-part of $\text{ord}({}_kg_2 \cdot {}_kg_3)$. Since the p'-part of $\text{ord}({}_kg_2 \cdot {}_kg_3)$ is unchanging with k, the first statement in the proposition gives $e(\boldsymbol{p}_{k+1}/\boldsymbol{p}_k) = p$.

To see why $k_p(2,3) = 1$, use that the action of $\mathcal{Q}''$ expresses the cusp width also from $({}_kg_4, {}_kg_1)$ (§2.4.1). The result must be the same, using an analogous expression $k_p(1,4)$. Since $({}_kg_4 \cdot {}_kg_1)^{-1} = {}_kg_2{}_kg_3$, then $k_p(2,3) = k_p(1,4)$. Now if both are nontrivial, it means

$$Z_p({}_kg_2, {}_kg_3) \geq ({}_kg_2 \cdot {}_kg_3)^{\text{ord}({}_kg_2 \cdot {}_kg_3)/p} \leq Z_p({}_kg_4, {}_kg_1).$$

Since ${}_kg_1, {}_kg_2, {}_kg_3, {}_kg_4$ generate G_k, this implies $Z_p(G_k)$ is nontrivial.

For $p = 2$, the computation works similarly, except for factors of 2-power order (bounded by 4) in $e_2(\boldsymbol{p}_k/\infty)$ from the action of $\mathcal{Q}''$ and the distinction between $o = o'$ and $o = 2 \cdot o'$ given in (3.5). These are, however, regular behaviors. Observations like those about $\mathcal{Q}''$ in §3.2.2, allow replacing $k >> 0$ by a more precise statement. □

***Principle 3.6* (Frattini Princ. 2)**. — *The definition of p' and g-p' cusp doesn't depend on its rep. in* $(\boldsymbol{g})\text{Cu}_4$ [**FS06**, Prop. 5.1]. *If* ${}_0\boldsymbol{g} \in \text{Ni}(G_0, \mathbf{C})$ *represents a g-p' cusp, then above it there is a g-p' cusp branch* $\{{}_k\boldsymbol{g} \in \text{Ni}(G_k, \mathbf{C})\}$.

Proof. — Use (g_1, g_2, g_3, g_4) for ${}_0\boldsymbol{g}$. Let $H \leq {}_0G$ be a p' group. Then, consider the pullback $\psi^{-1}(H)$ in ${}_p\tilde{G}$. The profinite version of Schur-Zassenhaus says the extension $\psi^{-1}(H) \to H$ splits [**FJ86**, 20.45]. Apply this to each p' group $H_{1,4}({}_0\boldsymbol{g})$ and $H_{2,3}({}_0\boldsymbol{g})$. This gives $H'_{1,4}, H'_{2,3} \leq {}_p\tilde{G}$, defined up to conjugation by $\tilde{P}_p$, mapping one-one to their counterparts modulo reduction by ${}_p\tilde{P}$.

Let $g'_1, g'_4 \in H'_{1,4}$ (resp. $g'_2, g'_3 \in H'_{2,3}$) be the elements over $g_1, g_4 \in H_{1,4}$ (resp. $g_2, g_3 \in H_{2,3}$). Then, $g'_2g'_3$ is conjugate to $(g'_1g'_4)^{-1}$ by some $h \in {}_p\tilde{P}$. Replace $H'_{1,4}$ by its conjugate by h to find $\boldsymbol{g}' = (g'_1, \dots, g'_4) \in \text{Ni}({}_p\tilde{G}, \mathbf{C})$ lying over ${}_0\boldsymbol{g}$. The images of $\boldsymbol{g}'$ in each $\text{Ni}(G_k, \mathbf{C})$ give the desired g-p' cusp branch. □

***Example 3.7* (sh of an H-M rep)**. — §2.4.1 has the definition of the shift **sh**. A H(arbater)-M(umford) rep. in the reduced Nielsen class $\text{Ni}(G, \mathbf{C})^{\text{rd}}$ (applies to inner or absolute equivalence) has the shape $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$. Then, $(\boldsymbol{g})\mathbf{sh}$ is clearly a g-p' cusp. It has width 1 or 2. A formula distinguishes between the cases (proof of Prop. 3.5). Typically our examples have $H_{2,3}(\boldsymbol{g}) \cap H_{1,4}(\boldsymbol{g}) = \langle 1\rangle$, or else $G = \langle g_1, g_2\rangle$ has a nontrivial cyclic p' kernel dividing the orders of $\langle g_i\rangle$, $i = 1, 2$.

3.2.2. Consequences of fine reduced moduli. — The reduced spaces of the levels of a component branch are moduli spaces. Using them as moduli spaces behooves us to know when they have (reduced) fine moduli: objects that represent points do so in a unique way. There isn't a prayer they have fine (reduced) moduli unless the corresponding unreduced spaces $\mathcal{H}(G_k, \mathbf{C})^{\mathrm{in}}$ have fine moduli. For that the if and only if criterion, given that G_0 is p-perfect, is that G_0 has no center [**BF02**, Prop. 3.21].

Given this, [**BF02**, Prop. 4.7] gives the if and only if criterion for level k_0 of a branch to have fine moduli. This says: Two computational conclusions hold from the action of H_4 and $\bar{M}_4$ on the corresponding level k_0 braid orbit Ni'_{k_0}:

(3.6a) $\mathcal{Q}''$ has all its orbits on Ni'_{k_0} of length 4; and

(3.6b) both γ'_{0,k_0} and γ'_{1,k_0} act without fixed point.

Both Thm. 5.1 and §6.2.3, on the *Branch Frattini Propery*, use Lem. 3.8.

Lemma 3.8. — *For any k, $\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k$ ramifies only over cusps (points over $j = \infty$) if and only if* (3.6b) *holds. If* (3.6b) *holds for $k = k_0$, then it holds also for $k \geq k_0$, and for each such k, p is the ramification index for each prime ramified in the cover $\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k$. So, this holds if the component branch B' has fine moduli (for $k = k_0$).*

Proof. — The cover $\bar{\mathcal{H}}'_k \to \mathbb{P}^1_j$ ramifies only over $j = 0, 1, \infty$. The lengths of the disjoint cycles for $\gamma'_{0,k}$ (resp. $\gamma'_{1,k}$) on Ni'_k correspond to the orders of ramification of the points of $\bar{\mathcal{H}}'_k$ lying over 0 (resp. 1).

Apply multiplicativeness of ramification to $\bar{\mathcal{H}}'_{k+1} \xrightarrow{\psi_{k+1,k}} \bar{\mathcal{H}}'_k \xrightarrow{\psi_k} \mathbb{P}^1_j$. If $\boldsymbol{p}_{k+1} \in \bar{\mathcal{H}}'_{k+1}$, denote $\psi_{k+1,k}(\boldsymbol{p}_{k+1})$ by $\boldsymbol{p}_k$. Then, $\boldsymbol{p}_{k+1}/\psi_k \circ \psi_{k+1,k}(\boldsymbol{p}_{k+1})$ has ramification index

$$e(\boldsymbol{p}_{k+1}/\psi_k \circ \psi_{k+1,k}(\boldsymbol{p}_{k+1})) = e(\boldsymbol{p}_{k+1}/\boldsymbol{p}_k)e(\boldsymbol{p}_k/\psi_k \circ \psi_{k+1,k}(\boldsymbol{p}_{k+1})). \tag{3.7}$$

If $\psi_k \circ \psi_{k+1,k}(\boldsymbol{p}_{k+1}) = 0$, then $e(\boldsymbol{p}_{k+1}/0) = 1$ and $e(\boldsymbol{p}_k/0) = 1$ are each either 1 or 3 (§2.4.2). Conclude from (3.7):

$$e(\boldsymbol{p}_{k+1}/0) = 3 \text{ and } e(\boldsymbol{p}_{k+1}/\psi_k \circ \psi_{k+1,k}(\boldsymbol{p}_{k+1})) = 1$$

both hold if and only if $e(\boldsymbol{p}_k/0) = 3$.

Statement (3.6b) for γ'_{0,k_0} says $e(\psi_{k_0}/0)$ is 3 for each ψ_{k_0} lying over 0. This inductively implies no point of $\bar{\mathcal{H}}_{k+1}$ lying over $0 \in \mathbb{P}^1_j$ ramifies over $\bar{\mathcal{H}}_k$ if $k \geq k_0$. The same argument works for γ_{1,k_0} and concludes the lemma. □

***Example 3.9* (When reduced fine moduli holds)**. — For all the examples of [**BF02**, Chap. 9], reduced fine moduli holds with $k_0 = 1$ in Lem. 3.8. [**Fri06b**] shows for $p = 2$ any H-M component branch has fine moduli. We hope to expand that considerably before publishing a final version. If Conj. 1.5 is true, then that implies any (infinite) component branch of any of the many A_4 and A_5 ($p = 2$ and any type of $2'$ conjugacy classes) **MT**s have reduced fine moduli.

***Example 3.10* ((3.6b) can hold without fine moduli)**. — Here again, we have a modular curve comparison with a highlight from [**BF02**, §4.3.2]. While there is a one-one map (onto) map $\mathcal{H}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\mathrm{in}} \to Y_1(p^{k+1})$ (§4.1.4), the spaces, as moduli spaces, are not exactly the same. The latter has fine moduli, but the former does not. The distinction is that the moduli problem for $\mathcal{H}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\mathrm{in}}$ is *finer* than that for $Y_1(p^{k+1})$: There are "more" genus 1 Galois covers of $\mathbb{P}^1_z$ with $D_{p^{k+1}}$ monodromy than there are corresponding elliptic curve isogenies. Still, (3.6b) holds.

3.2.3. Relations between g-p' and o-p' cusps. — For our arithmetic conjectures we only care about infinite K component branches (§1.2.1) where K is some number field. For this discussion we accept Conj. 1.5. That means in dealing with the possibility of o-p' cusp branches, we only need to consider those that appear on a g-p' component branch. Since o-p' cusp branches are so important, we hope thereby to be as explicit with them as with g-p' cusps.

This occurs, for example, if an o-p' cusp is over a g-p' cusp. To simplify, start with a g-p' cusp ${}_0\boldsymbol{g}$ at level 0 with $({}_0\boldsymbol{g})\mathbf{mp} \overset{\mathrm{def}}{=} v$ of order c. Prop. 3.12 shows the conditions of (3.8) sometimes hold (though not for shifts of H-M reps., Ex. 3.7).

Expressions in (3.8) are in additive notation in $M_0 = \ker(G_1 \to G_0)$; the group ring $\mathbb{Z}/p[G_0]$ acts on the right. For $g \in G_0$ and $m \in M_0$, denote the subspace of M_0 that commutes with g (on which g acts trivially) by Cen_g, and its translate by m by $\mathrm{Cen}_g - m$. Denote $1 + v + \cdots + v^{c-1} : M_0 \to M_0$ by $L(v)$.

Proposition 3.11. — *Suppose $\boldsymbol{g}' \in \mathrm{Ni}(G_1, \mathbf{C})$ lying over ${}_0\boldsymbol{g}$ is neither a g-p', nor a p (so is an o-p'), cusp. Let $\boldsymbol{g} \in \mathrm{Ni}(G_1, \mathbf{C})$ be a g-p' cusp over ${}_0\boldsymbol{g}$ as in the conclusion of Princ. 3.6. Then, with no loss we may assume*

$$\boldsymbol{g}' = ((m^*)^{-1} g_1 m^*, g_2, m_3 g_3 m_3^{-1}, (m_4 m^*)^{-1} g_4 (m_4 m^*))$$

with $m^*, m_3, m_4 \in M_k$ and (g_2, g_3) is not conjugate to (g_2, g_3').

Then, the order of $(\boldsymbol{g}')\mathbf{mp}$ is c and the following are equivalent to $\boldsymbol{g}'$ being o-p'.

(3.8a) *Product-one: $m_3({}_0g_3 - 1) + m_4({}_0g_4 - 1) + m^*(v - 1) = 0$.*

(3.8b) *p' middle-product: $m_3({}_0g_3 - 1)$ is an element of $M_0(v - 1)$.*

(3.8c) *Not g-p': It does not hold that $m_3({}_0g_2 - 1) \in \mathrm{Cen}_{g_3}({}_0g_2 - 1)$.*

Proof. — Since $\boldsymbol{g}'$ is an o-p' cusp, we may assume $H_{2,3}(\boldsymbol{g}')$ is not a p' group. Characterize this by saying $H_{2,3}(\boldsymbol{g})$ is not conjugate to $H_{2,3}(\boldsymbol{g}')$. By conjugating, we may assume $g_2 = g_2'$ and $g_3' = m_3 g_3 m_3^{-1}$ for some $m_3 \in M_k \setminus \{0\}$. For (g_2', g_3') to be conjugate to (g_2, g_3) is equivalent to some $m \in M_0 \setminus \{0\}$ commutes with g_2 while $m - m_3$ commutes with g_3. The other normalization conditions are similar. Then, (3.8a) is $\Pi(\boldsymbol{g}') = 1$ in additive notation.

Compute $(g_2 m_3 g_3 m_3^{-1})^c = (g_2 g_3 m_3^{g_2} m_3^{-1})^c$ to get $(g_2 g_3)^c = 1$ times an element $u \in M_0$. That u, in additive notation, is just

$$(m_3)({}_0g_3 - 1)(1 + v + v^2 + \cdots v^{c-1}) = (m_3)({}_0g_3 - 1)L(v).$$

Since $g_2 g_3$ is assumed p', that gives $u = 0$, or $(m_3)({}_0g_3 - 1)$ is in the kernel of $L(v)$. As, however, v has p' order, the characteristic polynomial $x^c - 1$ of v has no repeated roots. So, M_0 decomposes as a direct sum $\mathbb{Z}/p[x]/(x-1) \oplus \mathbb{Z}/p[x]/L(x)$ with v acting in each factor as multiplication by x. Thus, the kernel of $L(v) : M_0 \to M_0$ is exactly the image of $(1-v)$. That is, v having p' order is equivalent to $m_3({}_0g_3 - 1)$ is an element of $M_0(v-1)$. That completes showing (3.8b).

Finally, suppose there is $m \in M_0$ that conjugates (g_2', g_3') to (g_2, g_3). Compute to see this is equivalent to $\mathrm{Cen}_{g_2} - m_3 \cap \mathrm{Cen}_{g_3} = \varnothing$. An m in this overlap would satisfy $m_3({}_0g_2 - 1) = m({}_0g_2 - 1)$. Statement (3.8c) says there is no such m. □

Apply (3.8a) to the shift of an H-M cusp. Then, $c = 1$ and m_3 commutes with g_3, contrary to assumption. So the shift of an H-M rep. cannot have an o-p' cusp over it. Still, Prop. 3.12 shows some g-p' cusp branches produce a profusion of o-p' cusps over g-p' cusps.

Proposition 3.12. — *Let $\{{}_k\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})\}_{k=0}^{\infty}$ represent a g-p' cusp branch from Princ. 3.6. Let c_i be the order of ${}_0g_i$, $i = 1,2,3,4$. Assume*

(3.9) *$O_p'(G_0)$ is trivial and $1/c_2 + 1/c_3 + 1/c_4 + 1/c < 1$.*

Then, for k large, an o-p' cusp ${}_{k+1}\boldsymbol{g}' \in \mathrm{Ni}(G_{k+1}, \mathbf{C})$ lies over $\boldsymbol{g}_k$.

Proof. — Use notation of Prop. 3.11, starting with a g-p' cusp ${}_0\boldsymbol{g}$ at level 0. Take $m^* = 1$. Consider what (3.8) forces on $\boldsymbol{g}' = (g_1, g_2, m_3g_3m_3^{-1}, m_4^{-1}g_4m_4)$ to force it to be an o-p' cusp in $\mathrm{Ni}(G_1, \mathbf{C})$. Condition (3.8b) says:

$$(\boldsymbol{g}')\mathbf{mp} \text{ is } p' \Leftrightarrow m_3({}_0g_3 - 1) \in M_0(v-1).$$

Also we must assure $m_3({}_0g_2 - 1)$ is not in $\mathrm{Cen}_{g_3}({}_0g_2 - 1)$.

Combine all conditions of (3.8). Then, there is an o-p' cusp if and only if

$$M_0({}_0g_3 - 1) \cap M_0({}_0g_4 - 1) \cap M_0(v-1) \setminus \mathrm{Cen}_{g_3}({}_0g_2 - 1) \neq \varnothing. \tag{3.10}$$

By the *relative* codimension or dimension of a subspace of M_k, we mean the codimension or dimension divided by the dimension of M_k. While we can't expect (3.10) to hold at level 0, we show it holds with conditions (3.9) if we substitute ${}_k\boldsymbol{g}$ for ${}_0\boldsymbol{g}$ (and M_k for M_0) for k large.

If the relative codimension of

$$M_0({}_0g_3 - 1) \cap M_0({}_0g_4 - 1) \cap M_0(v-1)$$

plus the relative dimension of Cen_{g_3} is asymptotically less than 1, then (3.10) holds for $k >> 0$. Prop. 2.4 (using $O_{p'}(G) = \{1\}$) gives this for $k >> 0$ if (3.9) holds. So, these conditions imply an o-p' cusp over ${}_k\boldsymbol{g}$ for k large. □

***Example 3.13* (Case satisfying (3.9)).** — Let G_0 be the alternating group A_7 and let $p = 7$. Define the Nielsen class selecting ${}_0\boldsymbol{g}$ with $g_2, g_3 \in A_5$ both 5-cycles generating A_5 and having $v = g_2g_3$ a 3-cycle. From [**BF02**, Princ. 5.13] there is just one choice (up to conjugation) if g_1 and g_2 are in the two different conjugacy classes of order 5: $g_2 = (5\,4\,3\,2\,1)$ and $g_3 = (2\,4\,3\,5\,1)$, and $g_2g_3 = (5\,3\,4)$. Now choose g_1 and g_4 analogously as 5-cycles acting on $\{3,4,5,6,7\}$ so g_4g_1 is $(4\,3\,5)$. Here, $H_{2,3}({}_0\boldsymbol{g})$ and

$H_{1,4}({}_0\boldsymbol{g})$ are both copies of A_5. All the c_is are 5, while $c=3$. The inequality (3.9) holds: $1/5+1/5+1/3+1/5=14/15<1$.

4. Finer graphs and infinite branches in $\mathcal{C}_{G,\mathbf{C},p}$ and $\mathcal{T}_{G,\mathbf{C},p}$

We don't know what contribution o-p' cusps in §3.2.1 make to the genera of components at level k on a **MT**. Are they like g-p' cusps in defining projective systems of components through o-p' cusps. Or, if you go to a suitably high level are all the cusps above them p cusps? Conj. 1.6 says the latter holds. §4.6 consists of support for and implications of this.

Schur multipliers of quotients of the universal p-Frattini cover ${}_p\tilde{G}$ of G are at the center of these conclusions in the form of lifting invariants (§4.2). We must deal with these many Schur multipliers when considering graphs finer than $\mathcal{C}_{G,\mathbf{C},p}$ and $\mathcal{T}_{G,\mathbf{C},p}$.

4.1. Limit Nielsen classes. — For a full analysis of higher rank **MT** examples such as in (§4.1.4), §4.1.1 extends the previous component and cusp branch notions. This extension uses all quotients of the universal p-Frattini cover (not just characteristic quotients). Given the definition of cusps from [**Fri05a**, Lect. 4] for arbitrary values of r, the concepts of this section work there, too.

4.1.1. Extending graphs to include any quotients of ${}_p\tilde{G}$. — Let $\mathcal{G}_{G,p}$ be all finite covers $G'\to G$ through which ${}_p\tilde{G}\to G$ factors. Given $(G,\mathbf{C},p)$, consider components and cusps of $\{\mathrm{Ni}(G',\mathbf{C})^{\mathrm{in}}\}_{G'\in\mathcal{G}_{G,p}}$. As in previous cases, they form directed graphs $\mathcal{C}^f_{G,\mathbf{C},p}$ and $\mathcal{T}^f_{G,\mathbf{C},p}$ (the f superscript for *full*) with maps between them.

Now, however, there may be many kinds of maximal directed paths (*branches*) not just distinguishing finite from infinite). Also, among undirected paths there could be loops because there may be several chief series for the Krull-Schmidt decomposition of $\ker(G_{p,k+1}\to G_{p,k})$ into irreducible $G_{p,k}$ modules. This doesn't happen for $G_{2,1}(A_n)\to A_n$ for $n=4,5$, but does for $G_{2,2}(A_4)\to G_{2,1}(A_4)$ [**BF02**, Cor. 5.7].

A directed path on $\mathcal{C}^f_{G,\mathbf{C},p}$ is defined by $\{(\boldsymbol{g}_{H_i})\mathrm{Cu}_4\}_{i\in I}$ with I a directed set, H_i a quotient of ${}_p\tilde{G}$ and $\boldsymbol{g}_{H_i}\in\mathrm{Ni}(H_i,\mathbf{C})$. If $i'>i$, then ${}_p\tilde{G}\to H_i$ factors through $H_{i'}$ sending $(\boldsymbol{g}_{H_{i'}})\mathrm{Cu}_4$ to $(\boldsymbol{g}_{H_i})\mathrm{Cu}_4$. This path defines a unique braid orbit in $\mathrm{Ni}(H_i,\mathbf{C})$ for $i\in I$: A cusp path (resp. branch) defines a component path (resp. branch).

Lemma 4.1. — *A directed path on $\mathcal{T}^f_{G,\mathbf{C},p}$ defines a set of directed paths on $\mathcal{C}^f_{G,\mathbf{C},p}$: Each node from any of the latter sits on a corresponding node of the former (with the obvious converse). If $\{(\boldsymbol{g}_{H_i})\mathrm{Cu}_4\}_{i\in I}$ is a directed path, then we can choose its cusp representatives $\boldsymbol{g}_{H_i}$ to also be a projective system.*

Proof. — A directed path on $\mathcal{T}^f_{G,\mathbf{C},p}$ is defined by a directed system $\{H_i\}_{i\in I}$. For each i there is a node consisting of $\mathcal{H}_{H_i}$, a component of $\mathcal{H}(H_i,\mathbf{C})^{\mathrm{in,rd}}$. The points R_{H_i} of the nonsingular $\mathcal{H}_{H_i}$ over $j=\infty$ have ramification degrees adding to the degree of

$\bar{\mathcal{H}}_{H_i} \to \mathbb{P}^1_j$. For $i' \ge i$, the natural map $R_{H_{i'}} \to R_{H_i}$ defines a projective system of finite nonempty sets. So, the set of limits is nonempty, and each defines a directed cusp path. Let $\{(\boldsymbol{g}_{H_i})\mathrm{Cu}_4\}_{i\in I}$ be one of these (as in the correspondence of §2.4.2).

