Séminaires & Congrès
15, 2007, p. 47–118

FOUR LAMBDA STORIES, AN INTRODUCTION TO COMPLETELY INTEGRABLE SYSTEMS

by

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Abstract. — Among all non-linear differential equations arising in Physics or in geometry, completely integrable systems are exceptional cases, at the concurrence of miraculous symmetry properties. This text proposes an introduction to this subject, through a list of examples (the sinh-Gordon, Toda, Korteweg-de Vries equations, the harmonic maps, the anti-self-dual connections on the four-dimensional space). The leading thread is the parameter lambda, which governs the algebraic structure of each of these systems.

***Résumé* (Quatre histoires de lambda, une introduction aux systèmes complètement intégrables)**

Parmi toutes les équations différentielles non linéaires venant de la physique ou de la géométrie, les systèmes complètement intégrables sont des cas exceptionnels, où se conjuguent des propriétés de symétries miraculeuses. Ce texte propose une introduction à ce sujet, à travers une liste d'exemples (les équations de sh-Gordon, de Toda, de Korteweg-de Vries, les applications harmoniques, les connexions anti-auto-duales sur l'espace de dimension quatre). Le fil conducteur est le paramètre lambda, qui gouverne la structure algébrique de chacun de ces systèmes.

Introduction

Completely integrable systems are non linear differential equations or systems of differential equations which possess so much symmetry that it is possible to construct by quadratures their solutions. But they have something more: in fact the appellation 'completely integrable' helps to summarize a concurrence of miraculous properties which occur in some exceptional situations. Some of these properties are: a Hamiltonian structure, with as many conserved quantities and symmetries as the number of degrees of freedom, the action of Lie groups or, more generally, of affine Lie algebras, a reformulation of the problem by a *Lax equation*. One should also add

2000 ***Mathematics Subject Classification.*** — 37K10.
Key words and phrases. — Completely integrable systems, Korteweg-de Vries equations, harmonic maps, anti-self-dual connections, twistors theory.

that, in the best cases, these non linear equations are converted into linear ones after a transformation which is more or less the Abel map from a Riemann surface to a Jacobian variety, and so on. Each one of these properties captures an essential feature of completely integrable systems, but not the whole picture.

Hence giving a complete and concise definition of an integrable system seems to be a difficult task. And moreover the list of known completely integrable systems is quite rich today but certainly still not definitive. So in this introduction text I will just try to present different examples of such systems, some are ordinary differential equations, the other ones are partial differential equations from physics or from differential geometry. I will unfortunately neglect many fundamental aspects of the theory (such as the spectral curves, the R-matrix formulation and its relation to quantum groups, the use of symplectic reduction, etc.) and privilege one point of view: in each of these examples a particular *character*, whose presence was not expected at the beginning, appears and plays a key role in the whole story. Although the stories are very different you will recognize this *character* immediately: *his* name is λ and *he* is a complex parameter.

In the first section we outline the Hamiltonian structure of completely integrable systems and expound the Liouville–Arnold theorem. In the second section we introduce the notion of *Lax equation* and use ideas from the Adler–Kostant–Symes theory to study in details the Liouville equation $\frac{d^2}{dt^2}q + 4e^{2q} = 0$ and an example of the Toda lattice equation. We end this section by a general presentation of the Adler–Kostant–Symes theory. Then in the third section, by looking at the sinh–Gordon equation $\frac{d^2}{dt^2}q + 2\sinh(2q) = 0$, we will meet for the first time λ: here this parameter is introduced *ad hoc* in order to converte infinite dimensional matrices to finite dimensional matrices depending on λ.

The second λ story is about the KdV equation $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0$ coming from fluid mechanics. There λ comes as the eigenvalue of some auxiliary differential operator involved in the Lax formulation and hence is often called the *spectral parameter*. We will see also how the Lax equation can be translated into a zero-curvature condition. A large part of this section is devoted to a description of the Grassmannian of G. Segal and G. Wilson and of the τ-function of M. Sato and may serve for instance as an introduction before reading the paper by Segal and Wilson [**29**].

The third λ story concerns constant mean curvature surfaces and harmonic maps into the unit sphere. Although the discovery of the completely integrable structure of these problems goes back to 1976 [**27**], λ was already observed during the ninetenth century by O. Bonnet [**7**] and is related somehow to the existence of conjugate families of constant mean curvature surfaces, a well-known concept in the theory of minimal surfaces through the Weierstrass representation. This section is relatively short since the Author already wrote a monograph on this subject [**18**] (see also [**17**]).

The fourth λ story is part of the twistor theory developped by R. Penrose and his group during the last 40 years. The aim of this theory was initially to understand relativistic partial differential equations like the Einstein equation of gravity and the Yang–Mills equations for gauge theory in dimension 4, through complex geometry. Eventually this theory had also application to elliptic analogues of these problems on Riemannian four-dimensional manifolds. Here λ has also a geometrical flavor. If we work with a Minkowski metric then λ parametrizes the light cone directions or the celestial sphere through the stereographic projection. In the Euclidean setting λ parametrizes complex structures on a 4-dimensional Euclidean space. Here we will mainly focus on anti-self-dual Yang–Mills connections and on the Euclidean version of Ward's theorem which characterizes these connections in terms of holomorphic bundles.

A last general remark about the meaning of λ is that for all equations with Lax matrices which are polynomial in λ, the characteristic polynomial of the Lax matrix defines an algebraic curve, called the *spectral curve*, and λ is then a coordinate on this algebraic curve. Under some assumptions (e.g. for *finite gap* solutions of the KdV equation or for *finite type* harmonic maps) the Lax equation linearizes on the Jacobian of this algebraic curve.

The Author hopes that after reading this text the reader will feel the strong similarities between all these different examples. It turns out that these relationships can be precised, this is for instance the subject of the books [**22**] or [**21**]. Again the aim of this text is to present a short introduction to the subject to non specialists having a basic background in analysis and differential geometry. The interested reader may consult [**10**], [**13**], [**14**], [**17**], [**19**], [**23**] [**24**], [**32**] for more refined presentations and further references.

1. Finite dimensional integrable systems: the Hamiltonian point of view

Let us consider the space $\mathbb{R}^{2n}$ with the coordinates $(q,p) = (q^1, \cdots, q^n, p_1, \cdots, p_n)$. Many problems in Mechanics (and in other branches of mathematical science) can be expressed as the study of the evolution of a point in such a space, governed by the **Hamilton system of equations**

$$\left\{ \begin{array}{rcl} \dfrac{dq^i}{dt} & = & \dfrac{\partial H}{\partial p_i}(q(t),p(t)) \\ \dfrac{dp_i}{dt} & = & -\dfrac{\partial H}{\partial q^i}(q(t),p(t)), \end{array} \right.$$

where we are given a function $H : \mathbb{R}^{2n} \longmapsto \mathbb{R}$ called **Hamiltonian function**.

For instance paths $x : [a,b] \longrightarrow \mathbb{R}^3$ which are solutions of the Newton equation $m\ddot{x}(t) = -\nabla V(x(t))$ are critical points of the Lagrangian functional

$\mathcal{L}[x] := \int_a^b [\frac{m}{2}|\dot{x}(t)|^2 - V(x(t))]dt$. And by the Legendre transform they are converted into solutions of the Hamilton system of equations in $(\mathbb{R}^6, \omega)$ for $H(q,p) := \frac{|p|^2}{2m} + V(q)$.

We can view this system of equations as the flow of the **Hamiltonian vector field** defined on $\mathbb{R}^{2n}$ by

$$\xi_H(q,p) := \sum_i \frac{\partial H}{\partial p_i}(q,p)\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}(q,p)\frac{\partial}{\partial p_i}.$$

A geometrical, coordinate free, characterization of ξ_H can be given by introducing the **canonical symplectic form** on $\mathbb{R}^{2n}$

$$\omega := \sum_{i=1}^n dp_i \wedge dq^i.$$

Indeed ξ_H is the unique vector field which satisfies the relations

$$\forall (q,p), \mathbb{R}^{2n}, \forall X = \sum_i V^i \frac{\partial}{\partial q^i} + W_i \frac{\partial}{\partial p_i}, \quad \omega_{(q,p)}(\xi_H(q,p), X) + dH_{(q,p)}(X) = 0.$$

A notation is convenient here: given a vector $\xi \in \mathbb{R}^{2n}$ and for any $(q,p) \in \mathbb{R}^{2n}$, we denote by $\xi \lrcorner\, \omega_{(q,p)}$ the 1-form defined by $\forall X \in \mathbb{R}^{2n}$, $\xi \lrcorner\, \omega_{(q,p)}(X) = \omega_{(q,p)}(\xi, X)$. Then the preceding relation is just that $\xi_H \lrcorner\, \omega + dH = 0$ everywhere.

We call $(\mathbb{R}^{2n}, \omega)$ a **symplectic space**. More generally, given a smooth manifold $\mathcal{M}$, a *symplectic form* ω on $\mathcal{M}$ is a 2-form such that: (i) ω *is closed*, i.e., $d\omega = 0$, and (ii) ω *is non degenerate*, i.e., $\forall x \in \mathcal{M}$, $\forall \xi \in T_x\mathcal{M}$, if $\xi \lrcorner\, \omega_x = 0$, then $\xi = 0$. Note that the property (ii) implies that the dimension of $\mathcal{M}$ must be even. Then $(\mathcal{M}, \omega)$ is called a **symplectic manifold**.

1.1. The Poisson bracket. — We just have seen a rule which associates to each smooth function $f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ a vector field ξ_f (i.e., such that $\xi_f \lrcorner\, \omega + df = 0$). Furthermore for any pair of functions $f, g : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ we can define a third function called the **Poisson bracket** of f and g

$$\{f, g\} := \omega(\xi_f, \xi_g).$$

One can check easily that

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}.$$

In classical (i.e., not quantum) Mechanics the Poisson bracket is important because of the following properties:

1. if $\gamma = (q,p) : [a,b] \longrightarrow \mathbb{R}^{2n}$ is a solution of the Hamilton system of equations with the Hamiltonian H and if $f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is a smooth function, then

$$\frac{d}{dt}\left(f(\gamma(t))\right) = \{H, f\}(\gamma(t)).$$

This can be proved by a direct computation, either in coordinates:

$$\begin{aligned}\frac{d}{dt}(f\circ\gamma) &= \sum_i \frac{\partial f}{\partial p_i}(\gamma)\frac{dp_i}{dt}+\frac{\partial f}{\partial q^i}(\gamma)\frac{dq^i}{dt}\\ &= \sum_i \frac{\partial f}{\partial p_i}(\gamma)\left(-\frac{\partial H}{\partial q^i}(\gamma)\right)+\frac{\partial f}{\partial q^i}(\gamma)\left(\frac{\partial H}{\partial p_i}(\gamma)\right)\\ &= \{H,f\}\circ\gamma.\end{aligned}$$

or by a more intrinsic calculation:

$$\frac{d}{dt}(f\circ\gamma)=df_\gamma(\dot\gamma)=df_\gamma(\xi_H(\gamma))=-\omega_\gamma(\xi_f(\gamma),\xi_H(\gamma))=\{H,f\}\circ\gamma.$$

A special case of this relation is when $\{H,f\}=0$: we then say that H and f are **in involution** and we find that $f(\gamma(t))$ is constant, i.e., is a first integral. This can be viewed as a version of Noether's theorem which relates a continuous group of symmetry to a conservation law. In this case the vector field ξ_f is the infinitesimal symmetry and '$f(\gamma(t))=$ constant' is the conservation law.

2. The Lie bracket of two vector fields ξ_f and ξ_g is again a Hamiltonian vector field, more precisely

$$[\xi_f,\xi_g]=\xi_{\{f,g\}}.$$

This has the consequence that again if f and g are in involution, i.e., $\{f,g\}=0$, then the flows of ξ_f and ξ_g commute.

Both properties together implies the following: assume that $\{f,H\}=0$ and that (at least locally) df does vanish, which is equivalent to the fact that ξ_f does not vanish. Then we can reduce the number of variable by 2. A first reduction is due

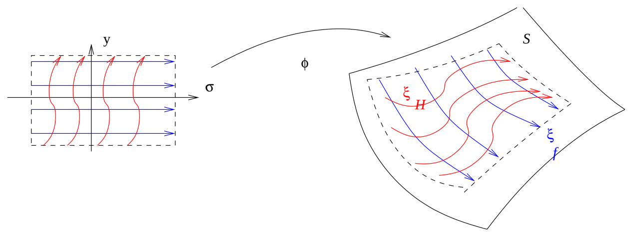

FIGURE 1. The symplectic reduction

to the first remark: the conservation of f along the integral curves of ξ_H can just be reformulated by saying that each integral curve of ξ_H is contained in a level set of f, i.e., the hypersurface $\mathcal{S}=\{m\in\mathbb{R}^{2n}|\ f(m)=C\}$. But also $\mathcal{S}$ is foliated by integral curves of the flow of ξ_f (a consequence of $\{f,f\}=0$). So for any point $m_0\in\mathcal{S}$ by the flow box theorem we can find a neighborhood $\mathcal{S}_{m_0}$ of m_0 in $\mathcal{S}$ and a diffeomorphism

$$\begin{array}{rccc}\varphi: & (-\varepsilon,\varepsilon)\times B^{2n-2}(0,r) & \longrightarrow & \mathcal{S}_{m_0}\\ & (\sigma,y) & \longmapsto & m\end{array}$$

so that $\frac{\partial\varphi}{\partial\sigma} = \xi_f \circ \varphi$. Now the second remark comes in: in the coordinates (σ, y) ξ_f is just $\frac{\partial}{\partial\sigma}$ and $[\xi_f, \xi_H] = 0$ reads that the coefficients of ξ_H are independent of σ, so they only depend on y. We conclude: locally the motion is equivalent to a Hamilton system of equations in $2n-2$ variables, namely the variables y. This is called a **symplectic reduction**.

1.2. The Liouville–Arnold theorem. — We can imagine a situation where we have a collection of n smooth functions $f_1, \cdots, f_n$ on an open subset Ω of $\mathbb{R}^{2n}$ which satisfies the following properties

1. the functions $f_1, \cdots, f_n$ are independent, i.e., we have everywhere
$$(df_1, \cdots, df_n) \text{ is of rank } n \quad \Longleftrightarrow \quad (\xi_{f_1}, \cdots, \xi_{f_n}) \text{ is of rank } n$$
2. the functions $f_1, \cdots, f_n$ are in involution, i.e.,
$$\forall i, j \in [\![1, n]\!], \quad \{f_i, f_j\} = 0.$$
3. there exists a function h of n real variables $(a_1, \cdots, a_n)$ such that $H = h(f_1, \cdots, f_n)$. Remark that this implies that
$$\{H, f_j\} = \sum_{i=1}^{n} \frac{\partial h}{\partial a_i}(f_1, \cdots, f_n)\{f_i, f_j\} = 0, \quad \forall j \in [\![1, n]\!].$$

Then it is possible to operate the above symplectic reduction n times: we get a local change of coordinates
$$\Phi : (\theta^i, I_i) \longmapsto (q^i, p_i)$$
such that
$$\Phi^* \left(\sum_{i=1}^{n} dp_i \wedge dq^i \right) = \sum_{i=1}^{n} dI_i \wedge d\theta^i \quad \text{and} \quad f_i \circ \Phi = I_i, \quad \forall i \in [\![1, n]\!]$$
And our Hamiltonian is now $h(I_1, \cdots, I_n)$. It means that the Hamilton equations in these coordinates read
$$\left\{ \begin{array}{lclcl} \dfrac{d\theta^i}{dt} & = & \dfrac{\partial h}{\partial I_i}(I) & =: & c^i \\[2ex] \dfrac{dI_i}{dt} & = & -\dfrac{\partial h}{\partial \theta^i}(I) & = & 0. \end{array} \right.$$
The second group of equation implies that the I_i's are constant and so are the c^i's, hence the first system implies that the θ^i's are affine functions of time. This result is the content of the *Liouville theorem* [**3**]. A more global conclusion can be achieved if one assume for instance that the functions f_i's are proper: then one proves that the level sets of $f = (f_1, \cdots, f_n)$ are tori, the coordinates transversal to the tori are called the *action variables* I_i, the coordinates on the tori are called the *angle variables* θ^i. This result is called the *Liouville–Arnold theorem* (see [**3**]) and can be generalized to symplectic manifolds.

A first possible definition of a so-called *completely integrable system* could be: *an evolution equation which can be described by a Hamiltonian system of equations for which the Liouville–Arnold theorem can be applied.* Indeed this theorem can then be used to integrate such finite dimensional dynamical systems by quadratures. However the Liouville–Arnold property covers only partially the features of *completely integrable systems*, which are also governed by sophisticated algebraic structures. Moreover these extra algebraic properties are particularly useful for the integration of infinite dimensional integrable systems: they will be expounded in the next sections and they will play a more and more important role in our presentation.

2. The Lax equation

In this section we will address the following question: how to cook up the conserved quantities? as a possible answer we shall see here a particular class of differential equations which possess a natural family of first integrals.

