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SOME LIMITING SITUATIONS FOR SEMILINEAR ELLIPTIC EQUATIONS

by

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Abstract. — The objective of this mini-course is to take a look at a standard semilinear partial differential equation $-\Delta u = \lambda f(u)$ on which we show the use of some basic tools in the study of elliptic equation. We will mention the maximum principle, barrier method, blow-up analysis, regularity and boot-strap argument, stability, localization and quantification of singularities, Pohozaev identities, moving plane method, etc.

***Résumé* (Quelques situations limites pour les équations semi-linéaires elliptiques)**
L'objectif de ce mini-cours est de jeter un coup d'œil sur une équation aux dérivées partielles standard $-\Delta u = \lambda f(u)$, avec laquelle nous allons montrer quelques outils de base dans l'étude des équations elliptiques. Nous mentionnerons le principe du maximum, la méthode de barrière, l'analyse de blow-up, la régularité, l'argument de boot-strap, la stabilité, la localisation et quantification de singularités, les identités de Pohozaev, la méthode du plan mobile, etc.

1. Introduction

We consider the following semilinear partial differential equation:

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and f is a smooth positive, nondecreasing and convex function over $\mathbb{R}_+$. For getting a positive solution u, necessarily λ is positive.

The convexity of f implies that

- $\lim_{t\to\infty} f(t)/t = a \in \mathbb{R}_+ \cup \{\infty\}$ exists.
- If $a \in \mathbb{R}_+$, then $\lim_{t\to\infty} f(t) - at = l \in \mathbb{R} \cup \{-\infty\}$ exists.

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Since the case $a = 0$ is trivial ($f \equiv$ *constant*), we will suppose that $a > 0$. Thus we can divide the study of problem (P_λ) into two different situations: the quasilinear case when $a \in (0, \infty)$ and superlinear case when $a = \infty$. We will see that the first case is rather well understood, while many questions are remained open for the second one.

In the following, $\|\cdot\|_p$ denotes the standard L^p norm for $1 \leq p \leq \infty$. $W^{1,p}(\Omega)$ is the Sobolev space of functions f such that f and $\nabla f \in L^p(\Omega)$. When $p = 2$, we use for simplicity $H^1(\Omega)$ to denote $W^{1,2}(\Omega)$, $H_0^1(\Omega)$ is the space of functions $f \in H^1(\Omega)$ verifying $f = 0$ on $\partial\Omega$. The symbol C means always a positive constant independent of λ.

2. Quasilinear situation

We begin with the quasilinear case where $a \in (0, \infty)$. Many results presented here are obtained by Mironescu & Rădulescu in [**27**].

2.1. Minimal solution and stability. — Since $f(u) \leq au + f(0)$ in this case, then if $u \in L^1(\Omega)$ is a weak solution of (P_λ) in the sense of distribution, we get easily that u is always a classical solution by standard boot-strap argument.

Lemma 2.1. — *For $\lambda > 0$, if (P_λ) is resolvable, then a minimal solution u_λ exists in the sense that any solution v of (P_λ) verifies $v \geq u_\lambda$ in Ω. Moreover, $(P_{\lambda'})$ is resolvable for any $\lambda' \in (0, \lambda)$.*

Proof. — We will use the barrier method. Remark that for $\lambda > 0$, $w_0 \equiv 0$ is a sub solution of (P_λ) since $f(0) > 0$. Now we define for any $n \in \mathbb{N}$,

$$-\Delta w_{n+1} = \lambda f(w_n) \text{ in } \Omega, \quad w_{n+1} = 0 \text{ on } \partial\Omega. \tag{1}$$

Using maximum principle, $w_1 > w_0 \equiv 0$ in Ω. On the other hand, let v be any solution of (P_λ), by monotonicity of f, we obtain

$$-\Delta(w_1 - v) = \lambda\,[f(0) - f(v)] \leq 0 \text{ in } \Omega, \quad w_1 - v = 0 \text{ on } \partial\Omega.$$

Thus $w_1 \leq v$ in Ω. We can prove by induction that the sequence $\{w_n\}$ verifies $w_n \leq w_{n+1} \leq v$ in Ω for any n, so $u_\lambda = \lim_{n\to\infty} w_n$ is well defined, and u_λ is a solution of (P_λ) by passing to the limit in (1). Moreover, $u_\lambda \leq v$. Notice that the definition of u_λ is independent of the choice of v, it is the minimal solution claimed.

If (P_λ) has a solution u, it is a super solution for $(P_{\lambda'})$ when $0 < \lambda' < \lambda$. As $\omega_0 \equiv 0$ is always a sub solution, the barrier method will solve as above $(P_{\lambda'})$. □

Let λ_1 be the first eigenvalue of $-\Delta$ on Ω with the Dirichlet boundary condition, we define φ_0 to be the first eigenfunction such that $\varphi_0 > 0$ in Ω and $\|\varphi_0\|_2 = 1$.

Lemma 2.2. — *If we denote $r_0 = \inf_{t>0} f(t)/t$, then (P_λ) has no solution for $\lambda > \lambda_1/r_0$. On the other hand, (P_λ) is resolvable for $\lambda > 0$ small enough.*

Proof. — Let $\xi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the solution of $-\Delta\xi = 1$ in Ω. It is easy to see that ξ is a super solution of (P_λ) for $\lambda \le f(\|\xi\|_\infty)^{-1}$. Applying the barrier method, we get a solution of (P_λ) for such λ.

Now we suppose that u is a solution of (P_λ) for some $\lambda > 0$, using φ_0 as test function and integrating by parts, we get

$$\lambda_1 \int_\Omega \varphi_0 u dx = -\int_\Omega u\Delta\varphi_0 dx = -\int_\Omega \varphi_0 \Delta u dx = \lambda \int_\Omega f(u)\varphi_0 dx.$$

As $f(u) \ge r_0 u$ in Ω, we have then

$$(\lambda_1 - \lambda r_0)\int_\Omega \varphi_0 u dx \ge 0.$$

Recalling that φ_0 and u are positive in Ω, the lemma is proved. □

Combining these two lemmas, we can claim

Theorem 2.3. — *There exists a critical value $\lambda^* \in (0,\infty)$ for the parameter λ, such that for any $\lambda > \lambda^*$, no solution exists for the problem (P_λ) while for any $\lambda \in (0,\lambda^*)$, a unique minimal solution u_λ exists for (P_λ). Furthermore the mapping $\lambda \mapsto u_\lambda$ is increasing with λ.*

It is natural to ask if we can determine the exact value of λ^* and what happens when $\lambda = \lambda^*$. Before considering these two questions, we show another characterization of the minimal solution u_λ, its stability. A solution u of (P_λ) is called stable if and only if the linearized operator associated to the equation, $-\Delta - \lambda f'(u)$ is nonnegative. More precisely,

$$\lambda \int_\Omega f'(u)\varphi^2 dx \le \int_\Omega |\nabla\varphi|^2 dx, \quad \text{for any} \quad \varphi \in H_0^1(\Omega). \tag{2}$$

Theorem 2.4. — *Let $\lambda \in (0,\lambda^*)$, the minimal solution u_λ is the unique stable solution of (P_λ).*

Proof. — First we prove that u_λ is stable. If it is not true, the first eigenvalue η_1 of $-\Delta - \lambda f'(u_\lambda)$ is negative, then there exists an eigenfunction $\psi \in H_0^1(\Omega)$ such that

$$-\Delta\psi - \lambda f'(u_\lambda)\psi = \eta_1\psi \ \text{ in } \Omega, \quad \psi > 0 \ \text{ in } \Omega.$$

Consider $u^\varepsilon = u_\lambda - \varepsilon\psi$, a direct calculation gives

$$-\Delta u^\varepsilon - \lambda f(u^\varepsilon) = -\eta_1\varepsilon\psi - \lambda\left[f(u_\lambda - \varepsilon\psi) - f(u_\lambda) + \varepsilon f'(u_\lambda)\psi\right] = \varepsilon\psi\left[-\eta_1 + o_\varepsilon(1)\right].$$

Since $\eta_1 < 0$, then $-\Delta u^\varepsilon - \lambda f(u^\varepsilon) \ge 0$ in Ω for $\varepsilon > 0$ small enough. Otherwise, using Hopf's lemma, we know that $u_\lambda \ge C\psi$ in Ω for some $C > 0$. Thus $u^\varepsilon \ge 0$ is a super solution of (P_λ) for $\varepsilon > 0$ small enough. As before, we can get a solution u such that $u \le u^\varepsilon$ in Ω, which contradicts the minimality of u_λ. So $\eta_1 \ge 0$.