The collections $(\boldsymbol{g}_{H_i})\mathrm{Cu}_4$, $i \in I$ also form a projective system of finite nonempty sets in the set of subsets of Nielsen classes. So, they too have projective limits. Each is a projective system of the form $\{\boldsymbol{g}_{H_i}\}_{i\in I}$. That gives the final statement. □

Definition 4.2 **(F paths, branches, ...)**. — For F a field the notion of F cusp path (resp. cusp branch), component path (resp. component branch) on $\mathcal{C}^f_{G,\mathbf{C},p}$ or $\mathcal{T}^f_{G,\mathbf{C},p}$ extends naturally that for $\mathcal{C}_{G,\mathbf{C},p}$ or $\mathcal{T}_{G,\mathbf{C},p}$ (as in §1.2.1).

4.1.2. Limit Groups. — Our next definitions use notation from Lem. 4.1.

Definition 4.3. — A directed path from a projective system $\{\boldsymbol{g}_{H_i}\}_{i\in I}$ has an attached group $\lim_{\infty\leftarrow i\in I} H_i = G^*$. Call this a *limit group* (of $(G,\mathbf{C},p)$) if the directed path is maximal. Then, $\mathrm{Ni}(G^*,\mathbf{C})$ is the *limit Nielsen class* attached to the maximal path, and $\lim_{\infty\leftarrow i} \boldsymbol{g}_{H_i} \in \mathrm{Ni}(G^*,\mathbf{C})$ represents the *limit braid orbit* of the path.

We might also call G^* the limit group of the braid orbit of $\boldsymbol{g}_G$, or of the component of $\mathcal{H}(G,\mathbf{C})$ attached to this orbit, etc.

Definition 4.4. — Suppose $\{\boldsymbol{g}_{H_i}\}_{i\in I}$ defines a maximal path. Then, for each $k \ge 0$ we can ask if $H_i = G_k$, for some i. If so, we say the path goes through level k of the $\mathbf{MT}$(and through braid orbit $O_{\boldsymbol{g}_{H_i}}$). If k_0 is the biggest integer with $\{\boldsymbol{g}_{H_i}\}_{i\in I}$ going through level k, then call the $\mathbf{MT}$ *obstructed* along the path at level k_0.

Obvious variants on Def. 4.4 refer to a braid orbit $O_{\boldsymbol{g}}$ at level k being obstructed: Every path through $O_{\boldsymbol{g}}$ is obstructed at level k, etc.

If O_* is the limit braid orbit in $\mathrm{Ni}(G^*,\mathbf{C})$ defined by a maximal path, then we say the path is obstructed at O_*. We also use variations on this. Any quotient G' of ${}_p\tilde{G}$ (possibly a limit group) has attached component and cusp graphs, $\mathcal{C}^f_{G,\mathbf{C},p}(G')$ and $\mathcal{T}^f_{G,\mathbf{C},p}(G')$, by running over Nielsen classes corresponding to quotients of G'.

4.1.3. Setup for the (strong) Main Conjecture. — Suppose F_u is free of rank u and J is finite acting faithfully on F_u. Consider $F_u \times^s J$, and let $\mathbf{C} = (\mathrm{C}_1,\dots,\mathrm{C}_r)$ be conjugacy classes in J. (Our examples use $r = 4$.)

Form $\tilde{F}_{u,p}$, the pro-p, pro-free completion of F_u. Then $\Phi^t = \Phi^t_p$ is the tth Frattini subgroup of $\tilde{F}_{u,p}$ (§1.1.2). Consider two sets $P_{\mathbf{C}}$ and $P'_{\mathbf{C}}$ of primes, with each consisting of those p with $\tilde{F}_{u,p}/\Phi^1 \times^s J$ not p-perfect, or p has this (respective) property:

- $P_{\mathbf{C}}$: $p \mid (p,|J|) \ne 1$.
- $P'_{\mathbf{C}}$: $p|\mathrm{ord}(g)$ some $g \in \mathbf{C}$.

For $p \notin P_{\mathbf{C}}$, denote (finite) J quotients of $\tilde{F}_{u,p}$ covering $(\mathbb{Z}/p)^u$ by $\mathcal{V}_p(J)$.

Problem 4.5. — Which $\mathrm{Ni}(V \times^s J,\mathbf{C})^{\mathrm{in}}$ are nonempty, $p \notin P_{\mathbf{C}}$ and $V \in \mathcal{V}_p(J)$.

For $p \notin P_{\mathbf{C}}$, form the collection $\mathcal{G}_{J,p}$ of limit groups over nonempty Nielsen classes (Def. 4.3). The $P'_{\mathbf{C}}$ version of this forms characteristic p-Frattini quotients of $F_u \times^s J$ where p may divide the order of J, but not the orders of elements in $\mathbf{C}$.

By taking F_u of rank 0 ($u = 0$), the $P'_{\mathbf{C}}$ version includes the weak Main Conjecture as a special case of the strong Main Conjecture 6.1.

We also must consider the finite J quotients V of $\tilde{F}_{u,p}$ where we ask only that V is nontrivial. Denote this set by $\mathcal{V}'_p(J)$.

Problem 4.6. — What are the $G^* \in \mathcal{G}_{J,p}$, $p \notin P'_{\mathbf{C}}$ (or just in $P_{\mathbf{C}}$)? What are the H_4 (braid) orbits on $\mathrm{Ni}(G^*, \mathbf{C})^{\mathrm{in}}$?

We say $G^* \in \mathcal{G}_{J,p}$ is a $\mathbf{C}$ *p-Nielsen limit*. If O is a braid orbit in $\mathrm{Ni}(G, \mathbf{C})$ we may consider only maximal paths (branches) over O. Then, maximal groups are p-Nielsen limits *through* O ($\mathbf{C}$ is now superfluous). So a cusp or component branch through O defines a p-Nielsen limit through O. Extend this to consider p-Nielsen limits through any nonempty braid orbit on $\mathrm{Ni}(G', \mathbf{C})$, G' any p-Frattini cover of G.

4.1.4. Examples: $u = 2$, $|J|$ is 2 or 3. — Take $F_u = \langle x_1, x_2\rangle$, Our two examples in (4.1) illustrate limit Nielsen classes, and the questions we pose.

(4.1a) $\mathbb{Z}/2$ case: $J = J_2 = \mathbb{Z}/2 = \{\pm 1\}$; -1 acts on generators of F_2 by $x_i \mapsto x_i^{-1}$, $i = 1, 2$; and $\mathbf{C} = \mathbf{C}_{2^4}$ is 4 repetitions of -1.

(4.1b) $J_3 = \mathbb{Z}/3 = \langle \alpha \rangle$; α maps $x_1 \mapsto x_2^{-1}$, $x_2 \mapsto x_1 x_2^{-1}$; and $\mathbf{C} = \mathbf{C}_{\pm 3^2}$ is two repetitions each of α, α^{-1}.

The apparent simplicity of (4.1a) is misleading: It is the Nielsen class behind Serre's Open Image Theorem ([**Fri05b**, §6] explains this). The result (in App. A) is that $\mathrm{Ni}(V \times^s J_2, \mathbf{C})$ is nonempty precisely when $V \in \mathcal{V}'_{\mathbf{C}_{2^4}}$ is abelian.

App. B shows all Nielsen classes in (4.1b) are nonempty because they contain H-M reps. (a special case of Princ. 3.6). That is, there are infinite component branches. Yet, it remains a challenge to Prob. 4.6.

Problem 4.7. — Let K be any number field. Are all infinite K component branches of $\mathcal{T}_{(\mathbb{Z}/p)^2 \times^s \mathbb{Z}/3, \mathbf{C}_{\pm 3^2}, p \neq 3}$, case (4.1b), defined by H-M rep. cusp branches?

Prop. B.1 gives an infinite limit group not equal to $\tilde{F}_{2,2} \times^s J_3$: H-M cusp branches don't give all infinite component branches of $\mathcal{T}^f_{(\mathbb{Z}/p)^2 \times^s \mathbb{Z}/3, \mathbf{C}_{\pm 3^2}, p \neq 3}$.

Remark 4.8. — It is essential for the RIGP (§1.1.3) that we consider questions like Prob. 4.6 for all r, based on Conj. 1.5.

4.2. The small lifting invariant.

— Let G be finite, $\psi : R \to G$ a Frattini central extension, and $\mathbf{C}$ conjugacy classes of G with elements of order prime to $|\ker(\psi)|$.

For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, we have a *small lifting invariant* $s_\psi(\boldsymbol{g}) = s_{R/G}(\boldsymbol{g}) = s_R(\boldsymbol{g})$ (notation of §2.1): Lift $\boldsymbol{g}$ to $\hat{\boldsymbol{g}} \in \mathbf{C}$ regarded as conjugacy classes in R and form

$\Pi(\hat{\boldsymbol{g}}) \in \ker(R \to G)$. It is an invariant on the braid orbit $O = O_{\boldsymbol{g}}$ of $\boldsymbol{g}$ which we call $s_R(O)$ [**Fri95**, Part III]. When $\ker(R \to G) = \mathrm{SM}_{G,p}$, denote this $s_{G,p}(O)$.

At times we regard $\ker(R \to G)$ as a multiplicative (resp. additive) group: So, $s_{G,p}(O) = 1$ (resp. $s_{G,p}(O) = 0$) when the invariant is trivial.

4.2.1. Component branch obstructions. — Consider a nontrivial Frattini central cover $R' \to G'$ through which ${}_p\tilde{G} \to G_0$ factors. Then, $\ker(R' \to G')$ is a quotient of the Schur multiplier of G' (§2.1). Denote the collection of such covers $\mathcal{SM}_{G,p}$, and the subcollection of $R' \to G'$ that are a subfactor of $G_{k+1} \to G_k$ with the notation $\mathcal{SM}_{G,p,k}$. Suppose $(G, \mathbf{C}, p)$ satisfies the usual **MT** conditions.

Lemma 4.9. — *In the above notation for $R' \to G' \in \mathcal{SM}_{G,p,k}$ these are equivalent:*

- *the injection from braid orbits in $\mathrm{Ni}(R', \mathbf{C})$ to braid orbits in $\mathrm{Ni}(G', \mathbf{C})$ has $\boldsymbol{g} \in \mathrm{Ni}(G', \mathbf{C})$ in its image;*
- *and $s_{R'}(\boldsymbol{g}) = 1$.*

For each $k \geq 0$, braid orbits in $\mathrm{Ni}(G_{k+1}, \mathbf{C})$ map onto compatible systems of braid orbits O on $\mathrm{Ni}(G', \mathbf{C})$ with $R' \to G' \in \mathcal{SM}_{G,p,k}$ and $s_{R'}(O) = 1$.

Similarly, infinite branches of $\mathcal{T}_{G,\mathbf{C},p}$ map onto compatible systems of braid orbits O in $\mathrm{Ni}(G', \mathbf{C})$ with $R' \to G' \in \mathcal{SM}_{G,p}$ and $s_{R'}(O) = 1$; and this is one-one.

Comments. — Given $\boldsymbol{g} \in \mathrm{Ni}(G', \mathbf{C})$ there is a unique lift to $\hat{\boldsymbol{g}} \in (R')^r \cap \mathbf{C}$, and $\hat{\boldsymbol{g}} \in \mathrm{Ni}(R', \mathbf{C})$ if and only if $s_{R'}(\boldsymbol{g}) = 1$. This shows the first paragraph statement.

Consider any cover $H'' \to H'$ through which ${}_p\tilde{G} \to G$ factors. We can always refine it into a series of covers to assume $\ker(H'' \to H') = M'$ is irreducible (as an H' module). For asking when braid orbits on $\mathrm{Ni}(H'', \mathbf{C})$ map surjectively to braid orbits on $\mathrm{Ni}(H', \mathbf{C})$ it suffices to assume M' is irreducible. [**FK97**, Obst. Lem. 3.2] says the map $\mathrm{Ni}(H'', \mathbf{C}) \to \mathrm{Ni}(H', \mathbf{C})$ is surjective unless M' is the trivial H' module. So, we have only to check surjectivity in those cases, using the lifting invariant. That establishes the second paragraph statement.

§6.4.5 uses $k = 1$ for $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$ to show the braid orbit map of the second paragraph is not necessarily one-one. This is from their being two orbits of H-M reps. in $\mathrm{Ni}(G_1(A_4), \mathbf{C}_{\pm 3^2})$.

The only point needing further comment is why the onto map of the last paragraph statement is one-one. That is because the collection of G' with $R' \to G' \in \mathcal{SM}_{G,p}$ is cofinal in all quotients of ${}_p\tilde{G}$: Prop. 2.4. □

Frattini Princ. 4.24 relates cusp branches (on $\mathcal{C}_{G,\mathbf{C},p}$) to component branches. This is a tool for considering if there is an o-p' cusp branch lying over a given o-p' cusp. Resolving Conj. 1.6 is crucial to deciding what are the infinite **MT** component branches. Though elementary, Lem. 3.1 is a powerful principle.

Principle 4.10. — *Suppose B' is a component branch on $\mathcal{T}_{G,\mathbf{C},p}$. The only way we can now prove G_F has a finite orbit on B' (the hypothesis of* (3.1)*) is to find a cusp* *branch B that defines B' for which, modulo braiding, G_F has a finite orbit on B. Further, all successes here are with g-p' branches.*

§4.6.2 has **MT**s with no g-p' cusps where we don't yet know if they have infinite component branches. Conj. 1.6 says they should not. Prob. 4.7 ($\mathbb{Z}/3$ rank 2 **MT**) has similar challenges for Conj. 1.5: Do g-p' cusps define all infinite component branches.

4.2.2. Replicating obstructed components. — Thm. 4.12 gives **MT**s with at least two components at every level. One is an H-M component with an obstructed component (Def. 4.4) lying above it at the next level ($k \geq 0$).

Suppose $\psi_0 : R_0 \to G_0$ is a Frattini central extension of G_0 with $\ker(\psi_0) = \mathbb{Z}/p$: a $\mathbb{Z}/p$ quotient as in §1.3.3. Further, suppose $\psi_1 : R_1 \to G_1$ is a Frattini central extension of G_1 with $\ker(\psi_1)$ also a $\mathbb{Z}/p$ quotient, but *antecedent* to $\ker(R_0 \to G_0)$. This means: $\ker(\psi_1) = \langle \tilde{a}^p \rangle$ with $\tilde{a} \in \ker({}_p\tilde{G} \to G_0)$ a lift of a generator of $\ker(\psi_0)$.

The idea of antecedents generalizes in the following technical lemma. It will seem less technical from the proof by recognizing M'_k interprets as M_0 multiplied by p^k.

Lemma 4.11. — *Then,* $\ker(R_1 \to G_0)$ *is a* $\mathbb{Z}/p^2[G_0]$ *module. For all* $k \geq 1$, *there is a Frattini cover* $\psi^*_k : R^*_k \to G_k$ *with* $\mathbb{Z}/p^2[G_0]$ *acting on* $\ker(\psi^*_k)$ *isomorphic to its action on* $\ker(R_1 \to G_0)$. *Also,* ψ^*_k *factors through a cover* $G^*_k \to G_k$ *with* G_0 *acting on* $\ker(G^*_k \to G_k) = M^*_k$ *as it does on* M_0. *Further:*

(4.2a) M^*_k *is a quotient of* M_k *(§1.3.3) on which* G_k *acts through* G_0*; and*

(4.2b) (4.2a) *extends to a* $\mathbb{Z}/p^2[G_k]$ *action on* $\ker(R^*_k \to G_k)$ *that factors through* $\mathbb{Z}/p^2[G_0]$ *acting on* $\ker(R_1 \to G_0)$.

Proof. — The condition that $\ker(R_1 \to G_1)$ is a $\mathbb{Z}/p^2[G_0]$ module is the main condition for an antecedent Schur multiplier, part of the characterization of that condition in [**Fri02**, Prop. 4.4].

The lemma says the $\mathbb{Z}/p^2[G_0]$ module $\ker(R_1 \to G_0)$ "replicates" at all levels. It comes from forming the *abelianization* ${}_p\tilde{G}/(\ker_0, \ker_0) \overset{\text{def}}{=} {}_p\tilde{G}'$ of ${}_p\tilde{G} \to G_0$ (as in §5.3.2 and used many times in such places as [**BF02**, §4.4.3]).

Denote the characteristic Frattini quotients of ${}_p\tilde{G}'$ by $\{G'_k\}_{k=0}^{\infty}$. Then, M_0 still identifies naturally with $\ker(G'_1 \to G_0)$. Since $\ker_0 / (\ker_0, \ker_0) = \ker'_0$ is abelian, taking all pth powers (additively: image of multiplication by p^k) in $\ker'_0$ gives the kth iterate of its Frattini subgroup $\ker'_k$. Then, M'_k is the 1st Frattini quotient of $\ker'_k$. Since G_0 acts on $\ker'_k$ this induces an action on M'_k. As $\ker(R_1 \to G_0)$ is also abelian, this replicates at level k as R'_k, also by "multiplication by p^k."

The conclusion of the lemma follows from recognizing, inductively from the universal p-Frattini property, that $R_{k+1} \to G_k$ must factor through $R'_{k+1} \to G'_k$, giving R^*_{k+1} as the pullback over G_k of $R'_{k+1} \to G'_k$, etc. □

Continue the notation of Lem. 4.11. We use it to replicate the event of having two components, one an H-M component, at one **MT** level to higher tower levels.

***Theorem 4.12*.** — *Let* ${}_0\boldsymbol{g} = ({}_0g_1^{-1}, {}_0g_1, {}_0g_2, {}_0g_2^{-1}) \in \mathrm{Ni}(G_0, \mathbf{C})$ *be an H-M rep. As in Princ. 3.6, take* $\{{}_k\boldsymbol{g}\}_{k=0}^{\infty}$ *to define an H-M cusp branch above* ${}_0\boldsymbol{g}$*. Assume there is a level 1 braid orbit represented by* ${}_1\boldsymbol{g}' \in \mathrm{Ni}(G_1, \mathbf{C})$ *with these properties:*

$$ {}_1\boldsymbol{g}' \mapsto {}_0\boldsymbol{g} \text{ and } s_{R_1}({}_1\boldsymbol{g}') \neq 1. $$

Then, there is a sequence $\{{}_k\boldsymbol{g}' \in \mathrm{Ni}(G_k, \mathbf{C})\}_{k=1}^{\infty}$ *with* ${}_k\boldsymbol{g}'$ *lying over* ${}_{k-1}\boldsymbol{g}$ *and* $s_{R_k^*}({}_k\boldsymbol{g}') \neq 1$*. Finally, we don't need to start these statements at level 0; they apply for* $k \geq k_0$*, if the hypotheses hold replacing* $(G_0, M_0, R_0, R_1))$ *with* $(G_{k_0}, M_{k_0}, R_{k_0}, R_{k_0+1})$*.*

Proof. — As in [**BF02**, §9], with no loss assume

$$ {}_1\boldsymbol{g}' = ({}_1g_1^{-1}, a_1({}_1g_1)a_1^{-1}, a_2^{-1}({}_1g_2)a_2, {}_1g_2^{-1}) $$

with $a_1, a_2 \in M_0$ the images of $\hat{a}_1, \hat{a}_2 \in R_1$ lying respectively over them. A restatement of $s_{R_1/G_1}({}_1\boldsymbol{g}') \neq 1$ (multiplicative notation) is this:

$$ (4.3) \quad a_1^{{}_0g_1} a_1^{-1} a_2^{-1} a_2^{{}_0g_2^{-1}} = 1, \text{ but } \hat{a}_1^{{}_0g_1} \hat{a}_1^{-1} \hat{a}_2^{-1} \hat{a}_2^{{}_0g_2^{-1}} \neq 1. $$

Now let a_1, a_2 represent their respective images in M_k^* and replace ${}_1g_1$ and ${}_1g_2$ in (4.3) by ${}_kg_1$ and ${}_kg_2$. This produces the ${}_k\boldsymbol{g}'$ in the theorem's statement. The corresponding expressions in (4.3) hold because we have a $\mathbb{Z}/p^2[G_k]$ isomorphism of $\ker(\psi_k^*)$ with $\ker(\psi_0^*)$.

The final statement applies the general principle that we can start a **MT** at any level we want just by shifting the indices. □

***Example 4.13* (Several components at high levels).** — [**BF02**, Prop. 9.8] shows level 1 of the $(A_5, \mathbf{C}_{3^4}, p = 2)$ **MT** has exactly two components, and these satisfy the hypotheses of Thm. 4.12 (more in Ex. B.2). Thus, each level $k \geq 1$ of this **MT** has at least two components. (Level 0 has just one.)

Level 1 of the $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$ **MT** has two H-M and four other components, each over the H-M component (from two at level 0; see §6.4.5). Thm. 4.12 lets us select whatever H-M cusp representatives we want over ${}_0\boldsymbol{g}$. So, suppose there are several braid orbits of H-M branches, and the hypothesis at one level holds. Then, each braid orbit of an H-M cusp branch through that level gives a pair of components at higher levels. Thus, Thm. 4.12 says each level $k \geq 2$ of the $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$ **MT** has at least eight components.

4.3. Weigel's p-Poincaré Duality Theorem.

— Let $\varphi : X \to \mathbb{P}_z^1$, with branch points $\boldsymbol{z}$, be a Galois cover in $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$ representing a braid orbit O.

With $U_{\boldsymbol{z}} = \mathbb{P}_z^1 \setminus \{\boldsymbol{z}\}$, use *classical generators* $x_1, \dots, x_r$ to describe the fundamental group $\pi_1(U_{\boldsymbol{z}}, z_0)$: $x_1, \dots, x_r$ (in order corresponding to branch points of φ, $z_1, \dots, z_r$) freely generate it, modulo the product-one relation $\prod_{i=1}^{r} x_i$ [**BF02**, §1.2]. Restrict φ off $\boldsymbol{z}$ to give $\varphi^0 : X^0 \to U_{\boldsymbol{z}}$. Let $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ be the corresponding branch cycles giving a representing homomorphism $\pi_1(U_{\boldsymbol{z}}, z_0) \to G$ by $x_i \mapsto g_i$, $i = 1, \dots, r$.

Denote the pro-p completion of the fundamental group of the (compact) Riemann surface X by $\pi_1(X)^{(p)}$. [**BF02**, Prop. 4.15] produces a quotient M_φ of $\pi_1(U_{\boldsymbol{z}}, z_0)$ with $\ker(M_\varphi \to G)$ identifying with $\pi_1(X)^{(p)}$ (proof of Lem. 4.14).