Suppose that some ordinary differential equation can be written

$$\frac{dL}{dt} = [L, M(L)], \tag{2.1}$$

where the unknown function is a $\mathcal{C}^1$ function

$$\begin{array}{rccc} L: & \mathbb{R} & \longrightarrow & M(n,\mathbb{R}) \\ & t & \longmapsto & L(t) \end{array}$$

and

$$\begin{array}{rccc} M: & M(n,\mathbb{R}) & \longrightarrow & M(n,\mathbb{R}) \\ & L & \longmapsto & M(L) \end{array}$$

is a $\mathcal{C}^1$ function on the set $M(n,\mathbb{R})$ of $n\times n$ real matrices (note that one could replace here $\mathbb{R}$ by $\mathbb{C}$ as well). Equation (2.1) is called the **Lax equation**. In the following two examples the map M is a projection onto the set of $n \times n$ real skew-symmetric matrices:

$$\mathfrak{so}(n) := \{A \in M(n,\mathbb{R})|\ A^t + A = 0\}.$$

Example 1. — *On $\mathbb{R}^2$ with the coordinates (q,p) and the symplectic form $\omega = dp \wedge dq$, we consider the Hamiltonian function $H(q,p) = |p|^2/2 + 2e^{2q}$. The associated Hamiltonian vector field is*

$$\xi_H(q,p) = p\frac{\partial}{\partial q} - 4e^{2q}\frac{\partial}{\partial p}.$$

Thus the corresponding Hamilton system of equations reads

$$\frac{dq}{dt} = p \quad ; \quad \frac{dp}{dt} = -4e^{2q}, \tag{2.2}$$

which is equivalent to $\frac{dq}{dt} = p$ *plus the condition that* $t \longmapsto q(t)$ *is a solution of the* Liouville equation*:*

$$\frac{d^2q}{dt^2} + 4e^{2q} = 0. \tag{2.3}$$

Then one can check that $t \longmapsto (q(t), p(t))$ *is a solution of (2.2) if and only if*

$$\frac{d}{dt}\begin{pmatrix} p/2 & e^q \\ e^q & -p/2 \end{pmatrix} = \left[\begin{pmatrix} p/2 & e^q \\ e^q & -p/2 \end{pmatrix}, \begin{pmatrix} 0 & e^q \\ -e^q & 0 \end{pmatrix}\right]. \tag{2.4}$$

The latter condition means that by choosing

$$L := \begin{pmatrix} p/2 & e^q \\ e^q & -p/2 \end{pmatrix} \quad \text{and} \quad \begin{array}{cccc} M: & M(2,\mathbb{R}) & \longrightarrow & \mathfrak{so}(2) \\ & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \end{array},$$

then $t \longmapsto L(t)$ *is a solution of the Lax equation (2.1).*

Example 2. — *A generalization of the previous example is the following: on* $\mathbb{R}^{2n}$ *with the coordinates* $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ *and the symplectic form* $\omega = dp_1 \wedge dq^1 + \cdots + dp_n \wedge dq^n$ *we consider the Hamiltonian function* $H(q,p) = \sum_{i=1}^n (p_i)^2/2 + \sum_{i=1}^{n-1} e^{2(q^i - q^{i+1})}$. *The associated Hamilton system of equations for maps* $(q,p) \longmapsto (q(t), p(t))$ *into* $\mathbb{R}^{2n}$ *is the* Toda lattice system of equations

$$\left\{\begin{array}{ccc} \dot{q}^1 & = & p_1 \\ & \vdots & \\ \dot{q}^i & = & p_i \\ & \vdots & \\ \dot{q}^n & = & p_n \end{array}\right. , \quad \left\{\begin{array}{cccc} \dot{p}_1 & = & & -2e^{2(q^1-q^2)} \\ & \vdots & & \\ \dot{p}_i & = & 2e^{2(q^{i-1}-q^i)} & -2e^{2(q^i-q^{i+1})}, \quad \forall 1 < i < n \\ & \vdots & & \\ \dot{p}_n & = & 2e^{2(q^{n-1}-q^n)} & \end{array}\right.$$

Then this system is equivalent to the condition $\frac{d}{dt}\left(\sum_{i=n}^n q^i\right) = \sum_{i=n}^n p_i$ *plus*[(1)] *the Lax equation (2.1) by letting*

$$L = \begin{pmatrix} p_1 & e^{(q^1-q^2)} & & & \\ e^{(q^1-q^2)} & p_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & p_{n-1} & e^{(q^{n-1}-q^n)} \\ & & & e^{(q^{n-1}-q^n)} & p_n \end{pmatrix}, \tag{2.5}$$

(1) Actually the Hamiltonian H is in involution with $f(q,p) := \sum_{i=1}^n p_i = \operatorname{tr} L$, so that a symplectic reduction can be done. The reduced symplectic space is the set of all trajectories of ξ_f contained in a given level set of f and is symplectomorphic to $\mathbb{R}^{2n-2}$ with its standard symplectic form. Hence the Lax equation is here equivalent to the image of the Toda system by this reduction.

and

$$\begin{array}{cccc} M: & M(n,\mathbb{R}) & \longrightarrow & \mathfrak{so}(n) \\ & \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & m_{12} & \cdots & m_{1n} \\ -m_{12} & 0 & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ -m_{1n} & -m_{2n} & \cdots & 0 \end{pmatrix} \end{array}$$

Note that the Hamiltonian function can also be written as

$$H(q,p) = \frac{1}{2} tr\ L^2. \tag{2.6}$$

In the case where $n = 2$ one recovers the Liouville equation by assuming $q^1 + q^2 = p_1 + p_2 = 0$ and by posing $q := q^1 - q^2$ and $p := p_1 - p_2$.

Of course the dynamical systems which can be written in the form (2.1) are exceptions. Moreover given a possibly completely integrable Hamiltonian system, the task of finding its formulation as a Lax equation may be nontrivial.

2.1. A recipe for producing first integrals

Theorem 1. — *Let $L \in \mathcal{C}^1(\mathbb{R}, M(n,\mathbb{R}))$ be a solution of the Lax equation (2.1). Then the eigenvalues of $L(t)$ are constant.*

Before proving this result we need the following

Lemma 1. — *Let $I \subset \mathbb{R}$ be some interval and $B : I \longrightarrow GL(n,\mathbb{R})$ be a $\mathcal{C}^1$ map. Then*

$$\frac{d}{dt}(\det B(t)) = (\det B(t))\,\mathrm{tr}\,\left(B(t)^{-1}\frac{dB}{dt}(t)\right). \tag{2.7}$$

Proof of Lemma 1. — Let $C \in \mathcal{C}^1(I, GL(n,\mathbb{R}))$, then

$$\det C = \sum_{\sigma\in\Sigma_n} (-1)^{|\sigma|} C_1^{\sigma(1)} \cdots C_n^{\sigma(n)}$$

implies that

$$\frac{d}{dt}(\det C) = \sum_{\sigma\in\Sigma_n} (-1)^{|\sigma|} \sum_{j=1}^n \frac{dC_j^{\sigma(j)}}{dt} C_1^{\sigma(1)} \cdots \widehat{C_j^{\sigma(j)}} \cdots C_n^{\sigma(n)},$$

where the symbol $\widehat{\cdot}$ just means that the quantity under the hat is omitted. Now assume that for $t = 0$ we have $C(0) = 1_n$. Then the above relation simplifies and gives

$$\frac{d}{dt}(\det C)(0) = \sum_{j=1}^n \frac{dC_j^j}{dt}(0) = \mathrm{tr}\ \frac{dC}{dt}(0). \tag{2.8}$$

Now consider $B \in \mathcal{C}^1(I, GL(n,\mathbb{R}))$ and an arbitrary value of t, say t_0, for which $B(t_0)$ is not necessarily equal to 1_n. We set

$$C(t) := B(t_0)^{-1}B(t+t_0),$$

so that $C(0) = 1_n$. Then on the one hand $\det C(t) = (\det B(t_0))^{-1} \det B(t_0+t)$ implies that

$$\frac{d}{dt}(\det C)(0) = (\det B(t_0))^{-1} \frac{d}{dt}(\det B)(t_0).$$

And on the other hand

$$\mathrm{tr}\ \frac{dC}{dt}(0) = \mathrm{tr}\ \left(B(t_0)^{-1}\frac{dB}{dt}(t_0)\right),$$

so by substitution in the relation (2.8) we exactly get relation (2.7) for $t = t_0$. □

Proof of Theorem 1. — Consider $L : I \longrightarrow M(n,\mathbb{R})$, a solution of the Lax equation (2.1) then, for any real or complex constant λ we obviously have $[L - \lambda 1_n, M(L)] = [L, M(L)]$ and so

$$\frac{d}{dt}(L - \lambda 1_n) = [L - \lambda 1_n, M(L)].$$

Fix some time t_0 and consider n distinct values $\lambda_1, \cdots, \lambda_n$ which are not eigenvalues of $L(t_0)$ (so that $\det(L(t_0) - \lambda_j) \neq 0$, $\forall j = 1, \cdots, n$). Then, because of the continuity of L there exists some $\varepsilon > 0$ such that $\det(L(t) - \lambda_j 1_n) \neq 0$, $\forall j = 1, \cdots, n$, $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Hence we can apply the previous lemma to $B = L - \lambda_j 1_n$, for all j and $I = (t_0 - \varepsilon, t_0 + \varepsilon)$: we obtain

$$\begin{array}{rcl}
\dfrac{d}{dt}\left(\det(L - \lambda_j 1_n)\right) & = & \det(L - \lambda_j 1_n)\,\mathrm{tr}\ \left((L - \lambda_j 1_n)^{-1}\dfrac{d(L - \lambda_j 1_n)}{dt}\right) \\
& = & \det(L - \lambda_j 1_n)\,\mathrm{tr}\ \big((L - \lambda_j 1_n)^{-1}[L - \lambda_j 1_n, M(L)]\big) \\
& = & \det(L - \lambda_j 1_n)\,\mathrm{tr}\ \big(M(L) - (L - \lambda_j 1_n)^{-1}M(L)(L - \lambda_j 1_n)\big) \\
& = & 0.
\end{array}$$

So $\det(L(t) - \lambda_j 1_n)$ is constant on I. Since this is true for n distinct values λ_j, we deduce that $\det(L(t) - \lambda 1_n)$ is constant on I, for all λ. Hence the characteristic polynomial is constant for all times. This proves Theorem 1. □

2.2. The search for a special ansatz. — This property leads us to the following. Assume for instance that the eigenvalues of $L(t)$ are all distinct. Then the matrix $L(t)$ is diagonalizable for all times, i.e., for all times t there exists an invertible matrix $P(t)$ such that

$$L(t) = P(t)^{-1}DP(t), \tag{2.9}$$

where D is a time independent diagonal matrix and the columns of $P(t)^{-1}$ are the eigenvectors of $L(t)$.

A related question (which makes sense even if $L(t)$ is not diagonalizable) is to find some map S into $GL(n,\mathbb{R})$ such that

$$L(t) = S(t)^{-1} L_0 S(t), \tag{2.10}$$

where $L_0 := L(0)$. Note that in the case where $L(t)$ is diagonalizable, i.e., if equation (2.9) has a solution, then in particular we have also $L(0) = P(0)^{-1} D P(0)$, so that

$$L(t) = P(t)^{-1} \left(P(0)L(0)P(0)^{-1}\right) P(t),$$

and hence $S(t) := P(0)^{-1} P(t)$ is a solution to (2.10).

Our approach here will be based on solving directly (2.10). For that purpose we will look for a differential equation on S which will be a sufficient condition for (2.10) to be true. We derivate L:

$$\begin{array}{rcl}\dfrac{dL}{dt} & = & \dfrac{dS^{-1}}{dt} L_0 S + S^{-1} L_0 \dfrac{dS}{dt} \\ & = & \left(-S^{-1}\dfrac{dS}{dt}S^{-1}\right) L_0 S + S^{-1} L_0 \dfrac{dS}{dt} \\ & = & -\left(S^{-1}\dfrac{dS}{dt}\right)\left(S^{-1}L_0S\right) + \left(S^{-1}L_0S\right)\left(S^{-1}\dfrac{dS}{dt}\right) \\ & = & \left[S^{-1}L_0S, S^{-1}\dfrac{dS}{dt}\right] = \left[L, S^{-1}\dfrac{dS}{dt}\right].\end{array}$$

A comparison with the Lax equation (2.1) shows that relation (2.10) holds for all times if and only if $\left[L, M(L) - S^{-1}\frac{dS}{dt}\right] = 0$ for all times. The simplest choice is to take the unique solution of

$$\left\{\begin{array}{rcl}\dfrac{dS}{dt} & = & SM(L), \quad \forall t \\ S(0) & = & 1_n.\end{array}\right. \tag{2.11}$$

Conversely we have

Proposition 1. — *Let $L \in \mathcal{C}^1(I, M(n,\mathbb{R}))$ be a solution of (2.1). Consider $S \in \mathcal{C}^1(I, GL(n,\mathbb{R}))$ the solution of (2.11). Then, denoting $L_0 := L(0)$, we have*

$$L(t) = S(t)^{-1} L_0 S(t), \quad \forall t. \tag{2.12}$$

Proof. — We just compute by using first (2.11) and then (2.1) that

$$\frac{d}{dt}\left(SLS^{-1}\right) = S\left(\frac{dL}{dt} + [M(L), L]\right) S^{-1} = 0.$$

So SLS^{-1} is constant. Since it is equal to L_0 for $t = 0$, the conclusion follows. □

The method to solve equation (2.1) that we are going to see (under some further hypotheses) is based on the study of the system (2.1) **and** (2.11). Even more we will adjoin to these two systems a third one:

$$\left\{\begin{array}{rcl}\dfrac{dT}{dt} & = & (L - M(L))T, \quad \forall t \\ T(0) & = & 1_n.\end{array}\right. \tag{2.13}$$

Then we have the following tricky computation. Start with the identity

$$L = M(L) + L - M(L),$$

true for all times. Multiply on the left by S and on the right by T:

$$SLT = SM(L)T + S(L - M(L))T$$

and use (2.12) on the left hand side and (2.11) and (2.13) and the right hand side

$$S\left(S^{-1}L_0S\right)T = \frac{dS}{dt}T + S\frac{dT}{dt}$$

to obtain

$$L_0(ST) = \frac{d}{dt}(ST).$$

Hence we deduce, using the fact that $S(0)T(0) = 1_n$, that

$$S(t)T(t) = e^{tL_0}.$$

So we observe that if we were able to extract the factor $S(t)$ from e^{tL_0} we would be able to deduce $L(t)$ by using (2.12). Fortunately it is possible in many examples (actually it corresponds to cases where the theory of Adler–Kostant–Symes can be applied, see below).

2.3. The decomposition of e^{tL_0}. — Let us first consider Example 1. Then

$$M\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}0 & \beta\\ -\beta & 0\end{pmatrix} \quad \text{and} \quad (Id - M)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ \beta+\gamma & \delta\end{pmatrix},$$

and we see that the two maps M and $Id - M$ are linear projection onto two supplementary subspaces of $M(2,\mathbb{R})$, namely $\mathfrak{so}(2)$ and the subset of lower triangular matrices

$$\mathfrak{t}^-(2,\mathbb{R}) := \left\{t = \begin{pmatrix}t_1^1 & 0\\ t_1^2 & t_2^2\end{pmatrix} \mid t_1^1, t_1^2, t_2^2 \in \mathbb{R}\right\}.$$

Since $M(2,\mathbb{R}) = \mathfrak{so}(2) \oplus \mathfrak{t}^-(2,\mathbb{R})$ there are indeed two natural projection maps π_L (onto $\mathfrak{so}(2)$) and π_R (onto $\mathfrak{t}^-(2,\mathbb{R})$) and $M = \pi_L$ and $1_2 - M = \pi_R$. This has the following consequences. First equation (2.11) and the fact that $\pi_L(L(t)) = M(L(t))$ takes values in $\mathfrak{so}(2)$ implies that $S(t)$ takes values in the rotation group $SO(2) := \{R \in M(2,\mathbb{R}) |\ R^tR = RR^t = 1_2\}$. Indeed, by using $\pi_L(L) + \pi_L(L)^t = 0$,

$$\frac{d}{dt}\left(SS^t\right) = \frac{dS}{dt}S^t + S\frac{dS^t}{dt} = S\pi_L(L)S^t + S\pi_L(L)^tS^t = 0.$$

Second equation (2.13) and the fact that $\pi_R(L(t)) = L(t) - M(L(t))$ takes values in $\mathfrak{t}^-(2,\mathbb{R})$ implies that $T(t)$ takes values in the group of lower triangular matrices with positive diagonal

$$T^-(2,\mathbb{R}) := \left\{T = \begin{pmatrix}T_1^1 & 0\\ T_1^2 & T_2^2\end{pmatrix} \mid T_1^1, T_2^2 \in (0,\infty), T_1^2 \in \mathbb{R}\right\}.$$

Indeed by writing $L - M(L) = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ then one can check that (2.13) implies that

$$T(t) = \begin{pmatrix} A(t) & 0 \\ D(t)\int_0^t \gamma(s)\frac{A(s)}{D(s)}ds & D(t) \end{pmatrix}, \quad \forall t,$$

where $A(t) := e^{\int_0^t \alpha(s)ds}$ and $D(t) := e^{\int_0^t \delta(s)ds}$. Lastly we observe that $\det e^{tL_0} > 0$, i.e., e^{tL_0} takes values in the subgroup $GL^+(2,\mathbb{R})$ of matrices with positive determinants (we even have $\det e^{tL_0} = e^{t\mathrm{tr} L_0}$, a consequence of Lemma 1).

Now we see that extracting $S(t)$ from e^{tL_0} just consists in solving the problem

$$\left\{ \begin{array}{rcl} S(t)T(t) & = & e^{tL_0} \in GL^+(2,\mathbb{R}) \\ S(t) & \in & SO(2) \\ T(t) & \in & T^-(2,\mathbb{R}) \end{array} \right., \quad \forall t. \tag{2.14}$$

Standard results from linear algebra tell us indeed that for each time t there is a unique solution $(S(t), T(t))$ to (2.14): it is given by the Gram–Schmidt orthonormalisation process. For 2×2 matrices we can easily write it explicitly: assume that for some t

$$e^{tL_0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$S(t) = \frac{1}{\sqrt{b^2+d^2}}\begin{pmatrix} d & b \\ -b & d \end{pmatrix}, \quad T(t) = \frac{1}{\sqrt{b^2+d^2}}\begin{pmatrix} ad-bc & 0 \\ ab+cd & b^2+d^2 \end{pmatrix}.$$

***Example 3* (Example 1 continued)**. — *We solve here the system (2.2) by using that method. Let q_0 and p_0 denote the initial value of q and p respectively at $t = 0$ and consider the matrix*

$$L_0 := \begin{pmatrix} p_0/2 & e^{q_0} \\ e^{q_0} & -p_0/2 \end{pmatrix}.$$

The first task is to compute e^{tL_0} and, for that purpose, we need to diagonalize L_0:

$$L_0 = \frac{1}{2e^{q_0}\epsilon_0}\begin{pmatrix} e^{q_0} & e^{q_0} \\ \epsilon_0 - p_0/2 & -\epsilon_0 - p_0/2 \end{pmatrix}\begin{pmatrix} \epsilon_0 & 0 \\ 0 & -\epsilon_0 \end{pmatrix}\begin{pmatrix} \epsilon_0 + p_0/2 & e^{q_0} \\ \epsilon_0 - p_0/2 & -e^{q_0} \end{pmatrix},$$

where $\epsilon_0 := \sqrt{(p_0)^2/4 + e^{2q_0}}$. Then

$$e^{tL_0} = \begin{pmatrix} \cosh(\epsilon_0 t) + \frac{p_0}{2\epsilon_0}\sinh(\epsilon_0 t) & \frac{e^{q_0}}{\epsilon_0}\sinh(\epsilon_0 t) \\ \frac{e^{q_0}}{\epsilon_0}\sinh(\epsilon_0 t) & \cosh(\epsilon_0 t) - \frac{p_0}{2\epsilon_0}\sinh(\epsilon_0 t) \end{pmatrix}.$$

We now compute $S(t)$ such that the decomposition $e^{tL_0} = S(t)T(t)$ holds:

$$S(t) = \frac{1}{\sqrt{\Delta(t)}}\begin{pmatrix} \cosh(\epsilon_0 t) - \frac{p_0}{2\epsilon_0}\sinh(\epsilon_0 t) & \frac{e^{q_0}}{\epsilon_0}\sinh(\epsilon_0 t) \\ -\frac{e^{q_0}}{\epsilon_0}\sinh(\epsilon_0 t) & \cosh(\epsilon_0 t) - \frac{p_0}{2\epsilon_0}\sinh(\epsilon_0 t) \end{pmatrix},$$

where $\Delta(t) := \cosh(2\epsilon_0 t) - \frac{p_0}{2\epsilon_0}\sinh(2\epsilon_0 t)$. *Lastly we compute* $L(t) = S(t)^{-1}L_0S(t)$*:*

$$L(t) = \frac{1}{\Delta(t)}\begin{pmatrix} \frac{p_0}{2}\cosh(2\epsilon_0 t) - \epsilon_0\sinh(2\epsilon_0 t) & e^{q_0} \\ e^{q_0} & -\frac{p_0}{2}\cosh(2\epsilon_0 t) + \epsilon_0\sinh(2\epsilon_0 t) \end{pmatrix}$$

and deduce:

$$\begin{cases} q(t) &= q_0 - \ln\left(\cosh(2\epsilon_0 t) - \dfrac{p_0}{2\epsilon_0}\sinh(2\epsilon_0 t)\right) \\ p(t) &= \dfrac{p_0\cosh(2\epsilon_0 t) - 2\epsilon_0\sinh(2\epsilon_0 t)}{\cosh(2\epsilon_0 t) - \frac{p_0}{2\epsilon_0}\sinh(2\epsilon_0 t)}. \end{cases}$$

We remark that $q(t) = q_0 - \ln\Delta(t)$ *and* $p(t) = -\frac{\dot{\Delta}(t)}{\Delta(t)}$.