Now we prove that (P_λ) has at most one stable solution. Suppose the contrary, there exists another stable solution $v \ne u_\lambda$. Define $\varphi = v - u_\lambda$, we get

$$\lambda\int_\Omega f'(v)\varphi^2 dx \le \int_\Omega |\nabla\varphi|^2 dx = -\int_\Omega \varphi\Delta\varphi dx = \lambda\int_\Omega \left[f(v) - f(u_\lambda)\right]\varphi dx,$$

so
$$\int_{\Omega}[f(v)-f(u_\lambda)-f'(v)(v-u_\lambda)]\,\varphi dx \geq 0.$$
By maximum principle, we know that $\varphi > 0$ in Ω. The convexity of f yields that the term in the bracket is non positive, so the only possibility is $f(v)-f(u_\lambda)-f'(v)(v-u_\lambda)\equiv 0$ in Ω, which means f is affine over $[u_\lambda(x), v(x)]$ for any $x\in\Omega$. Thus $f(x)=\bar{a}x+b$ in $[0,\max_\Omega v]$ and we get two solutions u and v of $-\Delta w=\bar{a}w+b$. This implies that
$$0=\int_{\Omega}u_\lambda\Delta v - v\Delta u_\lambda dx = b\int_{\Omega}(v-u_\lambda)dx = b\int_{\Omega}\varphi dx,$$
which is impossible since $b=f(0)>0$ and φ is positive in Ω. So we are done. □

An immediate consequence of Theorem 2.4 is

Proposition 2.5. — *For any $\lambda\in(0,\lambda_1/a)$, (P_λ) has one and unique solution u_λ.*

Proof. — Remark first $a=\sup_{\mathbb{R}_+}f'(t)$ by convexity of f. Thanks to the definition of λ_1, it is clear that each solution is stable if $\lambda\in(0,\lambda_1/a)$, so we get the uniqueness by that for stable solution. For the existence, we can consider the minimization problem $\min_{H_0^1(\Omega)} J(u)$ where
$$J(u)=\frac{1}{2}\int_{\Omega}|\nabla u|^2dx-\lambda\int_{\Omega}F(u)dx$$
with
$$F(u)=\int_0^{u^+}f(s)ds,\quad u^+=\max(u,0).$$
If $\lambda\in(0,\lambda_1/a)$, there exist ε, $A>0$ depending on λ such that $2\lambda F(t)\leq(\lambda_1-\varepsilon)t^2+A$ over $\mathbb{R}$. Thus $J(u)$ is coercive, bounded from below and weakly lower semi-continuous in $H_0^1(\Omega)$, the infimum of J is reached then by a function $u\in H_0^1(\Omega)$, so also by $u^+\in H_0^1(\Omega)$ since $J(u^+)\leq J(u)$. This critical point $u\geq 0$ of J gives a solution of (P_λ). □

2.2. Estimate of λ^*. — By Proposition 2.5, we know that $\lambda^*\geq\lambda_1/a$. The following result in [**27**] gives us more precise information for λ^*.

Theorem 2.6. — *We have three equivalent assertions:*

(i) $\lambda^*=\lambda_1/a$.
(ii) *No solution exists for (P_{λ^*}).*
(iii) $\lim_{\lambda\to\lambda^*}u_\lambda=\infty$ *u.c. in Ω. (u.c. means "uniformly on each compact set")*

Proof. — *(i) implies (ii)*. If (P_{λ^*}) has a solution u, then $u_\lambda\leq u$ in Ω for any $\lambda\in(0,\lambda^*)$, using the monotonicity of u_λ, $u^*=\lim_{\lambda\to\lambda^*}u_\lambda$ is well defined and u^* is clearly a stable solution of (P_{λ^*}) by limit. Consider the operator $G(u,\lambda)=-\Delta u-\lambda f(u)$, if the first eigenvalue η_1 of $-\Delta-\lambda^*f'(u^*)$ is positive, then we can apply the Implicit Function Theorem to get a solution curve in a neighborhood of λ^*, but this contradicts the definition of λ^*, so $\eta_1=0$. Thus, there exists $\psi\in H_0^1(\Omega)$ satisfying

$$-\Delta\psi-\lambda^*f'(u^*)\psi=0\ \text{ and }\ \psi>0\ \text{ in }\Omega. \tag{3}$$

Using φ_0 as test function and integrating by parts, we get

$$\int_\Omega [\lambda_1 - \lambda^* f'(u^*)]\, \psi\varphi_0 dx = 0.$$

As $\lambda_1 - \lambda^* f'(u^*) \geq 0$, we get $f'(u^*) \equiv a$ in Ω so that $f(t) = at + b$ in $[0, \max_\Omega u^*]$. But $b > 0$ deduces that no positive solution in $H_0^1(\Omega)$ can exist for the equation $-\Delta u = \lambda_1 u + b\lambda_1/a$ (we can use again φ_0), so the hypothesis is not true.

(ii) implies (iii). Here we mention a result of Hörmander (see [**22**]) as follows: *For a sequence of nonnegative super-harmonic functions $\{v_n\}$ in Ω, either v_n converges u.c. to ∞; or there exists a subsequence which converges in $L^1_{loc}(\Omega)$.* We need just to prove that the second case cannot occur for u_λ. Suppose the contrary, there exist $u_k = u_{\lambda_k}$ which converges in $L^1_{loc}(\Omega)$ to u^* with $\lambda_k \to \lambda^*$. We claim that $\|u_k\|_2 \leq C$. If it is false, we define $u_k = l_k w_k$ with $\|w_k\|_2 = 1$ and $\lim_{k\to\infty} l_k = \infty$ (up to subsequence). Since $f(t) \leq at + f(0)$,

$$-\Delta w_k = \frac{\lambda_k f(u_k)}{l_k} \leq a\lambda_k w_k + \frac{\lambda_k f(0)}{l_k} \leq a\lambda_k w_k + C \quad \text{in } \Omega,$$

it is easy to see that w_k is bounded in $H_0^1(\Omega)$, so that up to a subsequence, w_k converges weakly in H_0^1 and strongly in L^2 to some $w \in H_0^1$. Meanwhile, $-\Delta w_k$ tends to zero in $L^1_{loc}(\Omega)$ since $f(u_k) \leq au_k + b$ and l_k tends to ∞, this implies $-\Delta w = 0$ in Ω. Hence $w \equiv 0$, which is impossible because $\|w\|_2 = \lim_{k\to\infty} \|w_k\|_2 = 1$. So $\{u_k\}$ is bounded in $L^2(\Omega)$, hence in $H_0^1(\Omega)$ by equation. We prove readily that u^* is a solution of (P_{λ^*}) which contradicts (ii).

(iii) implies (ii). Any solution u of (P_{λ^*}) should satisfy $u \geq u_\lambda$, $\forall\, \lambda < \lambda^*$.

(ii)$\oplus$(iii) implies (i). Clearly $\lim_{\lambda\to\lambda^*} \|u_\lambda\|_2 = \infty$. Take $u_\lambda = l_\lambda w_\lambda$ with $\|w_\lambda\|_2 = 1$, then we have a subsequence w_k which converges weakly in H_0^1, strongly in L^2 and almost everywhere to $w \geq 0$. Moreover, in the sense of distribution,

$$-\Delta w = -\lim_{D'(\Omega)} \Delta w_k = \lim \frac{\lambda_k f(l_k w_k)}{l_k} = \lambda^* a w \geq \lambda_1 w \quad \text{a.e.}$$

Taking again φ_0 as test function, we see that the last inequality must be an equality, so $\lambda^* = \lambda_1/a$. ◻

Remark that when $f(t) \geq at$ in $\mathbb{R}_+$, we cannot get a solution for $\lambda = \lambda_1/a$ since $f(t) > at$ in a neighborhood of 0 (using always φ_0 as test function), we obtain an important consequence of Theorem 2.6 and Proposition 2.5.

Corollary 2.7. — *If we have $\lim_{t\to\infty} f(t) - at = l \geq 0$, then $\lambda^* = \lambda_1/a$, and a unique solution u_λ exists for (P_λ) for $\lambda \in (0, \lambda^*)$ while no solution exists for $\lambda \geq \lambda^*$.*

Moreover, the following result is established in [**27**].

Proposition 2.8. — *If $\lim_{t\to\infty} f(t) - at = l < 0$, then $\lambda_1/a < \lambda^* < \lambda_1/r_0$. A unique solution $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ exists for (P_{λ^*}). Furthermore, for any $\lambda \in (\lambda_1/a, \lambda^*)$, we have a second solution v_λ for (P_λ), such that v_λ tends u.c. to ∞ in Ω when $\lambda \downarrow \lambda_1/a$.*

Sketch of Proof. — A second solution v_λ is obtained by the standard Mountain-pass theory. We check here just $\lambda^* < \lambda_1/a$ and the uniqueness of solution for (P_{λ^*}). If $\lambda^* = \lambda_1/a$, by Theorem 2.6, $u_\lambda \to \infty$ u.c. to ∞ as λ tends to λ^*. Taking the first eigenfunction φ_0, for $\lambda < \lambda^*$, as $\lambda a \le \lambda_1$

$$0 = \int_\Omega \varphi_0 \left[\Delta u_\lambda + \lambda f(u_\lambda)\right] dx = \int_\Omega \varphi_0 \left[\lambda f(u_\lambda) - \lambda_1 u_\lambda\right] dx \le \lambda \int_\Omega \varphi_0 \left[f(u_\lambda) - a u_\lambda\right] dx.$$

Passing λ to λ^*, we get

$$0 \le \lambda l \int_\Omega \varphi_0 dx < 0,$$

which is absurd, hence $\lambda^* > \lambda_1/a$ if $l < 0$.