We sometimes denote M_φ by $M_{\boldsymbol{g}}$ when given $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ defined by classical generators. Lem. 4.14 says —up to braiding —$M_{\boldsymbol{g}} \to G$ is independent of $\boldsymbol{g}$. Since $\ker(M_{\boldsymbol{g}} \to G_k)$ is a pro-p group, the notation $\mathrm{Ni}(M_{\boldsymbol{g}}, \mathbf{C})$ makes sense (as in §1.1.2).

Lemma 4.14. — *The action of H_r on $\boldsymbol{g}$ is compatible with its action on $x_1, \dots, x_r$. This gives a braid orbit of homomorphisms starting with $M_{\boldsymbol{g}} \to G$. As abstract group extensions they are isomorphic.*

Also, p-Nielsen limits through O are maximal among quotients of ${}_p\tilde{G}$ through which $M_{\boldsymbol{g}} \to G$ factors (up to conjugation by $\ker(M_{\boldsymbol{g}} \to G)$). So, O starts a component branch of $\mathcal{T}_{G,\mathbf{C},p}$ if and only if, running over $R' \to G' \in \mathcal{SM}_{G,p}$ (as in Lem. 4.9), each $\psi_{G'} : M_{\boldsymbol{g}} \to G'$ extending $M_{\boldsymbol{g}} \to G$ extends to $\psi_{R'} : M_{\boldsymbol{g}} \to R'$.

The obstruction to extending $\psi_{G'}$ to $\psi_{R'}$ is the image in $H^2(M_{\boldsymbol{g}}, \ker(R' \to G'))$ by inflation of $\alpha \in H^2(G', \ker(R' \to G'))$ defining the extension $R' \to G'$.

Comments. — Let W be the normal subgroup of $\pi_1(U_{\boldsymbol{z}}, z_0)$ generated by $x_i^{\mathrm{ord}(g_i)}$, $i = 1, \dots, r$. Identify $U = \ker(\pi_1(U_{\boldsymbol{z}}, z_0)/W \to G)$ with π_1 of X; what Weigel calls a *finite index surface group* [**Wei05**, Proof of Prop. 5.1]. (If φ is not a Galois cover, then it is more complicated to describe $\pi_1(X)$ by *branch cycles* [**Fri89**, p. 75–77].)

In Weigel's notation, $\Gamma = \pi_1(U_{\boldsymbol{z}}, z_0)/W$. Form $M_{\boldsymbol{g}}$ by completing Γ with respect to Γ normal subgroups in U of index (in U) a power of p. For more details see §4.4.1. Then $M_{\boldsymbol{g}}$ has a universal property captured in the second paragraph of the lemma.

In a characteristic 0 smooth connected family of covers the isomorphism class of the monodromy group does not change. That is, the braiding of $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ to $\boldsymbol{g}'$ from a deformation of the cover with branch point set $\boldsymbol{z}_0$ over a path in $\pi_1(U_r, \boldsymbol{z}_0)$ produces another copy of G. The same is true if you apply this to a profinite family of covers defining a cofinal family of quotients of $M_{\boldsymbol{g}}$. This shows that braiding induces an isomorphism on $M_{\boldsymbol{g}}$ as said in the first paragraph of the lemma.

This gives the first paragraph statement. The final paragraph statement is likely well-known. See, for example, [**Fri95**, Prop. 2.7] or [**Wei05**, Prop. 3.2]. □

We continue notation of Lem. 4.14. The following translates [**Wei05**] for our group $M_{\boldsymbol{g}}$. We explain terminology and module conditions for later use.

Theorem 4.15. — *$M_{\boldsymbol{g}}$ is a dimension 2 oriented p-Poincaré duality group.*

Comments. — The meaning of the phrase (dimension 2) p-Poincaré duality is in [**Wei05**, (5.8)]. It expresses an exact cohomology pairing

$$H^k(M_{\boldsymbol{g}}, U^*) \times H^{2-k}(M_{\boldsymbol{g}}, U) \to \mathbb{Q}_p/\mathbb{Z}_p \overset{\mathrm{def}}{=} I_{M_{\boldsymbol{g}},p} \tag{4.4}$$

where U is any abelian p-power group that is also a $\Gamma = M_{\boldsymbol{g}}$ module, U^* is its dual with respect to $I_{M_{\boldsymbol{g}},p}$ and k is any integer. [**Ser97a**, I.4.5] has the same definition, though that assumes in place of $M_{\boldsymbol{g}}$ a pro-p-group. By contrast, $M_{\boldsymbol{g}}$ is p-perfect, being generated by p' elements (Lem. 2.1). In the extension problems of §4.4, the quotients of $M_{\boldsymbol{g}}$ that interest us are Frattini covers of G, so also p-perfect.

[**Ser97a**, p. 38] points to Lazard's result that a p-adic analytic group of dimension d (compact and torsion-free) is a Poincaré group of dimension d. Since, however, our group is residually pro-free, it isn't even residually p-adic analytic.

Weigel's result is for general Fuchsian groups Γ, and the *dualizing module* $I_{\Gamma,p}$, may not be the same as in this example. It is classical that $\pi_1(X)$ (and $\pi_1(X)^{(p)}$) satisfies Poincaré duality. [**Bro82**, Chap. VIII, §3, Remark] interprets this exactly as the discussion of §2.1 suggests for group cohomology. [**Ser97a**, Prop. 18, p. 25] applies Shapiro's Lemma to show a dualizing module that works for Γ also works for every open subgroup. Most of Weigel's proof establishes the converse: That the $I_{\pi_1(X),p}$ used here does act as a dualizing module for $M_{\boldsymbol{g}}$. □

***Remark 4.16* (Addendum to Lem. 4.14)**. — Suppose two extensions $M_{\boldsymbol{g}_i} \to G$, arise from $\boldsymbol{g}_i \in \mathrm{Ni}(G, \mathbf{C})$, $i = 1, 2$. Further, assume they are isomorphic. Then, it is still possible they are not braid equivalent, though examples aren't easy to come by. We allude to one in (6.11a): Two extensions corresponding to the two H-M components called $\mathcal{H}_1^{+,\beta}$, $\mathcal{H}_1^{+,\beta^{-1}}$. The group G in this case is $G_1(A_4)$. It has an automorphism mapping $\boldsymbol{g}_1$ to $\boldsymbol{g}_2$, giving elements in different braid orbits. Since these are H-M components, Princ. 3.6 gives isomorphic extensions $M_{\boldsymbol{g}_i} \to {}_p\tilde{G}$, $i = 1, 2$ (Princ. 3.6) in distinct braid orbits.

4.4. Criterion for infinite branches on $\mathcal{T}_{G,\mathbf{C},p}$.

— Cor. 4.19 reduces finding infinite component branches on $\mathcal{T}_{G,\mathbf{C},p}$ through a braid orbit (as in §1.3.3) to a sequence of small lifting invariant checks from the Schur multiplier of each G_k, $k \geq 1$. Cor. 4.20 is our major test for when we have a limit group.

4.4.1. One lifting invariant checks unobstructed braid orbits. — This subsection regards the small lifting invariant in additive notation. Let $O_k \leq \mathrm{Ni}(G_k, \mathbf{C})$ be a braid orbit and ${}_k\boldsymbol{g}$ a representative of this orbit. The cardinality of the fiber in (1.10b) over O_k is the degree of a level $k+1$ **MT** component over its level k image defined by O_k. This is a braid invariant. Cor. 4.19 is (at present) our best test for when it is nonempty, unless $\boldsymbol{g}$ braids to a g-p' representative (Princ. 3.6).

We may consider $M_{\boldsymbol{g}}$ as a completion of a group, $D_{\bar{\boldsymbol{\sigma}}}$, presented as $\langle \bar\sigma_1, \dots, \bar\sigma_r\rangle$ modulo the normal subgroup generated by $\bar{\boldsymbol{\sigma}} \stackrel{\text{def}}{=} \{\bar\sigma_i^{\mathrm{ord}(g_i)}, i = 1, \dots, r,\ \text{and}\ \bar\sigma_1 \cdots \bar\sigma_r\}$. Let $K_{\bar{\boldsymbol{\sigma}}^*}$ by the group from removing the quotient relation $\bar\sigma_1 \cdots \bar\sigma_r = 1$. Denote corresponding generators of it by $\bar\sigma_1^*, \dots, \bar\sigma_r^*$. Then, the cyclic groups $\langle \bar\sigma_i^* \rangle / (\bar\sigma_i^*)^{\mathrm{ord}(g_i)}$, $i = 1, \dots r$, freely generate $K_{\boldsymbol{\sigma}}$.

Complete $K_{\bar{\boldsymbol{\sigma}}^*}$ with respect to p-power index subgroups of $\ker(K_{\bar{\boldsymbol{\sigma}}^*} \to G)$, normal in $K_{\boldsymbol{\sigma}}$, calling the result $\tilde{K}_{\bar{\boldsymbol{\sigma}}^*}$ (forming a natural surjection $\psi_{\bar{\boldsymbol{\sigma}}^*} : \tilde{K}_{\bar{\boldsymbol{\sigma}}^*} \to M_{\boldsymbol{g}}$).

Lemma 4.17. — *Mapping the $\tilde{K}_{\bar{\boldsymbol{\sigma}}}$ generators $\bar{\sigma}_1^*, \dots, \bar{\sigma}_r^*$, in order, to entries of ${}_k\boldsymbol{g}$, gives a homomorphism $\mu_k : \tilde{K}_{\bar{\boldsymbol{\sigma}}^*} \to G_k$. If $h_1^*, \dots, h_r^* \in \mathbf{C} \cap G_{k+1}^r$ lie respectively over entries of ${}_k\boldsymbol{g}$, then the surjective homomorphism $\mu_{k+1} : \tilde{K}_{\bar{\boldsymbol{\sigma}}^*} \to G_{k+1}$ mapping $\bar{\sigma}_i^* \mapsto h_i^*$, $i = 1, \dots, r$, extends μ_k.*

Comments. — The construction is geometric: Remove an additional point z' from $U_{\boldsymbol{z}}$ to get $\pi_1(U_{\{\boldsymbol{z},z'\}}, z_0)$. We can identify this with notation coming from $D_{\boldsymbol{\sigma}}$, as the group freely generated by $\bar{\boldsymbol{\sigma}}$. This identifies $K_{\bar{\boldsymbol{\sigma}}^*}$ with its description above. It also identifies $\ker(K_{\bar{\boldsymbol{\sigma}}^*} \to G_0)$ with the fundamental group of $X' \stackrel{\text{def}}{=} X \setminus \{\varphi^{-1}(z')\}$. As X' is a projective curve with a nonempty set of punctures, this is a free group. □

***Remark 4.18* (Addendum to proof of Lem. 4.17)**. — The group $M_{\boldsymbol{g}}$ is not p-projective. Yet, here is why its cover $\tilde{K}_{\bar{\boldsymbol{\sigma}}^*}$ is. For P a p-Sylow of G, we can identify a p-Sylow of $\tilde{K}_{\bar{\boldsymbol{\sigma}}^*}$ with the pro-p completion of the free group $\pi_1(X'/P)$. A profinite group with pro-p p-Sylow is p-projective ([**FJ86**, Prop. 22.11.08], in new edition).

4.4.2. Two obstruction corollaries. — Continue the discussion of §4.4.1. If $\boldsymbol{g} \in O_k$, then it defines a cover $\psi_{\boldsymbol{g}} : M_{\boldsymbol{g}} \to G_k$. A paraphrase of Cor. 4.19 is that if $\psi_{\boldsymbol{g}}$ is *obstructed* at level k then it is by some $\mathbb{Z}/p$ quotient of $\ker(G_{k+1} \to G_k)$. Cor. 4.20 tells us precisely what are the exponent p Frattini extensions of a limit group.

Corollary 4.19. — *The fiber over O_k is empty if and only if there is some central Frattini extension $R \to G_k$ with kernel isomorphic to $\mathbb{Z}/p$ for which $\psi_{\boldsymbol{g}}$ does not extend to $M_{\boldsymbol{g}} \to R \to G$.*

Proof. — In the notation of §2.5 we only need to show this: If the fiber of (1.10b) is empty, then $s_{R/G_k}(\boldsymbol{g}) \neq 0$ for some $\mathbb{Z}/p$ quotient R/G_k of the first Loewy layer of M_k. [**Fri95**, Prop. 2.7] says $H^2(G_k, M_k) = \mathbb{Z}/p$: It is 1-dimensional. Lem. 4.14 says the obstruction to lifting ψ to G_{k+1} is the inflation of some fixed generator $\alpha \in H^2(G_k, M_k)$ to $\tilde{\alpha} \in H^2(M_{\boldsymbol{g}}, M_k)$.

Though $\tilde{\alpha}$ may seem abstract, the homomorphism μ_{k+1} of Lem. 4.17 allows us to form an explicit cocycle for the obstruction to lifting $M_{\boldsymbol{g}} \to G$. For each $\bar{g} \in M_{\boldsymbol{g}}$ choose $h_{\bar{g}} \in G_k$ as the image in G_k of one of the elements of $\tilde{K}_{\bar{\boldsymbol{\sigma}}^*}$ over $\bar{g}$. Now compute from this the 2-cocycle

$$\tilde{\alpha}(\bar{g}_1, \bar{g}_2) = h_{\bar{g}_1} h_{\bar{g}_2} (h_{\overline{g_1 g_2}})^{-1}, \bar{g}_1, \bar{g}_2 \in M_{\boldsymbol{g}}$$

describing the obstruction. Since $\psi_{\bar{\boldsymbol{\sigma}}^*}$ is a homomorphism, the only discrepancy between $\alpha(\bar{g}_1, \bar{g}_2)$ and the identity is given by the leeway in representatives for $h_{\overline{g_1 g_2}}$ lying over $\overline{g_1 g_2}$. So, the cocycle $\tilde{\alpha}(\bar{g}_1, \bar{g}_2)$ consists of words in the kernel of $K_{\bar{\boldsymbol{\sigma}}^*} \to M_{\boldsymbol{g}}$, and it vanishes if and only if it is possible to choose $(h_1^*, \dots, h_r^*)$ (as in the statement of Lem. 4.17) to satisfy $h_1^* \cdots h_r^* = 1$.

By (4.4) duality, $H^2(M_{\boldsymbol{g}}, M_k)$ has a perfect pairing with $H^0(M_{\boldsymbol{g}}, M_k^*)$, that initially goes into $H^2(M_{\boldsymbol{g}}, I_{M_{\boldsymbol{g}},p})$ by applying an element of $H^0(M_{\boldsymbol{g}}, M_k^*)$ to the values of a 2-cycle in $H^2(M_{\boldsymbol{g}}, M_k)$. Identify $H^0(M_{\boldsymbol{g}}, M_k^*)$ with

$$H_0(M_{\boldsymbol{g}}, D \otimes M_k) \simeq D \otimes_{\mathbb{Z}/p[M_{\boldsymbol{g}}]} M_k,$$

with $D = \mathbb{Z}/p$ the duality module for $\mathbb{Z}/p[M_{\boldsymbol{g}}]$ (on which it acts trivially). Hence, the tensor product $D \otimes_{\mathbb{Z}/p[M_{\boldsymbol{g}}]} M_k$ is canonically isomorphic to the maximal quotient of M_k on which $M_{\boldsymbol{g}}$ (and therefore G_k) acts trivially [**AW67**, p. 98]. That is, $D \otimes_{\mathbb{Z}/p[M_{\boldsymbol{g}}]} M_k$ identifies with the kernel of the maximal central exponent p extension of G_k.

Now we check the value of the pairing of $\tilde{\alpha}(\bullet, \bullet) \in H^2(M_{\boldsymbol{g}}, M_k)$ against an element $\beta \in H^0(M_{\boldsymbol{g}}, M_k^*)$. Further, regard $\beta \overset{\text{def}}{=} \beta_R$ as the linear functional on M_k from $\ker(G_{k+1} \to R)$, with $R \to G_k$ a central extension defining a $\mathbb{Z}/p$ quotient, as above.

Being very explicit, this says the value of β_R on $\tilde{\alpha}$ is the lifting invariant $s_R(\boldsymbol{g})$ for the image $\boldsymbol{g}$ of $(h_1^*, \dots, h_r^*)$ in $\mathrm{Ni}(G_k, \mathbf{C})$. Since the pairing is perfect, conclude the corollary: The obstruction for extending $M_{\boldsymbol{g}} \to G_k$ to $M_{\boldsymbol{g}} \to G_{k+1}$ is trivial if and only if $s_R(\boldsymbol{g})$ is trivial running over all such $R \to G_k$. □

The proof of the last result also applies to limit groups.

Corollary 4.20. — *If G^* is a limit group in a Nielsen class and a proper quotient of ${}_p\tilde{G}$, then G^* has exactly one nonsplit extension by a $\mathbb{Z}/p[G^*]$ module, and that module must be trivial.*

Proof. — Suppose $\boldsymbol{g}^* \in \mathrm{Ni}(G^*, \mathbf{C})$ represents the braid orbit giving G^* as a limit group (Def. 4.3). From the proof of Cor. 4.19, we have only to show there cannot be two $\mathbb{Z}/p$ quotients of the exponent p part of the Schur multiplier of G^*.

Suppose $R_i \to G^*$, $i = 1, 2$, are two distinct central extensions defining $\mathbb{Z}/p$ quotients. So, their kernels generate a 2-dimensional quotient of the Schur multiplier of G^*. Since G^* is a limit group, $s_{R_i/G^*}(\boldsymbol{g}^*) \neq 0$ generates $\ker(R_i \to G^*)$, $i = 1, 2$.

Apply Thm. 4.15: $H^2(M_{\boldsymbol{g}}, \mathbb{Z}/p) = \mathbb{Z}/p$. Let $\alpha_i \in H^2(M_{\boldsymbol{g}}, \mathbb{Z}/p) = \mathbb{Z}/p$ be the inflation of the element of $H^2(G^*, \mathbb{Z}/p)$ defining R_i, $i = 1, 2$. So there are p' integers a_i, $i = 1, 2$, with $a_1\alpha_1 + a_2\alpha_2 = 0$. Also, $a_1 s_{R_1/G^*}(\boldsymbol{g}^*) + a_2 s_{R_2/G^*}(\boldsymbol{g}^*) \neq 0$ defines a $\mathbb{Z}/p$ quotient of the Schur multiplier of G^*.

This gives a central extension $R^* \to G^*$, and the inflation of an element of $H^2(G^*, \mathbb{Z}/p)$ to $H^2(M_{\boldsymbol{g}^*}, \mathbb{Z}/p)$ defining it is 0. Thus Lem. 4.14 contradicts that G^* is a limit group since it says $M_{\boldsymbol{g}^*} \to G^*$ extends to $M_{\boldsymbol{g}^*} \to R^*$. □

4.4.3. Why Cor. 4.19 is a global result. — Consider two (braid inequivalent) extensions of $\psi_i : M_{\boldsymbol{g}} \to G_{k+1}$, $i = 1, 2$, of $\psi : M_{\boldsymbol{g}} \to G_k$. Assume, hypothetically, the following holds (it does not in general):

(4.5) There is an extension of ψ_1 to $\psi_1' : M_{\boldsymbol{g}} \to G_{k+2}$ if and only if there is an extension of ψ_2 to $\psi_2' : M_{\boldsymbol{g}} \to G_{k+2}$.

Applying Princ. 3.6 would then give the following (false) conclusion from (4.5).

(4.6) If $\boldsymbol{g}$ is a g-p' cusp, then any component branch of $\mathcal{T}_{G,\mathbf{C},p}$ through the braid orbit of $\boldsymbol{g}$ is infinite.

Cor. 4.19 works with G^*, any group through which ${}_p\tilde{G} \to G_0$ factors, replacing G_k and with any G^* quotient M^* of $\ker(G_1(G^*) \to G^*)$ replacing M_k. (Reminder: $G_1(G^*)$ is the 1st characteristic p-Frattini cover of G^*.) So, given an hypothesis like (4.5), one might try to reduce the proof of Cor. 4.19 to where M^* is simple. This would allow stronger conclusions, eschewing considering one integer k at-a-time.

This, however, is a variant of the false conclusion (4.6). Examples 4.21 and 4.22 show (4.6) is false. They explain why applying Cor. 4.19 to detect an infinite branch can't be done by just testing the lifting invariant at one level. These examples — based on [**BF02**, Chap. 9] — help understand this subtle argument.

Also, for a given **MT** level k, and $R' \to G' \in \mathcal{SM}_{G,p,k}$ (Lem. 4.9), precise genera formulas for **MT** branches require knowing if braid orbits achieve other lift values than the trivial one. Again, these examples illustrate. They rely on *centralizer condition* (4.7). So, we don't yet know how to generalize them to, say, replace A_n by $G_k(A_n)$ for k large, even for the antecedent Schur multiplier because (4.7) doesn't hold.

***Example 4.21* (Level 1 of the $(A_5, \mathbf{C}_{3^4}, p = 2)$ MT).** — Here $\mathbf{C}_{3^4}$ is four repetitions of the 3-cycle conjugacy class in A_5. [**BF02**, Prop. 9.8] shows there are exactly two braid orbits O_1 and O_2 on $\mathrm{Ni}(G_1(A_5), \mathbf{C}_{3^4})$ where $p = 2$, both over the unique braid orbit O on $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})$. The 2-part, $\mathrm{SM}_{G_1(A_5),2}$, of the Schur multiplier of $G_1(A_5)$ is $\mathbb{Z}/2$. Let $R_1 \to G_1$ be the $\mathbb{Z}/2$ quotient it defines. Then, $s_{R_1/G_1}(O_1) = 0$ and $s_{R_1/G_1}(O_2) \neq 0$. In fact, O and O_1 are orbits of H-M reps. So, at level 1 (but not at level 0) all possible lift invariants are assumed. This pure module argument used a strong condition:

(4.7) The rank of the centralizer in $M_0 = \ker(G_1(A_5) \to A_5)$ of $g \in \mathrm{C}_3$ is the same as the rank of $\mathrm{SM}_{G_1,2}$, and $R' \to G_1(A_5)$ is antecedent (§2.5).

Example 4.22. — [**FS06**] notes (4.7) also holds for $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$ (see §6.3; $R' \to G' = G_1(A_4)$ is the antecedent Schur multiplier). The Schur multiplier of $G_1(A_4)$ is $(\mathbb{Z}/2)^2$. Ad hoc arguments show we achieve the other two values of the lifting invariant running over $R'' \to G' = G_1(A_4)$, with $R'' \to G$ the two non-antecedent central Frattini extensions giving $\mathbb{Z}/p$ quotients.