A straightforward generalization of the preceding method works for solving Example 2, as follows. Let $\mathfrak{t}^-(n,\mathbb{R})$ be the set of $n\times n$ real lower triangular matrices. Then the splitting $M(n,\mathbb{R}) = \mathfrak{so}(n)\oplus\mathfrak{t}^-(n,\mathbb{R})$ leads us to a pair of projection mappings $\pi_L : M(n,\mathbb{R}) \longrightarrow \mathfrak{so}(n)$ and $\pi_R : M(n,\mathbb{R}) \longrightarrow \mathfrak{t}^-(n,\mathbb{R})$. Let $t\longmapsto L(t)$ be a $\mathcal{C}^1$ map which is a solution of $\frac{dL}{dt}(t) = [L(t),\pi_L(L(t))]$. Then set $L_0 := L(0)$ and consider the system

$$\begin{cases} \forall t,\ \dfrac{dL}{dt}(t) &= [L(t),\pi_L(L(t))] \quad \text{and} \quad L(0) = L_0 \\ \forall t,\ \dfrac{dS}{dt}(t) &= S(t)\pi_L(L(t)) \quad \text{and} \quad S(0) = 1_n \\ \forall t,\ \dfrac{dT}{dt}(t) &= \pi_R(L(t))T(t) \quad \text{and} \quad T(0) = 1_n. \end{cases} \tag{2.15}$$

Then by the same calculation as above one proves that

1. $\forall t\in\mathbb{R},\quad L(t) = S(t)^{-1}L_0S(t)$
2. $\forall t\in\mathbb{R},\quad S(t)T(t) = e^{tL_0}$
3. $S(t)$ takes values in $SO(n)$ and $T(t)$ takes values in $T^-(n,\mathbb{R})$, where $T^-(n,\mathbb{R})$ is the group of lower diagonal matrices with positive coefficients on the diagonal
4. e^{tL_0}takes values in $GL^+(n,\mathbb{R})$, where $GL^+(n,\mathbb{R})$ is the subgroup of matrices in $GL(n,\mathbb{R})$ with positive determinant
5. the map
$$\begin{array}{ccc} SO(n)\times T^-(n,\mathbb{R}) & \longrightarrow & GL^+(n,\mathbb{R}) \\ (R,T) & \longmapsto & RT, \end{array}$$
is a diffeomorphism. Actually the inverse of this map can be computed algebraically by using the Gram–Schmidt orthonormalization process.

So again we can compute the solution $L(t)$ by first computing e^{tL_0}, second by using Step 5 extracting from that matrix its $SO(n)$ part, namely $S(t)$ and third use the relation $L(t) = S(t)^{-1}L_0S(t)$.

2.4. Lie algebras and Lie groups. — The preceding method can actually be generalized to other group of matrices, or more generally in the framework of Lie groups. This can be seen by analyzing the five properties used in the previous subsection. Properties 1 and 2 just come from the equations, i.e., from system (2.15). Properties 3 and 4 have natural generalizations in the framework of Lie algebras.

A (real or complex) **Lie algebra** is (real or complex) vector space $\mathfrak{g}$ endowed with a bilinear map

$$\begin{array}{rccc} [\cdot,\cdot]: & \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & (\xi,\eta) & \longmapsto & [\xi,\eta] \end{array}$$

called **Lie bracket** which is skewsymmetric, i.e., which satisfies $[\xi,\eta]+[\eta,\xi]=0$ and which satisfies the *Jacobi identity* $[\xi,[\eta,\psi]]+[\psi,[\xi,\eta]]+[\eta,[\psi,\xi]]=0$. For simplicity the reader may consider that Lie algebras are vector spaces of matrices, i.e., subspaces of $M(n,\mathbb{R})$ or $M(n,\mathbb{C})$, which are endowed with the Lie bracket $[\xi,\eta]:=\xi\eta-\eta\xi$ and stable under this bracket.

A **Lie group** is a group and a manifold in a compatible way. It means that if $\mathfrak{G}$ is a Lie group then it is a smooth manifold endowed with a group law

$$\begin{array}{ccc} \mathfrak{G}\times\mathfrak{G} & \longrightarrow & \mathfrak{G} \\ (a,b) & \longmapsto & ab \end{array}$$

which is a smooth map. Here also the reader can figure out Lie groups as set of matrices, i.e., subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. If $e\in\mathfrak{G}$ is the unity then the tangent space to $\mathfrak{G}$ at e, $\mathfrak{g}=T_e\mathfrak{G}$, has a natural structure of Lie algebra. Indeed first we can associate to each $g\in\mathfrak{G}$ the **adjoint map**

$$\begin{array}{rccc} \mathrm{Ad}_g: & \mathfrak{G} & \longrightarrow & \mathfrak{G} \\ & a & \longmapsto & gag^{-1}. \end{array}$$

Since Ad_g is smooth we can consider its differential $d\left(\mathrm{Ad}_g\right)_e$ at e which maps linearly $\mathfrak{g}=T_e\mathfrak{G}$ to itself, since $\mathrm{Ad}_g(e)=e$. We will simply denote this map by $\mathrm{Ad}_g:\mathfrak{g}\longrightarrow\mathfrak{g}$. For matrices we can write $\mathrm{Ad}_g\eta=g\eta g^{-1}$. Now if we assume that $t\longmapsto g(t)$ is a smooth curve such that $g(0)=e$ and $\frac{dg}{dt}(0)=\xi\in T_e\mathfrak{G}$ we can consider the differential $\mathrm{ad}_\xi:=(d\mathrm{Ad}_{g(t)}/dt)(0)$ of $\mathrm{Ad}_{g(t)}$ at $t=0$ and set

$$\begin{array}{rccc} \mathrm{ad}_\xi: & \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & \eta & \longmapsto & \mathrm{ad}_\xi\eta=\frac{d\mathrm{Ad}_{g(t)}\eta}{dt}(0). \end{array}$$

Then it turns out that the bilinear map

$$\begin{array}{ccc} \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ (\xi,\eta) & \longmapsto & [\xi,\eta]:=\mathrm{ad}_\xi\eta \end{array}$$

is skewsymmetric and satisfies the Jacobi identity and so is a Lie bracket. The Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ encodes in a concise way the lack of commutativity of the Lie group and the Jacobi identity is the infinitesimal expression of the associativity of the group law on $\mathfrak{G}$. As an exercise the reader can check by himself that when dealing with subgroup of matrices we have $\mathrm{ad}_\xi\eta = \xi\eta - \eta\xi$, so that we recover the standard Lie bracket on matrices.

Lastly, for any $g \in \mathfrak{G}$, consider the smooth *left action* map $L_{g^{-1}} : \mathfrak{G} \longrightarrow \mathfrak{G}$, $h \longmapsto g^{-1}h$. Its differential at g is a linear map $d\left(L_{g^{-1}}\right)_g : T_g\mathfrak{G} \longrightarrow \mathfrak{g}$, and, for $\xi \in T_g\mathfrak{G}$, we will simply denote $d\left(L_{g^{-1}}\right)_g(\xi)$ by $g^{-1}\xi$, since it is exactly the expression that we obtain for matrix groups. We define an analogous map $T_g\mathfrak{G} \longrightarrow \mathfrak{g}$ by using the right action of g^{-1}, that we denote by $\xi \longmapsto \xi g^{-1}$. Then, for any $\alpha \in \mathcal{C}^1(\mathbb{R}, \mathfrak{g})$, we can consider the equation $S(t)^{-1}\frac{dS}{dt}(t) = \alpha(t)$, where $S \in \mathcal{C}^1(\mathbb{R}, \mathfrak{G})$, it is easy to show that this equation has a unique solution if we are given an initial condition $S(0) = S_0$. Similarly, for any $\beta \in \mathcal{C}^1(\mathbb{R}, \mathfrak{g})$ and given some $T_0 \in \mathfrak{G}$ there exists a unique solution $T \in \mathcal{C}^1(\mathbb{R}, \mathfrak{G})$ to the equation $\frac{dT}{dt}(t)T(t)^{-1} = \beta(t)$ with the initial condition $T(0) = T_0$.

Now assume that we are given Lie group $\mathfrak{G}$ with its Lie algebra $\mathfrak{g}$ and that $\mathfrak{g} = \mathfrak{g}_L \oplus \mathfrak{g}_R$, where $\mathfrak{g}_L$ and $\mathfrak{g}_R$ are the **Lie algebras** of respectively some **Lie subgroups** $\mathfrak{G}_L$ and $\mathfrak{G}_R$. We then define the projections mappings π_L and π_R onto the two factors and we consider the system (2.15). Automatically the analogues of Conditions 1, 2, 3 and 4 are satisfied (replacing $SO(n)$ by $\mathfrak{G}_L$, $T^-(n,\mathbb{R})$ by $\mathfrak{G}_R$ and $GL^+(n,\mathbb{R})$ by $\mathfrak{G}$). Hence if the analogue of Condition 5, i.e., that

$$\begin{array}{ccc} \mathfrak{G}_L \times \mathfrak{G}_R & \longrightarrow & \mathfrak{G} \\ (R,T) & \longmapsto & RT \end{array} \quad \text{is a diffeomorphism,}$$

is satisfied, we can solve the equation $\frac{dL}{dt} = [L, \pi_L(L)]$ by the same method as before, due to W. Symes [**31**]. Note that this last condition can be seen as the nonlinear version for groups of the splitting $\mathfrak{g} = \mathfrak{g}_L \oplus \mathfrak{g}_R$. In most examples one of the two sub Lie algebras, say $\mathfrak{g}_R$ is *solvable*: it means that if we consider $[\mathfrak{g}_R, \mathfrak{g}_R] := \{[\xi,\eta] | \,\xi,\eta \in \mathfrak{g}_R\}$ and then $[[\mathfrak{g}_R, \mathfrak{g}_R], [\mathfrak{g}_R, \mathfrak{g}_R]] := \{[\xi,\eta] | \,\xi,\eta \in [\mathfrak{g}_R, \mathfrak{g}_R]\}$, etc. then these subspaces will be reduced to 0 after a finite number of steps. The basic example of a solvable Lie algebra is the set of lower (or upper) triangular matrices $\mathfrak{t}^-(n,\mathbb{R})$. If so the splitting $\mathfrak{G} = \mathfrak{G}_L \cdot \mathfrak{G}_R$ is called an **Iwasawa** decomposition.

2.5. The Adler–Kostant–Symes theory. — The Hamiltonian structure was absent in our presentation. In order to understand how it is related to the previous method one needs the deeper insight provided by the *Adler–Kostant–Symes theory* [**1, 20, 31**]. The key ingredients are:

1. a Lie algebra $\mathfrak{g}$ which admits the vector space decomposition $\mathfrak{g} = \mathfrak{g}_L \oplus \mathfrak{g}_R$, where $\mathfrak{g}_L$ and $\mathfrak{g}_R$ are *Lie subalgebras*;
2. an $\mathrm{ad}^*_\mathfrak{g}$-invariant function on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

The first ingredient provides us with the *phase space*: the Poisson manifold $\mathfrak{g}^*_R$ (see below), whereas the second one helps us to build the *Hamiltonian function*. However we first need to introduce some extra notions in particular to clarify the meaning of the second assumption.

2.5.1. Poisson manifolds. — A *Poisson manifold* $\mathcal{M}$ is a smooth manifold endowed with a *skew-symmetric bilinear* map

$$\begin{array}{rccc}\{\cdot,\cdot\}: & \mathcal{C}^\infty(\mathcal{M})\times\mathcal{C}^\infty(\mathcal{M}) & \longrightarrow & \mathcal{C}^\infty(\mathcal{M})\\ & (f,g) & \longmapsto & \{f,g\}\end{array}$$

which satisfies the *Leibniz rule* $\{fg,h\} = f\{g,h\} + g\{f,h\}$ and the *Jacobi identity* $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$. Then $\{\cdot,\cdot\}$ is called a *Poisson bracket.* Symplectic manifolds endowed with the bracket $\{f,g\} = \omega(\xi_f,\xi_g)$ are examples of Poisson manifolds. Another important example, which goes back to S. Lie, is the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$: for any functions $f,g\in\mathcal{C}^\infty(\mathfrak{g}^*)$ we let $\{f,g\}_{\mathfrak{g}^*}\in\mathcal{C}^\infty(\mathfrak{g}^*)$ be defined by

$$\forall\alpha\in\mathfrak{g}^*,\quad \{f,g\}_{\mathfrak{g}^*}(\alpha) := ({}^{\mathfrak{g}^*}\alpha,[C_{\mathfrak{g}}df_\alpha, C_{\mathfrak{g}}dg_\alpha])_{\mathfrak{g}},$$

where $({}^{\mathfrak{g}^*}\cdot,\cdot)_{\mathfrak{g}} : \mathfrak{g}^*\times\mathfrak{g}\longrightarrow\mathbb{R}$ is the duality product and $C_{\mathfrak{g}}:\mathfrak{g}^{**}\longrightarrow\mathfrak{g}$ is the canonical isomorphism. In most cases we shall drop $C_{\mathfrak{g}}$ and simply write $\{f,g\}_{\mathfrak{g}^*}(\alpha) := ({}^{\mathfrak{g}^*}\alpha,[df_\alpha,dg_\alpha])_{\mathfrak{g}}$. The *co-adjoint action of* $\mathfrak{g}$ *on* $\mathfrak{g}^*$ is defined by associating to all $\xi\in\mathfrak{g}$ the linear map $\mathrm{ad}^*_\xi:\mathfrak{g}^*\longrightarrow\mathfrak{g}^*$ such that

$$\forall\alpha\in\mathfrak{g}^*,\forall\eta\in\mathfrak{g},\quad ({}^{\mathfrak{g}^*}\mathrm{ad}^*_\xi\alpha,\eta)_{\mathfrak{g}} := ({}^{\mathfrak{g}^*}\alpha,\mathrm{ad}_\xi\eta)_{\mathfrak{g}} = ({}^{\mathfrak{g}^*}\alpha,[\xi,\eta])_{\mathfrak{g}}.$$

Note that $\mathfrak{g}^*$ is not a symplectic manifold, however a result of A. A. Kirillov asserts that the integral manifolds of the distribution spanned by the vector fields $\alpha\longmapsto\mathrm{ad}^*_\xi\alpha$, for $\xi\in\mathfrak{g}$, (in fact the orbits of the co-adjoint action of a Lie group $\mathfrak{G}$ whose Lie algebra is $\mathfrak{g}$) are symplectic submanifolds. The symplectic structure on these orbits induces a Poisson bracket which coincides with the restriction of the Poisson bracket $\{f,g\}_{\mathfrak{g}^*}$.

2.5.2. Embedding $\mathfrak{g}_R^*$ in $\mathfrak{g}^*$. — As announced the phase space is the Poisson manifold $\mathfrak{g}_R^*$. However we will use the decomposition $\mathfrak{g} = \mathfrak{g}_L\oplus\mathfrak{g}_R$ to embedd $\mathfrak{g}_R^*$ in $\mathfrak{g}^*$. Let us define

$$\mathfrak{g}_L^\perp := \{\alpha\in\mathfrak{g}^* |\ \forall\xi\in\mathfrak{g}_L, ({}^{\mathfrak{g}^*}\alpha,\xi)_{\mathfrak{g}} = 0\}\subset\mathfrak{g}^*$$

and similarly

$$\mathfrak{g}_R^\perp := \{\alpha\in\mathfrak{g}^* |\ \forall\xi\in\mathfrak{g}_R, ({}^{\mathfrak{g}^*}\alpha,\xi)_{\mathfrak{g}} = 0\}\subset\mathfrak{g}^*.$$

We first observe that $\mathfrak{g}_R^*\simeq\mathfrak{g}^*/\mathfrak{g}_R^\perp$ and the quotient mapping $Q:\mathfrak{g}^*\longrightarrow\mathfrak{g}_R^*$ coincides with the restriction mapping $\alpha\longmapsto\alpha|_{\mathfrak{g}_R}$. Furthermore $\mathfrak{g}^* = \mathfrak{g}_R^\perp\oplus\mathfrak{g}_L^\perp$, so that we can define the associated projection mappings $\pi_R^\perp:\mathfrak{g}^*\longrightarrow\mathfrak{g}_R^\perp\subset\mathfrak{g}^*$ and $\pi_L^\perp:\mathfrak{g}^*\longrightarrow\mathfrak{g}_L^\perp\subset\mathfrak{g}^*$. However the restriction of $\pi_L^\perp$ to each fiber of Q is constant, hence there exists a unique map $\sigma:\mathfrak{g}_R^*\longrightarrow\mathfrak{g}_L^\perp\subset\mathfrak{g}^*$ such that the factorization $\pi_L^\perp = \sigma\circ Q$ holds: σ is the embedding of $\mathfrak{g}_R^*$ that we shall use.

A second task is to characterize the image $\{\cdot,\cdot\}_{\mathfrak{g}_L^\perp}$ of the Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{g}_R^*}$ by σ, defined by:

$$\forall\varphi,\psi\in\mathcal{C}^\infty(\mathfrak{g}_L^\perp),\quad \{\varphi,\psi\}_{\mathfrak{g}_L^\perp}\circ\sigma = \{\varphi\circ\sigma,\psi\circ\sigma\}_{\mathfrak{g}_R^*}. \tag{2.16}$$

Note that any functions $\varphi,\psi\in\mathcal{C}^\infty(\mathfrak{g}_L^\perp)$ can be considered as restrictions to $\mathfrak{g}_L^\perp$ of respectively functions $f,g\in\mathcal{C}^\infty(\mathfrak{g}^*)$ and it is convenient to have an expression of

$\{\varphi,\psi\}_{\mathfrak{g}_L^\perp}$ in terms of f and g. For that purpose we first need to precise the relationship between $d(\varphi\circ\sigma)_\alpha$ and $df_{\sigma(\alpha)}$, for all $\alpha\in\mathfrak{g}_R^*$, if $f\in\mathcal{C}^\infty(\mathfrak{g}^*)$ and $\varphi:=f|_{\mathfrak{g}_L^\perp}$: for any $\alpha\in\mathfrak{g}_R^*$,

$$d(\varphi\circ\sigma)_\alpha\circ Q = d(f\circ\sigma)_\alpha\circ Q = df_{\sigma(\alpha)}\circ\sigma\circ Q = df_{\sigma(\alpha)}\circ\pi_L^\perp = \left(\pi_L^\perp\right)^* df_{\sigma(\alpha)}.$$

Now let us introduce the two projection mappings $\pi_L:\mathfrak{g}\longrightarrow\mathfrak{g}_L\subset\mathfrak{g}$ and $\pi_R:\mathfrak{g}\longrightarrow\mathfrak{g}_R\subset\mathfrak{g}$ associated to the splitting $\mathfrak{g}=\mathfrak{g}_L\oplus\mathfrak{g}_R$. Observe that $\pi_L^\perp:\mathfrak{g}^*\longrightarrow\mathfrak{g}^*$ is the adjoint map of $\pi_R:\mathfrak{g}\longrightarrow\mathfrak{g}$, thus

$$d(\varphi\circ\sigma)_\alpha\circ Q = \pi_R^{**}df_{\sigma(\alpha)}.$$

Hence, since $Q:\mathfrak{g}^*\longrightarrow\mathfrak{g}_R^*$ is dual to the inclusion map $\iota:\mathfrak{g}_R\longrightarrow\mathfrak{g}$,

$$\iota\circ C_{\mathfrak{g}_R}\left(d(\varphi\circ\sigma)_\alpha\right) = C_{\mathfrak{g}}\left(d(\varphi\circ\sigma)_\alpha\circ Q\right) = C_{\mathfrak{g}}\left(\pi_R^{**}df_{\sigma(\alpha)}\right) = \pi_R C_{\mathfrak{g}} df_{\sigma(\alpha)},$$

or more simply, by dropping tautological maps, $d(\varphi\circ\sigma)_\alpha = \pi_R df_{\sigma(\alpha)}$. Hence,

$$\begin{array}{rcl}\forall\alpha\in\mathfrak{g}_R^*,\quad \{\varphi\circ\sigma,\psi\circ\sigma\}_{\mathfrak{g}_R^*}(\alpha) & := & ({}^{\mathfrak{g}_R^*}\alpha,[d(\varphi\circ\sigma)_\alpha, d(\psi\circ\sigma)_\alpha])_{\mathfrak{g}_R}\\ & = & ({}^{\mathfrak{g}_R^*}\alpha,[\pi_R df_{\sigma(\alpha)},\pi_R dg_{\sigma(\alpha)}])_{\mathfrak{g}_R}\\ & = & ({}^{\mathfrak{g}^*}\sigma(\alpha),[\pi_R df_{\sigma(\alpha)},\pi_R dg_{\sigma(\alpha)}])_{\mathfrak{g}}.\end{array}$$