As we have $\lambda^* > \lambda_1/a$, a solution v exists for (P_{λ^*}) by Theorem 2.6. By the proof of Theorem 2.6 (see step *(i) implies (ii)*), we can claim that $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is a solution of (P_{λ^*}), $u^* \le v$ and $\eta_1(-\Delta - \lambda^* f'(u^*)) = 0$. If $v \ne u^*$, consider $w = v - u^*$, we have $w > 0$ in Ω and

$$-\Delta w = \lambda^*\big[f(v) - f(u^*)\big] \ge \lambda^* f'(u^*) w \quad \text{in } \Omega.$$

Using the eigenfunction ψ verifying (3) (with $\eta_1 = 0$),

$$0 = \int_\Omega \big(\psi\Delta w - w\Delta\psi\big) dx \le \lambda^* \int_\Omega \big[f'(u^*) w\psi - f'(u^*)\psi w\big] dx = 0.$$

Therefore, we must have the equality $f(v) - f(u^*) = f'(u^*)w$ in Ω, which yields that f is linear in $[0, \max_\Omega v]$ and leads to a contradiction as in the proof of Theorem 2.4. □

It is also proved in [**27**] that if $l \ge 0$, the normalized family $w_\lambda = u_\lambda/\|u_\lambda\|_2$ converges to φ_0 in $H_0^1(\Omega)$ as $\lambda \uparrow \lambda^* = \lambda_1/a$. In the case $l > 0$ or some special cases for $l = 0$, they showed a first order expansion of the norm $\|u_\lambda\|_2$ in function of $(\lambda^* - \lambda)$. If $l < 0$, similar results were obtained for v_λ when $\lambda \downarrow \lambda_1/a$. In conclusion, all these results give us a rather clear schema of solutions for the quasilinear case.

3. Superlinear situation

From now on, we suppose that

(4) $$f \text{ is a positive, smooth, nondecreasing and convex function such that } \lim_{t\to\infty} \frac{f(t)}{t} = \infty.$$

With minor changes, Theorems 2.3 and 2.4 hold still, so we can ask always the questions as how to determinate λ^* or how to understand the asymptotic behavior of u_λ when λ tends to λ^*. In quasilinear case, we can prove by standard regularity theory that any weak solution in $L^1(\Omega)$ is a classical solution, and when λ tends to λ^*, either u_λ tends to infinity on each point of Ω, or u_λ tends to a classical solution for the limiting problem (P_{λ^*}). We will see that it is no longer true for the superlinear case, unbounded weak solutions can exist. In [**7**], Brezis *et al.* have proposed the following definition.

Definition 3.1. — *A function $\xi \in L^1(\Omega)$ is called a weak solution of (P_λ), if $f(\xi)d(x,\partial\Omega) \in L^1(\Omega)$ and*

$$-\int_\Omega \xi\Delta\varphi dx = \lambda\int_\Omega f(\xi)\varphi dx \tag{5}$$

for any $\varphi \in C^2(\overline{\Omega}) \cap \{\varphi|_{\partial\Omega} = 0\}$.

They proved then

Theorem 3.2. — *$u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is always a weak solution of (P_{λ^*}).*

Proof. — For any $\lambda \in [\lambda^*/2, \lambda^*)$, taking φ_0 as test function,

$$\lambda_1\int_\Omega \varphi_0 u_\lambda dx = \lambda\int_\Omega f(u_\lambda)\varphi_0 dx \geq \frac{\lambda^*}{2}\int_\Omega f(u_\lambda)\varphi_0 dx. \tag{6}$$

Since f is superlinear, there exists $C > 0$ such that $\lambda_1 t \leq \lambda^* f(t)/4 + C$ in $\mathbb{R}_+$. Using (6), we get

$$\int_\Omega f(u_\lambda)\varphi_0 dx \leq C, \quad \text{for any } \frac{\lambda^*}{2} \leq \lambda < \lambda^*.$$

Now let $-\Delta\xi = 1$ in Ω and $\xi = 0$ on $\partial\Omega$, we have

$$\int_\Omega u_\lambda dx = -\int_\Omega u_\lambda\Delta\xi dx = \lambda\int_\Omega f(u_\lambda)\xi dx \leq C_1\lambda^*\int_\Omega f(u_\lambda)\varphi_0 dx \leq C_2.$$

We have used the fact $\xi \leq C\varphi_0$ (by Hopf's lemma) to get the first inequality. Tending λ to λ^*, we obtain $u^* \in L^1(\Omega)$ and $f(u^*)d(x,\partial\Omega) \in L^1(\Omega)$, since $\varphi_0 \geq Cd(x,\partial\Omega)$. Now it is easy to verify (5) for u^*. □

On the other hand, it is proved in [**7**] that for any $\lambda > \lambda^*$, no weak solution exists for (P_λ). Later on, Martel proved in [**25**] that u^* is the unique weak solution for (P_{λ^*}), so u^* is really the extreme solution on the right in the schema (λ, u), we call it the *extremal solution*.

3.1. Regularity of u^*. — By classical examples, we know that u^* can be either a classical solution or not. The most well known cases are exponential and polynomial situations (see [**19**], [**23**], [**11**], [**26**] and [**9**]).

- For $f(u) = e^u$, u^* is smooth when $N \leq 9$. If $N \geq 10$ and Ω is the unit ball $B_1(0)$, $u^* = -2\log|x|$ is the extremal solution, hence no longer bounded.
- For $f(u) = (u+1)^p$ with $p > 1$, if

  $$N < N_p = 6 + \frac{4}{p-1} + \frac{4\sqrt{p(p-1)}}{p-1},$$

  u^* is smooth, and for $N \geq N_p$, $u^* = |x|^{-\frac{2}{p-1}} - 1$ is the extremal solution on $B_1(0)$. An immediate consequence is that for any $p > 1$ and $N \leq 10$, u^* is a smooth solution.

When $f(u) = e^u$ and $N \geq 3$, we can verify that $U(x) = -2\log|x|$ is always a weak solution of (P_λ) with $\Omega = B_1(0) \subset \mathbb{R}^N$ and $\lambda = 2(N-2)$. So not all unbounded weak solutions are extremal solutions u^*. Two natural questions are raised.

– For general superlinear nonlinearity f satisfying (4), when is the extremal solution u^* smooth?
– How can we know whether an unbounded weak solution is just u^*?

The key of the second question is the stability of u^*. Since u_λ are stable, passing to the limit, we know that (2) holds always for u^*. We look at the example where $f(u) = e^u$ and $\Omega = B_1(0)$. Consider $U(x) = -2\log|x|$, a necessary condition to have $U(x) = u^*$ is then the positivity of the operator $-\Delta - 2(N-2)r^{-2}$ where $r = |x|$. On the other hand, we have Hardy's inequality for $H_0^1(\Omega)$, which is optimal:

$$\int_\Omega |\nabla\varphi|^2 dx \geq \frac{(N-2)^2}{4}\int_\Omega \frac{\varphi^2}{r^2}dx, \quad \forall\, \varphi \in H_0^1(\Omega).$$

Thus we need to have $2(N-2) \leq (N-2)^2/4$ which is just equivalent to $N \geq 10$.

Brezis & Vázquez showed the following general result, whose proof is similar to that for the uniqueness of stable solution in Theorem 2.4.

Theorem 3.3. — *If $v \in H_0^1(\Omega)$ is an unbounded solution for (P_λ) such that the stability condition* (2) *is satisfied, then $\lambda^* = \lambda$ and $u^* = v$.*

For the regularity of extremal solution u^*, we can remark that u^* is smooth for low dimensions in general. By standard boot-strap argument, in order to show that u^* is a classical solution, it is sufficient to prove $u^* \in L^\infty(\Omega)$, therefore it suffices to prove that $\|u_\lambda\|_\infty$ remains uniformly bounded. The first result was obtained by Crandall & Rabinowitz [**11**]:

Theorem 3.4. — *Let f verify* (4). *Suppose moreover there exist t_0, β, $\mu > 0$ such that $\mu \leq \beta < 2 + \sqrt{\mu} + \mu$ and $\beta f'^2(t) \geq f(t)f''(t) \geq \mu f'^2(t)$ for $t \geq t_0$. Then $\|u_\lambda\|_\infty$ is uniformly bounded in $(0, \lambda^*)$ if $N < 4 + 2\mu + 4\sqrt{\mu}$.*

Recently, Nedev proved a remarkable result in [**31**]:

Theorem 3.5. — *Let f verify* (4), *then if $N = 2$ or 3, u^* is smooth solution for (P_{λ^*}). Furthermore, if $N \geq 4$, $u^* \in L^q(\Omega)$ for any $q < \frac{N}{N-4}$ and $f(u^*) \in L^q(\Omega)$ for all $q < \frac{N}{N-2}$.*

The main idea is always to make use of the stability of u_λ. Let φ, ψ be two smooth functions satisfying $\varphi(0) = \psi(0) = 0$ and $\psi' = \varphi'^2$, take $\varphi(u_\lambda)$ as test function in (2), we get

$$\begin{aligned}\lambda\int_\Omega f'(u_\lambda)\varphi^2(u_\lambda)dx \leq \int_\Omega |\nabla\varphi(u_\lambda)|^2 dx &= \int_\Omega \psi'(u_\lambda)\nabla u_\lambda \nabla u_\lambda dx \\ &= \lambda\int_\Omega f(u_\lambda)\psi(u_\lambda)dx.\end{aligned} \tag{7}$$

Thus for any $\lambda \in (0, \lambda^*)$,

$$\int_\Omega f'(u_\lambda)\varphi^2(u_\lambda)dx \leq \int_\Omega f(u_\lambda)\psi(u_\lambda)dx. \tag{8}$$

Now we need to choose suitable φ which leads to some interesting estimates of $f(u_\lambda)$ or u_λ. For example, Nedev used just the test function $\varphi(t) = f(t) - f(0)$.