4.5. Weigel branches in $\mathcal{C}_{G,\mathbf{C},p}$ and Frattini Princ. 3.

— [**Fri05a**, Lect. 4] generalizes g-p' reps. to all r. We believe having a g-p' cusp branch B is necessary for an infinite component branch in $\mathcal{T}_{G,\mathbf{C},p}$ (Conj. 1.5). Here we approach Conj. 1.6 using multiplicative notation for the small lifting invariant (§4.2).

4.5.1. Set up for o-p' cusps. — We introduce a practicum for deciding if a given o-p' cusp $\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})$ has an o-p' cusp $\boldsymbol{g}' \in \mathrm{Ni}(G_{k+1}, \mathbf{C})$ over it. (Compare with the more restrictive search for an o-p' cusp over a g-p' cusp in §3.2.3.) From this comes Def. 4.26 of a *Weigel cusp*. Prop. 3.12 says there are **MT**s where o-p' cusps appear at all high levels. Still, the examples we know do not produce Weigel branches (projective sequences of such cusps), so they do not contradict Conj. 1.6.

Assume $\boldsymbol{g} = (g_1, g_2, g_3, g_4) \in \mathrm{Ni}(G, \mathbf{C})$ is an o-p' cusp rep. As p' elements generate $H_{2,3} = \langle g_2, g_3\rangle = H$ it is p-perfect (Lem. 2.1). Consider diagram (4.8). The bottom (resp. top) row has the sequence for the p-representation cover R'_p of H (resp. G). Pullback of H in R_p is a central extension of H. So, a unique map $\beta_H : R_{p'} \to R_p$ makes (4.8) commutative:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{SM}_{G,p} \arrow[r] & R_p \arrow[r] & G \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{SM}_{H,p} \arrow[r] \arrow[u] & R'_p \arrow[r] \arrow[u, "\beta_H"'] & H \arrow[r] \arrow[u, "\mathrm{inj}"'] & 1.
\end{tikzcd} \tag{4.8}$$

Unlike its Lem. 2.5 analog, β may not be an embedding. Example: Let H be simple, with $\mathrm{SM}_{G,p} \neq \{1\}$ (p odd), and embed it in an alternating group. The following lemma summarizes this to show compatibility of (4.8) with Lem. 2.5.

Lemma 4.23. — *Properties of* (4.8) *apply to any p-perfect (or p') subgroup $H \le G$. Further, the map β_H is compatible with the map $\beta : {}_p\tilde{H} \to R_p$ defined in Rem. 2.6.*

4.5.2. The 3rd Frattini Principle. — Princ. 4.24 relates cusp types and lifting invariants for component branches. Assume ${}_0\boldsymbol{g} = \boldsymbol{g} = (g_1, g_2, g_3, g_4) \in \mathrm{Ni}(G, \mathbf{C})$ is an o-p' cusp rep. Denote a 5th p' conjugacy class containing $(g_2g_3)^{-1}$ by C_5. Similarly, its inverse is C_5^{-1}. Denote the collection $\mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_5$ (resp. $\mathrm{C}_1, \mathrm{C}_4, \mathrm{C}_5^{-1}$) by $\mathbf{C}_{2,3}$ (resp. $\mathbf{C}_{1,4}$). Also:

$$(g_2, g_3, (g_2g_3)^{-1}) = {}_0\boldsymbol{g}' \text{ and } ((g_4g_1)^{-1}, g_4, g_1)) = {}_0\boldsymbol{g}'',$$

and let $O_{\boldsymbol{g}}$, $O_{{}_0\boldsymbol{g}'}$ and $O_{{}_0\boldsymbol{g}''}$ be the respective braid orbits of the corresponding Nielsen class representatives.

Assume for some $k \ge 0$, ${}_k\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})$. Let $R_{G_k} \to G_k$ be the central extension of G_k with $\ker(R_{G_k} \to G_k)$ the maximal quotient of M_k on which G_k acts trivially. Then, we have similar notation with

$$H_{2,3}({}_k\boldsymbol{g}) = H_{2,3} \text{ and } H_{1,4}({}_k\boldsymbol{g}) = H_{1,4}$$

replacing G_k. Diagram (4.8), with $H = H_{2,3}$, induces maps $\beta_{2,3} : R_{H_{2,3}(\boldsymbol{g})} \to R_G$ from Lem. 4.23 as the situation deserves.

***Principle 4.24* (Frattini Principle 3)**. — *With the previous hypotheses*

$$s_{G,p}(\boldsymbol{g}) = \beta_{1,4}(s_{R_{H_{1,4}},p}((g_4g_1)^{-1}, g_4, g_1))\beta_{2,3}(s_{R_{H_{2,3}},p}(g_2, g_3, (g_2g_3)^{-1})). \tag{4.9}$$

Suppose ${}_k\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})$ is an o-p' cusp. Consider:

$$\begin{aligned}
{}_k\boldsymbol{g}' &= ({}_kg_2, {}_kg_3, ({}_kg_2{}_kg_3)^{-1}) \in \mathrm{Ni}(G_k(H_{2,3}(\boldsymbol{g})), \mathbf{C}_{2,3}) \textit{ and}\\
{}_k\boldsymbol{g}'' &= (({}_kg_4{}_kg_1)^{-1}, {}_kg_4, {}_kg_1)) \in \mathrm{Ni}(G_k(H_{1,4}(\boldsymbol{g})), \mathbf{C}_{1,4}).
\end{aligned}$$

Suppose $s_{R_{H_{2,3}({}_k\boldsymbol{g})}}({}_k\boldsymbol{g}') = 1$ *and* $s_{R_{H_{1,4}({}_k\boldsymbol{g})}}({}_k\boldsymbol{g}'') = 1$*. Then, there is an o-*$p'$ *cusp* ${}_{k+1}\boldsymbol{g} \in \mathrm{Ni}(G_{k+1}, \mathbf{C})$ *over* ${}_k\boldsymbol{g}$*.*

Assume there is an infinite component branch on the $(H_{2,3}(\boldsymbol{g}), \mathbf{C}_{2,3}, p)$ **MT** *over* $O_{{}_0\boldsymbol{g}'}$*, and also such a component branch on the* $(H_{1,4}(\boldsymbol{g}), \mathbf{C}_{1,4}, p)$ **MT** *over* $O_{{}_0\boldsymbol{g}''}$*. Then, an o-*p' *cusp branch gives an infinite component branch on the* **MT** *over* $O_{\boldsymbol{g}}$*.*

Proof. — Consider the 6-tuple, $\boldsymbol{g}^* = ((g_4g_1)^{-1}, g_4, g_1, g_2, g_3, (g_2g_3)^{-1})$. This is a juxtaposition of two product-one 3-tuples. Since $(g_4g_1)^{-1}(g_2g_3)^{-1} = 1$, we easily see $s_{G,p}(\boldsymbol{g}^*) = s_{G,p}(\boldsymbol{g})$. So, (4.9) follows from direct computation and the compatibility of the maps $\beta_{2,3}$ and $\beta_{1,4}$ defined in different places. Lem. 2.5 lets us assume $G_{p,k}(H_{2,3})$ and $G_{p,k}(H_{1,4})$ are in $G_{p,k}(G)$. Over ${}_k\boldsymbol{g}'$ (resp. ${}_k\boldsymbol{g}''$) Lem. 4.14 produces

$$ {}_{k+1}\boldsymbol{g}' \in \mathrm{Ni}(G_{k+1}(H_{2,3}(\boldsymbol{g})), \mathbf{C}_{2,3}) \text{ (resp. } {}_{k+1}\boldsymbol{g}'' \in \mathrm{Ni}(G_{k+1}(H_{1,4}(\boldsymbol{g})), \mathbf{C}_{1,4})). $$

Use Schur-Zassenhaus to produce $h \in \ker(G_{k+1} \to G_k)$ that conjugates

$$ ({}_{k+1}g'_2\, {}_{k+1}g'_3)^{-1} \in \mathrm{C}_5 \text{ to } {}_{k+1}g'_4 {}_{k+1}g'_1 \in \mathrm{C}_5. $$

Replace $({}_{k+1}g_2, {}_{k+1}g_3, ({}_{k+1}g_2, {}_{k+1}g_3)^{-1})$ with its conjugate by the image of h. So, with no loss, $({}_{k+1}g_1, {}_{k+1}g_2, {}_{k+1}g_3, {}_{k+1}g_4)$ has product-one, is in $\mathrm{Ni}(G_{p,k+1}(G), \mathbf{C})$ and lies over ${}_0\boldsymbol{g}$. This concludes the proof.

The final paragraph is a simple induction on the previous argument. □

If Conj. 1.6 holds, then the 3rd paragraph hypotheses of Princ. 4.24 can't hold.

***Remark 4.25* (Extend Princ. 4.24).** — [**Fri06b**] has a stronger version of the 2nd paragraph of Princ. 4.24: If $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ is a rep. for an o-p' cusp with any two of $s_{R_{H_{2,3}},p}({}_0\boldsymbol{g}')$, $s_{R_{H_{1,4}},p}({}_0\boldsymbol{g}'')$ and $s_{R_G,p}(\boldsymbol{g})$ equal 1, then the third is also 1.

4.6. Evidence for and consequences of no Weigel cusp branches.

— This subsection considers both evidence for and challenges to Conj. 1.6.

***Definition 4.26* (Weigel branch).** — If ${}_k\boldsymbol{g}$ satisfies the hypotheses of Princ. 4.24, 2nd paragraph, then we call $({}_k\boldsymbol{g})\mathrm{Cu}_4$ a *level k Weigel cusp*. A cusp branch which for large k consists of Weigel cusps is a *Weigel branch*.

We also refer to the component branch in $\mathcal{T}_{G,\mathbf{C},p}$ defined by a Weigel cusp branch as a Weigel component branch.

4.6.1. Example disappearances of o-p' *cusps.* — For $g \in A_n$ of odd order, let $w(g)$ be the sum of $(l^2 - 1)/8 \mod 2$ over all disjoint cycle lengths l in g ($l \not\equiv \pm 1 \mod 8$ contribute). [**Fri06a**, Cor. 2.3] has a short proof of Prop. 4.27 based on when $\mathbf{C} = \mathbf{C}_{3^r}$ is r repetitions of the 3-cycle class (guiding the original statement in [**Ser90**]).

Proposition 4.27. — *Suppose* $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ *with* $G \le A_n$ *transitive, and* $\mathbf{C}$ *consists of conjugacy classes in* G *with elements of respective odd orders* $d_1, \dots, d_r$*. Assume also* *the genus of a degree n cover $\varphi : X \to \mathbb{P}^1_z$ with branch cycles $\boldsymbol{g}$ from this embedding has genus 0. Then, $s_{\mathrm{Spin}_n}(\boldsymbol{g}) = (-1)^{\sum_{i=1}^r w(g_i)}$.*

At level 0 of the $(A_5, \mathbf{C}_{3^4})$ **MT** ($p = 2$), no cusps are 2 cusps: Widths are 1,1, 3,3, 5, 5 ([**BF02**, §2.9.3]; shifts of the cusps of width 1 are H-M reps.). By level 1, all o-2′ cusps disappear, leaving only g-2′ cusps (shifts of H-M reps.) as non-2 cusps [BFr02; §9.1]. Combine this with the comment before Prop. 3.12 for the following.

Proposition 4.28. — *The only infinite cusp branches on the $\mathcal{C}_{A_5,\mathbf{C}_{3^4},p=2}$ cusp tree are g-p' and p cusp branches.*

Problem 4.29. — Are there component branches on $\mathcal{T}_{A_5,\mathbf{C}_{3^4},p=2}$ that contain only p cusp branches?

4.6.2. Some Weigel cusps and challenges to Conj. 1.6. — We give an example Weigel cusp in a Nielsen class containing no g-p' cusps. Use notation from Ex. 3.13 and the representative for the Nielsen class $\mathrm{Ni}(A_5, \mathbf{C}_{\pm 53})$ given by $\boldsymbol{g} = (g_1, g_2, g_3)$ with $g_1 = (5\,4\,3\,2\,1)$ and $g_2 = (2\,4\,3\,5\,1)$, and $g_3 = (4\,3\,5)$.

There are two conjugacy classes of 5-cycles in A_5: C_{+5} and C_{-5}. Further, if $g \in \mathrm{C}_{+5}$, then so is g^{-1}. Let $\mathbf{C}_{\pm 53}$ denote the collection of conjugacy classes consisting of C_{+5}, C_{-5} and C_3 (class of a 3-cycle). [**BF02**, Princ. 5.15] shows $\mathrm{Ni}(A_5, \mathbf{C}_{\pm 53})$ (absolute or inner) has one braid orbit with lifting invariant +1. By Riemann-Hurwitz, the genus g of absolute covers (degree 5 over $\mathbb{P}^1_z$) in this Nielsen class is 1, from $2(5 + g - 1) = 10$. So Prop. 4.27 doesn't apply directly. Rather, [**BF02**, §5.5.2] shows how to compute beyond the genus 0 limitation. Now, take $p = 2$.

This Nielsen class clearly contains no g-2′ rep. Further, similar examples work for any $r \geq 3$ conjugacy classes. For $r \geq 5$: juxtapose $\boldsymbol{g} \in \mathrm{Ni}(A_5, \mathbf{C}_{\pm 53})$ with (g, g^{-1}) or (g, g, g) ($g \in \mathrm{C}_3$) appropriately. For $r = 4$, replace $\mathbf{C}_{\pm 53}$ by $\mathbf{C}_{\pm 53^2}$. Call the shift (resp. conjugacy classes) of one of these reps. $\boldsymbol{g}'$ (resp. $\mathbf{C}'$).

Result 4.30. — *For $\mathbf{C}' = \mathbf{C}_{\pm 53^2}$, the natural map $\mathrm{Ni}(G_1(A_5), \mathbf{C}') \to \mathrm{Ni}(A_5, \mathbf{C}')$ is onto: no level 0 braid orbit is obstructed. The cusp represented by*

$$\boldsymbol{g}'' = ((3\,4\,5), (5\,4\,3\,2\,1), (2\,4\,3\,5\,1), (3\,4\,5))$$

has an o-p' cusp in $\mathrm{Ni}(G_1(A_5), \mathbf{C}')$ over it. So, $\boldsymbol{g}''$ is a Weigel cusp.

Comments. — With $R \to A_5$ the Spin_5 cover of A_5, $s_R(\boldsymbol{g}'') = s_R((\boldsymbol{g}'')\mathbf{sh}) = 1$ as we explained above. The only appearance of $\mathbf{1}_{A_5}$ in $M_0 = \ker(G_1(A_5) \to A_5)$ is from $\ker(R \to A_5)$ ([**BF02**, Cor. 5.7] or [**Fri95**, Part II])). So, the hypotheses of Princ. 4.24, 2nd paragraph, with $k = 0$ apply; and the conclusion does also. □

If Conj. 1.6 holds for $\mathrm{Ni}(G_1(A_5), \mathbf{C}' = \mathbf{C}_{\pm 53^2})$ in Res. 4.30, then the conclusion to Prob. 4.31 is affirmative.

Problem 4.31. — Are all o-p' cusps gone at high levels of the $\mathrm{Ni}(G_1(A_5), \mathbf{C}_{\pm 53^2})$ $\mathbf{MT}$? Is it even possible this $\mathbf{MT}$ is empty at high levels (agreeing with nonexistence of infinite component branches having only p cusp branches as in §1.2.2)?

***Example 4.32* ($\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})$ with $p = 2$, see §6.3).** — There is an o-p' cusp:

$$\boldsymbol{g} = ((1\,2\,4), (1\,2\,3), (1\,3\,4), (1\,2\,4)).$$

Apply the proof of Res. 4.30 here. A direct application of Prop. 4.27 —since the genus 0 hypotheses holds —shows $\beta_{2,3}(s({}_0\boldsymbol{g}')) = -1$ while $\beta_{1,4}(s({}_0\boldsymbol{g}'')) = +1$ (in analogous notation). So, the 2nd paragraph Prop. 4.24 conclusion is that the left side of (4.9) is -1, and $\boldsymbol{g}$ is not in the image from $\mathrm{Ni}(\mathrm{Spin}_4, \mathbf{C}_{\pm 3^2})$.

5. Nub of the (weak) Main Conjecture

Use notation, especially for genera, around (3.2). Assume $B' = \{\mathcal{H}'_k\}_{k=0}^{\infty}$ is an infinite branch of $\mathcal{T}_{G,\mathbf{C},p}$ defined over a number field K. From Prop. 3.3, to consider the Main Conj. we may assume $G = G_0$ has the p-part of its center trivial. We make that assumption throughout this section. This lets us use the 2nd part of Princ. 3.5.

We show the Main Conj. 1.2 (for $r = 4$) holds unless we are in one of three cases. These we stipulate by listing how $\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k$ ramifies when $k >> 0$:

– either it doesn't ramify over cusps;

– it is equivalent to a degree p polynomial;

– or it is equivalent to a degree p rational function branched only at two points.

5.1. There should be no $\mathcal{T}_{G,\mathbf{C},p}$ genus 0 or 1 branches.

— We must consider two possibilities that would contradict the Main Conjecture:

(5.1a) $g_{\bar{\mathcal{H}}'_k} = 0$ for all $0 \le k < \infty$ (B' has genus 0; $\mathrm{Ge}_{B'}$ consists of 0's); or

(5.1b) For k large, $g_{\bar{\mathcal{H}}'_k} = 1$ (B' has genus 1; almost all of $\mathrm{Ge}_{B'}$ is 1's).

5.1.1. Reduction of the Main Conj. to explicit cases. — An elementary corollary of Riemann-Hurwitz says for $k >> 0$, (5.1b) implies $\bar{\mathcal{H}}'_{k+1} \to \bar{\mathcal{H}}'_k$ doesn't ramify. From Princ. 3.5 this says:

(5.2) For no value of k does $\bar{\mathcal{H}}'_k$ have a p cusp.

Now assume, contrary to (5.2), $\boldsymbol{p}'_k \in \mathcal{H}'_k$ is a p cusp for some k. Denote the degree of $\mathcal{H}'_{k+1}/\mathcal{H}'_k$ by ν_k and the number of primes $\boldsymbol{p}'_{k+1} \in \mathcal{H}'_{k+1}$ over $\boldsymbol{p}'_k$ by u_k. Thm. 5.1 says possibilities for (5.1a) that [**Fri06b**] must eliminate are these. For $k >> 0$, $\nu_k = p$, $u_k = 1$ and $\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k$ is equivalent (as a cover over K) to either:

(5.3a) a degree p *polynomial map*; or

(5.3b) a degree p rational function ramified precisely over two K conjugate points.

Theorem 5.1. — *If neither* (5.2) *nor* (5.3) *hold for the component branch B', then B' satisfies the conclusion of Main Conj. 1.2: High levels of B' have no K points.*

For B' with (3.6b) *holding (full elliptic ramification; including when B' has fine reduced moduli —§3.2.2) for $k >> 0$, the Main Conj. holds unless* (5.3b) *holds.*

Proof. — Assume (5.2) doesn't hold and $g'_k = 0$ for large k. That is,

(5.4) $\quad 2(\deg(\bar{\mathcal{H}}'_k/\mathbb{P}^1_j) - 1) = \text{ind}(\gamma'_{0,k}) + \text{ind}(\gamma'_{1,k}) + \text{ind}(\gamma'_{\infty,k}) :$ (5.1a) holds.

Consider now what would allow g'_{k+u}, $u \geq 0$ to also be 0.

Denote the cardinality of the p cusps on $\mathcal{H}'_k$ by t_k. For each p cusp, $\boldsymbol{p}'_k \in \mathcal{H}'_k$, Princ. 3.5 says the following.

(5.5a) Each $\boldsymbol{p}'_{k+1}$ over $\boldsymbol{p}'_k$ ramifies with index p and $\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k$ has degree $\nu_k = p \cdot u_k$.

(5.5b) Also, $t_{k+1} \geq t_k \cdot u_k$.

Apply (5.5a), by replacing k by $k+1$, to any $\boldsymbol{p}'_{k+2} \in \mathcal{H}'_{k+2}$ over a $\boldsymbol{p}'_{k+1}$. Conclude:

(5.6a) there is an index contribution of $t_k \cdot u_k \cdot u_{k+1} \cdot (p-1)$ from all $\boldsymbol{p}'_{k+2}$s to Riemann-Hurwitz from $\bar{\mathcal{H}}'_{k+2}$ to $\bar{\mathcal{H}}'_{k+1}$; and

(5.6b) Riemann-Hurwitz applied to $\bar{\mathcal{H}}'_{k+2} \to \bar{\mathcal{H}}'_{k+1}$ contradicts (5.4) if

$$t_k \cdot u_k \cdot u_{k+1} \cdot (p-1) > 2(p \cdot u_{k+1} - 1).$$

Suppose $t_k \geq 2$. Then, we contradict (5.4) if $(u_k - 1) \cdot p \geq u_k$. This happens unless $u_k = 1$ or $u_k = 2 = p$. In the latter case, with $t_k = 2$, we would have $t_{k+1} = 4$ from (5.5b). Then, putting $p = 2$ you see a contradiction by shifting k to $k+1$. So, the argument forces (with $t_k \geq 2$) $u_k = 1$, $t_k = 2$, and no ramification outside these two cusps. Further, under these assumptions (and (5.1a)), (3.6b) must hold for $k >> 0$.

On the other hand, if $t_k = 1$ for $k >> 0$, then (with (5.1a)), (5.5b) forces $u_k = 1$. That means $\mathcal{H}'_{k+1}/\mathcal{H}_k$ is a cover of genus 0 curves of degree p with one place totally ramified. This is equivalent to a cover represented by a polynomial (see Prop. 5.4). □

***Result* 5.2**. — *A branch B' of $\mathcal{T}_{G,\mathbf{C},p}$ contradicts case* (5.1a) *if there is a p cusp at level k and $\bar{\mathcal{H}}'_{k+u+1}/\bar{\mathcal{H}}'_{k+u}$ has degree $\geq p+1$. For B' to contradict* (5.1b)*, we only need one p cusp at a high level k: Princ. 3.5 forces $\mathcal{H}'_{k+1}/\mathcal{H}'_k$ to ramify.*

5.1.2. Why (5.2) *or* (5.3) *would contradict Conj. 1.2.* — Prop. 5.4 shows the exceptional cases in §5.1.1 are serious.

***Lemma* 5.3**. — *For any projective genus 1 curve X over a number field K, we can extend K to assume $X(K)$ is an elliptic curve with infinitely many points.*

Proof. — Extend K to assume $X(K) \neq \varnothing$, and use one of those points as an origin to assume X is an elliptic curve. Now form $\mu_K : G_K \to \text{GL}_2(\hat{\mathbb{Z}})$, the action of G_K on all division points of X. Put X in Weierstrass normal form, so its affine version has the shape

$$\{(x,y) \mid y^2 = x^3 - u_2 x - u_3\}.$$

Next we show $X(L)$ cannot be finite for each number field L/K. Suppose it is.