Thus in view of (2.16) we are led to set:

$$\forall\alpha\in\mathfrak{g}_L^\perp,\quad \{\varphi,\psi\}_{\mathfrak{g}_L^\perp}(\alpha) := ({}^{\mathfrak{g}^*}\alpha,[\pi_R df_\alpha,\pi_R dg_\alpha])_{\mathfrak{g}} \tag{2.17}$$

Then given a function $\varphi\in\mathcal{C}^\infty(\mathfrak{g}_L^\perp)$, its Hamiltonian vector field is the vector field ξ_φ on $\mathfrak{g}_L^\perp$ such that $\forall\psi\in\mathcal{C}^\infty(\mathfrak{g}_L^\perp)$, $d\psi(\xi_\varphi)=\{\varphi,\psi\}_{\mathfrak{g}_L^\perp}$. If φ is the restriction of some $f\in\mathcal{C}^\infty(\mathfrak{g}^*)$ then one computes by using again the identity $\pi_L^\perp=\pi_R^*$ that

$$\forall\alpha\in\mathfrak{g}_L^\perp,\quad \xi_\varphi(\alpha)=\pi_L^\perp\mathrm{ad}^*_{\pi_R df_\alpha}\alpha. \tag{2.18}$$

2.5.3. The $\mathrm{ad}^*_{\mathfrak{g}}$*-invariant functions on* $\mathfrak{g}^*$. — Our Hamiltonian functions on $\mathfrak{g}_L^\perp$ shall be restrictions of functions $f\in\mathcal{C}^\infty(\mathfrak{g}^*)$ which are *invariant under the co-adjoint action of* $\mathfrak{g}$, i.e., such that

$$\forall\alpha\in\mathfrak{g}^*,\forall\xi\in\mathfrak{g},\quad df_\alpha(\mathrm{ad}^*_\xi\alpha)=0. \tag{2.19}$$

However this relation means that $\forall\alpha\in\mathfrak{g}^*$, $\forall\xi\in\mathfrak{g}$,

$$0=\ ({}^{\mathfrak{g}^*}\mathrm{ad}^*_\xi\alpha, df_\alpha)_{\mathfrak{g}} = ({}^{\mathfrak{g}^*}\alpha,[\xi,df_\alpha])_{\mathfrak{g}} = -({}^{\mathfrak{g}^*}\alpha,[df_\alpha,\xi])_{\mathfrak{g}} = -({}^{\mathfrak{g}^*}\mathrm{ad}^*_{df_\alpha}\alpha,\xi)_{\mathfrak{g}},$$

and hence that

$$\forall\alpha\in\mathfrak{g}^*,\quad \mathrm{ad}^*_{df_\alpha}\alpha = \mathrm{ad}^*_{\pi_L df_\alpha}\alpha+\mathrm{ad}^*_{\pi_R df_\alpha}\alpha=0. \tag{2.20}$$

Thus in view of (2.18) and (2.20), for an $\mathrm{ad}^*_{\mathfrak{g}}$-invariant function f,

$$\forall\alpha\in\mathfrak{g}_L^\perp,\quad \xi_\varphi(\alpha)=-\pi_L^\perp\mathrm{ad}^*_{\pi_L df_\alpha}\alpha = -\mathrm{ad}^*_{\pi_L df_\alpha}\alpha, \tag{2.21}$$

where we used the fact that $\pi_L df_\alpha\in\mathfrak{g}_L$ and $\alpha\in\mathfrak{g}_L^\perp$ imply that $\mathrm{ad}^*_{\pi_L df_\alpha}\alpha\in\mathfrak{g}_L^\perp$. All that can be translated if we are given a symmetric nondegenerate bilinear form

$\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{R}$ which is $\mathrm{ad}_{\mathfrak{g}}$-invariant (i.e., such that $\langle[\xi,\eta],\zeta\rangle + \langle\eta,[\xi,\zeta]\rangle = 0$): this induces an isomorphism $\mathfrak{g}\longrightarrow\mathfrak{g}^*$, $\xi\longmapsto\xi^\sharp$ defined by $({}^{\mathfrak{g}^*}\xi^\sharp,\eta)_{\mathfrak{g}} = \langle\xi,\eta\rangle$ and:

$$({}^{\mathfrak{g}^*}\mathrm{ad}^*_\xi\eta^\sharp,\zeta)_{\mathfrak{g}} = ({}^{\mathfrak{g}^*}\eta^\sharp,[\xi,\zeta])_{\mathfrak{g}} = \langle\eta,[\xi,\zeta]\rangle = -\langle[\xi,\eta],\zeta\rangle = -({}^{\mathfrak{g}^*}[\xi,\eta]^\sharp,\zeta)_{\mathfrak{g}}.$$

Thus $\mathrm{ad}^*_\xi\eta^\sharp = -[\xi,\eta]^\sharp$. Hence the vector field defined by (2.21) is equivalent to:

$$X_f(\xi) = [\pi_L\nabla f_\xi,\xi],$$

so that its flow is a Lax equation!

Moreover **the whole family of $\mathrm{ad}^*_{\mathfrak{g}}$-invariant functions on $\mathfrak{g}^*$ gives by restriction on $\mathfrak{g}_L^\perp$ functions in involution**, as we will see[(2)]. This is a consequence of the following identity, valid for any functions $f,g\in\mathcal{C}^\infty(\mathfrak{g}^*)$ which are $\mathrm{ad}^*_{\mathfrak{g}}$-invariant:

$$(2.22)\quad \forall\alpha\in\mathfrak{g}^*,\quad \Delta_{f,g}(\alpha) := ({}^{\mathfrak{g}^*}\alpha,[\pi_R df_\alpha,\pi_R dg_\alpha])_{\mathfrak{g}} - ({}^{\mathfrak{g}^*}\alpha,[\pi_L df_\alpha,\pi_L dg_\alpha])_{\mathfrak{g}} = 0.$$

This can be proved by a direct computation:

$$\begin{array}{rcl} \Delta_{f,g}(\alpha) & = & ({}^{\mathfrak{g}^*}\alpha,[\pi_R df_\alpha,\pi_R dg_\alpha])_{\mathfrak{g}} + ({}^{\mathfrak{g}^*}\alpha,[\pi_L dg_\alpha,\pi_L df_\alpha])_{\mathfrak{g}} \\ & = & ({}^{\mathfrak{g}^*}\mathrm{ad}^*_{\pi_R df_\alpha}\alpha,\pi_R dg_\alpha)_{\mathfrak{g}} + ({}^{\mathfrak{g}^*}\mathrm{ad}^*_{\pi_L dg_\alpha}\alpha,\pi_L df_\alpha)_{\mathfrak{g}} \\ & \overset{(2.20)}{=} & -({}^{\mathfrak{g}^*}\mathrm{ad}^*_{\pi_L df_\alpha}\alpha,\pi_R dg_\alpha)_{\mathfrak{g}} - ({}^{\mathfrak{g}^*}\mathrm{ad}^*_{\pi_R dg_\alpha}\alpha,\pi_L df_\alpha)_{\mathfrak{g}} \\ & = & -({}^{\mathfrak{g}^*}\alpha,[\pi_L df_\alpha,\pi_R dg_\alpha])_{\mathfrak{g}} - ({}^{\mathfrak{g}^*}\alpha,[\pi_R dg_\alpha,\pi_L df_\alpha])_{\mathfrak{g}} \\ & = & 0. \end{array}$$

Hence we deduce from (2.22) that if $f,g\in\mathcal{C}^\infty(\mathfrak{g}^*)$ are $\mathrm{ad}^*_{\mathfrak{g}}$-invariant and if $\alpha\in\mathfrak{g}_L^\perp$, then

$$\forall\alpha\in\mathfrak{g}_L^\perp,\quad \{f,g\}_{\mathfrak{g}_L^\perp}(\alpha) = ({}^{\mathfrak{g}^*}\alpha,[\pi_R df_\alpha,\pi_R dg_\alpha])_{\mathfrak{g}} = ({}^{\mathfrak{g}^*}\alpha,[\pi_L df_\alpha,\pi_L dg_\alpha])_{\mathfrak{g}} = 0.$$

2.5.4. Integration by the method of Symes. — We assume that $\mathfrak{g}_L$, $\mathfrak{g}_R$ and $\mathfrak{g}$ are respectively the Lie algebras of Lie groups $\mathfrak{G}_L$, $\mathfrak{G}_R$ and $\mathfrak{G}$ and consider functions $f\in\mathcal{C}^\infty(\mathfrak{g}^*)$ which are $\mathrm{Ad}^*_{\mathfrak{G}}$-invariant, i.e., such that

$$(2.23)\qquad \forall g\in\mathfrak{G},\forall\alpha\in\mathfrak{g}^*\quad f(\mathrm{Ad}^*_g\alpha) = f(\alpha),$$

where $\mathrm{Ad}^*_g:\mathfrak{g}^*\longrightarrow\mathfrak{g}^*$ is defined by $({}^{\mathfrak{g}^*}\mathrm{Ad}^*_g\alpha,\xi)_{\mathfrak{g}} = ({}^{\mathfrak{g}^*}\alpha,\mathrm{Ad}_g\xi)_{\mathfrak{g}}$, $\forall\alpha\in\mathfrak{g}^*$, $\forall\xi\in\mathfrak{g}$. Note that (2.23) is equivalent to (2.19) if $\mathfrak{G}$ is connected. We will use the following two observations. First if $f\in\mathcal{C}^\infty(\mathfrak{g}^*)$ is $\mathrm{Ad}^*_{\mathfrak{G}}$-invariant, then

$$(2.24)\qquad \forall g\in\mathfrak{G},\forall\alpha\in\mathfrak{g}^*\quad df_\alpha = \mathrm{Ad}_g df_{\mathrm{Ad}^*_g\alpha}.$$

This is proved by deriving the relation (2.23) with respect to α, which gives

$$df_\alpha = df_{\mathrm{Ad}^*_g\alpha}\circ\mathrm{Ad}^*_g = C_{\mathfrak{g}}^{-1}\circ\mathrm{Ad}_g\circ C_{\mathfrak{g}}\circ df_{\mathrm{Ad}^*_g\alpha}\simeq\mathrm{Ad}_g\circ df_{\mathrm{Ad}^*_g\alpha}.$$

Second for any $g\in\mathcal{C}^1(\mathbb{R},\mathfrak{G})$ and $\alpha\in\mathcal{C}^1(\mathbb{R},\mathfrak{g}^*)$, if we let $\alpha_0 := \alpha(0)$, then

$$(2.25)\qquad \forall t\in\mathbb{R},\quad \dot\alpha(t) = \mathrm{ad}^*_{g^{-1}(t)\dot g(t)}\alpha(t)\quad\Longrightarrow\quad\forall t\in\mathbb{R},\quad \alpha(t) = \mathrm{Ad}^*_{g(t)}\alpha_0.$$

(2) In fact $\mathrm{ad}^*_{\mathfrak{g}}$-invariant functions on $\mathfrak{g}^*$ are in involution for the Poisson structure $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$, but their flows are trivial and, hence, are not interesting. The point here is that they induced non-trivial flows on $\mathfrak{g}_L^\perp$.

The proof of (2.25) is left to the reader (hint: prove that, for all $\xi_0 \in \mathfrak{g}$, $({}^{\mathfrak{g}^*}\alpha(t), \mathrm{Ad}_{g^{-1}(t)}\xi_0)_{\mathfrak{g}}$ is time independent), note that the converse is also true. Now let $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ be an $\mathrm{Ad}^*_{\mathfrak{G}}$-invariant and consider a solution $\alpha \in \mathcal{C}^1(\mathbb{R}, \mathfrak{g}^*)$ of the flow of (2.21), i.e.,

$$\dot\alpha = \xi_f(\alpha) = -\mathrm{ad}^*_{\pi_L df_\alpha}\alpha, \quad \alpha(0) = \alpha_0. \tag{2.26}$$

We associate to α the solutions $S \in \mathcal{C}^1(\mathbb{R}, \mathfrak{G}_L)$ and $T \in \mathcal{C}^1(\mathbb{R}, \mathfrak{G}_R)$ of the following two equations:

$$S^{-1}\dot S = -\pi_L df_\alpha, \quad S(0) = 1_{\mathfrak{G}}, \tag{2.27}$$

and

$$\dot T T^{-1} = -\pi_R df_\alpha, \quad T(0) = 1_{\mathfrak{G}}. \tag{2.28}$$

Then by (2.26) and (2.27), $\dot\alpha = \mathrm{ad}^*_{S^{-1}\dot S}\alpha$, which implies thanks to (2.25) that $\alpha = \mathrm{Ad}^*_S\alpha_0$. Hence by using successively (2.24), (2.27) and (2.28), we deduce that:

$$\begin{aligned} df_{\alpha_0} &= \mathrm{Ad}_S df_{\mathrm{Ad}^*_S\alpha_0} = \mathrm{Ad}_S df_\alpha = -\mathrm{Ad}_S(S^{-1}\dot S + \dot T T^{-1}) \\ &= -\dot S S^{-1} - \mathrm{Ad}_S(\dot T T^{-1}) = -(\dot{ST})(ST)^{-1}. \end{aligned}$$

Thus $ST = e^{-tdf_{\alpha_0}}$. Hence we can deduce the solution to (2.26) if the splitting $\mathfrak{G}_R \cdot \mathfrak{G}_R = \mathfrak{G}$ holds.

2.5.5. An example. — We let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, the set of $n\times n$ real matrices with vanishing trace, $\mathfrak{g}_L = \mathfrak{so}(n)$ and $\mathfrak{g}_R = \mathfrak{st}^-(n, \mathbb{R})$, the set of $n \times n$ real lower triangular matrices with vanishing trace (i.e., $\mathfrak{st}^-(n, \mathbb{R}) := \mathfrak{t}^-(n, \mathbb{R}) \cap \mathfrak{sl}(n, \mathbb{R})$). Note that $\mathfrak{sl}(n, \mathbb{R})$ is the Lie algebra of $SL(n, \mathbb{R})$, $\mathfrak{so}(n)$ is the Lie algebra of $SO(n)$ and $\mathfrak{st}^-(n, \mathbb{R})$ is the Lie algebra of $ST^-(n, \mathbb{R})$, the subgroup of $SL(n, \mathbb{R})$ of lower triangular matrices with positive entries on the diagonal. We identify $\mathfrak{g}^*$ with $M(n, \mathbb{R})/\mathbb{R}1_n$, with the duality product

$$\forall\alpha \in M(n, \mathbb{R}), \forall\xi \in \mathfrak{sl}(n, \mathbb{R}), \quad ({}^{\mathfrak{g}^*}\alpha, \xi)_{\mathfrak{g}} := \mathrm{tr}\ (\alpha^t\xi).$$

Then $\mathfrak{g}_L^\perp = \mathfrak{so}(n)^\perp$ can be identified with $\mathrm{sym}(n, \mathbb{R})/\mathbb{R}1_n$ ($\mathrm{sym}(n, \mathbb{R})$ is the set of real $n \times n$ symmetric matrices) and $\mathfrak{g}_R^\perp = \mathfrak{st}^-(n)^\perp$ with $\mathfrak{t}_0^+(n, \mathbb{R})/\mathbb{R}1_n$, where $\mathfrak{t}_0^+(n, \mathbb{R})$ is the set of $n \times n$ real upper triangular matrices with vanishing entries on the diagonal. The co-adjoint action of $\mathfrak{G}$ on $\mathfrak{g}^*$ can be computed: $\forall\alpha \in \mathfrak{g}^*$, $\forall\xi \in \mathfrak{g}$, $\forall g \in \mathfrak{G}$,

$$({}^{\mathfrak{g}^*}\mathrm{Ad}^*_g\alpha, \xi)_{\mathfrak{g}} = ({}^{\mathfrak{g}^*}\alpha, \mathrm{Ad}_g\xi)_{\mathfrak{g}} = \mathrm{tr}\ (\alpha^t g\xi g^{-1}) = \mathrm{tr}\ ((g^t\alpha(g^t)^{-1})^t\xi) = ({}^{\mathfrak{g}^*}g^t\alpha(g^t)^{-1}, \xi)_{\mathfrak{g}}.$$

Hence

$$\mathrm{Ad}^*_g\alpha = g^t\alpha(g^t)^{-1}.$$

In particular all functions of the form $\alpha \longmapsto \mathrm{tr}\ \alpha^k$, for $k \in \mathbb{N}^*$, are $\mathrm{Ad}_{\mathfrak{G}}$-invariant. Moreover, through the identification $\mathfrak{st}^-(n, \mathbb{R})^* \simeq \mathrm{sym}(n, \mathbb{R})/\mathbb{R}1_n$ the co-adjoint action of $\mathfrak{G}_R = ST^-(n, \mathbb{R})$ on $\mathfrak{st}^-(n, \mathbb{R})^*$ reads $\forall g \in ST^-(n, \mathbb{R})$,

$$\mathrm{sym}(n, \mathbb{R}) \ni L_0 \longmapsto \mathrm{Ad}^*_g L_0 \simeq \pi_{\mathrm{sym}(n,\mathbb{R})}\left(g^t L_0 (g^t)^{-1}\right),$$

where the projection mapping $\pi_{\mathrm{sym}(n,\mathbb{R})}$ has the kernel $\mathfrak{t}_0^+(n,\mathbb{R})$. For instance if

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 1 & 0 & \ddots & & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & & \ddots & 0 & 1 \\ 0 & & 0 & 1 & 0 \end{pmatrix} \in \mathrm{sym}(n,\mathbb{R}),$$

and if

$$g^t = \begin{pmatrix} e^{-q^1} & a_1 & * & & * \\ 0 & e^{-q^2} & a_2 & & \\ & \ddots & \ddots & \ddots & * \\ & & 0 & e^{-q^{n-1}} & a_{n-1} \\ 0 & & & 0 & e^{-q^n} \end{pmatrix} \in ST^-(n,\mathbb{R}),$$

then $g^t L_0 (g^t)^{-1} = L$ with

$$L = \begin{pmatrix} -e^{q^2}a_1 & * & * & & * \\ e^{q^2-q^1} & a_1e^{q^2}-a_2e^{q^3} & \ddots & & \\ 0 & \ddots & \ddots & \ddots & * \\ & & \ddots & a_{n-2}e^{q^{n-1}}-a_{n-1}e^{q^n} & * \\ 0 & & 0 & e^{q^n-q^{n-1}} & a_{n-1}e^{q^n} \end{pmatrix}.$$

Hence by taking the image by $\pi_{\mathrm{sym}(n,\mathbb{R})}$ of this matrix we obtain a matrix of the type (2.5) with $\sum_{i=1}^n p_i = 0$.