We look again at the exponential case $f(u) = e^u$, let $\varphi(u) = e^{\alpha u} - 1$ with $\alpha > 0$, then $\psi(u) = \alpha(e^{2\alpha u} - 1)/2$. The inequality (8) gives

$$\int_\Omega e^{u_\lambda}\left(e^{\alpha u_\lambda} - 1\right)^2 dx \leq \int_\Omega \frac{\alpha e^{u_\lambda}}{2}\left(e^{2\alpha u_\lambda} - 1\right) dx,$$

hence

$$\left(1 - \frac{\alpha}{2}\right)\int_\Omega e^{(2\alpha+1)u_\lambda}dx \leq \int_\Omega 2e^{(\alpha+1)u_\lambda} + \left(1 - \frac{\alpha}{2}\right)e^{u_\lambda}dx \leq \varepsilon\int_\Omega e^{(2\alpha+1)u_\lambda}dx + C_\varepsilon$$

for any $\varepsilon > 0$. So if we take $\alpha < 2$ and $\varepsilon < 1 - \alpha/2$, we obtain

$$\int_\Omega e^{(2\alpha+1)u_\lambda}dx \leq C_\alpha.$$

This means that $\|f(u_\lambda)\|_p \leq C_p$ for any $p < 5$, so $\|u_\lambda\|_{W^{2,p}}$ is bounded for any $p < 5$. We know that $W^{2,p}(\Omega) \subset L^\infty(\Omega)$ for $p > N/2$, which means $\|u_\lambda\|_\infty \leq C$ if $N \leq 9$.

In [**36**], we have proved a general result under a weak additional condition on f.

Theorem 3.6. — *Let f satisfy (4), rewrite $f(t) = f(0) + te^{g(t)}$. Then $\lim_{t\to\infty} t^2 g'(t) = \infty$. Assume in addition that there exists $t_0 > 0$ such that $t^2 g'(t)$ is nondecreasing in $[t_0, \infty)$, we have then u^* is a smooth solution, for all $\Omega \subset \mathbb{R}^N$ with $N \leq 9$.*

This result is almost optimal, since by the example of e^u, we see that the result fails in general for $N \geq 10$. Moreover, our result is valid for any usual superlinear nonlinearity f, because the corresponding function $t^2 g'$ will not change infinitely its variation near ∞, so it works for weak superlinear functions as $f(t) = t\log\log\ldots\log t$ (for t near ∞), or for strong nonlinearities as $f(t) = e^{e^{\cdots e^t}}$.

Conversely, the worst situation is when f looks like piecewise affine. In other words, when f' changes infinitely its speed of acceleration, then we could never verify the condition required for g.

Proof. — Using (4), we can prove that $\lim_{t\to\infty}(tf' - f)/f' = \infty$. So we get

$$t^2 g' = \frac{t\,[tf' - f + f(0)]}{f - f(0)} \geq \frac{tf' - f}{f'} \longrightarrow \infty.$$

To prove the regularity of u^*, we need the following lemma, whose proof is given by boot-strap argument (see [**36**]).

Lemma 3.7. — *Assume that for $p > 1$, $\sigma \in [1, p)$, there exists $C > 0$ satisfying*

$$\int_\Omega f(u_\lambda)dx + \int_\Omega \frac{\widetilde{f}^p(u_\lambda)}{u_\lambda^{p-\sigma}}dx \leq C, \quad \forall\, \lambda \in (0, \lambda^*) \tag{9}$$

where $\widetilde{f}(t) = f(t) - f(0)$. *Then*

– *if* $p > N/2$, u_λ *is uniformly bounded in* $L^\infty(\Omega)$*;*
– *if* $p \leq N/2$, $\|u_\lambda\|_q \leq C$, $\forall\, q < \frac{\sigma N}{N-2p}$ *and* $\|f(u_\lambda)\|_q \leq C$, $\forall\, q < \frac{\sigma N}{N-2p+2\sigma}$.

Take now $\varphi(t) = te^{\alpha g(t)}$ $(\alpha \geq 0)$ and

$$\psi(t) = \int_0^t \varphi'^2(s)ds,$$

Using integration by parts and the monotonicity of $t^2 g'$ in $[t_0, \infty)$, we may claim

$$\psi(t) \leq C + \left[t + \frac{\alpha}{2}t^2 g'(t)\right] e^{2\alpha g(t)}, \quad \forall\, t \in \mathbb{R}_+ \tag{10}$$

Inserting this estimate in (8), we get

$$\left(1 - \frac{\alpha}{2}\right)\int_\Omega u^3 g'(u) e^{(2\alpha+1)g(u)} dx \leq \int_\Omega \left[C + Cue^{g(u)} + ue^{2\alpha g(u)} + \frac{\alpha}{2}u^2 g'(u) e^{2\alpha g(u)}\right] dx.$$

Since $\lim_{t\to\infty} e^{g(t)} = \lim_{t\to\infty} t^2 g'(t) = \infty$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left(1 - \frac{\alpha}{2} - \varepsilon\right)\int_\Omega u^3 g'(u) e^{(2\alpha+1)g(u)} dx \leq C_\varepsilon.$$

Thus for any $\alpha < 2$, there exists $C > 0$ such that

$$\int_\Omega \frac{\widetilde{f}(u_\lambda)^{2\alpha+1}}{u_\lambda^{2\alpha}} dx = \int_\Omega u^3 g'(u) e^{(2\alpha+1)g(u)} dx \leq C, \quad \text{for all } \lambda \in (0, \lambda^*).$$

By Lemma 3.7, the extremal solution u^* is bounded if $N \leq 9$. □

When u^* is just a weak solution, it is interesting to have its regularity in some Sobolev spaces, one motivation comes also from Theorem 3.3. In [**36**], we prove

Theorem 3.8. — *Let* f *verify* (4) *and rewrite* $f = e^g$*, assume that*

$$\limsup_{t\to\infty} \frac{-g''(t)}{g'^2(t)} = 1 - \mu \tag{11}$$

for $\mu \in (0,1)$. *Then* u^* *is bounded if* $N < 6 + 4\sqrt{\mu}$. *Furthermore, if* $N \geq 6 + 4\sqrt{\mu}$, *we have*

$$u^* \in L^q(\Omega), \ \ \forall\, q < \frac{2(1+\sqrt{\mu})N}{N - 6 - 4\sqrt{\mu}} \quad \textit{and} \quad f(u^*) \in L^q(\Omega), \ \ \forall\, q < \frac{2(1+\sqrt{\mu})N}{N-2}.$$

The condition (11) is equivalent to $\liminf_{t\to\infty} ff''/f'^2 = \mu$, which is a little bit stronger than the convexity of f, but much less restrictive than conditions in Theorem 3.4, since we do not need any upper bound for ff''/f'^2. Remark also that under the condition (11), we have always $u^* \in H^2(\Omega)$ since $f(u^*)$ belongs to $L^2(\Omega)$, so we can apply Theorem 3.3.

Furthermore, Theorem 3.8 shows also that if f is strongly nonlinear such that $\mu > 9/16$, then u^* is a classic solution when $N \leq 9$.

Another interesting question on the regularity of extremal solution u^* is to understand if it depends only on topological properties of the domain Ω or it depends also on geometrical properties of Ω. For example, the following question appeared in [**9**]: Let $f(u) = e^u$ and Ω be arbitrary smooth bounded domain in $\mathbb{R}^N$ with $N \geq 10$, do we have always $\|u^*\|_\infty = \infty$? Recently, Davila & Dupaigne have given a negative answer in [**14**].

4. Blow up analysis

When u^* is smooth, we know by Crandall-Rabinowitz's theory that (λ^*, u^*) is a turning point in the solution schema (λ, u), that is for $\lambda < \lambda^*$ but near λ^*, a second solution exists. We are interested in the behavior of this branch of unstable solutions. In this direction, no general conclusion can be obtained, since the behavior depends strongly on the nonlinearity f, on the topological or/and geometrical properties of the domain Ω.

We will concentrate our attention for the case $f(u) \sim e^u$ near ∞, which has many applications in geometry and physics. In fact, the equation $-\Delta u = \lambda e^u$ relates to the geometric problem of Riemannian surfaces with constant Gaussian curvature in dimension two. In higher dimension (when $N \geq 3$), it arises in the theory of thermionic emission, isothermal gas sphere, gas combustion and many other physical problems.