First we show μ_K is an embedding of G_K. Suppose not. Let $\sigma \in G_K$ with $\mu_K(\sigma) = 1$, but assuming $\sigma \neq 1$, there is a finite extension L/K on which σ acts nontrivially. Take a primitive generator x_0 for L/K (that is, $L = K(x_0)$). Solve for y_0 so that $(x_0, y_0) \in X(L')$, with $L' = K(x_0, y_0)$. By assumption (x_0, y_0) gives a division point on X, and clearly σ acts nontrivially on it.

That gives that μ_K is an embedding. Yet, a simple consequence of Hilbert's irreducibility Theorem is that there is a Galois L/K with group S_n for any large integer n. It is an elementary group observation that S_n for $n > 5$ large cannot embed in $\mathrm{GL}_2(\mathbb{Z}/N)$ for any value of N. This contradiction finishes the proof. □

Proposition 5.4. — *A* **MT** *for which either* (5.2) *or* (5.3) *holds fails the conclusion of Conj. 1.2.*

Proof. — Recall: We start with a component branch B' having definition field a number field K'. If B' satisfies (5.2), then Lem. 5.3 gives $k = k_0$, and K with $[K : K'] < \infty$ and $|\bar{\mathcal{H}}'_{k_0}(K)| = \infty$. Now we have a tower of elliptic curves, all isogenous. Each therefore has infinitely many rational points. Only finitely many of these can be cusps, and the rest will be rational points on $\mathcal{H}'_k$, for each k. That shows, if (5.2) holds, we do contradict Conj. 1.2.

Now consider (5.3a): $\bar{\mathcal{H}}'_{k+1} \to \bar{\mathcal{H}}'_k$ is a degree p cover (over K) of genus 0 curves with a distinguished totally ramified point $\boldsymbol{p}'_k \in \bar{\mathcal{H}}'_k$. Then, both $\boldsymbol{p}'_k$ and the unique point $\boldsymbol{p}'_{k+1}$ over it are K points. So, again $\mathcal{H}'_k(K)$ is infinite and if (5.3a) holds, then we contradict Conj. 1.2.

Finally, consider (5.3b). Suppose $X \to Y$ is a K map of genus 0 curves of degree N. Then, they both define elements of order 2 in the Brauer-Severi group $H^2(G_K, \bar{K}^*)$. Denote these $[X]$ and $[Y]$. Then, $N \cdot [X] = [Y]$ (in additive notation – see the argument of [**BF02**, Lem. 4.11] for example). In particular, if $N = p$ is odd, and K is large enough that X has a rational point, then $[X] = [Y] = 0$ and both have infinitely many rational points. The case for $N = 2$ is even easier for it is automatic that $2 \cdot [\bar{\mathcal{H}}'_{k+1}] = 0$ $(= [\bar{\mathcal{H}}'_k])$. For this case we immediately have a tower of degree 2 maps between $\mathbb{P}^1$ s. So, finishing (5.3b) reverts to the previous case. □

5.2. What we need to complete the Conj. 2.2 proof.

— The results of §5.1 show the main point in finishing the Main Conjecture for $r = 4$ is a p cusp at some high level. Better yet, if the lim sup of $\deg(\bar{\mathcal{H}}'_{k+u+1}/\bar{\mathcal{H}}'_{k+u})$ is *not* p, one such p cusp guarantees the p cusp count (at level k) is unbounded as $k \mapsto \infty$. Prop. 5.5 gives examples that show how to compute a (growing) lower bound to the p-cusp count with the levels.

5.2.1. Reducing to pure cusp branches. — §1.2.1 calls an infinite cusp branch B pure in cases (1.5a) and (1.5c) if these have no extraneous (finite) start strings of g-p' (possibly followed by a string of o-p') cusps. Continue that notation to define B by a sequence of cusp sets $({}_k\boldsymbol{g})\mathrm{Cu}_4 \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}$. We can assume k is large. That allows starting at any desired level. So we revert to where B is one of the pure infinite cusp branches B in $\mathcal{C}_{G,\mathbf{C},p}$ with representatives

$$\{{}_k\boldsymbol{g} = ({}_kg_1, \dots, {}_kg_4) \in \mathrm{Ni}'_k\}_{k=0}^{\infty}.$$

Here Ni'_k is the braid ($\bar{M}_4$ —§2.4.1) orbit on $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in,rd}}$ of ${}_k\boldsymbol{g}$. For all $k \ge 0$, either:

(5.7a) From Princ. 3.5, $p|({}_k\boldsymbol{g})\mathbf{mp}$; or

(5.7b) From Princ. 3.6, ${}_k\boldsymbol{g}$ is a g-p' rep.; or

(5.7c) From Princ. 4.24 (or Rem. 4.25), ${}_k\boldsymbol{g}$ is a Weigel cusp with

$$s_{R_{H_{2,3}}}({}_k\boldsymbol{g}) = 1 = s_{R_{H_{1,4}}}({}_k\boldsymbol{g}).$$

5.2.2. Using a g-p' cusp branch to get p cusps. — §6.2.3 [**BF02**, §9] does many cases of (5.7b), where $p = 2$ and there is a g-p' cusp that is the shift of an H-M rep. Here is what we learned, by example, about getting p cusps from it. Our example continues §4.6.1: the $(A_5, \mathbf{C}_{3^4}, p = 2)$ **MT** where level 0 had no 2 cusps.

Prop. 4.27 applies with the $\mathrm{Spin}_5 \to A_5$ cover to show both level 1 components have p cusps (with $p = 2$) [**BF02**, Cor. 8.3]. The full analysis says the component, $\mathcal{H}_+(G_1(A_5), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$, containing all the H-M cusps, has genus 12 and degree 16 over the unique component of $\mathcal{H}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$. It also has all the real (and so all the $\mathbb{Q}$) points at level 1 [**BF02**, §8.6]. Further, all except the shift of the H-M cusps are 2 cusps. The other component, $\mathcal{H}_-(G_1(A_5), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ is obstructed, so no full branch over it has ${}_2\tilde{G}(A_5)$ (the whole 2-Frattini cover of A_5) as a limit group.

Proposition 5.5. — *The number of p cusps at level k in any H-M component branch over $\mathcal{H}_+(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ is unbounded in k.*

Proof. — The argument has this abstract idea. Let $B = \{\boldsymbol{p}_k\}_{k=0}^{\infty}$ be a g-p' cusp branch. Suppose for $k \ge k_0$ you can braid $\boldsymbol{p}_k$ to a p cusp $\boldsymbol{p}'_k$ with ramification index exactly divisible by p. Then, Princ. 3.5 allows, with $k = k_0 + u$, inductively braiding $\boldsymbol{p}_k$ to a sequence of cusps $\boldsymbol{p}'_k(1), \dots, \boldsymbol{p}'_k(u)$ with $\boldsymbol{p}'_k(t)$ having ramification index exactly divisible by p^t, $u = 1, \dots, t$. From their ramification indices over $j = \infty$, these give u different p cusps at level $k_0 + u$.

For $\mathrm{Ni}(G_k(A_5), \mathbf{C}_{3^4})$ you can take $k_0 = 1$ and $\boldsymbol{p}'_k$ is produced as the *near* H-M rep. associated to $\boldsymbol{p}_k$ [**BF02**, Prop. 6.8]. □

5.2.3. Limit groups and field of moduli examples. — These examples show our progress in computing, and that the consequences are relevant to the abstract results.

Problem 5.6. — What are the limit groups of full component branches (§4.1) over $\mathcal{H}_-(G_1(A_5), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$?

***Example 5.7* (Continuing Prob. 5.6)**. — By contrast to examples in §A.2 and §B.1, we don't yet know the limit groups for $\mathcal{H}_-(G_1(A_5), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$. Example: Each space $\mathcal{H}(A_5, \mathbf{C}_{3^r})^{\mathrm{in,rd}}$, $r \ge 5$, has exactly two components $\mathcal{H}_\pm(A_5, \mathbf{C}_{3^r})$ [**Fri06a**, Thm. 1.3].

Also, $\mathcal{H}_+(A_5, \mathbf{C}_{3^r})$ is a g-2$'$ component. So, from Princ. 3.6 it has ${}_2\tilde{G}(A_5)$ as a limit group.

Further, $\mathcal{H}_-(A_5, \mathbf{C}_{3^r})$ has a unique limit group, just A_5. This is because the 1st Loewy layer (§A.2.1) of $M_0(A_5)$ consists of just the Schur multiplier $\mathbb{Z}/2$ of A_5 [**BF02**, Cor. 5.7]. We know the Schur multiplier of $G_1(A_5)$ is just $\mathbb{Z}/2$. Still, what if other A_5 irreducible modules appear in the first Loewy layer of the characteristic module M_1? Then, akin to Ex. B.3, the braid orbit corresponding to $\mathcal{H}_-(G_1(A_5), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ could have all limit groups larger than $G_1(A_5)$.

Problem 5.8. — [**Fri06a**, Thm. 1.3] says $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{in,rd}}$, $r \geq n$, always has exactly two components, which we can denote $\mathcal{H}_{n,r,\pm}$. When $p = 2$, $\mathcal{H}_{n,r,+}$ always has ${}_2\tilde{G}(A_n)$ as one limit group. Further, the limit groups of $\mathcal{H}_{n,r,-}$ never include ${}_2\tilde{G}(A_n)$. Still, as in Ex. 5.7, for which (n, r) is A_n a limit group? From [**FK97**, Obst. Lem. 3.2] (as in Lem. 4.9), the result only depends on n: Whether there is another irreducible in the 1st Loewy layer of $M_0(A_n)$. [**FK97**, Rem. 2.5] (based on [**Ben83**]) shows there is a Frattini cover of A_8 that doesn't factor through Spin_8. So, A_8 is never a limit group of $\mathcal{H}_{8,r,-}$. We know little about this for $n \notin \{4, 5, 8, 9\}$.

Our next example shows how significant are the cusps $\boldsymbol{p}'_k$ in the braid from $\boldsymbol{p}_k$ to $\boldsymbol{p}'_k$ in the proof of Prop. 5.5. The topic shows how one **MT** produces an infinite number of closely related situations contrasting the field of moduli and the field of definition of covers corresponding to points on tower levels.

***Example 5.9* (Moduli field versus definition field).** — Recall the cusps $\boldsymbol{p}'_k$ achieved from braiding from H-M cusps in the proof of Prop. 5.5. These and the H-M cusps are are the only real (coordinates in $\mathbb{R}$) cusps on the $(A_5, \mathbf{C}_{3^4}, p = 2)$ **MT** at level $k > 0$. Let $R_k \to G_k(A_5)$ be the representation cover antecedent (§4.2.2) to the Schur multiplier of A_5.

Regard the branch as defined over $\mathbb{R}$. Then, $\mathbb{R}$ points over any $1 < j < \infty$ in the real component abutting to $\boldsymbol{p}_k$ represent covers in $\mathrm{Ni}(R_k, \mathbf{C}_{3^4})$ whose field of definition is $\mathbb{R}$ equal to its field of moduli. By contrast, with similar words concluding "real component abutting to $\boldsymbol{p}'_k$" (not $\boldsymbol{p}_k$) here the moduli field is $\mathbb{R}$, but it is not a definition field [**BF02**, Prop. 6.8].

5.3. Chances for a genera formula. — Ques. 3.2 asks if a g-p' cusp branch represented by $B = \{{}_k\boldsymbol{g} \in \mathrm{Ni}'_k\}_{k=0}^{\infty}$ (notation like that of Princ. 3.5) can deliver an analytic expression for genera akin to that for a modular curve tower. Further, Prop. 5.5 supports why we expect to be able to braid from a g-p' cusp at level 0, in numbers increasing with k, a collection of p cusps resembling those on modular curve towers (as in [**Fri05a**, Talk 1]). §5.3.1 lists the challenges for this. §5.3.2 suggests simplifying to a, still valuable, abelianized version.

5.3.1. Challenging a genera formula. — Our examples show Ques. 3.2 is difficult.

(5.8a) Are there o-p' cusps in the orbit of ${}_k\boldsymbol{g}$?

(5.8b) For $k >> 0$ are there any p-cusps in the orbit of ${}_0\boldsymbol{g}$. If so, given how many there are at level 0; how many will there be at level k?

(5.8c) Can we separate the braid orbit of ${}_k\boldsymbol{g}$ from other braid orbits?

We comment on these challenges. Example of (5.8a): Prop. 3.12 gives a **MT** with related pairs of g-p' and o-p' cusps, represented respectively by ${}_k\boldsymbol{g}$ and ${}_k\boldsymbol{g}'$, at every level. Can you braid between ${}_k\boldsymbol{g}$ and ${}_k\boldsymbol{g}'$?

Here is an immediate case wherein we must distinguish between (5.8a) and (5.8b). If ${}_o\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$ is an H-M rep., there are two possibilities since $\langle g_1, g_2\rangle = G_0$: Either this is a p cusp or it is an o-p' cusp. For the latter, we guess at high levels that either the only cusps above it are p cusps. Princ. 4.24 presents this possibility (contrary to Conj. 1.6):

(5.9) There is an infinite branch on the **MT**, $(G_0, \mathbf{C}', p)$ with $\mathbf{C}'$ the conjugacy classes of g_1, g_2 and $g_1g_2^{-1}$.

Having such a branch is equivalent to having the homomorphism $\psi' : M_{\boldsymbol{g}'} \to G_0$ defined by $\boldsymbol{g}' = (g_1, g_2, (g_1g_2)^{-1})$ extending to $\tilde{\psi}' : M_{\boldsymbol{g}'} \to {}_p\tilde{G}$. [**Fri06b**] notes a necessary condition from the genus of the 3 branch point cover $X \to \mathbb{P}^1_z$ representing ψ'. It must exceed the rank of $\ker({}_p\tilde{G} \to G_0)$. Apply (5.9) to the example of §4.6.2, with $\mathrm{Ni}(A_5, \mathbf{C}_{\pm 53})$. The genus g of the corresponding X satisfies

$$2(60 + g - 1) = 2(60/5) \cdot 4 + (60/3) \cdot 2,$$

so $g = 9$, while the rank of $\ker({}_p\tilde{G} \to G_0)$ is 4.

Example of (5.8c): Thm. 4.12 gives examples with at least two components — one H-M — at each higher level of a **MT**. The cases we give *replicate* (in the sense of antecedent Schur multipliers) a two (or more) component situation at level 1. This regularity of behavior is what we *expect* with g-p' cusps. Yet, is it always like this?

5.3.2. Shimura-like levels and abelianized genera. — A level k **MT** component, $\mathcal{H}'_k$, has above it a tower one may compare with Shimura varieties. That goes like this.

Let $\ker_k = \ker({}_p\tilde{G} \to G_k)$ (§1.1.2). The sequence of spaces comes from forming ${}_p\tilde{G}/(\ker_k, \ker_k) = {}_{p,k}\tilde{G}$. This gives a p-Frattini extension of G_k by the abelian group $\ker_k/(\ker_k, \ker_k) = L_k$, as in the proof of Lem. 4.11. The lift of $g \in G_k$ to $\tilde{g} \in {}_{p,k}\tilde{G}$ gives an action of g on L_k by the conjugation by $\tilde{g}$.

Form the spaces $\{\mathcal{H}_{k,u}\}_{u\geq 0}^{\infty}$ corresponding to the Nielsen classes $\mathrm{Ni}({}_{p,k}\tilde{G}/p^uL_k, \mathbf{C})$, and denote by $\{\mathcal{H}'_{k,u}\}_{u\geq 0}^{\infty}$ those (abelianized) components over $\mathcal{H}'_k = \mathcal{H}'_{k,0}$.

Let $R'_k \to G_k$ (resp. $R_k \to G_k$) be maximal among central, p-Frattini (resp. exponent p Frattini) extensions of G_k. Then, $\ker(R'_k \to G_k)$ (resp. $\ker(R_k \to G_k)$) is the maximal p quotient (resp. exponent p) of G_k s Schur multiplier. Cor. 4.19 checks for an infinite branch above a given component by inductively checking Nielsen elements ${}_k\boldsymbol{g}$ for $s_{R_k/G_k}({}_k\boldsymbol{g}) = 0$ at successive levels for *all* k. §4.4.3 has examples that require successive checks. Finding, however, if there is a projective sequence of abelianized components requires only one lifting invariant check.

Theorem 5.10. — *For $u >> 0$, $\mathcal{H}'_{k,u}$ is nonempty if and only if $s_{R'_k/G_k}(O') = 0$ (just one test).*

Ques. 3.2 has this easier, yet very valuable, variant.

***Problem 5.11* (Abelianized Tower Genera).** — Label the precise ingredients needed to compute genera of the $\{\mathcal{H}'_{k,u}\}_{u\ge 0}^{\infty}$ components.

6. Strong Conjecture for $r = 4$

Our strong Main Conjecture 6.1 is an expectation that the best **MT**s are akin to those of modular curves. §6.1 shows how the **MT** cusp language applies to modular curves. §6.2 strengthens that, noting cusp branches defined by g-p' cusps and p cusps generalize projective sequences of modular curve cusps. Finally, §6.3 starts a discussion (continued in the appendix) on a non-modular curve **MT** whose low levels have genus 0 and 1 components with worthy applications.

6.1. Initial comparison of MTs with modular curves. — Let $D_{p^{k+1}}$ be the dihedral group of order $2\cdot p^{k+1}$ with p odd.

6.1.1. The strong Main Conjecture. — [**Fri05a**, Lect. 1] computes the genera of the modular curves $X_0(p^{k+1})$ and $X_1(p^{k+1})$ as **MT** levels. Example: $X_1(p^{k+1})$, defined by $\mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\mathrm{in,rd}}$ with C_2 the involution class, has these properties.

(6.1a) There is one $\bar{M}_4$ orbit.
(6.1b) We inductively compute all cusps at level k using an H-M rep. (width p^{k+1}), and the shift of H-M rep. cusps are g-p' cusps of width 1.
(6.1c) γ'_0 or γ'_1 have no fixed points.
(6.1d) $\mathcal{Q}''$ (§2.4.1) acts trivially at all levels.

[**FS06**, Prop. 8.4] generalizes (6.1c) and (6.1d). This is the **MT** version of Serre's abelian variety lemma: (roughly) among automorphisms, only the identity fixes many torsion points. Use the notation of §4.1.3 for a **MT** of rank $u \ge 0$. Again, assume $r = 4$ for these **MT**s.

***Conjecture 6.1* (Strong Main Conjecture).** — $P_{\mathbf{C}}$ Version: Over all $p \notin P_{\mathbf{C}}$, for only finitely many $V \in \mathcal{V}_p(J)$, does $\mathcal{H}(V \times^s J, \mathbf{C})^{\mathrm{in,rd}}$ have genus 0 or 1 components.

There is a $P'_{\mathbf{C}}$ version, though the weak Conjecture and Conj. 6.1 imply it.

***Conjecture 6.2* (Mazur-Merel Version of the strong Main Conjecture)**

With hypotheses of Conj. 6.1, over all $p \notin P_{\mathbf{C}}$, for only finitely many $V \in \mathcal{V}_p(J)$, does $\mathcal{H}(V \times^s J, \mathbf{C})^{\mathrm{in,rd}}$ have a rational point.

6.1.2. Comparison with the Strong Torsion Conjecture. — The following observation generalizing [**BF02**, Thm. 6.1] appears in [**Cad05b**, Prop. 4.10]. For the **MT** of $(G, \mathbf{C}, p)$, the (weak) Main Conjecture *for all values of* r follows from Conj. 6.3, called by [**Sil92**] and [**KW98**] the Strong Torsion Conjecture. Let K be a number field.

Conjecture 6.3 **(STC)**. — For $g, d \geq 1$, there exists $n(g, d) \geq 1$ with this property. If $n \geq n(d, g)$, then there are no dimension g abelian varieties A defined over K, with $[K : Q] \leq d$, and having a K torsion point of order n.

By contrast, the weak conjecture for the **MT** given by $(D_p, \mathbf{C}_{2^r}, p)$ (necessarily for a nonempty **MT**, $r = 2g + 2 \geq 4$ is even) is *equivalent* to the following.

Conjecture 6.4. — For k large there is no cyclic group $C \cong \mathbb{Z}/p^{k+1}$ of torsion on a hyperelliptic Jacobian of genus g for which G_K acts on C through its cyclotomic action on $\langle e^{2\pi i/p^{k+1}} \rangle$ [**DF94**, §5.2].

Further, the Strong Main Conjecture for a higher rank **MT** doesn't follow from Conj. 6.3 because the genus of the curves (and so the dimension of the Jacobians) in question grows with primes p.

6.2. Modular curve comparison for Serre's OIT.

— Principles 3.6 and 4.24 help toward describing all branches in $\mathcal{C}_{G,\mathbf{C},p}$. This guides the strong Conjecture in how it might effectively generalize Serre's Open Image Theorem (OIT) [**Ser98**].

6.2.1. Frattini properties in the OIT. — Here are significant OIT ingredients.

(6.2a) Acting by $G_{\mathbb{Q}_p}$ on projective systems of points in neighborhoods of H-M reps. on $\{X_1(p^{k+1})\}_{k=0}^{\infty}$ gives a transvection in the projective sequence of monodromy inertia groups.

(6.2b) The geometric monodromy group, $\mathrm{PSL}_2(\mathbb{Z}/p^{k+1})$, for $X_1(p^{k+1}) \to \mathbb{P}^1_j$ is a p-Frattini cover of the monodromy at level 0 if $p \neq 2$ or 3.

Here is how (6.2b) works (p is odd). Let $\{\boldsymbol{p}_k \in X_0(p^{k+1})\}_{k=0}^{\infty}$ be a projective sequence of points over $j' \in F$. Then G_F acts on these to give a map

$$G_F \xrightarrow{\psi_{2,j'}} \lim_{\infty \leftarrow k} \mathrm{GL}_2(\mathbb{Z}/p^{k+1})/\{\pm I_2\} = GL_2(\mathbb{Z}_p)/\{\pm I_2\} \xrightarrow{\mathrm{Det}} \mathrm{GL}_1(\mathbb{Z}_p).$$

The induced map $\psi_{1,j'} : G_F \to \mathrm{GL}_1(\mathbb{Z}_p)$ is onto an open subgroup because (essentially) all the roots of 1 are present in the field generated by the division points on elliptic curves. This deduction interprets from the Weil pairing on elliptic curves. This is an alternating pairing on p^{k+1} division points into p^{k+1}th roots of one — interpreted as the cup product pairing from 1st (ℓ-adic, but $\ell = p$) cohomology to the 2nd ℓ-adic cohomology. Rem. A.2 states the **MT** version of this.