3. The sinh–Gordon equation

We now consider another example of equation, the sinh–Gordon equation

$$\frac{d^2q}{dt^2} + 2\sinh(2q) = 0. \tag{3.29}$$

By setting $p := \frac{dq}{dt}$ we can see easily that (3.29) is equivalent to the Hamiltonian system of equations

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = -2\sinh(2q),$$

corresponding to the Hamiltonian function $H(q,p) = \frac{p^2}{2} + \cosh(2q)$. Equation (3.29) can be solved by using quadratures and more precisely by inverting an elliptic integral on a Riemann surface of genus one. Indeed first observe that $H(q,\dot q) = \dot q^2/2 + \cosh(2q)$

is a constant, say ϵ. From that we deduce that $dt = dq/\sqrt{2(\epsilon - \cosh(2q))}$. By posing $z = \cosh(2q)$ we obtain

$$t_1 - t_0 = \int_{\cosh(2q_0)}^{\cosh(2q_1)} \frac{dz}{\sqrt{P(z)}},$$

where $P(z) := 8(z^2-1)(\epsilon - z)$. The right hand side of this relation is an elliptic integral. The natural framework to understand it is tho consider the (compactification of the) Riemann surface $\{(x,y) \in \mathbb{C}^2 |\, y^2 = P(x)\}$. The inverse map to this integral can then be constructed by using theta functions. All these methods go back to Jacobi, Abel and Riemann. Note that the analogous method for Equation (2.3) $\frac{d^2q}{dt^2} + 4e^{2q} = 0$ gives $dt = dq/2\sqrt{\sigma^2 - e^{2q}}$, where $\sigma^2 := p^2/4 + e^{2q}$ and by posing $z = e^q/\sigma$ we get $dt = dz/2\sigma z\sqrt{1-z^2} = -\frac{1}{2\sigma} d\mathrm{Arg}\cosh(1/z)$. Thus in particular we do not need elliptic integral in this case. Hence Equation (3.29) is both similar to and more involved than the Liouville equation (2.3), so one should expect that it can be solved by similar method. This is true as we will see, but this requires a more general framework.

Here it turns out that (3.29) can be written as Lax equation by using infinite matrices! namely (3.29) is equivalent to $\dot{L} = [L, M(L)]$, where

$$L = \frac{1}{2}\begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & p & e^q & & & \\ & e^q & -p & e^{-q} & & \\ & & e^{-q} & p & e^q & \\ & & & e^q & -p & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

and

$$M(L) = \frac{1}{2}\begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & e^q & & & \\ & -e^q & 0 & e^{-q} & & \\ & & -e^{-q} & 0 & e^q & \\ & & & -e^q & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}.$$

We see that the linear map M is the projection onto skewsymmetric matrices parallel to lower triangular matrices, just as before. Such matrices are difficult to handle. One can represent them by constructing the linear operators acting on the Hilbert space $\ell^2(\mathbb{Z})$ whose matrices in the canonical base $(\cdots, e_{-2}, e_{-1}, e_0, e_1, e_2, \cdots)$ are L and $M(L)$ respectively. Using the same notations for the matrices and the operators we have

$$\left\{\begin{array}{lcl} Le_{2n} & = & \frac{1}{2}\left(e^{-q}e_{2n-1}+pe_{2n}+e^{q}e_{2n+1}\right) \\ Le_{2n+1} & = & \frac{1}{2}\left(e^{q}e_{2n}-pe_{2n+1}+e^{-q}e_{2n+2}\right) \end{array}\right. ,$$

$$\left\{\begin{array}{lcl} M(L)e_{2n} & = & \frac{1}{2}\left(e^{-q}e_{2n-1}-e^{q}e_{2n+1}\right) \\ M(L)e_{2n+1} & = & \frac{1}{2}\left(e^{q}e_{2n}-e^{-q}e_{2n+2}\right) \end{array}\right. .$$

One then check that $\frac{dL}{dt}e_m = [L, M(L)]e_m$, $\forall m \in \mathbb{Z}$ if and only if q and p are solutions of the Hamilton equations.

3.1. Introducing a complex parameter. — An alternative way is to identify the Hilbert space $\ell^2(\mathbb{Z})$ with (a subspace of the) *loops in* $\mathbb{C}^2$, $L^2(S^1, \mathbb{C}^2)$, through the Fourier transform. In the following we denote by $S^1 := \{\lambda \in \mathbb{C}^* | \, |\lambda| = 1\}$, for any $v \in L^2(S^1, \mathbb{C}^2)$ we denote by v_λ the value of v at $\lambda \in S^1$ and (ϵ_1, ϵ_2) is the canonical basis of $\mathbb{C}^2$. The subspace of $L^2(S^1, \mathbb{C}^2)$ that we are going to consider is the space of even loops[(3)], $L^2_{even}(S^1, \mathbb{C}^2) := \{v \in L^2(S^1, \mathbb{C}^2) | \, v_{-\lambda} = v_\lambda, \forall \lambda \in S^1\}$. This is done by the Fourier decomposition isomorphism

$$\begin{array}{cccc} \Phi: & \ell^2(\mathbb{Z}) & \longrightarrow & L^2_{even}(S^1, \mathbb{C}^2) \\ & v & \longmapsto & \Phi(v), \end{array}$$

where Φ is defined by

$$\left\{\begin{array}{lclcl} \Phi(e_{2n})_\lambda & = & \lambda^{2n-2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} & = & \lambda^{2n-2}\epsilon_2 \\ \Phi(e_{2n+1})_\lambda & = & \lambda^{2n}\begin{pmatrix} 1 \\ 0 \end{pmatrix} & = & \lambda^{2n}\epsilon_1. \end{array}\right.$$

Then it turns out that through the diffeomorphism Φ the action of the linear operators L and $M(L)$ translates into relatively simple operators.

Lemma 2. — *$\forall v \in L^2_{even}(S^1, \mathbb{C}^2)$, $\forall \lambda \in S^1$, we have*

$$\left(\Phi \circ L \circ \Phi^{-1}\right)(v)_\lambda = \frac{1}{2}\begin{pmatrix} -p & e^{-q}+\lambda^2 e^{q} \\ \lambda^{-2}e^{q}+e^{-q} & p \end{pmatrix} v_\lambda =: \widetilde{L}_\lambda v_\lambda,$$

$$\left(\Phi \circ M(L) \circ \Phi^{-1}\right)(v)_\lambda = \frac{1}{2}\begin{pmatrix} 0 & e^{-q}-\lambda^2 e^{q} \\ \lambda^{-2}e^{q}-e^{-q} & 0 \end{pmatrix} v_\lambda =: \widetilde{M}(\widetilde{L})_\lambda v_\lambda.$$

Proof. — Just compute. □

(3) We shall see later on the reason for choosing even maps.

The operators $\Phi \circ L \circ \Phi^{-1}$ and $\Phi \circ M(L) \circ \Phi^{-1}$ are very particular instances of operators acting on $L^2_{even}(S^1, \mathbb{C}^2)$ since they are characterized by mappings $S^1 \ni \lambda \longmapsto \widetilde{L}_\lambda \in \mathfrak{sl}(2,\mathbb{C})$ and $S^1 \ni \lambda \longmapsto \widetilde{M}(\widetilde{L})_\lambda \in \mathfrak{sl}(2,\mathbb{C})$ in such a way that the action on a given vector in $L\mathbb{C}^2$ is given by pointwise multiplication by these matrix valued maps.

Now we can deduce from the previous facts that the sinh–Gordon equation is equivalent to the Lax equation

$$\frac{d\widetilde{L}_\lambda}{dt} = \left[\widetilde{L}_\lambda, \widetilde{M}(\widetilde{L})_\lambda\right], \quad \forall \lambda \in S^1.$$

Actually we will not use this formulation since it causes some difficulties to use the Adler–Kostant–Symes method. We shall instead use the matrices

$$L_\lambda := P_\lambda^{-1} \widetilde{L}_\lambda P_\lambda \quad \text{and} \quad M(L)_\lambda := P_\lambda^{-1} \widetilde{M}(\widetilde{L})_\lambda P_\lambda,$$

where[(4)]

$$P_\lambda := \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}.$$

This gives us

$$L_\lambda = \frac{1}{2} \begin{pmatrix} -p & \lambda^{-1} e^{-q} + \lambda e^q \\ \lambda^{-1} e^q + \lambda e^{-q} & p \end{pmatrix},$$

and

$$M(L)_\lambda = \frac{1}{2} \begin{pmatrix} 0 & \lambda^{-1} e^{-q} - \lambda e^q \\ \lambda^{-1} e^q - \lambda e^{-q} & 0 \end{pmatrix}.$$

And (3.29) is equivalent to the Lax equation $\frac{dL_\lambda}{dt} = [L_\lambda, M(L)_\lambda]$.

3.2. Loop algebras and loop groups. — We now have to think about the objects $\widetilde{L}_\lambda$, L_λ, etc. They defined maps into (matrix) Lie algebras. We can observe indeed that each of these maps takes values into $\mathfrak{sl}(2,\mathbb{C}) := \{M \in M(2,\mathbb{C}) |\, \mathrm{tr} M = 0\}$ which is the Lie algebra of $SL(2,\mathbb{C}) := \{M \in GL(2,\mathbb{C}) |\, \det M = 1\}$. We will denote

$$L\mathfrak{sl}(2,\mathbb{C}) := \{\xi : S^1 \ni \lambda \longmapsto \xi_\lambda \in \mathfrak{sl}(2,\mathbb{C})\}.$$

We need to fix some regularity and some topology on this space in order to be able to define a Lie bracket on $L\mathfrak{sl}(2,\mathbb{C})$ by the rule

$$\forall \xi, \eta \in L\mathfrak{sl}(2,\mathbb{C}), \forall \lambda \in S^1, \quad [\xi, \eta]_\lambda = [\xi_\lambda, \eta_\lambda].$$

One instance of a topology which ensures us that $[\xi, \eta] \in L\mathfrak{sl}(2,\mathbb{C})$ is the L^∞ topology (on the set of continuous loops). With these preliminaries $L\mathfrak{sl}(2,\mathbb{C})$ has now the structure of an (infinite dimensional) Lie algebra, called a **(Lie) loop algebra**. So we will now think of L and $M(L)$ as maps into the loop algebra $L\mathfrak{sl}(2,\mathbb{C})$.

(4) The reader will observe that P_λ is only defined up to sign. However the conjugate matrices $P_\lambda^{-1} \widetilde{L}_\lambda P_\lambda$ and $P_\lambda^{-1} \widetilde{M}(\widetilde{L})_\lambda P_\lambda$ are unambiguously defined.

We will also be able to interpret $M(L)_\lambda$ as the image of L_λ by a suitable projection mapping of $L\mathfrak{sl}(2,\mathbb{C})$. We first remark (using in particular that if $\lambda \in S^1$ then λ^{-1} is the complex conjugate of λ) that

$$\forall \lambda \in S^1, \quad M(L)_\lambda^\dagger + M(L)_\lambda = 0,$$

where $A^\dagger := \overline{A}^t$. So $M(L)_\lambda \in \mathfrak{su}(2)$, $\forall \lambda \in S^1$, where $\mathfrak{su}(2) := \{\alpha \in M(2,\mathbb{C}) |\, \alpha^\dagger + \alpha = 0\}$. Hence we are led to introduce

$$L\mathfrak{su}(2) := \{\xi : S^1 \ni \lambda \longmapsto \xi_\lambda \in \mathfrak{su}(2)\} \subset \mathfrak{sl}(2,\mathbb{C}),$$

which is itself a loop (sub)algebra. We need to find a complementary vector subspace in $L\mathfrak{sl}(2,\mathbb{C})$ which will be another loop subalgebra. This rests on two constructions.

a) Finite dimensional splitting of $\mathfrak{sl}(2,\mathbb{C})$

We let

$$\mathfrak{b} := \left\{ \begin{pmatrix} t & 0 \\ s & -t \end{pmatrix} |\, t \in \mathbb{R}, s \in \mathbb{C} \right\} \subset \mathfrak{sl}(2,\mathbb{C}),$$

and we then observe that $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{b}$ and that $\mathfrak{b}$ is the Lie algebra of the group

$$\mathfrak{B} := \left\{ \begin{pmatrix} \tau & 0 \\ \sigma & 1/\tau \end{pmatrix} |\, \tau \in (0,\infty), \sigma \in \mathbb{C} \right\} \subset SL(2,\mathbb{C}).$$

b) Infinite dimensional splitting in $L\mathfrak{sl}(2,\mathbb{C})$

Any loop $\xi_\lambda = \sum_{k\in\mathbb{Z}} \widehat{\xi}_k \lambda^k$ can be decomposed as

$$\xi_\lambda = \left(\sum_{k<0} \widehat{\xi}_k \lambda^k - \sum_{k>0} \widehat{\xi}_{-k}^\dagger \lambda^k \right) + \widehat{\xi}_0 + \left(\sum_{k>0} \left(\widehat{\xi}_k + \widehat{\xi}_{-k}^\dagger \right) \lambda^k \right).$$

Here the first term on the right hand side takes values in $\mathfrak{su}(2)$ (by again using the fact that $\lambda \in S^1 \Longleftrightarrow \overline{\lambda} = \lambda^{-1}$) and the last term involves only positive powers of λ.

Now we shall use the first decomposition a) in order to deal with the middle term $\widehat{\xi}_0 \in \mathfrak{sl}(2,\mathbb{C})$ in the second decomposition b). Namely we split it according to the decomposition $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{b}$: $\widehat{\xi}_0 = \left(\widehat{\xi}_0\right)_{\mathfrak{su}(2)} + \left(\widehat{\xi}_0\right)_{\mathfrak{b}}$. This leads us to the decomposition

$$\xi_\lambda = \left(\sum_{k<0} \widehat{\xi}_k \lambda^k - \sum_{k>0} \widehat{\xi}_{-k}^\dagger \lambda^k + \left(\widehat{\xi}_0\right)_{\mathfrak{su}(2)} \right) + \left(\left(\widehat{\xi}_0\right)_{\mathfrak{b}} + \sum_{k>0} \left(\widehat{\xi}_k + \widehat{\xi}_{-k}^\dagger \right) \lambda^k \right),$$

where the first term is a loop in $L\mathfrak{su}(2)$. The second one belongs to

$$L_{\mathfrak{b}}^+\mathfrak{sl}(2,\mathbb{C}) := \{\xi : S^1 \ni \lambda \longmapsto \xi_\lambda \in \mathfrak{sl}(2,\mathbb{C}) |\, \xi_\lambda = \sum_{k=0}^{\infty} \widehat{\xi}_k \lambda^k, \widehat{\xi}_0 \in \mathfrak{b}\}.$$

The previous computation shows that

$$L\mathfrak{sl}(2,\mathbb{C}) = L\mathfrak{su}(2) \oplus L_{\mathfrak{b}}^+\mathfrak{sl}(2,\mathbb{C}),$$

so that we can define two projection mappings π_L and π_R to respectively $L\mathfrak{su}(2)$ and $L_{\mathfrak{b}}^+\mathfrak{sl}(2,\mathbb{C})$. But the nice thing is that $L_{\mathfrak{b}}^+\mathfrak{sl}(2,\mathbb{C})$ is also a Lie algebra!

As this point if we wish to extend the method described in the previous section we need to know whether the loop algebras that we have defined are the Lie algebras of Lie groups. This turns out to be true: we define the **loop (Lie) groups**

$$LSL(2,\mathbb{C}) := \{g : S^1 \ni \lambda \longmapsto g_\lambda \in SL(2,\mathbb{C})\},$$
$$LSU(2) := \{g : S^1 \ni \lambda \longmapsto g_\lambda \in SU(2)\} \subset SL(2,\mathbb{C}),$$

(where $SU(2) := \{g \in M(2,\mathbb{C}) |\, g^\dagger g = 1_2\}$) and

$$L^+_{\mathfrak{B}}SL(2,\mathbb{C}) := \Big\{g : S^1 \ni \lambda \longmapsto g_\lambda \in SL(2,\mathbb{C}) |\, g_\lambda = \sum_{k=0}^{\infty} \widehat{g}_k\lambda^k, \widehat{g}_0 \in \mathfrak{B}\Big\}.$$

These set are endowed with the product law

$$\forall g, h \in LSL(2,\mathbb{C}), \forall \lambda \in S^1, \quad (gh)_\lambda = (g_\lambda)\,(h_\lambda)\,.$$

So $LSL(2,\mathbb{C})$, $LSU(2)$ and $L^+_{\mathfrak{B}}SL(2,\mathbb{C})$ are Lie groups whose Lie algebras are respectively $L\mathfrak{sl}(2,\mathbb{C})$, $L\mathfrak{su}(2)$ and $L^+_{\mathfrak{b}}\mathfrak{sl}(2,\mathbb{C})$.

3.3. The solution of the sinh–Gordon equation by using the Adler–Kostant–Symes method. — We now summarize the previous steps. A map $q : \mathbb{R} \longrightarrow \mathbb{R}$ is a solution of the sinh–Gordon equation (3.29) if and only if the map $L : \mathbb{R} \longrightarrow L\mathfrak{sl}(2,\mathbb{C})$ defined by

$$\forall t \in \mathbb{R}, \forall \lambda \in S^1, \quad L_\lambda(t) = \frac{1}{2}\begin{pmatrix} -p(t) & \lambda^{-1}e^{-q(t)} + \lambda e^{q(t)} \\ \lambda^{-1}e^{q(t)} + \lambda e^{-q(t)} & p(t) \end{pmatrix}$$

(where $p(t) = \dot{q}(t)$) is a solution of the Lax equation

$$\frac{dL_\lambda}{dt}(t) = \left[L_\lambda(t), \pi_L(L(t))_\lambda\right],$$

where π_L is the projection on the first factor of $L\mathfrak{sl}(2,\mathbb{C}) = L\mathfrak{su}(2) \oplus L^+_{\mathfrak{b}}\mathfrak{sl}(2,\mathbb{C})$. We then consider π_R to be the projection on the second factor of this splitting and the two extra equations

$$\frac{dS_\lambda}{dt}(t) = S_\lambda(t)\pi_L(L(t))_\lambda, \quad S_\lambda(0) = 1,$$
$$\frac{dT_\lambda}{dt}(t) = \pi_R(L(t))_\lambda T_\lambda(t), \quad T_\lambda(0) = 1.$$

Then we know that automatically $S(t) \in LSU(2)$ and $T(t) \in L^+_{\mathfrak{B}}SL(2,\mathbb{C})$, $\forall t \in \mathbb{R}$. Moreover by repeating the previous computation we can prove that

$$L_\lambda(t) = \left(S_\lambda(t)\right)^{-1} L_\lambda(0) S_\lambda(t),$$
$$L_\lambda(0)\left(S_\lambda(t)T_\lambda(t)\right) = \frac{d}{dt}\left(S_\lambda(t)T_\lambda(t)\right).$$

So

$$S_\lambda(t)T_\lambda(t) = e^{tL_\lambda(0)}, \quad \forall t \in \mathbb{R}.$$

The computation of $e^{tL_\lambda(0)}$ is basically not very much complicated than the kind of computations done in the previous section (without the parameter λ). The key step is then to extract $S_\lambda(t)$ from $S_\lambda(t)T_\lambda(t)$. This step is now much harder: this is the content of the following result.

***Theorem 2* (Iwasawa decomposition)**. — *The mapping*

$$\begin{array}{ccc} LSU(2)\times L^+_{\mathfrak{B}}SL(2,\mathbb{C}) & \longrightarrow & LSL(2,\mathbb{C}) \\ (S,T) & \longmapsto & ST \end{array}$$

is a diffeomorphism.