4.1. Exponential case in dimension two. — Consider $(\lambda > 0)$

$$-\Delta u = \lambda e^u \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u = 0 \;\text{ on } \partial\Omega. \tag{12}$$

We know that the Moser-Trudinger inequality holds: there exists $C > 0$ such that

$$\int_\Omega e^{4\pi u^2/\|\nabla u\|_2^2} dx \leq C, \quad \text{for any } u \in H_0^1(\Omega).$$

Consequently, $e^{ku} \in L^1(\Omega)$ for all $k > 0$ if $u \in H_0^1(\Omega)$. Applying Mountain-pass theory, we can prove then for any $\lambda \in (0, \lambda^*)$, a second unstable solution v_λ exists. Moreover, the family v_λ satisfies

$$\lim_{\lambda \to 0} \|v_\lambda\|_\infty = \infty \quad \text{and} \quad \lambda \int_\Omega e^{v_\lambda} dx \leq C. \tag{13}$$

We would like to understand the blow-up of v_λ when $\lambda \to 0$. As $-\Delta v_\lambda$ is uniformly bounded in $L^1(\Omega)$, the standard regularity theory for elliptic equation (see [**21**]) shows that $\{v_\lambda\}$ is bounded in $W^{1,p}(\Omega)$, for any $p \in [1, 2)$. Then up to a subsequence, v_λ converges weakly in $W^{1,p}(\Omega)$ $(1 < p < 2)$ to a limit function v_0. We would like to determine the function v_0, which will permit us to understand more clearly the asymptotic behavior of v_λ. A first step was obtained by Brezis & Merle in [**8**], they proved the following $\varepsilon-$regularity result.

Theorem 4.1. — *Let v_λ be a family of functions satisfying $-\Delta v_\lambda = \lambda e^{v_\lambda}$ in $\Omega \subset \mathbb{R}^2$, if for some $\eta > 0$, $B_{2\eta}(x_0) \subset \Omega$ and $\|\lambda e^{v_\lambda}\|_{L^1(B_{2\eta}(x_0))} \leq \varepsilon < 4\pi$, then $\|v_\lambda\|_{L^\infty(B_\eta(x_0))}$ is uniformly bounded.*

Proof. — For simplicity, we omit the index λ and we use B_r to denote $B_r(x_0)$. Define $E(x) = \lambda e^{v(x)}$ and

$$w_1(x) = -\frac{1}{2\pi}\int_{B_{2\eta}} \log|x-y| \times E(y)dy.$$

So $-\Delta w_1 = \lambda e^v$ in $B_{2\eta}$. Apply Jensen's inequality,

$$\int_{B_{2\eta}} e^{\alpha w_1}dx \leq \iint_{B_{2\eta}\times B_{2\eta}} |x-y|^{-\frac{\alpha Q}{2\pi}}\frac{E(y)}{Q}dydx, \quad \text{for any } \alpha > 0,$$

where $Q = \|\lambda e^v\|_{L^1(B_{2\eta})}$. Thus $e^{\alpha w_1} \in L^1(B_{2\eta})$ if $\alpha Q < 4\pi$. Define

$$-\Delta w_2 = 0 \ \text{ in } B_{2\eta}, \quad w_2 = v - w_1 \ \text{ on } \partial B_{2\eta}.$$

Obviously $v = w_1 + w_2$ in $B_{2\eta}$. Using the well known properties for harmonic functions (see [**21**]), we have

$$\|w_2\|_{L^\infty(B_{3\eta/2})} \leq C\|w_2\|_{L^1(B_{2\eta})} \leq C\Big[\|v\|_{L^1(B_{2\eta})} + \|w_1\|_{L^1(B_{2\eta})}\Big]$$

If $Q \leq \varepsilon < 4\pi$, we can choose $\alpha \in (1, 4\pi/\varepsilon)$, then $e^v = e^{w_1+w_2}$ is uniformly bounded in $L^\alpha(B_{3\eta/2})$. Now we decompose v as $w_1' + w_2'$ in $B_{3\eta/2}$ with

$$\left\{\begin{array}{rll} -\Delta w_1' = & \lambda e^v & \text{in } B_{3\eta/2} \\ w_1' = & 0 & \text{on } \partial B_{3\eta/2} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rll} -\Delta w_2' = & 0 & \text{in } B_{3\eta/2} \\ w_2' = & v & \text{on } \partial B_{3\eta/2}. \end{array}\right.$$

We get easily that $\|w_1'\|_{W^{2,\alpha}(B_{3\eta/2})}$ and $\|w_2'\|_{L^\infty(B_\eta)}$ are uniformly bounded. Using the fact $W^{2,\alpha} \subset L^\infty$ in dimension two for $\alpha > 1$, the proof is completed. $\square$

Now for a family of solutions verifying (12) and (13), we define the blow-up set S as the set where v_λ is not uniformly bounded, that is

$$S = \{x \in \Omega \mid \exists\ \lambda_k \to 0,\ x_k \to x \text{ such that } v_{\lambda_k}(x_k) \to \infty\}. \tag{14}$$

Theorem 4.1 yields that if the L^1 norm of λe^{v_λ} is locally smaller then 4π, then λe^{v_λ} tends locally to zero and no blow-up can occur. Thus we can claim that $S = \Sigma$ where

$$\Sigma = \big\{x \in \Omega \mid \forall\ \eta > 0,\ \limsup_{\lambda\to 0}\|\lambda e^{v_\lambda}\|_{L^1(B_\eta(x)\cap\Omega)} \geq 4\pi\big\}. \tag{15}$$

From the boundedness of $\|\lambda e^{v_\lambda}\|_1$, up to a subsequence, we obtain

$$\#(S) < \infty, \quad \lambda e^{v_\lambda} \to \sum_{x_i\in S} m_i\delta_{x_i} \ \text{ in the sense of measures}$$

where $m_i \geq 4\pi$ and δ_x denotes the Dirac measure over the point x. Therefore,

Proposition 4.2. — *Let v_λ be a family of functions satisfying* (12) *and* (13)*, there exists a finite set $S \subset \Omega$ and $m_j \geq 4\pi$ such that up to a subsequence,*

$$v_\lambda \to v_0 = \sum_{x_j\in S} m_j G(x, x_j) \quad \text{in } W^{1,p}(\Omega), \quad \forall\ 1 \leq p < 2$$

where $G(x, x_j)$ is the Green function

$$-\Delta_x G(x,y) = \delta_y(x) \quad \text{in } \Omega \quad \text{and} \quad G(x,y) = 0 \quad \text{if} \quad x \in \partial\Omega.$$

Moreover, v_λ converges to v_0 in $C^k_{loc}(\overline{\Omega}\setminus S)$ for any $k \in \mathbb{N}$.

In fact, as indicated in [**30**], using the moving plane argument proved by Gidas, Ni & Nirenberg (see Proposition 4.6 and Appendix), we can show the existence of a *fixed* neighborhood U of $\partial\Omega$ such that no blow-up occurs in U (under the assumptions $\|\lambda e^{v_\lambda}\|_{L^1(\Omega)} = O(1)$ and $\Omega \subset \mathbb{R}^2$). The last assertion of Proposition comes from the fact that no blow-up appears out of S or near $\partial\Omega$, so v_λ are uniformly bounded locally in $\overline{\Omega} \setminus S$, which leads to the higher order convergence by boot-strap arguments.

The next step is to determine the quantities m_j and to localize the blow-up set S. For that, we will use the local analysis and Pohozaev identities.

Lemma 4.3. — *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, if $-\Delta u = f(u)$ in Ω, then*

$$\begin{aligned} &\int_\Omega \left[NF(u) - \frac{N-2}{2}uf(u)\right]dx \\ &= \int_{\partial\Omega}\left[(x\cdot\nabla u)\frac{\partial u}{\partial\nu} - (x\cdot\nu)\frac{|\nabla u|^2}{2} + (x\cdot\nu)F(u) + \frac{N-2}{2}u\frac{\partial u}{\partial \nu}\right]d\sigma \end{aligned} \tag{16}$$

where ν denotes the unit external normal vector on $\partial\Omega$ and $F(t) = \displaystyle\int_0^t f(s)ds$.

Return to our problem, let x_0 be any point in S, by translation, we can assume that $x_0 = 0$. Taking now $N = 2$, $f(u) = \lambda e^u$ and $\Omega = B_\eta = B_\eta(0)$ in (16), we get

$$2\int_{B_\eta}\lambda(e^{v_\lambda}-1)dx = \int_{\partial B_\eta}\left[\eta\left(\frac{\partial v_\lambda}{\partial\nu}\right)^2 - \frac{\eta|\nabla v_\lambda|^2}{2} + \eta\lambda(e^{v_\lambda}-1)\right]d\sigma. \tag{17}$$

We fix $\eta_0 > 0$ small enough such that $\overline{B}_{\eta_0} \cap S = \{0\}$. By Proposition 4.2, for any $\eta \in (0, \eta_0]$ fixed, when $\lambda \to 0$,

$$\lambda(e^{v_\lambda}-1)\chi_{B_\eta} \to m_0\delta_0 \quad\text{and}\quad v_\lambda \to -\frac{m_0}{2\pi}\log r + R_0(x) \ \text{ in } C^1(\partial B_\eta)$$

where $r = |x|$ and $R_0 \in C^1(\overline{B}_{\eta_0})$. Thus, the l.h.s. of (17) is equal to $2m_0 + o_\lambda(1)$ while

$$\text{r.h.s.} = \frac{m_0^2}{4\pi} + O(\eta) + o_\lambda(1), \quad \forall\, \eta \le \eta_0 \text{ fixed.}$$

Tending first λ to 0 and then η to 0, we get $2m_0 = m_0^2/4\pi$, that is $m_0 = 8\pi$.