Let G_F^0 be the kernel of $\psi_{1,j'}$. Consider the restriction $\psi_{2,j'}^0 : G_F^0 \to \mathrm{PSL}_2(\mathbb{Z}_p)$, and composite by going mod p to get $\psi_{2,j'}^0 \bmod p : G_F^0 \to \mathrm{PSL}_2(\mathbb{Z}/p)$.

Result 6.5. — *For $p \neq 2$ or 3, if $\psi^0_{2,j'} \bmod p$ is onto, then $\psi^0_{2,j'} : G^0_F \to \mathrm{PSL}_2(\mathbb{Z}_p)$ is onto. If $p = 3$ (also for $p = 2$), and $\psi^0_{2,j'} \bmod p^2$ is onto, then so is $\psi^0_{2,j'}$.*

Comments. — First: The mod p map $\mathrm{PSL}_2(\mathbb{Z}_p) \to \mathrm{PSL}_2(\mathbb{Z}/p)$ is a Frattini cover if $p \neq 2$ or 3 [**Ser98**, IV-23 Lem. 2]. It isn't, however, the universal p-Frattini cover of $\mathrm{PSL}_2(\mathbb{Z}/p)$, ever! For example, consider the case $p = 5$: $\mathrm{PSL}_2(\mathbb{Z}/p) = A_5$. Then, $M_0 = \ker(G_{5,1}(A_5) \to A_5)$ (notation of §1.1.2) is a rank 6, A_5 module. It fits in a nonsplit short exact sequence $0 \to M' \to M_0 \to M' \to 0$ with M' the adjoint representation of $\mathrm{PSL}_2(\mathbb{Z}/5)$ (on 2×2 trace 0 matrices [**Fri95**, Rem. 2.10]).

For $p = 3$, $\mathrm{PSL}_2(\mathbb{Z}/3)$ is not simple. Yet, $\mathrm{PSL}_2(\mathbb{Z}_3) \to \mathrm{PSL}_2(\mathbb{Z}/3^2)$ is Frattini.

[**Wei04**, Thm. C] computes the rank of $\ker(G_{p,1}(\mathrm{PSL}_2(\mathbb{F}_q)) \to \mathrm{PSL}_2(\mathbb{F}_q))$ when $\mathbb{F}_q$ is the finite field of order $q = p^u$. The adjoint representation appears a lot. This also computes this characteristic rank for the other primes dividing $|\mathrm{PSL}_2(\mathbb{F}_q)|$, giving important empirical data for effective computation of Frattini ranks.

Let R_q be the Witt vectors for $\mathbb{F}_q$. [**Völ95**, §4] notes that $\mathrm{GL}_n(R_q) \to \mathrm{GL}_n(\mathbb{F}_q)$ is a Frattini cover so long as $p > 2$ does not divide n, and if $p = 3$, $n \geq 4$. [**Vas03**, §4] uses this Frattini principle in the full context of Shimura varieties, continuing the tradition of [**Ser98**]. Those with Shimura variety experience know that the semi-simple groups that arise, generalizing the PSL_2 case (symplectic groups, for example), are from a moduli problem on abelian varieties. □

Remark 6.6. — It is elementary that $\psi^0_{2,j'} \bmod p$ (in Res. 6.5) is onto for a dense set j' in any number field. For $p \neq 2$ or 3, just apply Hilbert's Irreducibility Theorem to the irreducible cover $X_0(p) \to \mathbb{P}^1_j$ (for $p = 3$, to $X_0(p^2) \to \mathbb{P}^1_j$).

6.2.2. F(rattini)-quotients of $\mathbf{MT}$*s.* — Consider a rank u $\mathbf{MT}$ from $F_u \times^s J$ and 4 conjugacy classes in J (§4.1.3). For $p \notin P_{\mathbf{C}}$, assume $\tilde{G}^* = V^* \times^s J \in \mathcal{G}_{J,p}$ is a $\mathbf{C}$ p-Nielsen limit. That means there are projective systems of $\{\boldsymbol{g}_V \in \mathrm{Ni}(V \times^s J, \mathbf{C})\}'$ with $'$ indicating running over finite J quotients of V^* covering $\mathbb{Z}/p^u$. This projective system defines a cusp branch.

By taking braid orbits, these define a projective system of $\mathbf{MT}$ components on the full component graph $\mathcal{T}^f_{\mathbb{Z}/p^u \times^s J, \mathbf{C}, p}$. Use our previous notation B for a cusp branch and B' for the component branch B defines. For a J quotient V of V^* use B_V and B'_V for the corresponding cusp $\boldsymbol{g}_V$ and its component. Let $F_{\mathbf{C}}$ be the definition field of all the inner reduced Hurwitz spaces $\mathcal{H}(G_k((\mathbb{Z}/p)^u) \times^s J, \mathbf{C})^{\mathrm{in,rd}}$ as in §3.1.1. To simplify, assume $F_{\mathbf{C}} = \mathbb{Q}$.

Definition 6.7. — Suppose V_0 is a J quotient of $(\mathbb{Z}/p)^u$. We call the $\mathbf{MT}$ for $(V_0 \times^s J, \mathbf{C}, p)$ an F-quotient of the $\mathbf{MT}$ for $((\mathbb{Z}/p)^u \times^s J, \mathbf{C}, p)$. Then, there is a natural map from $\mathcal{T}^f_{\mathbb{Z}/p^u \times^s J, \mathbf{C}, p}$ to $\mathcal{T}^f_{V_0 \times^s J, \mathbf{C}, p}$ (on cusps also) induced by the map

$$\mathcal{H}((\mathbb{Z}/p)^u \times^s J, \mathbf{C})^{\mathrm{in,rd}} \to \mathcal{H}(V_0 \times^s J, \mathbf{C})^{\mathrm{in,rd}} \stackrel{\mathrm{def}}{=} \mathcal{H}_{V_0}.$$

We will refer to B'_V on branch B' as if it is the corresponding Hurwitz space. Also, for $V_1 \to V_2$ a homomorphism of J groups, denote the corresponding Hurwitz space map as $B'_{V_1} \to B'_{V_2}$. Let $\mathbf{G}_V$ be the geometric monodromy group of $B'_V \to \mathbb{P}^1_j$.

In the best circumstances for the cusp branch B, as in §1.3, we expect this.

(6.3a) Computable $\mathbb{Q}_p$ action: We can decipher the $G_{\mathbb{Q}_p}$ orbit on B.

(6.3b) Branch Frattini Property: Excluding finitely many V_2 corresponding to B'_{V_2} on the branch B', all the maps $\mathbf{G}_{V_1} \to \mathbf{G}_{V_2}$ are p-Frattini covers.

(6.3c) Smooth genera: The genera of B'_V should have a modular curve-like formula, coming from clear understanding of g-p' and p-cusps on B'.

§6.2.4 notes results on the $\mathbb{R}$ and $\mathbb{Q}_\ell$ nature of cusp branches, extending (6.3a).

6.2.3. More on Branch Frattini propery (6.3b). — A weaker version of (6.3b) would assert that $\mathbf{G}_{V_1} \to \mathbf{G}_{V_2}$ is a p-group. In turn this implies all ramification groups are p-groups, and Lem. 3.8 (condition (3.6b)) implies exactly that.

Property (6.3b) is an analog of Serre's use of the p-Frattini property. We expect something like it for all reasonable **MT**s. For example, suppose we have a g-p' (or even, shift of an H-M) cusp on a **MT**. Then, we expect the geometric monodromy groups $\mathbf{G}_k$ of $\bar{\mathcal{H}}(G_k(G), \mathbf{C}) \to \mathbb{P}^1_j$ to satisfy (6.3b).

That is, for k_0 large and $k \geq k_0$, $\mathbf{G}_k \to \mathbf{G}_{k_0}$ should be a p-Frattini cover. For certain, however, we can't always take $k_0 = 0$. For example, for the **MT** for $(A_5, \mathbf{C}_{3^4}, p = 2)$ we have these facts. This continues Ex. 4.13, Ex. 4.21, §4.6.1,§5.2.2, Ex. 5.9 and §6.2.3.

(6.4a) There is exactly one H-M component B'_1 at level 1.

(6.4b) the degree of $B'_1 \to B'_0$ is 16, but

(6.4c) $|H_{1,0}| = |\ker(\mathbf{G}_1 \to \mathbf{G}_0)| = 3 \cdot 2^6$ with an S_3 at the top [**BF02**, App. A].

So, $H_{1,0}$ is not even a two group. We use proofs, not **GAP** calculations, so we know why this is happening. Prob. 6.8 starts with a fixed g-p' branch (as in §B).

Problem 6.8. — Show $H_{k+1,k} = \ker(\mathbf{G}_{k+1} \to \mathbf{G}_k)$ is a 2-group (resp. p-group) for large k for the $(A_5, \mathbf{C}_{3^4}, p = 2)$ (resp. $((\mathbb{Z}/p)^2 \times^s \mathbb{Z}/3, \mathbf{C}_{\pm 3^2}, p \neq 3)$ **MT**.

My thinking (6.3b) might hold came from [**Iha86**] (even though Ihara has p-groups, the opposite of p-perfect groups).

Of course, if we knew explicitly the subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ defining the **MT** levels that would answer Prob. 6.8. Even one other case than modular curves where we could test these problems would be reassuring. In fact, [**Ber99**] almost includes the non-trivial F-quotient of $((\mathbb{Z}/p)^2 \times^s \mathbb{Z}/3, \mathbf{C}_{\pm 3^2}, p \equiv 1 \mod 3)$. Only, he has taken for $\mathbf{C}$ the repetition 3 times of one conjugacy class, and the other just once? He uses the Bureau representation of the braid group to effect his calculation. It promises answering such questions as Prob. B.5 for at least this non-modular curve situation.

6.2.4. Complete fields and tangential base points. — Suppose B is a cusp branch. Much work on the Inverse Galois Problem is appropriate for service to this problem.

Problem 6.9. — What do we need to know to detect when B is a projective sequence of $\mathbb{Q}_\ell$ cusps, $\ell \neq p$ (including $\ell = \infty$)?

The effective computation for $\mathbb{R}$ points on Hurwitz spaces in [**FD90**] works to analyze higher **MT** levels (as in [**BF02**, §6], especially see the use made in Ex. 5.9). The model for $\mathbb{Q}_\ell$ has followed this. It is necessary for a positive answer to Prob. 6.9 that the manifolds $\bar{\mathcal{H}}'_k$ have definition field $\mathbb{Q}_\ell$.

The basic proposition in that direction is [**Fri95**, Thm. 3.21]. It says: If all H-M reps. in the Nielsen classes for level k lie in one braid orbit (so all the H-M cusps lie on $\bar{\mathcal{H}}'_k$) then $\bar{\mathcal{H}}'_k$ has definition field $\mathbb{Q}$. Further, it gives a criterion for this to happen at level 0 that implies it automatically at all other levels. Then, Harbater patching applies to produces a projective sequence of $\mathbb{Q}_\ell$ cusps on $\{\bar{\mathcal{H}}'_k\}_{k=0}^{\infty}$. [**Dèb06**, Thm. 2.7] has a precise statement from [**DD04**].

[**DE06**] redoes the author's result using a more classical compactification. One problem: When $r = 4$, the criterion of [**Fri95**, Thm. 3.21] never applies. An example failure is the two H-M components at Level 1 in §6.4.5 (see Rem. 6.11).

So, we require deeper methods to analyze the definition field of a component branch and its cusps when $r = 4$. Based on [**IM95**] and [**Wew02**], [**BF02**, App. D.3] describes a method that *will* work with sufficient grasp of the group theory and use of an especially good cusp branch.

Again, B is a g-p' cusp branch, defining a component branch B' on a **MT**. The desired archetype for a tangential base point comes from $X_0(p^{k+1})$. We identify this space with $\mathcal{H}(\mathbb{Z}/p^{k+1} \times^s \mathbb{Z}/2, \mathbf{C}_{2^4})^{\mathrm{abs,rd}}$; the *absolute* reduced Hurwitz space related to the nontrivial F-quotient in Serre's OIT. The unique cusp of width p^{k+1} identifies with the unique H-M cusp, and so it has $\mathbb{Q}$ as definition field.

In the now classical picture, points on the space approaching this cusp preciously go to a controlled p-catastrophe. A p-adic power series representing j, parametrizes a Tate curve (p-adic torus) degenerating with $j \mapsto \infty$ (p-adically).

Generalizing such constructions to g-p' cusps cannot be trivial. Yet, the apparatus for exploiting them as Serre does in [**Ser98**, IV.29–IV.45] is already in the Grothendieck-Teichmüller motivated formulas of Ihara-Matsumoto-Wewers ([**IM95**], [**Wew02**]; [**BF02**, App. D] discusses this). Making it work, à la [**Nak99**], in our more general situation requires a dedicated project. Deciding the definition field of the two genus 1 components in (6.11b) is a practical example of its value.

The groups $H_{2,3}(\boldsymbol{g})$ and $H_{1,4}(\boldsymbol{g})$ give a *type* to g-p' cusps. [**Fri05a**, Lect. 4] defines g-p' rep. types in Nielsen classes for any r, making sense of Prob. 6.10 for all r.

Problem 6.10. — Show this analog of [**Fri95**, Thm. 3.21] for general g-p' cusp branches of a given type holds. If there are finitely many (resp. one) braid orbit of this type, then G_F has a finite orbit (resp. is fixed) on their component branch(s).

Remark 6.11. — Examples show that outer automorphisms of G_k can conjugate distinct H-M components on $\mathcal{H}(G_k, \mathbf{C})$ ((6.11) and [**BF02**, §9.1]). Is this is a general phenomenon? Nor do we know if there are *always*, modulo braiding, just finitely many G_F orbits of H-M reps. This consideration makes sense for all g-p' cusps.

6.3. $F_2 \times^s \mathbb{Z}/3$**,** $p = 2$**: Level 0, 1 components.** — Components on these levels bring up deeper aspects of complex multiplication and the inverse Galois problem. This example shows how such tools as the **sh**-*incidence matrix* can identify components at a **MT** level. We now explain why at level 0 there are two components:

$$\mathcal{H}(\tilde{F}_{2,2}/\Phi^1 \times^s J_3, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}} = \mathcal{H}_0^+ \cup \mathcal{H}_0^-.$$

Both have genus 0, and $\mathcal{H}_0^+$ is an H-M component. The other has nontrivial lifting invariant; there is nothing above it at level 1. Though both are families of genus 1 curves, and upper half plane quotients, neither is a modular curve.

6.3.1. Setting up reduced Nielsen classes. — This Nielsen class has $G = A_4$ with $\mathbf{C}_{\pm 3^2}$ as two pairs of 3-cycles in each of the conjugacy classes with order 3. First look at the situation with A_3 replacing A_4.

The total Nielsen class $\mathrm{Ni}(A_3, C_{\pm 3^2})^{\mathrm{in}}$ contains six elements corresponding to the six possible arrangements of the conjugacy classes. Since A_3 is abelian, the inner classes are the same. Also, the outer automorphism of A_n ($n = 3$ or 4) from conjugation by $(1\,2) \in S_n$ restricts to A_3 to send a conjugacy class arrangement to its complement. Here is a convenient list of the arrangements, and their complements:

$$\begin{array}{lll} [1]+-+- & [2]++-- & [3]+--+ \\ [4]-+-+ & [5]--++ & [6]-++-. \end{array}$$

The group $\mathcal{Q}'' = \langle q_1 q_3^{-1}, \mathbf{sh}^2 \rangle$ equates elements in this list with their complements. So, inner reduced classes and absolute (not reduced) classes are the same. Conclude: $\mathcal{H}(A_3, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}} \to \mathbb{P}^1_j$ is a degree three cover with branch cycles

$$(\gamma_0^*, \gamma_1^*, \gamma_\infty^*) = ((1\,3\,2), (2\,3), (1\,2)).$$

Check easily: If $(g_1, \dots, g_4)$ maps to [1], and (with no loss) $g_1 = (1\,2\,3)$, then either this is $\boldsymbol{g}_{1,1}$ (in (6.5)) or $g_1 g_2$ has order 2. Listing the four order 2 elements gives a total of five elements in the reduced Nielsen class $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ lying over [1].

6.3.2. Effect of γ_∞ *on* $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$. — Start with an H-M rep over [1] in A_3:

$$\boldsymbol{g}_{1,1} = ((1\,2\,3), (1\,3\,2), (1\,3\,4), (1\,4\,3)) \in \mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2}). \tag{6.5}$$

The middle twist squared on this conjugates the middle two by $(1\,4)(2\,3)$ to give

$$\boldsymbol{g}_{1,2} = ((1\,2\,3), (4\,2\,3), (4\,2\,1), (1\,4\,3)).$$

The result is a γ_∞ orbit of length 4. The middle twist squared on

$$\boldsymbol{g}_{1,3} = ((1\,2\,3),(1\,2\,4),(1\,4\,2),(1\,3\,2))$$

leaves it fixed, giving a γ_∞ orbit of length 2. Similarly, the square of the middle twist on $\boldsymbol{g}_{1,4} = ((1\,2\,3),(1\,2\,4),(1\,2\,3),(1\,2\,4))$ conjugates the middle pair by $(1\,3)(2\,4)$ producing $\boldsymbol{g}_{1,5} = ((1\,2\,3),(1\,2\,4),(2\,4\,3),(1\,4\,3))$. Again the middle twist gives an element of order 4 on reduced Nielsen classes.

The H-M rep. $\boldsymbol{g}_{3,1} = ((1\,2\,3),(1\,3\,2),(1\,4\,3),(1\,3\,4)) \in \mathrm{Ni}(A_4,\mathbf{C}_{\pm 3^2})$ maps to [3] in A_3. Applying γ_∞ gives $\boldsymbol{g}_{3,2} = ((1\,2\,3),(1\,2\,4),(1\,3\,2),(1\,3\,4))$, the same as conjugating on the middle two by $(2\,4\,3)$. The result is a length 3 γ_∞ orbit.

On Nielsen class representatives over [3], γ_∞ has one orbit of length 3 and two of length one. See this by listing the second and third positions (leaving $(1\,2\,3)$ as the first). Label these as

$$1' = ((1\,3\,2),(1\,4\,3)), 2' = ((1\,2\,4),(1\,3\,2)), 3' = ((1\,2\,4),(2\,3\,4)),$$
$$4' = ((1\,2\,4),(1\,2\,4)), 5' = ((1\,2\,4),(1\,4\,3)).$$

6.3.3. Using Wohlfahrt's Theorem. — For $\Phi^{\mathrm{rd}} : \mathcal{H}^{\mathrm{rd}} \to U_\infty$, one of our reduced Hurwitz space covers, let $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ define it as an upper half-plane quotient $\mathbb{H}/\Gamma$ (§2.3.1). Now let N_Γ be the least common multiple (lcm) of its cusp widths. Equivalently: N_Γ is the lcm of the ramification orders of points of the compactification $\bar{\mathcal{H}}^{\mathrm{rd}}$ over $j = \infty$; or the lcm of the orders of γ_∞ on reduced Nielsen classes.

Wohlfahrt's Theorem [**Woh64**] says Γ is congruence if and only if Γ contains the congruence subgroup, $\Gamma(N_\Gamma)$, defined by N_Γ. We have a situation with a modular curve-like aspect, though we find these j-line covers aren't modular curves by seeing the cusps fail Wohlfahrt's condition. Here is our procedure.

Compute γ_∞ orbits on $\mathrm{Ni}^{\mathrm{rd}}$. Then, check their distribution among $\bar{M}_4 = \langle \gamma_\infty, \mathbf{sh}\rangle$ orbits ($\mathcal{H}^{\mathrm{rd}}$ components). For each $\mathcal{H}^{\mathrm{rd}}$ component $\mathcal{H}'$, check the lcm of γ_∞ orbit lengths to compute N', the modulus if it were a modular curve. Then, see whether a permutation representation of $\Gamma(N')$ could produce $\Phi' : \mathcal{H}' \to \mathbb{P}^1_j$, and the type of cusps now computed. Denote Spin_4 (§2.1) by $\hat{A}_4$.

Use notation ending §6.3.2. Note: Neither of $\mathcal{H}_0^{\mathrm{in,rd},\pm}$ have reduced fine moduli. The Nielsen braid orbit for $\mathcal{H}_0^{\mathrm{in,rd},-}$ (resp. $\mathcal{H}_0^{\mathrm{in,rd},+}$) fails (6.6a) (resp. and also (6.6b)):

(6.6a) $\mathcal{Q}''$ has length 2 (not 4 as required in (3.6a)) orbits; and

(6.6b) γ_1 has a fixed point (Lem. 6.13; contrary to (3.6b)).

Proposition 6.12. — *Then, γ_∞ fixes $4'$ and $5'$ and cycles $1' \to 2' \to 3'$. So there are two $\bar{M}_4$ orbits on $\mathrm{Ni}(A_4,\mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$, Ni_0^+ and Ni_0^-, having respective degrees 9 and 6 and respective lifting invariants to $\hat{A}_4$ of $+1$ and -1. The first, containing all H-M reps., has orbit widths 2,4 and 3. The second has orbit widths 1,1 and 4. Neither defines a modular curve cover of $\mathbb{P}^1_j$.*

Denote the corresponding completed covers $\bar{\psi}_0^{\pm} : \bar{\mathcal{H}}_0^{\mathrm{in,rd},\pm} \to \mathbb{P}^1_j$. *Both* $\bar{\mathcal{H}}_0^{\mathrm{in,rd},\pm}$ *have genus 0. Both have natural covers* $\bar{\mu}^{\pm} : \bar{\mathcal{H}}_0^{\mathrm{in},\pm} \to \mathbb{P}^1_j$ *by completing the map*

$$\boldsymbol{p} \in \mathcal{H}_0^{\mathrm{in,rd},\pm} \mapsto \beta(\boldsymbol{p}) \overset{\mathrm{def}}{=} j(\mathrm{Pic}(X_{\boldsymbol{p}})^{(0)}) \in \mathbb{P}^1_j. \tag{6.7}$$

Then, this case's identification of inner and absolute reduced classes gives

$$\boldsymbol{p} \in \mathcal{H}_0^{\mathrm{in,rd},\pm} \mapsto (j(\boldsymbol{p}), j(\mathrm{Pic}(X_{\boldsymbol{p}})^{(0)})), \tag{6.8}$$

a birational embedding of $\bar{\mathcal{H}}_0^{\mathrm{in,rd},\pm}$ *in* $\mathbb{P}^1_j \times \mathbb{P}^1_j$.