In fact this result, which is proved in [**28**], requires a choice of a topology on $LSL(2,\mathbb{C})$ stronger than $L^\infty(S^1)$. We may for instance use the H^s-topology for $s > 1/2$, induced by the norm $||\xi||_{H^s} := (\sum_{k\in\mathbb{Z}}(1+k^{2s})|\widehat{\xi}_k|^2)^{1/2}$, where $\xi_\lambda = \sum_{k\in\mathbb{Z}}\widehat{\xi}_k\lambda^k$ is the Fourier decomposition of ξ. Once we have this result we know that, for any time t, there exists a unique $S(t)$ in $LSU(2)$ and a unique $T(t)$ in $L^+_{\mathfrak{B}}SL(2,\mathbb{C})$ such that $S_\lambda(t)T_\lambda(t) = e^{tL_\lambda(0)}$, $\forall\lambda\in S^1$. We deduce hence $L(t)$ by using the relation $L_\lambda(t) = (S_\lambda(t))^{-1}L_\lambda(0)S_\lambda(t)$. Note also that this method is more theoretical than practical since unfortunately there is no way to write down explicitly the Iwasawa decomposition. However one can recover the algebro-geometric solution obtained by quadratures by working on the complex curve $\det(L_\lambda - \mu 1_2) = \mu^2 - \frac{1}{4}(\lambda^{-2}+\lambda^2) - \frac{1}{2}H(p,q) = 0$, which encodes the constants of motion.

4. The Korteweg–de Vries equation

The most famous example of an infinite dimensional completely integrable system is the Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0, \tag{4.30}$$

where

$$\begin{array}{cccc} u: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,t) & \longmapsto & u(x,t). \end{array}$$

4.1. The Lax formulation. — We first view this partial differential equation as a mechanical problem in an infinite dimensional configuration space: we associate to each time $t\in\mathbb{R}$ the function

$$\begin{array}{cccc} u(\cdot,t): & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & u(x,t) \end{array}$$

and the two operators defined by

$$\begin{array}{rcl} L_{u(\cdot,t)} & := & -\dfrac{\partial^2}{\partial x^2} - u(\cdot,t), \\ P_{3;u(\cdot,t)} & := & -4\dfrac{\partial^3}{\partial x^3} - 3\left(u(\cdot,t)\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial x}u(\cdot,t)\right). \end{array}$$

These operators act on $\mathcal{C}^\infty(\mathbb{R}_x)$ by the rule: $\forall\varphi\in\mathcal{C}^\infty(\mathbb{R}_x)$,

$$\begin{array}{rcl} L_{u(\cdot,t)}\varphi(x) & = & -\varphi''(x)-u(x,t)\varphi(x)\\ P_{3;u(\cdot,t)}\varphi(x) & = & -4\varphi'''(x)-3\left(u(x,t)\varphi'(x)+(u(x,t)\varphi(x))'\right). \end{array}$$

Then it turns out that u is a solution of (4.30) if and only if

$$\frac{dL_{u(\cdot,t)}}{dt}=[P_{3;u(\cdot,t)},L_{u(\cdot,t)}]. \tag{4.31}$$

Historically it was for the KdV equation that such an equation was written[(5)] by Peter Lax (see [**12**]).

Let us think on the operator $L_{u(\cdot,t)}$ as being formally diagonalizable, then the Lax equation can be interpreted as the *integrability* condition of the following overdetermined system: find $\lambda\in\mathbb{C}$ and $\varphi_\lambda:\mathbb{R}^2\longrightarrow\mathbb{R}$ s.t.

$$\left\{\begin{array}{rcl} \dfrac{\partial\varphi_\lambda}{\partial t} & = & P_{3;u(\cdot,t)}\varphi_\lambda\\ L_{u(\cdot,t)}\varphi_\lambda & = & \dfrac{\lambda^2}{4}\varphi_\lambda. \end{array}\right. \tag{4.32}$$

Here an evolution equation is coupled with an eigenvalue equation. In other words we look for trajectories $t\longmapsto[x\longmapsto\varphi_\lambda(x,t)]$ in $\mathcal{C}^\infty(\mathbb{R})$ which are integral curves of the time dependant vector field $P_{3;u(\cdot,t)}$, but in such a way that at any time $x\longmapsto\varphi_\lambda(x,t)$ is also an eigenvector of the operator $L_{u(\cdot,t)}$ for the eigenvalue $\frac{\lambda^2}{4}$. For that reason the complex parameter λ is called the **spectral parameter**.

In the following we will restrict ourself to a class of functions $u\in\mathcal{C}^\infty(\mathbb{R}_x)$ which decay to 0 at $\pm\infty$. A consequence of that is that the equations (4.32) can be approximated by

$$\left\{\begin{array}{rl} \dfrac{\partial\varphi}{\partial t}+4\dfrac{\partial^3\varphi}{\partial x^3} & =0\\ \dfrac{\partial^2\varphi}{\partial x^2}+\dfrac{\lambda^2}{4}\varphi & =0. \end{array}\right.$$

in the neighborhood of $\pm\infty$. Hence in particular any solution φ is asymptotic to $\alpha e^{\frac{1}{2i}(\lambda x+\lambda^3t)}+\beta e^{-\frac{1}{2i}(\lambda x+\lambda^3t)}$ when $x\to-\infty$. In the following we shall consider the normalized solution φ_λ such that φ_λ is asymptotic to $e^{\frac{1}{2i}(\lambda x+\lambda^3t)}$ when $x\to-\infty$. Then any solution of (4.32) will be a linear combination of φ_λ and $\varphi_{-\lambda}$. This function φ_λ is called the *Baker–Akhiezer function.* Note that similar theories were developped for the study of *periodic in space variable* solutions to the KdV equation [**24**] or for *algebraic solutions with singularities* [**2**].

(5) As pointed out to us by the Referee the first known apparition of a 'Lax equation' with a spectral parameter λ is due to R. Garnier [**15**] in 1919.

4.2. A formulation as a first order system. — In the following we want to rewrite the system (4.32) with more symmetry between the space and time variables. A first step is to write down it as a first order condition.

Lemma 3. — *By posing*

$$\psi_\lambda(x,t) := \begin{pmatrix} \varphi_\lambda(x,t) \\ \frac{\partial \varphi_\lambda}{\partial x}(x,t) + i\frac{\lambda}{2}\varphi_\lambda(x,t) \end{pmatrix},$$

the system (4.32) is equivalent to

$$\left\{ \begin{array}{rcll} \dfrac{\partial \psi_\lambda}{\partial x}(x,t) & = & U_\lambda(x,t)\psi_\lambda(x,t) & \textit{(spectral constraint)} \\ \dfrac{\partial \psi_\lambda}{\partial t}(x,t) & = & V_\lambda(x,t)\psi_\lambda(x,t) & \textit{(evolution)} \end{array} \right. \tag{4.33}$$

where, by denoting $u_x := \frac{\partial u}{\partial x}$, $u_{xx} := \frac{\partial^2 u}{\partial x^2}$, *etc.*

$$U_\lambda(x,t) := \begin{pmatrix} -i\frac{\lambda}{2} & 1 \\ -u & i\frac{\lambda}{2} \end{pmatrix}, \tag{4.34}$$

$$V_\lambda(x,t) := \begin{pmatrix} u_x + i\lambda u - i\frac{\lambda^3}{2} & -2u + \lambda^2 \\ (u_{xx} + 2u^2) + i\lambda u_x - \lambda^2 u & -u_x - i\lambda u + i\frac{\lambda^3}{2} \end{pmatrix}. \tag{4.35}$$

Proof. — Assume that φ_λ is an arbitrary smooth function on $\mathbb{R}^2$ and pose

$$D_\lambda := \frac{\partial}{\partial x} + i\frac{\lambda}{2}, \quad \text{and} \quad \psi_\lambda = \begin{pmatrix} \varphi_\lambda \\ D_\lambda \varphi_\lambda \end{pmatrix}.$$

We also define:

$$T_\lambda := \frac{\partial \varphi_\lambda}{\partial t} - P_{3;u}\varphi_\lambda \quad \text{and} \quad S_\lambda := L_u\varphi_\lambda - \frac{\lambda^2}{4}\varphi_\lambda,$$

and remark that φ_λ is a solution of the system (4.32) if and only if $T_\lambda = S_\lambda = 0$. Thus the equivalence between (4.32) and (4.33) lies on the following identities, obtained through a lengthy but straightforward computation:

$$\frac{\partial \psi_\lambda}{\partial x} = U_\lambda \psi_\lambda + \begin{pmatrix} 0 \\ -S_\lambda \end{pmatrix},$$

$$\frac{\partial \psi_\lambda}{\partial t} = V_\lambda \psi_\lambda + \begin{pmatrix} T_\lambda + 4\frac{\partial S_\lambda}{\partial x} \\ D_\lambda T_\lambda - (\lambda^2 - 2u)S_\lambda + 4D_\lambda \frac{\partial S_\lambda}{\partial x} \end{pmatrix}. \qquad \square$$

As a consequence we can anticipate that a map u will be a solution of the KdV equation if and only if there exist nontrivial solutions to the system (4.33) for sufficiently enough different values of λ. This system is again overdetermined. We will see in the next paragraph a necessary and sufficient condition on U_λ and V_λ in order that (4.33) has nontrivial solutions.

4.3. The zero curvature condition. — Here we address the following question: given two $\mathcal{C}^1$ maps $U, V : \mathbb{R}^2 \longrightarrow M(n,\mathbb{C})$ what are the necessary and sufficient conditions on U and V in order that the system

$$\frac{\partial g}{\partial x} = Ug, \quad \frac{\partial g}{\partial t} = Vg \tag{4.36}$$

have solutions $g : \mathbb{R}^2 \longrightarrow GL(n,\mathbb{C})$?

Let us start by a very special and simple case, when $n = 1$: then U and V are just complex valued functions and we look for a map $g : \mathbb{R}^2 \longrightarrow \mathbb{C}^*$ such that $dg = Ugdx + Vgdt$. Since $\mathbb{R}^2$ is simply connected any map $g : \mathbb{R}^2 \longrightarrow \mathbb{C}^*$ can be written

$$g(x,t) = e^{f(x,t)}, \quad \forall (x,t) \in \mathbb{R}^2,$$

where $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$. Then the equation on g is

$$e^f df = Ue^f dx + Ve^f dt \quad \Longleftrightarrow \quad df = Udx + Vdt.$$

By Poincaré lemma, and still because $\mathbb{R}^2$ is simply connected, such an equation has a solution if and only if $d(Udx + Vdt) = 0$ or

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} = 0.$$

We shall see that a similar condition is necessary and sufficient for the general case where n is arbitrary. Let us first look for a necessary condition. Assume that $g : \mathbb{R}^2 \longrightarrow GL(n,\mathbb{C})$ is a solution of (4.36). Then, by using two times (4.36),

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial t}\right) = \frac{\partial (Ug)}{\partial t} - \frac{\partial (Vg)}{\partial x} \\
&= \frac{\partial U}{\partial t} g + U\frac{\partial g}{\partial t} - \frac{\partial V}{\partial x} g - V\frac{\partial g}{\partial x} \\
&= \left(\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x}\right) g + U(Vg) - V(Ug) \\
&= \left(\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U,V]\right) g.
\end{aligned}$$

Since g takes values in $GL(n,\mathbb{C})$ this forces

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U,V] = 0. \tag{4.37}$$

(Equation (4.37) is often called a *zero curvature* condition because the left hand side can be interpreted as the curvature of the connection form $-Udx - Vdt$.) The converse is true as claims the following.

Lemma 4. — *Let $U, V \in \mathcal{C}^1(\mathbb{R}^2, M(n,\mathbb{C}))$. Then for any $g_0 \in GL(n,\mathbb{C})$, there exists a unique map $g \in \mathcal{C}^2(\mathbb{R}^2, GL(n,\mathbb{C}))$ such that*

$$\left\{ \begin{array}{rcl} dg &=& Ug\,dx + Vg\,dt, \quad \text{on } \mathbb{R}^2 \\ g(0) &=& g_0. \end{array} \right.$$

if and only if U and V satisfy (4.37).

Proof. — It is a consequence of Frobenius' theorem, see [**18**], Appendix 5.1. □

Now we observe that if g is a solution of (4.36) any column vector of g is a nontrivial solution of the system

$$\frac{\partial\psi}{\partial x} = U\psi, \quad \frac{\partial\psi}{\partial t} = V\psi, \tag{4.38}$$

where $\psi \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{C}^n)$. Thus the preceding result can be formulated in a slightly different way: the system (4.38) has n linearly independent solutions if and only if relation (4.37) is satisfied.

4.4. Back to the KdV equation: the inverse scattering method. — As suggested by the preceding considerations the following is true: a smooth function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a solution of the KdV equation (4.30) if and only if the maps U_λ and V_λ defined respectively by (4.34) and (4.35) satisfy

$$\frac{\partial U_\lambda}{\partial t} - \frac{\partial V_\lambda}{\partial x} + [U_\lambda, V_\lambda] = 0, \quad \forall\lambda \in \mathbb{C}^*. \tag{4.39}$$

We are now going to describe the principle of a method for solving the KdV equation which works assuming that $u(x,t)$ and its derivatives tends to zero sufficiently quickly when $|x| \to \infty$. For simplicity we will assume a strongest hypothesis, namely that u vanishes outside the strip $S_R := \{(x,t) \in [-R,R] \times \mathbb{R}\}$. (This hypothesis is actually not valid since, even if we assume that for the time $t = 0$ the spatial support of u is contained in a compact interval then this will be non longer true for all other times in general, because the KdV equation is dispersive.) Then it turns out that on $\mathbb{R}^2 \setminus S_R$,

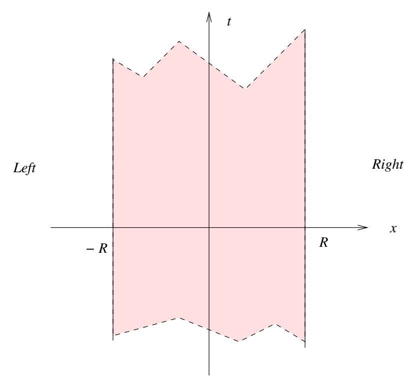

FIGURE 2. The support of u

U_λ and V_λ have very simple expressions:

$$U_\lambda(x,t) := \begin{pmatrix} -i\frac{\lambda}{2} & 1 \\ 0 & i\frac{\lambda}{2} \end{pmatrix}, \quad \forall (x,t) \in \mathbb{R}^2 \setminus S_R,$$

$$V_\lambda(x,t) := \begin{pmatrix} -i\frac{\lambda^3}{2} & \lambda^2 \\ 0 & i\frac{\lambda^3}{2} \end{pmatrix} = \lambda^2 U_\lambda(x,t), \quad \forall (x,t) \in \mathbb{R}^2 \setminus S_R.$$

Hence it is possible to integrate the overdetermined system

$$\frac{\partial \psi_\lambda}{\partial x} = U_\lambda \psi_\lambda, \quad \frac{\partial \psi_\lambda}{\partial t} = V_\lambda \psi_\lambda, \tag{4.40}$$

explicitly on each connected component of $\mathbb{R}^2 \setminus S_R$ (the left and the right ones): for any value of λ there exist (left) constants a_λ^L, b_λ^L and (right) constants a_λ^R, b_λ^R such that

$$\psi_\lambda(x,t) = \begin{pmatrix} 1 & -i\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_\lambda^L e^{\frac{1}{2i}(\lambda x + \lambda^3 t)} \\ b_\lambda^L e^{-\frac{1}{2i}(\lambda x + \lambda^3 t)} \end{pmatrix}, \quad \forall (x,t) \in (-\infty, -R] \times \mathbb{R},$$

$$\psi_\lambda(x,t) = \begin{pmatrix} 1 & -i\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_\lambda^R e^{\frac{1}{2i}(\lambda x + \lambda^3 t)} \\ b_\lambda^R e^{-\frac{1}{2i}(\lambda x + \lambda^3 t)} \end{pmatrix}, \quad \forall (x,t) \in [R, \infty) \times \mathbb{R}.$$

This shows that ψ_λ behaves in a very rigid way on each connected component of $\mathbb{R}^2 \setminus S_R$: the value of ψ_λ on each connected component is completely determined by the value on a single point of it.

4.4.1. The inverse scattering method. — This leads us to the following method to solve the KdV equation. Suppose that we know the value of u for $t = 0$. Then we can relate in principle the value of ψ_λ on the point $(x,t) = (R,0)$ in terms of its values on the point $(x,t) = (-R,0)$, for all values of λ. This is done by

$$\psi_\lambda(R,0) = G_\lambda^0(R)\psi_\lambda(-R,0), \tag{4.41}$$

where $G_\lambda^0 : [-R,R] \longrightarrow SL(2,\mathbb{C})$ is the unique solution of $G_\lambda^0(-R) = 1_2$ and $\frac{dG_\lambda^0}{dx}(x) = U_\lambda(x,0)G_\lambda^0(x)$. Observe that Relation (4.41) holds because $\chi_\lambda(x) := G_\lambda^0(x)\psi_\lambda(-R,0)$ satisfies the same equation $\frac{d\chi_\lambda}{dx} = U_\lambda \chi_\lambda$ on $[-R,R]$ and has the same initial condition $\chi_\lambda(-R) = \psi_\lambda(-R)$ as the restriction of ψ_λ on $[-R,R] \times \{0\}$. Moreover G_λ^0 takes values in $SL(2,\mathbb{C})$ because the matrices $U_\lambda(x,t)$ and $V_\lambda(x,t)$ take values in $\mathfrak{sl}(2,\mathbb{C})$. Hence using the **scattering matrix** $G_\lambda^0(R)$ we can relate the left data $(a_\lambda^L, b_\lambda^L)$ to the right data $(a_\lambda^R, b_\lambda^R)$ by a linear relation[(6)].

Dans la note : y?

Since on the other hand ψ_λ is completely rigid on the left and the right connected components, knowing $G_\lambda^0(R)$ it is very simple to deduce the analogue $G_\lambda^t(R)$ of $G_\lambda^0(R)$ for an arbitrary value of t. Thus if we are able to deduce the restriction of u on $[-R,R] \times \{t\}$ from the knowledge of $G_\lambda^t(R)$ we can in principle solve the KdV equation.

[(6)]y the normalization on φ_λ introduced previously we can assume without loss of generality that $(a_\lambda^L, b_\lambda^L) = (1,0)$.

This last task, known as the *inverse scattering problem* is quite delicate but can be done (see for instance [**12**] for details).

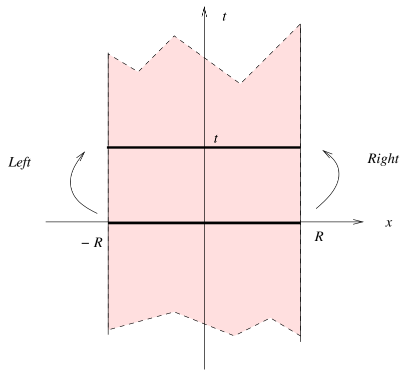

FIGURE 3. The inverse scattering method: the scattering of $u|_{[-R,R]\times\{0\}}$ is encoded by $G^0_\lambda(R)$ and the scattering of $u|_{[-R,R]\times\{t\}}$ is encoded by $G^t_\lambda(R)$.

To conclude this paragraph let us stress out again the fact that the proof that we briefly expounded was not correct, because we assumed that the support of u is contained in a strip bounded. The realistic situation is when u decreases to 0 when $|x|$ tends to ∞. Then ψ_λ does not have the rigid behavior that we did exploit here outside a strip but *asymptotically*, when x tends to $\pm\infty$, and the parameters a^L_λ, b^L_λ, a^R_λ and b^R_λ encode the asymptotic behavior of ψ_λ.