For the localization of x_j, we give here just the proof of single blow-up situation. The general case can be obtained in a similar way. Multiplying $-\Delta u = f(u)$ by ∇u and integrating by parts over Ω, we obtain a Pohozaev type identity:

$$\int_{\partial\Omega} F(u)\nu d\sigma = \int_{\partial\Omega}\left[\frac{|\nabla u|^2}{2}\nu - \frac{\partial u}{\partial\nu}\nabla u\right]d\sigma. \tag{18}$$

In our case, $v_\lambda = 0$ on $\partial\Omega$, so that $\nabla v_\lambda = (\partial_\nu v_\lambda)\nu$ on $\partial\Omega$. Hence

$$\int_{\partial\Omega}\left(\frac{\partial v_\lambda}{\partial\nu}\right)^2 \nu d\sigma = 0_{\mathbb{R}^2}. \tag{19}$$

If $S = \{x_0\}$, passing to the limit $\lambda \to 0$, we get

$$\int_{\partial\Omega} \left[\frac{\partial G}{\partial \nu}(x, x_0)\right]^2 \nu d\sigma = 0_{\mathbb{R}^2}. \tag{20}$$

We claim that x_0 is a critical point of the Robin function $\widetilde{H}$ associated to the Green function G. Thus the blow-up set S is localized by the Green function of the domain Ω. Indeed, this is a direct consequence of

Lemma 4.4. — *For any $x_0 \in \Omega$, the left hand side of* (20) *is just $-\nabla\widetilde{H}(x_0)$.*

Proof. — This lemma and the idea of its proof here are valid in any dimension, but we consider only the case of dimension two for simplicity. If $\Omega \subset \mathbb{R}^2$, we know that

$$G(x, y) = -\frac{\log|x - y|}{2\pi} + H(x, y) \tag{21}$$

where H is a smooth symmetric function in $\Omega \times \Omega$ and $\widetilde{H}(x) = H(x, x)$. Using (18) with $-\Delta_x G(x, x_0) = 0$ on $\Omega_\eta = \Omega \setminus \overline{B}_\eta(x_0)$ ($\eta > 0$ is small) and $G(x, x_0) = 0$ on $\partial\Omega$, so

$$-\frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial G}{\partial \nu}\right)^2 \nu d\sigma + \int_{\partial B_\eta} \frac{|\nabla G|^2}{2}\nu - \frac{\partial G}{\partial \nu}\nabla G d\sigma = 0.$$

Noticing that the unit normal vector ν on ∂B_η is just $-(x - x_0)/\eta$, we get

$$\begin{aligned}\int_{\partial B_\eta} \frac{|\nabla G|^2}{2}\nu d\sigma &= \int_{\partial B_\eta} \left[\frac{\nu}{8\pi^2\eta^2} + \frac{(x - x_0)\cdot\nabla H}{2\pi\eta^2}\nu + \frac{|\nabla H|^2}{2}\nu\right] d\sigma \\ &= -\int_{\partial B_\eta} \frac{\partial_\nu H}{2\pi\eta}\nu d\sigma + O(\eta).\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{\partial B_\eta} \frac{\partial G}{\partial \nu}\nabla G d\sigma &= \int_{\partial B_\eta} \left(\frac{1}{2\pi\eta} + \partial_\nu H\right)\left[\frac{(x - x_0)}{2\pi\eta^2} + \nabla H\right] d\sigma \\ &= -\int_{\partial B_\eta} \frac{\partial_\nu H}{2\pi\eta}\nu d\sigma + \frac{1}{2\pi\eta}\int_{\partial B_\eta} \nabla_x H(x, x_0) d\sigma + O(\eta).\end{aligned}$$

Finally

$$\int_{\partial\Omega} \left(\frac{\partial G}{\partial \nu}\right)^2 \nu d\sigma = -\frac{1}{\pi\eta}\int_{\partial B_\eta} \nabla_x H(x, x_0) d\sigma + O(\eta) = -2\nabla_x H(x_0, x_0) + O(\eta).$$

In conclusion, by passing η to 0, we obtain

$$\int_{\partial\Omega} \left[\frac{\partial G}{\partial \nu}(x, x_0)\right]^2 \nu d\sigma = -\nabla\widetilde{H}(x_0). \tag{22}$$

Here we used $\nabla\widetilde{H}(x_0) = 2\nabla_x H(x_0, x_0)$ by the symmetry of H. ∎

We can also get (22) by the Pohozaev identity. Using (16) for v_λ,

$$\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial v_\lambda}{\partial\nu}\right)^2 d\sigma = \int_\Omega 2F(v_\lambda)dx.$$

But if we translate the domain Ω and consider $v_\lambda(x+x_0)$ for any $x_0\in\mathbb{R}^2$, we have also

$$\frac{1}{2}\int_{\partial\Omega}(x-x_0)\cdot\nu\left(\frac{\partial v_\lambda}{\partial\nu}\right)^2 d\sigma = \int_\Omega 2F(v_\lambda)dx.$$

By difference, we obtain

$$\int_{\partial\Omega}(x_0\cdot\nu)\left(\frac{\partial v_\lambda}{\partial\nu}\right)^2 d\sigma = 0, \quad \forall\ x_0\in\mathbb{R}^2$$

which is just equivalent to (19). We remark that the exact form of f or F is not used, that's why for autonomous partial differential equation with $-\Delta$ and the Dirichlet boundary condition, the single blow-up lies often on the critical point of the Robin function.

In general, we have the following result proved in [**30**].

Theorem 4.5. — *Let v_λ be a family of solutions of* (12) *such that*

$$\lim_{\lambda\to 0}\int_\Omega \lambda e^{v_\lambda}dx = \ell\in\mathbb{R}_+\cup\{\infty\} \textit{ exists.}$$

Then we have the following alternatives:

(i) *$\ell=\infty$, then v_λ tends to ∞ u.c. in Ω;*
(ii) *$\ell=0$, v_λ tends to zero uniformly in $\overline{\Omega}$;*
(iii) *$\ell=8\pi m$ with $m\in\mathbb{N}^*$. Up to a subsequence, there exists $S=\{x_1,...,x_m\}\subset\Omega$ such that v_λ blows up on S, v_λ converges to $8\pi\sum_j G(x,x_j)$ in $W^{1,p}(\Omega)$ for any $1<p<2$. Moreover, $\lambda e^{v_\lambda}dx\to 8\pi\sum_j \delta_{x_j}$ in the sense of measure and $x=(x_1,...,x_m)$ is a critical point of*

$$\Psi(x) = \sum_{j=1}^m H(x_j,x_j) + \sum_{i\neq j} G(x_i,x_j).$$

For the proof of (i), we need a remarkable result which is proved in [**20**] by moving plane method (see Appendix), and which is only valid in dimension two.

Proposition 4.6. — *Let $\Omega\subset\mathbb{R}^2$ be a smooth bounded domain, there exists $\varepsilon_0>0$ depending only on Ω such that for any C^1 function f and $u\in C^2(\overline{\Omega})$ solution of*

$$-\Delta u = f(u),\ u>0 \ \text{ in } \Omega\subset\mathbb{R}^2, \quad u=0 \ \text{ on } \partial\Omega,$$

then u has no stationary point in the D_{ε_0}, where $D_\varepsilon=\{x\in\Omega,\ d(x,\partial\Omega)<\varepsilon\}$ denotes the open ε-neighborhood of $\partial\Omega$.

Associated to Hopf's lemma, we get then v_λ is decreasing with respect to $d(x, \partial\Omega)$ in a small neighborhood of the boundary independent of λ. This implies that if $\ell = \infty$, there exists $\varepsilon_1 > 0$ satisfying

$$\lim_{\lambda \to 0} \int_{\Omega \setminus D_{\varepsilon_1}} \lambda e^{v_\lambda} dx = \infty.$$

Therefore for any compact set $K \subset \Omega$, using the uniform positivity of the Green function G over $K \times (\Omega \setminus D_{\varepsilon_1})$, there exists $C > 0$ such that for any $x \in K$,

$$v_\lambda(x) = \int_\Omega G(x,y)\lambda e^{v_\lambda} dy \geq \int_{\Omega \setminus D_{\varepsilon_1}} G(x,y)\lambda e^{v_\lambda} dy \geq \int_{\Omega \setminus D_{\varepsilon_1}} C\lambda e^{v_\lambda} dy \to \infty.$$

The case (ii) comes from Theorem 4.1. The case (iii) can be proved by similar calculus as for the single bubble case using local analysis and Pohozaev's identities.