If we denote the corresponding H_4 *orbits on* $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in}}$ *by* $\mathrm{Ni}^{\mathrm{in},\pm}$, *then* Q'' *orbits on both have length 2.*

6.4. Proof of Prop. 6.12.

— This proof takes up the next four subsections.

6.4.1. γ_∞ *orbits on* $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$. — First: γ_∞ fixes $4'$ and it maps $5'$ to $((1\,2\,3),(2\,3\,4),(1\,2\,4),(3\,1\,2))$ (conjugate by $(1\,2\,3)$ to $5'$).

These computations establish the orbit lengths:

$$\begin{aligned}(g_{1,1})\gamma_\infty &= ((1\,2\,3),(1\,4\,2),(1\,3\,2),(1\,4\,3)) = (3')\mathbf{sh},\\ (g_{1,3})\gamma_\infty &= ((1\,2\,3),(1\,4\,2),(1\,2\,4),(1\,3\,2)) = (1')\mathbf{sh}.\end{aligned}$$

They put the H-M rep. in the $\bar{M}_4$ orbit with γ_∞ orbits of length 2,3 and 4 (in the orbit of the $1' \to 2' \to 3'$ cycle). Use Ni_0^+ for the Nielsen reps. in this $\bar{M}_4$ orbit.

6.4.2. Graphics and Computational Tools: **sh***-incidence.* — The **sh**-incidence matrix of Ni_0^+ comes from the following data. Elements $\boldsymbol{g}_{1,1}, \boldsymbol{g}_{1,2}, \boldsymbol{g}_{1,3}$ over [1] are permuted as a set by **sh**. They map by γ_∞ respectively to $\boldsymbol{g}_{2,1}, \boldsymbol{g}_{2,2}, \boldsymbol{g}_{2,3}$ over [2]. Under γ_∞ these map respectively to $\boldsymbol{g}_{1,2}, \boldsymbol{g}_{1,1}, \boldsymbol{g}_{1,3}$, while $\boldsymbol{g}_{3,1}, \boldsymbol{g}_{3,2}, \boldsymbol{g}_{3,3}$ cycle among each other. So, there are three γ_∞ orbits, $O_{1,1}$, $O_{1,3}$ and $O_{3,1}$ on Ni_0^+ named for the subscripts of a representing element.

The data above shows

$$|O_{1,1} \cap (O_{3,1})\mathbf{sh}| = 2, \ |O_{1,3} \cap (O_{3,1})\mathbf{sh}| = 1.$$

Compute: **sh** applied to $\boldsymbol{g}_{1,3}$ is $\boldsymbol{g}_{1,1}$ so $|O_{1,1} \cap (O_{1,3})\mathbf{sh}| = 1$. The rest has two sources:

- symmetry of the **sh**-incidence matrix, and;
- elements in a row (or column) add up to ramification index of the cusp labeling that row (or column).

TABLE 1. **sh**-Incidence Matrix for Ni_0^+

Orbit	$O_{1,1}$	$O_{1,3}$	$O_{3,1}$
$O_{1,1}$	1	1	2
$O_{1,3}$	1	0	1
$O_{3,1}$	2	1	0

Similarly, the $\mathbf{sh}$-incidence matrix of Ni_0^- comes from the following data. Elements $\boldsymbol{g}_{1,4}, \boldsymbol{g}_{1,5}$ over [1] map by γ_∞ respectively to $\boldsymbol{g}_{2,4}, \boldsymbol{g}_{2,5}$ over [2], and these map respectively to $\boldsymbol{g}_{1,5}, \boldsymbol{g}_{1,4}$, while γ_∞ fixes both $\boldsymbol{g}_{3,4}, \boldsymbol{g}_{3,5}$. So, there are three γ_∞ orbits, $O_{1,4}$, $O_{3,4}$ and $O_{3,5}$ on Ni_0^-.

Table 2. $\mathbf{sh}$-Incidence Matrix for Ni_0^-

Orbit	$O_{1,4}$	$O_{3,4}$	$O_{3,5}$
$O_{1,4}$	2	1	1
$O_{3,4}$	1	0	0
$O_{3,5}$	1	0	0

Lemma 6.13. — *In general, the* $\mathbf{sh}$*-incidence matrix is the same as the matrix obtained by replacing* $\mathbf{sh} = \gamma_1$ *by* γ_0*. Further, the only possible elements fixed by either lie in* γ_∞ *orbits* O *with* $|O \cap (O)\mathbf{sh} \neq 0|$.

On Ni_0^+ *(resp.* Ni_0^-*),* γ_1 *fixes 1 (resp. no) element(s), while* γ_0 *fixes none.*

Proof. — We explain the first paragraph. From $((\boldsymbol{g})\gamma_\infty^{-1})\gamma_0 = (\boldsymbol{g})\gamma_1$ on reduced Nielsen classes, the range of γ_0 and γ_1 are the same on any γ_∞ orbit. So, the $\mathbf{sh}$-incidence matrix is the same as the matrix obtained by replacing $\mathbf{sh} = \gamma_1$ by γ_0.

A fixed point of $\gamma_1 = \mathbf{sh}$ in O, a γ_∞ orbit, would contribute to $O \cap (O)\mathbf{sh}$. Since the $\mathbf{sh}$-incidence matrix is the same as that for replacing γ_1 by γ_0, 0's along the diagonal also imply there is no γ_0 fixed point.

We now show the statement about fixed points of $\gamma_1 = \mathbf{sh}$. Any fixed points must come from a nonzero entry along the diagonal of the $\mathbf{sh}$-incidence matrix. For Ni_0^+, there is precisely one reduced Nielsen class $\boldsymbol{g}$ in $O_{1,1} \cap (O_{1,1})\mathbf{sh}$. Write $\boldsymbol{g} = (\boldsymbol{g}')\mathbf{sh}$. Apply $\mathbf{sh}$ to both sides, and conclude $(\boldsymbol{g})\mathbf{sh} = \boldsymbol{g}'$. Therefore, as there is only one element with this property, $\boldsymbol{g} = \boldsymbol{g}'$. Now return to the example details.

Apply the above to Ni_0^-. Since $|O_{1,4} \cap (O_{1,4})\mathbf{sh} = 2|$, there are either two fixed points, or none. Since $\mathbf{sh}$ preserves the fiber over [1], we need only check if $(\boldsymbol{g}_{1,4})\mathbf{sh}$ is reduced equivalent to $\boldsymbol{g}_{1,4}$. Apply $q_1^{-1}q_3$ to $(\boldsymbol{g}_{1,4})\mathbf{sh}$: the result is $((1\,2\,3), (3\,4\,2), (1\,3\,4), (1\,2\,4))$. Conjugate this by $(1\,2\,3)^{-1}$ to get $\boldsymbol{g}_{1,5}$. So, γ_1 has no fixed points on Ni_0^-. Since γ_0 moves the fibers over [1], [2], [3] in a cycle, it fixes no Nielsen class elements. □

We know the degrees of $\bar\psi_o^\pm$ are respectively 9 and 6. Lem. 6.13 gives the genus $g_0^\pm$ of $\bar{\mathcal{H}}_0^{\mathrm{in},\pm}$ from Riemann-Hurwitz:

$$\begin{aligned} &2(9 + g_0^+ - 1) = 3 \cdot 2 + (9-1)/2 + (1+2+3) = 16, \text{ or } g_0^+ = 0; \\ &2(6 + g_0^- - 1) = 2 \cdot 2 + 6/2 + 3 = 10, \text{ or } g_0^- = 0. \end{aligned} \tag{6.9}$$

Remark 6.14. — In the $\bar M_4$ orbit on $\mathrm{Ni}_0^{\mathrm{in},-}$ there is a nonzero diagonal entry, though neither γ_0 nor γ_1 has a fixed point in the corresponding γ_∞ orbit.

6.4.3. Checking $s_{\hat{A}_4/A_4}$ of §4.2 on two $\bar{M}_4$ orbits. — Apply **sh** to $4'$. This shows $g_{1,4}, g_{1,5}, 4', 5'$ all lie in one $\bar{M}_4$ orbit. Any H-M rep. has lifting invariant $+1$, and since it is a $\bar{M}_4$ invariant, all elements in Ni_0^+ have lifting invariant $+1$. For the other orbit, we have only to check the lifting invariant on $4'$, written in full as

$$\boldsymbol{g}_{1,4} = ((1\,2\,3), (1\,2\,4), (1\,2\,4), (4\,3\,2)) = (g_1, \dots, g_4).$$

Compute the lifting invariant as $\hat{g}_1\hat{g}_2\hat{g}_3\hat{g}_4$. Since $g_2 = g_3$ (and their lifts are the same), the invariant is $\hat{g}_1\hat{g}_2^2\hat{g}_4$. Apply Prop. 4.27 (not necessary, though illuminating). The genus zero hypothesis for a degree 4 cover holds for $((1\,2\,3), (1\,4\,2), (4\,3\,2))$:

$$s_{\hat{A}_4/A_4}(\boldsymbol{g}_{1,4}) = (-1)^{3\cdot(3^2-1)/8} = -1.$$

6.4.4. Why $\mathcal{H}_0^\pm$ aren't modular curves. — From §6.3.3, if the degree nine cover is modular, the monodromy group of the cover is a quotient of $\mathrm{PSL}_2(12)$. If the degree 6 orbit is modular, the monodromy group is a quotient of $\mathrm{PSL}_2(4)$. Since $\mathrm{PSL}_2(\mathbb{Z}/4)$ modular curve has the λ-line as a quotient, with 2,2,2 as the cusp lengths, these cusp lengths are wrong for the second cover to correspond to the λ-line. Similarly, for the degree nine cover, as $\mathrm{PSL}_2(\mathbb{Z}/12)$ has both $\mathrm{PSL}_2(\mathbb{Z}/4)$ and $\mathrm{PSL}_2(\mathbb{Z}/3)$ as a quotient, the cusp lengths are wrong.

We can check the length of a $\mathcal{Q}''$ orbit on $\mathrm{Ni}_0^{\mathrm{in},+}$ and $\mathrm{Ni}_0^{\mathrm{in},-}$ by checking the length of the orbit of any particular element. If an orbit has an H-M rep. like $\boldsymbol{g}_{1,1}$ it is always convenient to check elements of $\mathcal{Q}''$ on it:

$$\begin{aligned}(\boldsymbol{g}_{1,1})\mathbf{sh}^2 &= (1\,3)(2\,4)\boldsymbol{g}_{1,1}(1\,3)(2\,4) \text{ and;}\\ (\boldsymbol{g}_{1,1})q_1q_3^{-1} &= (1\,3)\boldsymbol{g}_{1,1}(1\,3).\end{aligned} \tag{6.10}$$

The top line of (6.10) says $\mathbf{sh}^2$ fixes $\boldsymbol{g}_{1,1}$. The bottom line, however, says $(\boldsymbol{g}_{1,1})q_1q_3^{-1}$ is absolute, but not inner equivalent to $\boldsymbol{g}_{1,1}$. For $\mathrm{Ni}_0^{\mathrm{in},-}$, $\boldsymbol{g}_{1,4}$ is transparently fixed by $\mathbf{sh}^2$, and $(\boldsymbol{g}_{1,4})q_1q_3^{-1} = (3\,4)\boldsymbol{g}_{1,4}(3\,4)$. Conclude the orbit length of $\mathcal{Q}''$ on both $\mathrm{Ni}_0^{\mathrm{in},+}$ and $\mathrm{Ni}_0^{\mathrm{in},-}$ is 2.

We finish Prop. 6.12 by producing the map β in (6.8), and thereby concluding Prop. 6.15. Each $\boldsymbol{p} \in \mathcal{H}(\tilde{F}_{2,2}/\Phi^1 \times^s J_3, \mathbf{C}_{\pm 3^2})^{\mathrm{abs,rd}}$ gives a degree 4 cover $\varphi : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ with four 3-cycle branch points. From R-H, the genus g of $X_{\boldsymbol{p}}$ satisfies $2(4+g-1) = 8$, or $g = 1$. It may not, however, be an elliptic curve, though its degree 0 Picard variety $\mathrm{Pic}(X_{\boldsymbol{p}})^{(0)}$ is. Define β by taking its j-invariant.

Proposition 6.15. — *The absolute space $\mathcal{H}(\tilde{F}_{2,2}/\Phi^1 \times^s J_3, \mathbf{C}_{\pm 3^2})^{\mathrm{abs,rd}}$ at level 0 embeds in $\mathbb{P}^1_j \times \mathbb{P}^1_j$, but is not a Modular curve. So, André's Thm.* [**And98**] *says it contains at most finitely many Shimura-special points (unlike the J_2 case).*

Conjecture 6.16. — The conclusion of Prop. 6.15 is true for all other $p \neq 3$.

Yet, we have a problem: What does *Shimura special* mean when $p \neq 2$ or 3?

6.4.5. Level 1 of $(A_4, \mathbf{C}_{\pm 3^2}, p=2)$. — Level 1 of the **MT** covers $\mathcal{H}_0^+$:

$$\mathcal{H}(\tilde{F}_{2,2}/\Phi^2 \times^s J_3, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}} \to \mathcal{H}_0^+.$$

We know level 1 has two genus 0 components, $\mathcal{H}_1^{-,c}, \mathcal{H}_1^{-,c'}$, complex conjugate and *spin* obstructed; two genus 3 components, $\mathcal{H}_1^{+,3}, \mathcal{H}_1^{-,3}$, one spin obstructed, the other obstructed by another Schur multiplier; and two genus 1 components, $\mathcal{H}_1^{+,\beta}$, $\mathcal{H}_1^{+,\beta^{-1}}$ both H-M comps [**FS06**].

Significance of $\mathcal{H}_1^{+,\beta}$, $\mathcal{H}_1^{+,\beta^{-1}}$:

(6.11a) $\mathrm{Out}(\tilde{F}_{2,2}/\Phi^2 \times^s J_3)$ conjugates $\mathcal{H}_1^{+,\beta}$ to $\mathcal{H}_1^{+,\beta^{-1}}$.

(6.11b) The following are equivalent for $K \leq \mathbb{R}$ a number field [**BF02**, Ex. 9.2].
- There are ∞-ly many (reduced inequivalent – §2.3.1) 4 branch point, K regular realizations of the 2-Frattini extension $G_1(A_5)$ of A_5.
- $\mathcal{H}_1^{+,\beta}$ has ∞-ly many K points.

Appendix A
Nielsen classes for $F_2 \times^s \mathbb{Z}/2$

§A.1 does the Nielsen class version of all modular curves, by considering them coming from a rank 2 MT. Prop. A.1 shows there is a unique limit group $(\mathbb{Z}_p)^2 \times^s \mathbb{Z}/2$ — not the whole universal p-Frattini cover — for each $p \neq 2$. Then, §A.2 shows the Heisenberg group kernel acts here as a universal obstruction, running over all odd p.

A.1. Limit groups for the rank 2 MT of modular curves.

— Following §6.2.2, we consider the nonempty Nielsen classes of the form $\mathrm{Ni}(V \times^s \mathbb{Z}/2, \mathbf{C}_{2^4})$, $V \in \mathcal{V}_p'$ (a nontrivial $\tilde{F}_{2,p}$ quotient on which $\mathbb{Z}/2$ acts, as in §4.1.3). The following formalizes an argument of [**Fri95**, p. 114]. Form the projective completion of

$$K_4 = \langle \boldsymbol{\sigma} = \sigma_1, \dots, \sigma_4 \quad \mathrm{mod}\ \sigma_1\sigma_2\sigma_3\sigma_4 = 1 \text{ (product-one)}\rangle.$$

Denote the result by $\hat{K}_{\boldsymbol{\sigma}}$. Use the notation of §1.1.2.

Proposition A.1. — *Let $\hat{D}_{\boldsymbol{\sigma}}$ (compatible with Cor. 4.19) be the quotient of $\hat{K}_{\boldsymbol{\sigma}}$ by*

$$\sigma_i^2 = 1,\ i = 1,2,3,4 \ (\textit{so}\ \sigma_1\sigma_2 = \sigma_4\sigma_3).$$

Then, $\prod_{p\neq 2} \mathbb{Z}_p^2 \times^s J_2 \equiv \hat{D}_{\boldsymbol{\sigma}}$ and $\mathbb{Z}_p^2 \times^s J_2$ is the unique $\mathbf{C}_{2^4}$ p-Nielsen class limit.

The component graph of $\mathcal{C}^f_{(\mathbb{Z}/p)^2 \times^s \mathbb{Z}/2, \mathbf{C}_{\pm 3^2}, p}(\mathbb{Z}_p^2 \times^s J_2)$ (as in §4.1.1) is a principle homogeneous space for $G(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$.

Proof. — We show $\hat{D}_{\boldsymbol{\sigma}}$ is $\tilde{\mathbb{Z}}^2 \times^s J_2$; $\sigma_1\sigma_2$ and $\sigma_1\sigma_3$ are generators of $\tilde{\mathbb{Z}}^2$; and then that σ_1 acts on $\tilde{\mathbb{Z}}^2$ by multiplication by -1.

First: $\sigma_1(\sigma_1\sigma_2)\sigma_1 = \sigma_2\sigma_1$ shows σ_1 conjugates $\sigma_1\sigma_2$ to its inverse. Also,

$$(\sigma_1\sigma_2)(\sigma_1\sigma_3) = (\sigma_1\sigma_3)\sigma_3(\sigma_2\sigma_1)\sigma_3 = (\sigma_1\sigma_3)(\sigma_1\sigma_2)$$

shows the said generators commute. The maximal possible quotient is $\mathbb{Z}_p^2 \times^s \{\pm 1\}$.

Now we show for $G = V \times^s J_2$, V a nontrivial quotient of $\mathbb{Z}_p^2$, that $\mathrm{Ni}(G, \mathbf{C}_{2^4})$ is nonempty. Use a cofinal family of V s, $(\mathbb{Z}/p^{k+1})^2$, $p \neq 2$. Two proofs, one pure Nielsen class, the other with elliptic curves, appear in [**Fri05b**, §6.1.3]. That shows $\mathbb{Z}_p^2 \times^s \{\pm 1\}$ is a limit group.

Uniqueness of the limit group does follow if we know there is just one braid orbit on the respective inner Nielsen classes. Alas, that isn't so.

To finish we use absolute Nielsen classes as an aid. Apply the elementary divisor theorem to $(\mathbb{Z}_p)^2$: Up to change of basis we may assume $V = \mathbb{Z}_p/p^{u_1} \times \mathbb{Z}_p/p^{u_2}$ with $u_1 \le u_2$. If $u_1 = 0$, [**Fri78**, p. 156] shows there is just one braid orbit: in agreement with identifying $\mathcal{H}(D_{p^{u_2+1}}, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ with the irreducible modular curve $Y_1(p^{k+1})$.

This argument also applies to the general case to reduce to when $u_1 = u_2$. That case is the first two paragraphs of the proof of [**Fri05b**, Prop. 6.3]. Its essential gist, where abs refers to modding out by $\mathrm{GL}_2(\mathbb{Z}/p^{u+1})$ on Nielsen classes:

(A.1a) There is just one element in $\mathrm{Ni}((\mathbb{Z}/p^{u+1})^2 \times \mathbb{Z}/2, \mathbf{C}_{\pm 3^2})^{\mathrm{abs,rd}}$;

(A.1b) each of the $\varphi(p^{u+1})/2$ inner classes defines a unique component of $\mathcal{H}((\mathbb{Z}/p^{u+1})^2 \times \mathbb{Z}/2, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$; and

(A.1c) the classes of (A.1b) are conjugate under the action of $G(\mathbb{Q}(e^{2\pi i/p^{u+1}})/\mathbb{Q})$.

With u varying this gives the last statement of the result. □

Remark A.2 **(Comments on** (A.1b) **and** (A.1c)**).** — Use the notation above. Excluding multiplication by -1, the outer automorphisms $(\mathbb{Z}/p^{k+1})^2 \times^s (\mathbb{Z}/p^{k+1})^*$ of $(\mathbb{Z}/p^{k+1})^2 \times^s \{\pm 1\}$ act through $\mathrm{GL}_2/\mathrm{SL}_2$ on $(\mathbb{Z}/p^{k+1})^2$. By contrast the H_4 action is through $\mathrm{SL}_2(\mathbb{Z}/p^{k+1})$ (explicitly in the proof). That is why you can't braid between The components of $\mathcal{H}((\mathbb{Z}/p^{k+1})^2 \times^s \mathbb{Z}/2, \mathbf{C}_{2^4})^{\mathrm{in,rd}}$. Yet, they form a single orbit under $G(\mathbb{Q}(\cos(2\pi/p^{k+1}))/\mathbb{Q})$. This is the Hurwitz space interpretation of the *Weil pairing*.

The group $(\mathbb{Z}/p)^2 \times^s J_2$ has quotients of the form $\mathbb{Z}/p \times^s J_2 = G^*$. Corresponding to that $\mathbb{Z}_p^2 \times^s J_2$ has the universal p-Frattini cover $\mathbb{Z}/p \times^s J_2$ of G^* as a quotient. This is the source of the complex multiplication situation in Serre's OIT (§6.2).

A.2. Heisenberg analysis of modular curve Nielsen classes. — We briefly remind the reader of Loewy layers and apply Jenning's Thm. in §A.2.1. Then, §A.2.2 applies this to explain a universal obstruction from a Heisenberg group.

A.2.1. A Loewy layer example. — [**Ben91**, p. 3] explains Loewy layers of a $\mathbb{Z}/p[G]$ module M, though with no examples. Most readers won't realize they are almost always hard to compute (if $p||G|$).

Let $J_{G,p} = J$ be the intersection of the maximal left (or right) ideals of $\mathbb{Z}/p[G]$: The Jacobson radical of $\mathbb{Z}/p[G]$. The basic lemma is that $M/J_{G,p}M$, the *first* Loewy layer of M, is the maximal semi-simple quotient of M for the action of G. Then, to continue the series inductively apply this with $J_{G,p}M$ replacing M.

Usually, however, this is far less information than you want. [**Fri95**, Part II] is where I needed modular representations for the first time. This explains the following point: Knowing M from its Loewy layers requires adding info on the nonsplit subquotients M' of M of the form $0 \to S_1 \to M' \to S_2 \to 0$ with S_1 (resp. S_2) irreducibles in the $\ell+1$st) (resp. ℓth) Loewy layer. An arrow from the $\ell+1$st at S_1 to a copy of S_2 in the ℓth Loewy layer represents M'. These arrows give (anti-)directed paths from layer 1 to any other layer ℓ.