4.5. An alternative point of view on the inverse scattering method. — Let us come back to the first formulation of the KdV equation, based on the overdetermined system (4.32). In the following we consider a solution u to the KdV equation which decays to 0 when $x \to -\infty$. Then one can show (under reasonable assumptions on u) that for any $\lambda \in \mathbb{C}$ there exists a unique Baker–Akhiezer function φ_λ (i.e., a solution of (4.32) asymptotic to $e^{\frac{1}{2i}(\lambda x+\lambda^3 t)}$ when $x \to -\infty$).

4.5.1. A formal development of φ_λ. — We start by trying to write the Baker–Akhiezer function as the product of its (left) asymptotic value $e^{\frac{1}{2i}(\lambda x+\lambda^3 t)}$ by an asymptotic expansion in powers of λ^{-1}:

$$\varphi_\lambda(x,t) = e^{\frac{1}{2i}(\lambda x+\lambda^3 t)} \sum_{k=0}^{\infty} a_k(x,t)\left(\frac{2i}{\lambda}\right)^k. \tag{4.42}$$

A straightforward computation gives, denoting by $D = \frac{\partial}{\partial x}$:

$$L_u\varphi_\lambda - \frac{\lambda^2}{4}\varphi_\lambda = e^{\frac{1}{2i}(\lambda x+\lambda^3 t)}\left(i\lambda Da_0 + \sum_{k=0}^{\infty}(L_u a_k - 2Da_{k+1})\left(\frac{2i}{\lambda}\right)^k\right).$$

Hence this ansatz will give us a formal solution to our question if and only if $Da_0 = 0$ and $L_u a_k = 2Da_{k+1}$, $\forall k \in \mathbb{N}$. Because of the asymptotic condition at $-\infty$ it is natural to choose $a_0 = 1$ and then to construct all a_k recursively: each a_{k+1} is a primitive of $\frac{1}{2}L_u a_k$. In order to ensure the asymptotic condition it is also natural to assume that for all $k \geq 1$, a_k tends to 0 at $-\infty$. Thus we see that, if it exists (a condition that we will assume in the following), a solution to (4.32) of the form (4.42) and asymptotic to $e^{\frac{1}{2i}(\lambda x + \lambda^3 t)}$ at $-\infty$ is **unique**.

4.5.2. The construction of the Grassmannian of Sato. — We now look at the family of maps $(\varphi_\lambda)_{\lambda\in\mathbb{C}^*}$ as a single map from $\mathbb{R}^2$ with values in the loop space $L^2(S^1, \mathbb{C})$ by restricting λ to S^1 (like before, for the sinh–Gordon equation). Then intuitively the idea will consist in considering the Frenet framing of the curve $[x \longmapsto (\varphi_\lambda(x,t))_\lambda]$ in the infinite dimensional space $L^2(S^1, \mathbb{C})$. Imagine that we fix some point $(x,t) \in \mathbb{R}^2$ and consider the (a priori) infinite dimensional subspace

$$W(x,t) := \mathrm{Span}_{\mathbb{C}}\left(\varphi_\lambda(x,t), D\varphi_\lambda(x,t), D^2\varphi_\lambda(x,t), \cdots, D^k\varphi_\lambda(x,t), \cdots\right).$$

It turns out that there is a simple way to construct this space, based on the observation that:

$$D^2\varphi_\lambda(x,t) = \left(-\frac{\lambda^2}{4} - u(x,t)\right)\varphi_\lambda(x,t).$$

Hence one can show by recursion that $\forall k \in \mathbb{N}$,

$$D^k\varphi_\lambda(x,t) = A_k(x,t)(\lambda^2)\varphi_\lambda(x,t) + B_k(x,t)(\lambda^2)D\varphi_\lambda(x,t),$$

where A_k and B_k are polynomials whose coefficients are functions of (x,t) (of degree $[\frac{k}{2}]$ and $[\frac{k-1}{2}]$ respectively). These coefficients can be computed in principle: they are complicated algebraic functions of $u, Du, D^2u, \cdots$. But the precise expression of A_k and B_k has no importance for us. What is relevant is that we can propose a more tractable definition of W:

$$W(x,t) := \{\alpha(\lambda^2)\varphi_\lambda(x,t) + \beta(\lambda^2)D\varphi_\lambda(x,t) | \alpha, \beta \in L^2_+(S^1, \mathbb{C})\},$$

where $L^2_+(S^1, \mathbb{C}) \subset L^2(S^1, \mathbb{C})$ is the subspace of loops which have a holomorphic extension inside the unit disk in $\mathbb{C}$ (or in other words which has the expansion $\sum_{k=0}^{\infty} \widehat{\alpha}_k \lambda^k$).

Now it is simple to see that $W(x,t)$ does *not* depend on x: if we derivate any map $x \longmapsto \xi_\lambda(x) \in L^2(S^1, \mathbb{C})$ which satisfies $\xi_\lambda(x) \in W(x,t)$ for all x (for the moment we still fix t), then one sees immediately that $\frac{d\xi_\lambda}{dx} \in W(x,t)$. But actually the same is also true when we derivate with respect to t. This follows from the extra condition $\frac{\partial\varphi_\lambda}{\partial t} = P_{3;u}\varphi_\lambda$, since $P_{3;u}$ is a differential operator in x. So W is independent of (x,t). Note also that W is stable by the map $f(\lambda) \longmapsto \lambda^2 f(\lambda)$ from $L^2(S^1, \mathbb{C})$ to itself. We write $\lambda^2 W \subset W$ for this property.

We need now to define the set Gr of all subspaces of $L^2(S^1,\mathbb{C})$ which are comparable in some sense with the subspace

$$H^+ := L^2_+(S^1,\mathbb{C}) = \left\{\xi \in L^2(S^1,\mathbb{C})|\xi_\lambda = \sum_{k=0}^{\infty}\widehat{\xi}_k\lambda^k\right\}.$$

For that purpose let us also define

$$H^- := \left\{\xi \in L^2(S^1,\mathbb{C})|\xi_\lambda = \sum_{k=-\infty}^{-1}\widehat{\xi}_k\lambda^k\right\},$$

so that $L^2(S^1,\mathbb{C}) = H^+ \oplus H^-$. We can therefore define the projections π^+ and π^- on the first and the second factor of this splitting respectively. Then the Grassmannian manifold Gr is the set of subspaces $W \subset L^2(S^1,\mathbb{C})$ such that $\pi^+|_W : W \longrightarrow H^+$ is a Fredholm operator (i.e., has finite dimensional kernel and cokernel) and $\pi^-|_W : W \longrightarrow H^-$ is a compact operator. We will denote also by $Gr^{(2)}$ the subset of Gr of subspaces W which satisfies $\lambda^2 W \subset W$.

4.5.3. Linearizing the KdV equation. — Inspired by the work of M. Sato, G. Segal and G. Wilson have constructed an elegant way to picture a solution u to the KdV equation under the extra hypothesis that $e^{-\frac{1}{2i}(\lambda x+\lambda^3 t)}W \in Gr^{(2)}$, $\forall(x,t)$, where W is 'spanned' by φ_λ and $D\varphi_\lambda$. In the following we denote

$$\ell_\lambda(x,t) := \frac{1}{2i}(\lambda x + \lambda^3 t),$$

and

$$w_\lambda(x,t) := \sum_{k=1}^{\infty} a_k(x,t)\left(\frac{2i}{\lambda}\right)^k,$$

so that $\varphi_\lambda(x,t) = e^{\ell_\lambda(x,t)}\left(1 + w_\lambda(x,t)\right)$. We assume further that

– $\forall(x,t)$, $e^{-\ell_\lambda(x,t)}W \cap H^- = \{0\}$
– the index of the restriction of π^+ to $e^{-\ell_\lambda(x,t)}W$ is 0.

The main consequence of these two hypotheses is that the restriction of π^+ to $e^{-\ell_\lambda(x,t)}W$ is an isomorphism to H^+.

Two elementary but crucial observations are that for any $(x,t) \in \mathbb{R}^2$ we have

– $e^{-\ell_\lambda(x,t)}\varphi_\lambda(x,t) \in e^{-\ell_\lambda(x,t)}W$
– $e^{-\ell_\lambda(x,t)}\varphi_\lambda(x,t) = 1 + w_\lambda(x,t)$.

Since $[\lambda \longmapsto w_\lambda(x,t)] \in H^-$ the second property implies in particular that

$$e^{-\ell_\lambda(x,t)}\varphi_\lambda(x,t) \in 1 + H^-,$$

where $1 \in H^+$ is the constant loop and $1+H^- := \{1+f_\lambda | f \in H^-\}$. In a more geometrical language we can say that $e^{-\ell_\lambda(x,t)}\varphi_\lambda(x,t)$ lies at the intersection of $e^{-\ell_\lambda(x,t)}W$ and $1 + H^-$. But because of our hypotheses this intersection is reduced to a point: the unique one in $e^{-\ell_\lambda(x,t)}W$ which is the inverse image of 1 by π^+.

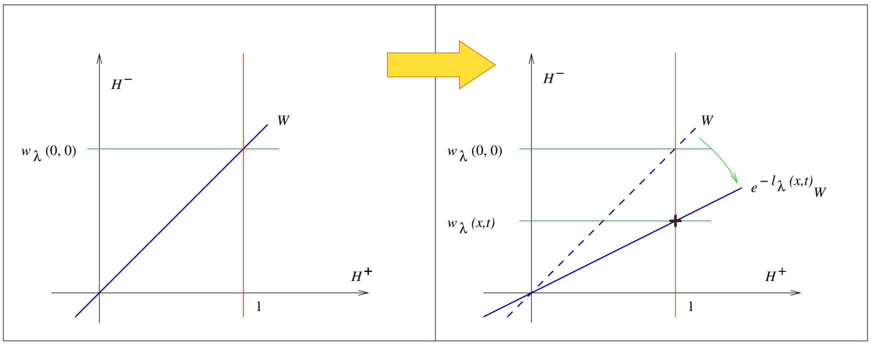

FIGURE 4. How the KdV flow arises from the action of a linear flow on the Grassmannian.

This leads us to a geometrical construction of solutions to the KdV equation. We start with some 'initial condition' $u_0 \in \mathcal{C}^\infty(\mathbb{R}_x)$ which decays at infinity and we associate to u_0 the subspace W. We assume that u_0 is so that W satisfies the preceding hypotheses. Then for any $(x,t) \in \mathbb{R}^2$ we consider the unique point in $e^{-\ell_\lambda(x,t)}W \cap (1 + H^-)$: it has the form $1 + w_\lambda(x,t)$. The expansion of $w_\lambda(x,t)$ in powers of λ^{-1} gives us the coefficients $a_k(x,t)$. Lastly because of the recursion relations between the a_k's we have $2Da_1 = -L_u a_0 = u$. So u is obtained by

$$u(x,t) = 2\frac{\partial a_1}{\partial x}(x,t).$$

4.6. Working with operators. — We now want to exploit the Lax equation in a way similar to the method expounded for finite dimensional integrable systems in section 2. Here a preliminary clarification will be useful: the variable x seems to play two different roles in the Lax formulation of the KdV equation. On the one hand x is a *(spatial) dynamical variable* and on the other hand x is used to construct the function space on which act L_u and P_u or the subspace W in the Sato Grassmannian. In order to remove this ambiguity we shall give two different names to this variable: t_1 for the *dynamical* variable and (still) x for the dumb variable used for the representation of the operators. We shall also denote $t_3 := t$ (we will see the reasons for these notations later on). We introduce the operator

$$P_1 := \frac{\partial}{\partial x}$$

acting on $\mathcal{C}^\infty(\mathbb{R}_x)$. We will work with maps

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}_x) \\ (t_1, t_3) & \longmapsto & v(t_1, t_3), \end{array}$$

and we denote

$$v(t_1, t_3; x) = v(t_1, t_3)(x).$$

We are interested in maps which are solutions of

$$\left\{\begin{array}{rcl}\dfrac{\partial L_{v(t_1,t_3)}}{\partial t_1} & = & \left[P_1, L_{v(t_1,t_3)}\right] \\ \dfrac{\partial L_{v(t_1,t_3)}}{\partial t_3} & = & \left[P_{3;v(t_1,t_3)}, L_{v(t_1,t_3)}\right].\end{array}\right. \tag{4.43}$$

The first equation is just

$$-\frac{\partial v(t_1,t_3)}{\partial t_1} \simeq \frac{\partial L_{v(t_1,t_3)}}{\partial t_1} = -P_1 v(t_1,t_3)$$

and implies that $v(t_1,t_3;x) = u(x+t_1,t_3)$ for some function u. This is the reason for the identification of t_1 with x. The second equation is then again the familiar Lax formulation of the KdV equation on u: $\frac{\partial u}{\partial t_3} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0$.

A straightforward generalization of the ideas encountered in section 2 is to look for a map $(t_1,t_3) \longmapsto K_{v(t_1,t_3)}$ into the space of invertible operators acting on $\mathcal{C}^\infty(\mathbb{R}_x)$ such that

$$L_{v(t_1,t_3)} = K_{v(t_1,t_3)} L_{u_0} K^{-1}_{v(t_1,t_3)}, \tag{4.44}$$

where u_0 could be the value of v for a particular value of (t_1,t_3). This is the analogue for operators of equation (2.10). Note that it is actually more suitable to choose u_0 to be the asymptotic value of v when $t_1 \to \pm\infty$, i.e., $u_0 = 0$. So we will just let $L_{u_0} = L_0 := -\frac{\partial^2}{\partial x^2}$. Now let us analyze equation (4.44): it implies (denoting $a = 1$ or 3, $L_v = L_{v(t_1,t_3)}$ and $K_v = K_{v(t_1,t_3)}$) that

$$\frac{\partial L_v}{\partial t_a} = \frac{\partial K_v}{\partial t_a} L_0 K_v^{-1} - K_v L_0 K_v^{-1} \frac{\partial K_v}{\partial t_a} K_v^{-1} = \left[\frac{\partial K_v}{\partial t_a} K_v^{-1}, L_v\right].$$

So because of the equation $\frac{\partial L_v}{\partial t_a} = [P_{a;v}, L]$ (where $P_{a;v} = P_{a;v(t_1,t_3)}$) we should have $\left[\frac{\partial K_v}{\partial t_a} K_v^{-1} - P_{a;v}, L_v\right] = 0$. By multiplying by K_v^{-1} on the left and by K_v on the right this equation we get

$$\left[K_v^{-1}\frac{\partial K_v}{\partial t_a} - K_v^{-1} P_{a;v} K_v, L_0\right] = 0. \tag{4.45}$$

At first glance the simplest choice would be to assume that $\frac{\partial K_v}{\partial t_a} = P_{a;v} K_v$. But there are other possibilities since L_0 commutes with all differential operators in the variable x with constant coefficients. And it will be more suitable in the following to look at K_v such that

$$K_v^{-1}\frac{\partial K_v}{\partial t_a} - K_v^{-1} P_{a;v} K_v = -P_{a;0},$$

(where here $P_{1;0} = \frac{\partial}{\partial x}$ and $P_{3;0} = -4\frac{\partial^3}{\partial x^3}$) which is equivalent to

$$\frac{\partial K_v}{\partial t_a} = P_{a;v} K_v - K_v P_{a;0}. \tag{4.46}$$

Indeed in the inverse scattering method one is interested in Baker–Akhiezer functions which are in our new setting (t_1,t_3)-dependant eigenvectors φ_λ of $L_{v(t_1,t_3)}$ for the eigenvalue $\frac{\lambda^2}{4}$, which satisfy $\frac{\partial\varphi_\lambda}{\partial t_a} = P_{a;v(t_1,t_3)}\varphi_\lambda$ and with the normalization $\varphi_\lambda(x) \simeq e^{\ell_\lambda(x+t_1,t_3)}$ when $x \to -\infty$. We use again the notation $\ell_\lambda(x+t_1,t_3) := \frac{1}{2i}(\lambda(x+t_1)+\lambda^3 t_3)$. It is natural to try to construct the Baker–Akhiezer function of the form

$$\varphi_\lambda(t_1,t_3;x) = K(t_1,t_3)e^{\ell_\lambda(x+t_1,t_3)}, \tag{4.47}$$

where $K(t_1,t_3)$ is an operator which preserves the asymptotic condition. Since $\frac{\partial}{\partial t_a}e^{\ell_\lambda} = P_{a;0}e^{\ell_\lambda}$, a substitution of the above expression for φ_λ in $\frac{\partial\varphi_\lambda}{\partial t_a} = P_{a;v}\varphi_\lambda$ gives us

$$\frac{\partial K}{\partial t_a}e^{\ell_\lambda} + KP_{a;0}e^{\ell_\lambda} = P_{a;v}Ke^{\ell_\lambda} \quad\Longleftrightarrow\quad \left(\frac{\partial K}{\partial t_a} + KP_{a;0} - P_{a;v}K\right)e^{\ell_\lambda} = 0.$$

So we are led to the relation (4.46). To summarize: if we assume that v is a solution of (4.43), if K_v satisfies (4.44) for some value of (t_1,t_3), say $(0,0)$, and if if K_v is a solution of (4.46) then K_v is a solution of (4.44) for all values of (t_1,t_3) (because one can compute that (4.46) implies $\frac{\partial(K_v^{-1}L_vK_v)}{\partial t_a} = \left[P_{a;0}, K_v^{-1}L_vK_v\right]$; hence the identity $K_v^{-1}L_vK_v = L_0$, obviously true for $(t_1,t_3)=(0,0)$, is actually true $\forall(t_1,t_3)$).

4.7. The KdV hierarchy. — In the following we shall be very formal and ignore all analytical subtleties. We construct an operator $K_{v(t_1,t_3)}$ which satisfies (4.44) as a pseudo-differential operator with the asymptotic expansion

$$K_{v(t_1,t_3)} = \sum_{k=0}^{\infty} a_k(t_1,t_3)D^{-k}, \tag{4.48}$$

where the coefficients $a_k(t_1,t_3)$ are functions of x, $D = \frac{\partial}{\partial x}$ and we adopt the convention $D^0 = Id$. For details see [**29**]. Observing that $L_v = -D^2 - v$ and $L_0 = -D^2$, a substitution of the above expression of K_v in equation (4.44) gives us $(-D^2 - v)(K_v\varphi) = K_v(-D^2\varphi)$, $\forall\varphi$, which is equivalent to

$$2Da_0D\varphi + \sum_{k=0}^{\infty}\left(2Da_{k+1} - L_va_k\right)D^{-k}\varphi = 0, \quad \forall\varphi.$$

Hence we find that $Da_0 := 0$ (we choose $a_0 = 1$) and $2Da_{k+1} = L_va_k$, $\forall k \geq 0$, so the coefficients a_k satisfy exactly the same conditions as the coefficients of the expansion of the Baker-Akhiezer function and hence are the same. Thus

$$K_{v(t_1,t_3)}e^{\ell_\lambda(x+t_1,t_3)} = e^{\ell_\lambda(x+t_1,t_3)}\left(1 + \sum_{k=1}^{\infty} a_k(x+t_1,t_3)\left(\frac{2i}{\lambda}\right)^k\right) \tag{4.49}$$

and we recover the same expression for the Baker–Akhiezer function.