Now a natural question is to understand the quantity 8π. Indeed, a gauge transformation will give us the answer. Take x_λ which realizes $\max_\Omega v_\lambda(x)$, define $A_\lambda = v_\lambda(x_\lambda)$, $\xi_\lambda^2 = e^{-A_\lambda - \log\lambda}$ and $w_\lambda(x) = v_\lambda(x_\lambda + \xi_\lambda x) + \log\lambda + 2\log\xi_\lambda$. It is easy to see that

$$-\Delta w_\lambda = e^{w_\lambda} \text{ in } \Omega_\lambda = \left\{y \in \mathbb{R}^2, x_\lambda + \xi_\lambda y \in \Omega\right\}, \quad w_\lambda(0) = 0 \quad \text{and} \quad w_\lambda(x) \leq 0 \text{ in } \Omega_\lambda.$$

Moreover

$$\int_\Omega \lambda e^{v_\lambda} dx = \int_{\Omega_\lambda} e^{w_\lambda} dx.$$

For v_λ verifying (13), we have ξ_λ tends to zero, otherwise v_λ is uniformly bounded and no blow-up occurs. Using $w_\lambda \leq 0$, $0 < e^{w_\lambda} \leq 1$ and Harnack's inequality, we can prove that up to a subsequence, w_λ converges locally uniformly to a function w, solution of

$$-\Delta w = e^w \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^w dx < \infty, \tag{23}$$

and $\max_{\mathbb{R}^2} w(x) = w(0) = 0$. Chen & Li proved in [**10**] that any solution of (23) satisfies

$$\int_{\mathbb{R}^2} e^w dx = 8\pi.$$

More precisely, w is a radial function

$$w(x) = -2\log\left(1 + \frac{|x|^2}{8}\right) \quad \text{in } \mathbb{R}^2.$$

The same analysis can be done near any blow-up point, which means that the function v_λ looks like locally as the concentration of a standard solution over $\mathbb{R}^2$ near each $x_i \in S$.

4.2. Further results and comments. — By Theorem 4.5, we know that the behavior of blow-up solutions of (12) is controlled by the Green function of the domain Ω. Conversely, we may ask if all configurations in Theorem 4.5 can really appear. Indeed, the answer depends on topological or/and geometrical properties of the domain Ω.

For simply connected domain, Mizoguchi & Suzuki proved that the case (i) can never happen. For example, when $\Omega = B_1 \subset \mathbb{R}^2$, we know that all solutions are radially symmetric (see Theorem 5.1 in Appendix), moreover u is decreasing along the radius by maximum principle, so the only possibility is the blow-up at the origin, thus only the case $m = 1$ occurs.

More generally, Baraket & Pacard showed in [**4**] that if $(x_i) \in \Omega^m$ is a nondegenerate critical point of Ψ, then there exists a family of solutions v_λ such that v_λ blows up exactly on these points x_i. But it is rather difficult to verify the nondegeneracy condition for a general critical point (x_i). Very recently, Del Pino, Kowalczyk & Musso prove that if the domain Ω is not simply connected, than for any $m \in \mathbb{N}^*$, we may construct a family of solutions which makes m bubbles, in particular, by diagonal process, we can find a family of solutions such that the case (i) of Theorem 4.5 appears.

On the other hand, Theorem 4.5 works for more general exponential like nonlinearities f. In fact, we need just to suppose that $\lim_{t\to\infty} f(t)e^{-t} = 1$, and similar result holds also for the equation $-\Delta u = \lambda V(x)e^u$ with a function $V \in C^1(\overline{\Omega})$ (see [**34, 24**]). The result in [**4**] was also generalized to functions like $f(u) = e^u + Ce^{\gamma u}$ with $\gamma < 1$ in [**5**], the case $f(x,u) = V(x)e^u$ and multiplicity result for blow-up solutions are obtained in [**16**].

In contrast with the situation in dimension two, the behavior of unstable solutions for $-\Delta u = \lambda e^u$ is far away to be understood for higher dimension cases ($N \geq 3$). The only situation well known is the case with $\Omega = B_1$, for which we can take advantage of the radial symmetry of solutions. For example, when $\Omega = B_1(0) \subset \mathbb{R}^3$, Gel'fand showed in [**19**] that the curve of solutions will make a form of corkscrew near another critical value $\lambda^{**} \in (0, \lambda^*)$, so the configuration is quite different from the case with $B_1 \subset \mathbb{R}^2$.

In [**35**] and [**33**], we have considered the following problem:

$$-\mathrm{div}(\zeta(x)\nabla u) = \lambda\zeta(x)e^u \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u = 0 \quad \text{on } \partial\Omega \tag{24}$$

where ζ is a positive smooth function over $\overline{\Omega}$. Our motivation are two fold. First, when we work with rotational symmetric solutions of $-\Delta u = \lambda e^u$ in dimension $N \geq 3$, we can find that the equation is reduced to (24). Fore example, let the torus be

$$\mathbb{T} = \left\{(x_i) \in \mathbb{R}^N, (\|\widehat{x}\| - 1)^2 + x_N^2 \leq R^2\right\}$$

where $R < 1$ and $\widehat{x} = (x_1, \ldots, x_{N-1})$. If we look for solutions in the form $u(x) = u(r, x_N)$ with $r = \|\widehat{x}\|$, a direct calculus shows that the problem $-\Delta u = \lambda e^u$ in Ω is transformed to

$$-\mathrm{div}(r^{N-2}\nabla u) = \lambda r^{N-2}e^u \ \text{ in } \Omega_{\mathbb{T}} = \{(r,z) : (r-1)^2 + z^2 < R^2\}, \quad u = 0 \ \text{ on } \partial\Omega_{\mathbb{T}}.$$

This is just a special case of (24). On the other hand, equation (24) is similar to (12), we may expect that similar results will hold. But this is not true.

It is not difficult to see the existence of critical value λ^* and that of minimal solutions u_λ for (24). In [**35**], we studied the asymptotic behavior of bubbling or unstable solutions v_λ of (24) when $\lambda \to 0$. We proved that if

$$\int_\Omega \lambda\zeta(x)e^{v_\lambda}dx \to \ell \quad \text{and} \quad \lim_{\lambda\to\infty} \|v_\lambda\|_\infty = \infty,$$

then $\ell \in 8\pi\mathbb{N}^*$. Furthermore, up to a subsequence, there exists a finite set $S = \{x_1, ..., x_k\} \subset \Omega$ such that $v_\lambda \to v_0$ in $W^{1,p}$ for any $p \in (1,2)$, where v_0 satisfies

$$\operatorname{div}(\zeta(x)\nabla v_0) + 8\pi \sum_i m_i\zeta(x_i)\delta_{x_i} = 0 \ \text{ in } \Omega, \quad v_0 = 0 \ \text{ on } \partial\Omega.$$

Here $m_i \in \mathbb{N}^*$ and each x_i must be a critical point of ζ. This is similar to the case (ii) in Theorem 4.5, but the blow-up set is determined now by the function ζ instead of the Green function.

We proved also that if $x \in S$ is a nondegenerate minimum point of ζ, then the corresponding m must be equal to 1, and a single bubble example ($k = m_1 = 1$) is constructed for the symmetric case $\Omega = B_1$ with radial ζ. However, we were not able to determine if each m_i is always equal to 1 in general, and we did not give a method to construct bubbling solutions for general ζ or Ω.

We give the answer to these questions in a very recent work [**33**].

Theorem 4.7. — *Let $\bar{x} \in \Omega$ be a strict local maximum point of ζ, i.e. there exists $\delta > 0$ such that*

$$\zeta(x) < \zeta(\bar{x}), \quad \forall\ x \in B_\delta(\bar{x})\backslash\{\bar{x}\}.$$

Then for any $m \in \mathbb{N}^*$, equation (24) has a family of solutions v_λ such that

$$\lambda\int_\Omega \zeta(x)e^{v_\lambda}dx \to 8\pi m\zeta(\bar{x}), \quad v_\lambda \to v_0 \quad \text{in} \quad C^2_{loc}(\overline{\Omega}\backslash\{\bar{x}\})$$

where v_0 satisfies

$$-\nabla(\zeta(x)\nabla v_0) = 8\pi m\zeta(\bar{x})\delta_{\bar{x}} \quad \text{in } \Omega, \quad v_0 = 0 \quad \text{on } \partial\Omega.$$

Thus near a strict maximum point of ζ, we obtain a family of multi-bubble solutions with any $m \in \mathbb{N}^*$. Therefore by diagonal process, we may have a family of solutions for (24), such that

$$\lim_{\lambda\to 0}\int_\Omega \lambda\zeta(x)e^{v_\lambda}dx = \infty, \tag{25}$$

even for *simply connected* domain. This is unexpected and new, comparing to the result in [**28**] for isotropic case or to the case of local minimum point for ζ for (24).

Furthermore, we can give precise expansion of blow-up solutions v_λ near $\bar{x}$ and show that near $\bar{x}$, the flatter the anisotropic coefficient ζ is, the larger are the distances between the bubbles. We prove also that if x_0 is a topologically non trivial critical point of ζ (see definition in [**15**]), we have always a family of single bubble solution which blows up at this point.

However, if we look at the problem on the torus with rotational symmetry, we see that the function $\zeta(x) = x_1^{N-2}$ does not have any critical point in $\Omega_{\mathbb{T}}$. Thus, a new situation occurs, the bubble must go to the boundary, while the boundary bubbles do not appear in the isotropic case, thanks to Proposition 4.6.

In this case, we show in a forthcoming work [**32**] some analysis of boundary blow-up for (24). For example, we show that the bubbles will be localized near critical points of $\zeta|_{\partial\Omega}$, i.e. at $x \in \partial\Omega$ verifying $\partial_\tau\zeta(x) = 0$. We prove that no bubble can exist near a nondegenerate maximum of ζ on $\partial\Omega$, and also the existence of solutions such that $\lambda\zeta(x)e^{v_\lambda}dx$ converges to $8\pi\delta_{x_0}$ when $\lambda \to 0$, if $x_0 \in \partial\Omega$ is a strict local minimum of ζ such that $\partial_\nu\zeta(x_0) < 0$.