For G a p-group, and $M = \mathbb{Z}/p[G]$, J is the augmentation ideal:

$$\ker\Big(\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g\Big).$$

Jenning's Thm. [**Ben91**, Thm. 3.14.6] (based on [**Qui68**]) gives Loewy layer dimensions with a Hilbert polynomial $H_G(t)$ (variable t). The only p-group irreducible is $\mathbf{1}_G$. So, add the Loewy arrows from levels $\ell+1$ to ℓ and we know everything.

Let $F^\dagger{}_u(G) = \{g \in G \mid g-1 \in J^u\}$. So, $F^\dagger{}_1(G) = G$. Then, the input for $H_G(t)$ consists of the dimensions $n_1, n_2, \dots, n_u, \dots$ of the graded pieces of a Lie algebra due to Jenning's. The uth graded piece is $F^\dagger{}_u/F^\dagger{}_{u+1}$. Part of the proof shows $F^\dagger{}_u$ is generated by commutators and pth powers from Fs with lower subscripts. In particular, if $G = (\mathbb{Z}/p)^n$, then $n_1 = n$ and $F^\dagger{}_u/F^\dagger{}_{u+1}$ is trivial for $u \ge 2$. So, the general expression $\prod_{u\ge1}(\frac{1-t^{pu}}{1-t^u})^{n_u}$ becomes just $H_{(\mathbb{Z}/p)^n}(t) = (\frac{1-t^p}{1-t})^n$.

Lemma A.3. — *Then, $H_{(\mathbb{Z}/p)^2}(t) = (1+t+\cdots+t^{p-1})^2$ and the respective Loewy layers of $\mathbb{Z}/p[(\mathbb{Z}/p)^2]$ have the dimensions $1, 2, \dots, p, p-1, \dots, 1$. Given generators x_1, x_2 of the $\mathbb{Z}/p$ module $(\mathbb{Z}/p)^2$, the symbols $x_1^\alpha x_2^{\ell-\alpha}$, $0 \le \alpha, \ell-\alpha < p$ represent generators of copies of $\mathbf{1}$ at Loewy layer ℓ. Arrows from $\mathbf{1}$ associated to $x_1^\alpha x_2^{\ell-\alpha}$ go to copies of $\mathbf{1}$ associated to $x_1^\alpha x_2^{\ell-1-\alpha}$ and to $x_1^{\alpha-1}x_2^{\ell-\alpha}$ under the above constraints.*

Proof. — Calculate the coefficients of $(1+t+\cdots+t^{p-1})^2$ to see the numerical series correctly expresses the dimensions. The Loewy arrow statements come from identifying those subquotients of $R = \mathbb{Z}/p[G]$ that are module extensions of $\mathbf{1}$ by $\mathbf{1}$. For this use the Poincaré-Birkoff-Witt basis for the universal enveloping algebra of R [**Ben91**, p. 88]. □

A.2.2. A Heisenberg obstruction. — The situation of Prop. A.1 is an example of Cor. 4.19. First, $(\mathbb{Z}\times\mathbb{Z}) \times^s \mathbb{Z}/2$ is an oriented p-Poincaré duality group if p is odd: the finite-index subgroup $\mathbb{Z}\times\mathbb{Z}$ is a surface group (the fundamental group of the torus). Denote the matrix $\left(\begin{smallmatrix}1&x&z\\0&1&y\\0&0&1\end{smallmatrix}\right)$ by $M(x,y,z)$ and consider

$$\mathbb{H}_{R,3} = \{M(x,y,z)\}_{x,y,z\in R},$$

the *Heisenberg group* with entries in the commutative ring R. Let $H \le S_n$. Then, there is a 1-dimensional $\mathbb{Z}/p[S_n]$ (so also a $\mathbb{Z}/p[H]$) module whose action is $m \mapsto$

$(m)g = (-1)^{\mathrm{Det}(g)}m$. Denote M by $\mathbf{1}^-$. This extends to a $\mathbb{Z}_p[H]$ action on $\mathbb{Z}_p$. Denote this module as $\mathbb{Z}_p^-$.

In our usual notation, let $G_0 = (\mathbb{Z}/2)^2 \times^s \mathbb{Z}/2$ and denote the 1st characteristic p-Frattini cover of G_0 by G_1. Prop. A.4 uses a universal Frattini extension. It specializes for all odd primes p to the $\mathbb{Z}/p$ quotient obstructing (as in Def. 4.4) the unique braid orbit in $\mathrm{Ni}(G_0, \mathbf{C}_{2^4})$ from lifting to $\mathrm{Ni}(G_1, \mathbf{C}_{2^4})$, as in Cor. 4.19. In fact, by pullback we see it as the limit group obstruction in Cor. 4.20.

Proposition A.4. — *The map $\mathbb{H}_{\mathbb{Z}/p,3} \to (\mathbb{Z}/p)^2$ by $M(x,y,z) \mapsto (x,y)$ is a Frattini extension. The p-Frattini module $M_0(G_0)$ of G_0 has $\mathbf{1}_{G_0} \oplus \mathbf{1}^-_{G_0} \oplus \mathbf{1}^-_{G_0}$ at its head. The extension defined by $\mathbf{1}_{G_0}$ gives the Heisenberg group, obstructing the* **MT** *at level 1. Still, it gives an infinite limit group $(\mathbb{Z}_p)^2 \times^s \mathbb{Z}/2$ by regarding $\mathbb{Z}_p \times \mathbb{Z}_p$ as $\mathbb{Z}_p^- \times \mathbb{Z}_p^-$.*

Proof. — The characteristic Frattini cover $\psi_{1,0} : G_1((\mathbb{Z}/p)^2) \to (\mathbb{Z}/p)^2$ factors through $\psi_{\mathrm{ab}} = (\mathbb{Z}/p^2)^2 \to (\mathbb{Z}/p)^2$ (modding out by p). The nontrivial element of $\mathbb{Z}/2$ acts by multiplication by -1 on $(\mathbb{Z}/p^2)^2$. In fact, ψ_{ab} is the maximal abelian extension through which $\psi_{1,0}$ factors.

Loewy layers of any $(\mathbb{Z}/p)^2 \times^s \mathbb{Z}/2$ module are copies of $\mathbf{1}$ and $\mathbf{1}^-$. So, any proper extension of ψ_{ab} through which $\psi_{1,0}$ factors, also factors through $\psi' : H \to (\mathbb{Z}/p)^2$ with $\ker(\psi')$ of dimension 3 and H not abelian.

We choose the Heller construction (in [**Fri95**, Part II], for example) to describe the characteristic module

$$M_0((\mathbb{Z}/p)^2 \times^s \mathbb{Z}/2) = \ker(G_1((\mathbb{Z}/p)^2 \times^s \mathbb{Z}/2) \to (\mathbb{Z}/p)^2 \times^s \mathbb{Z}/2)(p \text{ odd}).$$

Here is the rubric for this simple, though still nontrivial case. Suppose G_0 is p-split: $G_0 = P^* \times^s H$ with $(|H|, p) = 1$ and P^* the p-Sylow, as in our case. Use the Poincaré-Birkhoff-Witt basis of the universal enveloping algebra (from the proof of Lem. A.3) to deduce the action of H from its conjugation action on P^*. In our case, the ℓth Loewy layer of $\mathbb{Z}/p[P^*] \overset{\mathrm{def}}{=} P_{\mathbf{1}}$, with $P^* = (\mathbb{Z}/p)^2$ consists of sums of $\mathbf{1}$ (resp. $\mathbf{1}^-$) if ℓ is even (resp. odd) from 0 to $2p-2$ (resp. 1 to $2p-1$). That is the projective indecomposable module for $\mathbf{1}$.

Now list the Loewy display for the projective indecomposable modules for G_0 by tensoring the Loewy layers of the projective indecomposables for $\mathbf{1}$ with the semi-simple modules for H [**Sem**, p. 737]. In our case, the semi-simples for $\mathbb{Z}/2$ are just $\mathbf{1}$ and $\mathbf{1}^-$ giving $P_{\mathbf{1}}$ and $P_{\mathbf{1}^-}$ as the projective indecomposables, the latter having the same look as the former except you switch the levels with $\mathbf{1}$ with those with $\mathbf{1}^-$. Finally, M_0 is $\Omega_2 \overset{\mathrm{def}}{=} \ker(\psi_2 : P_{\mathbf{1}^-} \oplus P_{\mathbf{1}^-} \to \ker(P_{\mathbf{1}} \to \mathbf{1}))$ with this understanding: $\ker(P_{\mathbf{1}} \to \mathbf{1})$ has at its head $\mathbf{1}^- \oplus \mathbf{1}^-$ and ψ_2 is the map from the minimal projective $(P_{\mathbf{1}^-} \oplus P_{\mathbf{1}^-})$ that maps onto $\ker(P_{\mathbf{1}} \to \mathbf{1})$.

Using the arrows between Loewy layers that appear in Lem. A.3, we can be explicit about constructing ψ_2 (knowing the result is independent of our choices). For example, map the first copy of $P_{\mathbf{1}^-}$ in $P_{\mathbf{1}^-} \oplus P_{\mathbf{1}^-}$ so the image P' has $\mathbf{1}^-$ at its head coming from the 3rd layer of $P_{\mathbf{1}^-}$.

Then, map the second copy of $P_{\mathbf{1}^-}$ in $P_{\mathbf{1}^-} \oplus P_{\mathbf{1}^-}$ to see the head of the image in $\ker(P_{\mathbf{1}} \to \mathbf{1})/P'$ is $\mathbf{1} \oplus \mathbf{1}^-$. These summands come from the respective 2nd and 3rd Loewy layers of the copy of $P_{\mathbf{1}^-}$. That concludes the head of M_0. The rest follows by identifying $\mathbb{H}_{\mathbb{Z}/p,3} \times^s \mathbb{Z}/2$ with the quotient of G_1 that extends G_0 by $\mathbf{1}_{G_0}$. □

Appendix B
Nielsen classes for $F_2 \times^s \mathbb{Z}/3$

§6.3 used the $p = 2$ case of the **MT** with $\mathbb{Z}/3$ acting on F_2. §B.1 gives our present knowledge of limit groups here. Finally, Ex. B.3 shows the effect of Schur multiplier statements from §2.5: They account for much, but not all, of the six level 1 components for the case $p = 2$. §B.2 gives a meaning to complex multiplication by considering the F-quotient from §6.2.2 when $p \equiv 1 \mod 3$.

B.1. Limit groups for another rank 2 MT.

— The next result works by proving the existence of H-M reps. (whose shift gives example g-p' cusps as in Ex. 3.7). So, this produces $\tilde{F}_{2,p} \times^s J_3$ as a limit group for each $p \neq 3$ from Princ. 3.6.

Recall the action of α from (4.1b). It induces the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, with characteristic polynomial $x^2 + x + 1$, on the $(\mathbb{Z}/p)^2$ quotient of F_2. Denote $F_2/(F_2, F_2)$ by L_2 and its completion at p by $L_{2,p}$.

Proposition B.1. — *The $(A_4, \mathbf{C}_{\pm 2})$ **MT** for $p \neq 3$ has $\tilde{F}_{2,p} \times^s J_3$ as a limit group. For $p = 2$, the $(A_4, \mathbf{C}_{\pm 2})$ **MT** also has $L_{2,p} \times^s J_3$ as a limit group.*

Proof. — Let $G = G_p = (\mathbb{Z}/p)^2 \times^s J_3$: $\langle \alpha \rangle = J_3$. We find

$$g_1 = (\alpha, \boldsymbol{v}_1) \text{ and } g_2 = (\alpha, \boldsymbol{v}_2)$$

so that $\langle g_1, g_2 \rangle = G$. The H-M rep. $(g_1, g_1^{-1}, g_2, g_2^{-1})$ is in $\mathrm{Ni}(G, \mathbf{C}_{\pm 3^2})^{\mathrm{in}}$. Conjugate in G, to take a representative in the inner class with $\boldsymbol{v}_1 = \mathbf{0}$. Consider

$$g_1 g_2^{-1} = (1, -\boldsymbol{v}_2) \text{ and } g_1^2 g_2 = (1, \alpha^{-1}(v_2)).$$

So, g_1, g_2 generate precisely when $\langle -\boldsymbol{v}_2, \alpha^{-1}(v_2) \rangle = (\mathbb{Z}/p)^2$. Such a $\boldsymbol{v}_2$ exists because the eigenvalues of α are distinct. So $(\mathbb{Z}/p)^2$ is a cyclic $\langle \alpha \rangle$ module.

Now consider $\mathrm{Ni}(G, \mathbf{C}_{2^4})^{\mathrm{in}}$ with $G = U \times^s J_3$ and U (a $\mathbb{Z}/3$ module) having $(\mathbb{Z}/p)^2$ as a quotient. There is a surjective map $\psi : G \to (\mathbb{Z}/p)^2 \times^s J_3$: a Frattini cover. So, if g_1', g_2' generate $(\mathbb{Z}/p)^2 \times^s J_3$, then respective order 3 lifts of g_1', g_2' to $g_1, g_2 \in G$ automatically generate G. Princ. 3.6 now applies: For $p \neq 3$, an H-M cusp branch gives $\tilde{F}_{2,p} \times^s J_3$ as a limit group.

Now we turn to the case $p = 2$, and consider the other, not H-M rep., braid orbit on $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3})$ given in Prop. 6.12. [**BF02**, Cor. 5.7] gives this Loewy display for

$M_0 = \ker(G_1(A_4) \to A_4)$: $0 \to U_3 \to U_3 \oplus \mathbf{1}$ with U_3 the 2-dimensional irreducible for $\mathbb{Z}/2[A_4]$. In the augmented Loewy display, there is an arrow from the leftmost U_3 to each summand of $U_3 \oplus \mathbf{1}$.

Let $\boldsymbol{g}$ be a representative of the orbit Ni_0^- obstructed by $\hat{A}_4 \to A_4$. The completion at $p = 2$ of the quotient $F_2/(F_2, F_2) \times^s \mathbb{Z}/3$ is $L_{2,2} \times^s \mathbb{Z}/3$, a 2-Frattini cover of A_4. Notice that $\mathbf{1}_{A_4}$ is not a subquotient in this group. Therefore, Cor. 4.20 implies the map $M_{\boldsymbol{g}} \to A_4$ extends to $M_{\boldsymbol{g}} \to L_{2,2} \times^s \mathbb{Z}/3$. Indeed, it is a Nielsen limit group through the braid orbit of $\boldsymbol{g}$. □

***Example B.2* (The $(A_5, \mathbf{C}_{3^4}, p = 2)$ MT).** — We continue Ex. 4.21. Let O_2 be the non-H-M braid orbit of $\mathrm{Ni}(G_1(A_5), \mathbf{C}_{3^4})$. [**BF02**, Prop. 9.14] shows $G_1(A_5)$ embeds in A_N for several values of N (40, 60, 80, 120) with an additional property: With

$$\mathrm{Spin}_N \times_{A_N} G_1(A_5) \overset{\text{def}}{=} \mathrm{Spin}'_N \to G_1(A_5),$$

we have $s_{\mathrm{Spin}'_N}(O_2) = -1$.

Let $R'_k \to G_k$ be the $k-$1st antecedent to $\mathrm{Spin}'_N \to G_1(A_5)$ (§4.2.2). As noted in Ex. 4.13, the hypotheses of Thm. 4.12 hold for this example and each level $k \geq 1$ of the **MT** has an H-M component with at least two distinct limit groups.

***Example B.3* ($\mathrm{Ni}(G_1(A_4), \mathbf{C}_{\pm 3^2})$ braid orbits).** — Again $p = 2$. Similar to Ex. B.2, and again using Ex. 4.13, each level $k \geq 2$ has two H-M components, and each such component has at least four distinct limit groups.

***Problem B.4*.** — Let $\mathcal{H}'_k$ be one of the H-M components in Ex. B.3. Is the number of limit groups through $\mathcal{H}'_k$ bounded with k?

B.2. Complex multiplication for the $\mathbb{Z}/3$ case. — Use the notation above. If $p \neq 2, 3$, α on $(\mathbb{Z}/p)^2$ has eigenvalues defined over $\mathbb{Z}/p$ precisely when -3 is a square $\mod p$. From quadratic reciprocity, these are the $p \equiv 1 \mod 3$. Exactly then, $\tilde{F}_{2,p} \times^s J_3$ has quotients of the form $\mathbb{Z}/p \times^s J_3 = G^*$. Corresponding to that, the universal p-Frattini cover $\mathbb{Z}_p \times^s J_3$ of G^* is a quotient of $\mathbb{Z}_p^2 \times^s J_3$.

***Problem B.5*.** — When $p \equiv 1 \mod 3$, does a $\mathbb{Z}_p \times^s J_3$ quotient of $\tilde{F}_{2,p} \times^s J_3$ correspond to "complex multiplication case" for special values $j' \in \mathbb{P}^1_j$ (as in the J_2 case in Rem. A.2)? For all $j' \in \mathbb{P}^1_j$ over a number field, does this give a full analog of Serre's OIT in the J_3 case?

The nontrivial F-quotient when $p \equiv 1 \mod 3$ is like that for modular curves, a **MT** case where $M'_k = \ker(G_{k+1}(\mathbb{Z}/p \times^s J_3) \to G_k(\mathbb{Z}/p \times^s J_3))$ has rank 1 (as in Prop. 2.4). What we know of $M_k = \ker(G_{k+1}((\mathbb{Z}/p)^2 \times^s J_3) \to G_k((\mathbb{Z}/p)^2 \times^s J_3))$ (as a G_k module, to which the conclusion of Prop. 2.4 applies) is from Semmen's thesis [**Sem**]. Such information is significant in analyzing (6.3b).

Appendix C

Related Luminy talks and typos from [BF02]

Other Luminy talks contain material whose perusal simplifies our explaining the use of the Hurwitz monodromy group and the background for this paper §C.1. Our approach to explaining progress on **MT**s is to use [**BF02**] as a reference book in translating between geometric and arithmetic statements until the completion of [**Fri07**]. Our web site version of the former has typos corrected as they appear.

C.1. Conference talks that explain significant background points. — Expositional elements of the following papers support their use in **MT**s.

- Matthieu Romagny and Stefan Wewers introduced Nielsen classes and material on Hurwitz spaces.
- Kay Magaard introduced braids acting (through Hurwitz monodromy H_r; §2.4.1) on Nielsen classes, necessary for computations.
- Pierre Dèbes defined a (rank 0) Modular Tower (**MT**), comparing that with modular curves.
- The (weak; rank 0) Main Conjecture is that there are no rational points at suitably high tower levels. Pierre's talk reduced this conjecture, for four branch point towers, to showing the genus rises with the levels.
- Darren Semmen presented the profinite Frattini category. This showed how Schur multipliers control properties of the Modular Tower levels.

C.2. Typos from the printed version of [BF02]

- p. 55, line 4 of 2nd paragraph: to the near H-M and H-M [not H -M] p. 87, line 4. It also explains H-M [not H -M] and near H-M p. 87, line 8. *complements* of H-M and near H-M [not H -M] p. 89, after (8.6): H-M or near H-M [not H -M] rep. is p. 180, 3rd line of 2nd par.: [not H -M]
- p. 92: It said: "The cusp pairing for $r = 4$ should extend to the case $r \geq 5$, though we don't yet know how."

  We knew how to do that by the time the paper was complete, though we forgot to delete this line. It now says: "The cusp pairing for $r = 4$ extends to the case $r \geq 5$ (§2.10.2)."
- p. 93: 1st par. §1.4.7 (end): Change Merel-Mazur to Mazur-Merel.
- p. 94: (and image of $g_1^{-1}g_2$ in A_5 of order 5)
- p. 103–104. Use of $\mathcal{Q}''$ in Def. 2.12 on p. 103 precedes its definition on p. 104.
- Bottom of p. 107: $|\mathrm{Ni}_k^{\mathrm{in}}| = (p^{k+1} + p^k)\varphi(p^k)/2$ should be

$$|\mathrm{Ni}_k^{\mathrm{in}}| = (p^{k+1} + p^k)\varphi(p^{k+1})/2.$$

- Statement of Prop. 2.17. [States the condition $o(g_1, g_2)$ is odd, after it uses that condition.] It should say this. Let $g_1g_2 = g_3$, and $g_2g_1 = g_3'$. Let

$o(g_1, g_2) = o$ (resp. $o'(g_1, g_2) = o'$) be the length of the orbit of γ^2 (resp. γ) on (g_1, g_2). If $g_1 = g_2$, then $o = o' = 1$.
Proposition 2.17 *Assume $g_1 \neq g_2$. The orbit of γ^2 containing (g_1, g_2) is $(g_3^j g_1 g_3^{-j}, g_3^j g_2 g_3^{-j})$, $j = 0, \dots, \mathrm{ord}(g_3){-}1$. So,*

$$o = \mathrm{ord}(g_3)/|\langle g_3\rangle \cap Z(g_1, g_2)| \overset{\mathrm{def}}{=} o(g_1, g_2).$$

Then, $o' = 2 \cdot o$, unless o is odd, and with $x = (g_3)^{(o-1)/2}$ and $y = (g_3')^{(o-1)/2}$ (so $g_1 y = x g_1$ and $y g_2 = g_2 x$), $y g_2$ has order 2. Then, $o' = o$.

- p. 129: Title of Section 4 should be: [Moduli] and reduced Modular Towers (change “Modular” to “Moduli”).
- p. 140: Reference to [Fr01] changed to [Fr02]: and a more precise quote: [Fr02, Prop. 2.8]: M. Fried, Moduli of relatively nilpotent extensions, Institute of Mathematical Science Analysis 1267, June 2002, Communications in Arithmetic Fundamental Groups, 70–94.
- p. 160, line 22: as $(u, v) = (\mp(\boldsymbol{g}), \mathbf{wd}(bg))$ should be, as
$$(u, v) = (\mp(\boldsymbol{g}), \mathbf{wd}(\boldsymbol{g})).$$
- p. 172: 1st par. of Prop. 8.12, change “(and $g_1^{-1} g_2$ of order 5)” to “(and image of $g_1^{-1} g_2$ in A_5 of order 5).”
- p. 180: 1st line of 2nd paragraph of §9: Orbits of $\langle\gamma_1, q_2\rangle$ should be [Orbits of $\langle\gamma_1, \gamma_\infty\rangle$], to emphasize here we view q_2 as in $\bar{M}_4$.
- Bottom p. 184: G_{k+1} [acts] trivially on ...
- p. 188: Def. 9.11: ${}^T\hat{H}$ should be ${}^T\hat{G}$.
- Ex. 9.19: The 3rd sentence should be: For this case, $\mathrm{tr}(T_{H'}(m)) = 4 = \mathrm{tr} T_{H'}(m')$ and $\mathrm{tr}(T_{H'}(mm')) = 8$: $m, m' \in C_{18}$ and $mm' \in C_{16}$.

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