Returning to the situation on $\mathbb{T}$, this result will enable us to get a new family of solutions for $-\Delta u = \lambda e^u$ which blows up on a $(N-2)$ dimensional submanifold on $\partial\mathbb{T}$.

5. Appendix : Moving plane method

The moving plane method goes back to Alexandrov's famous paper on constant mean curvature hypersurface. It has known a great development in the study of partial differential equations since the work of Gidas, Ni & Nirenberg [**20**]. The main idea is to move a hyperplane $\Sigma_q = \{x \cdot n = q\}$ following a fixed direction n from far away, it will reach for a first time some points of $\partial\Omega$, then cut a little domain T_q; we will try to compare the value of solution over T_q with that in the reflecting domain of T_q by the hyperplane Σ_q, and we hope to push the hyperplane as far as we can.

5.1. Classical method. — We consider a solution u of

$$-\Delta u = f(u),\ u > 0 \ \text{ in } B_1 \subset \mathbb{R}^N, \quad u = 0 \ \text{ on } \partial B_1 \tag{26}$$

where $B_1 = B_1(0)$ is the unit ball. The classical method suppose that u is a C^1 solution and f is locally Lipschitz. Then by Hopf's lemma, we know that $\partial_\nu u < 0$ on the boundary, in particular, $\partial_1 u(e_1) < 0$ where $e_1 = (1, 0, \dots, 0)$. Hence $\partial_1 u$ is negative in a neighborhood of e_1, which means that u is decreasing in the x_1 direction near e_1. Therefore, if we denote Σ_q the hyperplane defined by $\{x = (x_i) \in \mathbb{R}^N |\ x_1 = q\}$, $T_q = \{x \in B_1,\ x_1 > q\}$, then there exists $\varepsilon > 0$ such that

$$u(x) < u(2q - x_1, \widetilde{x}), \quad \text{for any } x \in T_q \ \text{ and } \ q \in (1-\varepsilon, 1).$$

Here $\widetilde{x} = (x_2, \dots, x_N)$, so $(2q - x_1, \widetilde{x})$ is just the reflecting point of x with respect to Σ_q. Consider now

$$q_0 = \inf\left\{q > 0, \text{ s.t. } u(x) \le u(2b - x_1, \widetilde{x}) \ \text{ in } T_b,\ \forall\, b \in (q, 1)\right\}.$$

Clearly such q_0 is well defined.

We claim then $q_0 = 0$. If it is not the case, by continuity, we get $u(x) \le v_{q_0}(x) = u(2q_0 - x_1, \widetilde{x})$ in T_{q_0}, and $u \not\equiv v_{q_0}$ since $u(x) = 0$ for $x \in \partial B_1 \cap \partial T_{q_0}$ while the reflecting point of such x lies in B_1. Thus by the invariance of Laplacian under reflection, $w_{q_0} = v_{q_0} - u$ is a nontrivial solution of

$$-\Delta w_{q_0} = f(v_{q_0}) - f(u) = c(x)w_{q_0}, \ w_{q_0} \geq 0 \ \text{ in } T_{q_0}; \quad w_{q_0} = 0 \ \text{ on } \partial T_{q_0} \cap \Sigma_{q_0}.$$

Using the strong maximum principle, we obtain $w_{q_0} > 0$ in T_{q_0} and $\partial_1 w_{q_0} > 0$ on $\partial T_{q_0} \cap \Sigma_{q_0}$. Moreover, if we look at $w_q(x) = v_q(x) - u(x) = u(2q - x_1, \widetilde{x}) - u(x)$, it is easy to see that

$$\frac{\partial w_q}{\partial x_1} = -2\frac{\partial u}{\partial x_1} \quad \text{on } \partial T_q \cap \Sigma_q.$$

So we get $\partial_1 u(x) < 0$ on $\partial T_{q_0} \cap \Sigma_{q_0}$. The same argument works for all $b \in [q_0, 1]$, finally $\partial_1 u(x) < 0$ in $\overline{T}_{q_0}$. This implies that we can push a little bit the hyperplane Σ_q for some $q < q_0$, and we still have $u(x) - u(2q - x_1, \widetilde{x}) < 0$, which is a contradiction with the definition of q_0. Thus $q_0 = 0$.

Finally, $q_0 = 0$ means $u(x) \leq u(-x_1, \widetilde{x})$ for any $x \in B_1$ and $x_1 \leq 0$. But if we do the same work with the opposite direction, the inverse inequality is also true, so $u(x) = u(-x_1, \widetilde{x})$ in B_1, i.e. we obtain the symmetry of u with respect to x_1. Now, as we can proceed with any direction, we conclude that u is a radial function in B_1.

We remark that the central argument is the invariance of Laplacian under reflection and the strong maximum principle. But we need here a nice regularity of u.

5.2. Idea of Berestycki & Nirenberg. — In [**6**], Berestycki & Nirenberg weakened a lot the condition on u by remarking that the first eigenvalue of $-\Delta$ is large for a domain with small Lebesgue measure. They used also the Harnack type inequalities to replace the classical strong maximum principle.

More precisely, let u be a solution of (26) in $C^0(\overline{\Omega}) \cap H^1(\Omega)$, we fix $A > 0$ large enough such that $g(x) = f(x) - Ax$ is decreasing in $[0, \max_\Omega u]$. We know that $\lambda_1(-\Delta)$ associated to the Dirichlet boundary condition tends to ∞ when $|\Omega'|$, the Lebesgue measure of Ω', goes to zero. Therefore, there exists $\varepsilon_0 > 0$ such that the operator $L = -\Delta - A$ is coercive in $H_0^1(\Omega')$, if $|\Omega'| \leq \varepsilon_0$. It is the case for T_q when q is near 1. Using the same notation as above, we get

$$Lw_q = g(v_q) - g(u) \ \text{ in } T_q, \quad w_q^- = \min(0, w_q) = 0 \ \text{ on } \partial T_q.$$

Using w_q^- as test function, we get

(27)
$$0 \leq \int_{T_q} \Big[|\nabla w_q^-|^2 - A(w_q^-)^2 \Big] dx = \int_{T_q} w_q^- L(w_q) dx = \int_{T_q} [g(v_q) - g(u)] w_q^- dx \leq 0.$$

Hence $w_q^- \equiv 0$ which means $u(x) \leq u(2q - x_1, \widetilde{x})$ in T_q for q near 1.

Moreover, as $g(u) - g(v_q) = c(x)w_q$ with c uniformly bounded, using the equation for the nonnegative H^1 function w_q, we have the following Harnack inequality:

$$\text{(28)} \qquad \exists\, r_0, \ p, \ C > 0 \quad \text{s.t. } \|w_q\|_{L^p(B_r(x))} \leq C \inf_{\overline{B}_{2r}(x)} w_q, \ \forall\, r \leq r_0, \ \overline{B}_{2r}(x) \subset T_q.$$

As T_q is connected, either $w_q \equiv 0$ or $w_q > 0$ in T_q.

Now we define q_0 as above, we claim again $q_0 = 0$. Suppose the contrary, we have $w_{q_0} \geq 0$ in T_{q_0} and $w_{q_0} > 0$ on $\partial B_1 \cap \partial T_{q_0}$, so $w_{q_0} > 0$ in $\overline{T}_{q_0} \setminus \Sigma_{q_0}$. We cut then the domain T_{q_0} into two parts

$$K_1 = T_{q_0} \cap \left\{x \in \mathbb{R}^N,\ x_1 \in (q_0, q_0 + \eta)\right\}, \quad K_2 = T_{q_0} \setminus \overline{K}_1$$

where $\eta > 0$ is small enough such that $|K_1| \leq \varepsilon_0/2$. It is easy to see that $\min_{\overline{K}_2} w_{q_0} > 0$. By the continuity of u, $\min_{\overline{K}_2} w_q > 0$ for q near q_0. Otherwise, for q near q_0 such that $|T_q \setminus K_2| \leq \varepsilon_0$, we have $w \geq 0$ on $\partial(T_q \setminus K_2)$. Similarly as in (27), we obtain then $w_q \geq 0$ in $T_q \setminus K_2$. Finally we conclude that $w_q \geq 0$ in T_q for q less than, but near q_0. We reach again a contradiction, which leads to

Theorem 5.1. — *Let f be a locally Lipschitz function in $\mathbb{R}_+$, let u be a solution of* (26) *in $C^0(\overline{\Omega}) \cap H^1(\Omega)$. Then u is radially symmetric.*

5.3. Further remarks. — The moving plane method is based essentially on the invariance of elliptic operator with respect to some symmetry transformation, Berestycki & Nirenberg's idea takes advantage of small domain to begin this method, and the Harnack type estimates lead to push the hyperplane to the limiting position.

So the idea of moving plane method can be generalized to many other situations, it can be applied with other symmetric domains and manifolds, with other transformations or hypersurfaces (for example moving sphere under Kelvin transformation), with more general elliptic operators provided their invariance under the corresponding transformation, or work with the whole space under suitable condition on the behavior of solution at infinity. We refer the readers to [**3, 2, 17**] and references therein for some recent developments.

The idea of weak regularity required for u is very important, since it incites us to generalize the method to other type of degenerate operators as p-Laplacian, or some degenerate operators in Carnot-Caratheodory spaces, for which the solutions are generally less smooth than for $-\Delta$, see for example [**12, 13, 18**].

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