The fact that the expression (4.48) is a solution of (4.46) has other interesting consequences. Indeed we can rewrite (4.46) as

$$
\begin{array}{rcccc}
 & K_v P_{a;0} K_v^{-1} & = & P_{a;v} & -\dfrac{\partial K_v}{\partial t_a} K_v^{-1} \\
\text{i.e.,} & \displaystyle\sum_{k=-\infty}^{a} (\,\cdot\,) D^k & = & \displaystyle\sum_{k=0}^{a} (\,\cdot\,) D^k & - \displaystyle\sum_{k=-\infty}^{-1} (\,\cdot\,) D^k.
\end{array}
\tag{4.50}
$$

In the second line we have stressed out the powers of D used in the expansion of each pseudo-differential operator. It is clear that this relation coincides with the splitting $T = T_+ + T_-$ of a pseudo-differential operator $T = \sum_{k=-\infty}^{a} T_k D^k$ into the sum of a differential operator $T_+ = \sum_{k=0}^{a} T_k D^k$ and of a pseudo-differential operator of negative order $T_- = \sum_{k=-\infty}^{-1} T_k D^k$. Hence we can write

$$P_{a;v} = \left(K_v P_{a;0} K_v^{-1}\right)_+ .$$

This relation was discovered by L. A. Dikki and I. M. Gel'fand [**9**]. It suggests us that $P_{1;v}$ and $P_{3;v}$ are actually two members of a whole family of differential operators. For any $a \in \mathbb{N}$ we set

$$P_{a;0} := (2i)^{a-1} D^a \quad \text{and} \quad P_{a;v} := \left(K_v P_{a;0} K_v^{-1}\right)_+ .$$

We remark that for a even, i.e., $a = 2p$, we have

$$
\begin{aligned}
P_{2p;v} &= \left(K_v P_{2p;0} K_v^{-1}\right)_+ \\
&= (2i)^{2p-1} \left(K_v \left(D^2\right)^p K_v^{-1}\right)_+ \\
&= (2i)^{2p-1} \left(\left(-K_v L_0 K_v^{-1}\right)^p\right)_+ \\
&= (-1)^p (2i)^{2p-1} \left(L_v^p\right)_+ = -2^{2p-1} i L_v^p,
\end{aligned}
$$

where we have used the relation (4.44) in the fourth line. In particular $P_{2;v} = -2iL_v$. Moreover if we set:

$$Q := K_v D K_v^{-1} = K_v P_{1;0} K_v^{-1},$$

then $-Q^2 = -K_v D^2 K_v^{-1} = K_v L_0 K_v^{-1} = L_v$, i.e., Q is a square root of $-L_v$. And more generally $P_{a;v} = (2i)^{a-1} \left(Q^a\right)_+$. So we can associate an infinite countable family of differential equations (called the *KdV hierarchy*) to the KdV equation: we let v to be a function of an infinite number of variables $t_1, t_2, t_3, t_4, \cdots$, with values in $\mathcal{C}^\infty(\mathbb{R}_x)$ and we write the system

$$\frac{\partial L_v}{\partial t_a} = \left[P_{a;v}, L_v\right], \quad \forall a \in \mathbb{N}. \tag{4.51}$$

Note that all equations with respect to even variables t_{2p} are trivial since $[P_{2p;v}, L_v] = -2^{2p-1} i \left[L_v^p, L_v\right] = 0$. This is the reason why we do not write down the t_{2p} variables. One can show that it is possible to integrate all these equations simultaneously, i.e., that all these flows commute (see for instance [**36**]). The beautiful thing is that all the previous constructions can be extended to these flows (although the concrete expression of the operators P_a for large a's can be very complicated). In particular the Baker–Akhiezer function can be expanded as

$$\varphi_\lambda(t_1,t_3,t_5,\cdots) = e^{\ell_\lambda(t_1,t_3,t_5,\cdots)}\left(1+\sum_{k=0}^{\infty} a_k(t_1,t_3,t_5,\cdots)\left(\frac{2i}{\lambda}\right)^k\right),$$

where

$$\ell_\lambda(t_1,t_3,t_5,\cdots) = \frac{1}{2i}\left(\lambda t_1+\lambda^3 t_3+\lambda^5 t_5+\cdots\right).$$

And the flow can be pictured geometrically through the action of the operator of multiplication by $e^{-\ell_\lambda(t_1,t_3,t_5,\cdots)}$ on an element W of the Grassmannian $Gr^{(2)}$.

Lastly we can remark that each equation of the system (4.51), which can also be written as $\frac{\partial L_v}{\partial t_a} = (2i)^{a-1}\left[\left((-L_v)^{a/2}\right)_+, L_v\right]$, can be understood by using the Adler–Kostant–Symes theory. Here the Lie algebra of pseudo-differential operators is splitted as a sum of two Lie subalgebras according to the decomposition (4.50).

4.8. The τ-function. — An alternative method to construct the Baker–Akhiezer function out of the action of the operator of multiplication by $e^{-\ell_\lambda}$ on the Grassmannian has been developed by Sato. It is based on the so-called *τ-function*. To explain this object imagine first that we are looking at a finite dimensional complex vector space E with the splitting $E = E^+ \oplus E^-$, where $\dim E^+ = p$ and $\dim E^- = q$. We denote by Gr the Grassmannian manifold of p-dimensional subspaces W of E and

$$Gr_* := \{W \in Gr | W \cap H^- = \{0\}\}$$

the open dense subset of Gr of p-dimensional subspaces which are transverse to H^-. Then if we denote by $\pi^+ : E \longrightarrow E^+$ the projection parallel to E^-, for any $W \in Gr_*$, the restriction of π^+ to W, $(\pi^+)|_W : W \longrightarrow H^+$, is an isomorphism. Let $\mathfrak{G}$ be the subgroup of $GL(E)$ which preserves H^+, i.e.,

$$\mathfrak{G} := \{g \in GL(E)|\, gH^+ \subset H^+\}.$$

Now if we fix some $W \in Gr_*$, we define $\mathfrak{G}_W := \{g \in \mathfrak{G}|\, g^{-1}W \in Gr_*\}$. The τ-function associated to W is a map

$$\tau_W : \mathfrak{G}_W \longrightarrow \mathbb{C},$$

defined as follows. Let $(e_1^+,\cdots,e_p^+)$ and $(e_1^-,\cdots,e_q^-)$ be bases of respectively E^+ and E^-. Let $(\alpha_+^1,\cdots,\alpha_+^p,\alpha_-^1,\cdots,\alpha_-^q)$ be the dual basis to $(e_1^+,\cdots,e_p^+,e_1^-,\cdots,e_q^-)$ and $\alpha_+ := \alpha_+^1\wedge\cdots\wedge\alpha_+^p$. Lastly let $(u_1,\cdots,u_p)$ be a basis of W. Then

$$\forall g \in \mathfrak{G}_W, \quad \tau_W(g) := \frac{\alpha_+(g^{-1}u_1,\cdots,g^{-1}u_p)}{\alpha_+(g^{-1}(\pi^+u_1),\cdots,g^{-1}(\pi^+u_p))}.$$

It is clear that this expression is independent from the choice of the basis of W. For instance if $p=2$ and $q=1$, one could imagine that E^+ represents the surface of the (approximatively flat) earth, E^- represents the vertical direction. The sun is at the vertical and we imagine a piece of surface C contained in W: its shadows is just $\pi^+(C)$. When we let g^{-1} act on the space E the body is moved and its shadow also.

The value of τ_W at g is then the ratio between the area of the shadow of $g^{-1}(C)$ by the area of the image by g^{-1} of the shadow of C, i.e., $\frac{\text{area}\pi^+(g^{-1}(C))}{\text{area}g^{-1}(\pi^+(C))}$.

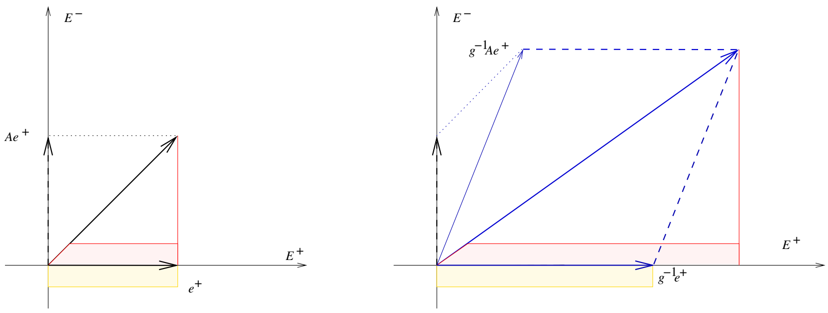

FIGURE 5. A picture of the τ-function

The τ-function has the following property which will be useful later.

Proposition 2. — *We have the relation*

$$\forall g \in \mathfrak{G}_W, \forall \gamma \in \mathfrak{G}_{g^{-1}W}, \quad \tau_{g^{-1}W}(\gamma) = \frac{\tau_W(g\gamma)}{\tau_W(g)}. \tag{4.52}$$

Proof. — Let $(u_1, \dots u_p)$ be a basis of W. Then $(g^{-1}u_1, \cdots, g^{-1}u_p)$ is a basis of $g^{-1}W$, hence

$$\tau_{g^{-1}W}(\gamma) = \frac{\alpha_+(\gamma^{-1}g^{-1}u_1, \cdots, \gamma^{-1}g^{-1}u_p)}{\alpha_+(\gamma^{-1}\pi^+g^{-1}u_1, \cdots, \gamma^{-1}\pi^+g^{-1}u_p)} = AB,$$

where

$$A := \frac{\alpha_+((g\gamma)^{-1}u_1, \cdots, (g\gamma)^{-1}u_p)}{\alpha_+((g\gamma)^{-1}\pi^+u_1, \cdots, (g\gamma)^{-1}\pi^+u_p)} = \tau_W(g\gamma),$$

and

$$B := \frac{\alpha_+(\gamma^{-1}g^{-1}\pi^+u_1, \cdots, \gamma^{-1}g^{-1}\pi^+u_p)}{\alpha_+(\gamma^{-1}\pi^+g^{-1}u_1, \cdots, \gamma^{-1}\pi^+g^{-1}u_p)}.$$

In the expression for B we can simplify by $\det\big((\gamma^{-1})|_{E^+}\big)$ in the numerator and the denominator to get

$$B = \frac{\alpha_+(g^{-1}\pi^+u_1, \cdots, g^{-1}\pi^+u_p)}{\alpha_+(\pi^+g^{-1}u_1, \cdots, \pi^+g^{-1}u_p)} = \frac{\alpha_+(g^{-1}\pi^+u_1, \cdots, g^{-1}\pi^+u_p)}{\alpha_+(g^{-1}u_1, \cdots, g^{-1}u_p)} = \frac{1}{\tau_W(g)},$$

where we have used the fact that $(\pi^+)^*\alpha^+ = \alpha^+$. Hence the result follows. □

It will be useful to give an algebraic expression of that by using the matrix

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

of g^{-1} in the basis $(e_1^+, \cdots, e_p^+, e_1^-, \cdots, e_q^-)$. Here the 0 is due to the fact that $g^{-1}H^+ = H^+$ and, by using an identification between operators and matrices we have $a \in GL(E^+)$, $b \in L(E^-, E^+)$ and $d \in GL(E^-)$. We also describe W as the graph of some linear map $A : H^+ \longrightarrow H^-$. Then we can choose $u_j = e_j^+ + Ae_j^+$, $\forall j = 1, \cdots, p$ and compute

$$\tau_W(g) = \frac{\det \begin{pmatrix} 1_p & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1_p \\ A \end{pmatrix}}{\det \begin{pmatrix} 1_p & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1_p \\ 0 \end{pmatrix}} = \frac{\det(a + bA)}{\det a} = \det(1_p + a^{-1}bA).$$

Let us now go to the infinite dimensional case and replace $E = E^+ \oplus E^-$ by $H = H^+ \oplus H^-$, where $H = L^2(S^1, \mathbb{C})$. Our discussion will be mainly formal, our scope being to give an intuitive idea of the theory. In particular we will not study the conditions in order to guarantee that all the determinants written here make sense, we just point out that the determinant of an operator T exists if and only if $T - Id$ is of trace class and we refer to [**29**] for details. We will be particularly concerned with the group of linear maps obtained by multiplication by a nonvanishing function in H^+ (with the standard product law on the set of functions on S^1). We define

$$G := \{f \in L^2(S^1, \mathbb{C}) |\, [\lambda \longmapsto f_\lambda] \text{ has a holomorphic extension inside } \overline{D(0,1)},$$
$$\forall \lambda \in \overline{D(0,1)}\, f_\lambda \neq 0 \text{ and } f_0 = 1\} \subset H^+.$$

This set has a structure of Abelian group for the multiplication of functions. It also occurs naturally in the KdV hierarchy since the map $(t_1, t_2, t_3, \cdots) \longmapsto e^{\frac{1}{2i}(t_1\lambda + t_2\lambda^2 + t_3\lambda^3 + \cdots)}$ is a parametrization of G.

To each element $g \in G$ we associate the linear operator

$$\begin{array}{rccc} [g]: & H & \longrightarrow & H \\ & \varphi & \longmapsto & [g]\varphi, \end{array}$$

where

$$\forall \lambda \in S^1 \subset \mathbb{C}^*, \quad ([g]\varphi)_\lambda := g_\lambda \varphi_\lambda.$$

Note that we need in principle to assume further regularity conditions in the definition of G, for instance in order that the operator $[g]$ be continuous(7) on H. We denote $[G] := \{[g] |\, g \in G\}$. Observe that, as for the finite dimensional case, $\forall [g] \in [G]$, $[g]H^+ \subset H^+$.

We can give a matrix representation of operators in $[G]$ by introducing

$$e_n = [\lambda \longmapsto \lambda^n] \in H,$$

the vectors which compose the Hilbertian basis of H given by the Fourier transform. Then any map $g \in G$ can be decomposed as $g = \sum_{n=0}^{\infty} \widehat{g}_n e_n$ (which is equivalent to

(7) The assumption that $g \in L^\infty(S^1, \mathbb{C})$ is necessary and sufficient to guarantee the continuity of $[g]$. But one need to assume actually stronger hypotheses, like for instance $g \in H^s(S^1, \mathbb{C})$, for $s > \frac{1}{2}$ in order that the properties that $\pi^+ : H \longrightarrow H^+$ is Fredholm and $\pi^- : H \longrightarrow H^-$ is Hilbert–Schmidt be preserved by the action of $[g]$, see [**28**].

$g_\lambda = \sum_{n=0}^\infty \widehat{g}_n \lambda^n$), where $\widehat{g}_n = g_0 = 1$. And the matrix of $[g]$ in the basis $(e_n)_{n\in\mathbb{Z}}$ has the form

$$[g] = \left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & \widehat{g}_1 & \widehat{g}_2 & \widehat{g}_3 & \\ \ddots & 0 & 1 & \widehat{g}_1 & \widehat{g}_2 & \ddots \\ \hline \ddots & 0 & 0 & 1 & \widehat{g}_1 & \ddots \\ & 0 & 0 & 0 & 1 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{array}\right),$$

where the block decomposition has the meaning $\left(\begin{array}{c|c} {\scriptstyle H^+\to H^+} & {\scriptstyle H^-\to H^+} \\ \hline {\scriptstyle H^+\to H^-} & {\scriptstyle H^-\to H^-} \end{array}\right)$. Now we define the τ-function as follows: for any $W \in Gr$, let $A : H^+ \longrightarrow H^-$ be the unique operator such that W is the graph of A, i.e., $W = \{(v, Av)|\, v \in H^+\}$. Let $[G]_W := \{[g] \in [G]|\, (\pi^+)|_{[g^{-1}]W} : [g^{-1}]W \longrightarrow H^+$ is an isomorphism$\}$. Then $\forall [g] \in [G]_W$

$$\tau_W([g]) := \det(Id_{H^+} + a^{-1}bA), \quad \text{where} \quad [g^{-1}] = \left(\begin{array}{c|c} a & b \\ \hline 0 & d \end{array}\right).$$

4.8.1. The relation with the Baker–Akhiezer function. — We introduce an extra complex parameter $\zeta \in \mathbb{C}$ such that $|\zeta| > 1$ and define for each value of ζ the map

$$\begin{array}{cccc} q^{(\zeta)} : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & \lambda & \longmapsto & 1 - \frac{\lambda}{\zeta}. \end{array}$$

We observe that, because of the condition $|\zeta| > 1$, $q^{(\zeta)} \in G$.

Lemma 5. — *Assume that $W \cap (e_0 + H^-)$ is reduced to one point, that we denote by $\psi = e_0 + \sum_{k=1}^\infty a_k (2i)^k e_{-k}$ (so that $\psi_\lambda = 1 + \sum_{k=1}^\infty a_k \left(\frac{2i}{\lambda}\right)^k$). Then $\forall \zeta \in \mathbb{C}$ such that $|\zeta| > 1$,*

$$\tau_W\left(\left[q^{(\zeta)}\right]\right) = 1 + \sum_{k=1}^\infty a_k \left(\frac{2i}{\zeta}\right)^k = \psi_\zeta.$$

Proof. — We need to compute $\tau_W\left(\left[q^{(\zeta)}\right]\right) = \det(Id_{H^+} + a^{-1}bA)$, where

$$\left(\begin{array}{c|c} a & b \\ \hline 0 & d \end{array}\right) = \left[q^{(\zeta)}\right]^{-1} = \left[\left(q^{(\zeta)}\right)^{-1}\right],$$

and W is the graph of $A : H^+ \longrightarrow H^-$. Observe that $e_0 + Ae_0$ is precisely the intersection point of W with $e_0 + H^-$. Hence if we write $Ae_n = \sum_{k=1}^\infty A_n^{-k} e_{-k}$, $\forall n \in \mathbb{N}$ or:

$$A = \left(\begin{array}{cccc|} \cdots & A_2^{-1} & A_1^{-1} & A_0^{-1} \\ \cdots & A_2^{-2} & A_1^{-2} & A_0^{-2} \\ & \vdots & \vdots & \vdots \end{array}\right)$$

then $A_0^{-k} = (2i)^k a_k$, $\forall k \in \mathbb{N}^*$.

Computation of $\left[q^{(\zeta)}\right]^{-1}$. — $\forall\psi\in H$, $\left[q^{(\zeta)}\right]^{-1}\psi=\frac{\psi}{q^{(\zeta)}}$ or:

$$\left(\left[q^{(\zeta)}\right]^{-1}\psi\right)_\lambda=\frac{\psi_\lambda}{q^{(\zeta)}_\lambda}=\psi_\lambda\sum_{k=0}^{\infty}\lambda^k\zeta^{-k}.$$

In particular, $\forall n\in\mathbb{Z}$,

$$\left(\left[q^{(\zeta)}\right]^{-1}e_n\right)_\lambda=\lambda^n\sum_{k=0}^{\infty}\lambda^k\zeta^{-k}=\sum_{k=n}^{\infty}\lambda^k\zeta^{n-k}=\left(\sum_{k=n}^{\infty}\zeta^{n-k}e_k\right)_\lambda.$$

As a matrix,

$$\left[q^{(\zeta)}\right]^{-1}=\left(\begin{array}{ccc|ccc}\ddots&\ddots&\ddots&\ddots&&\\ \ddots&1&\zeta^{-1}&\zeta^{-2}&\zeta^{-3}&\\ \ddots&0&1&\zeta^{-1}&\zeta^{-2}&\ddots\\ \hline \ddots&0&0&1&\zeta^{-1}&\ddots\\ &0&0&0&1&\ddots\\ &&\ddots&\ddots&\ddots&\ddots\end{array}\right)=\left(\begin{array}{c|c}a&b\\ \hline 0&d\end{array}\right).$$

Computation of b. — Its matrix is given by the preceding expression. Alternatively it is possible to compute it as follows: $\forall n\in\mathbb{Z}$, such that $n\leq-1$,

$$(be_n)_\lambda=\left(\sum_{k=n}^{\infty}\lambda^k\zeta^{n-k}\right)_+=\sum_{k=0}^{\infty}\lambda^k\zeta^{n-k}=\zeta^n\sum_{k=0}^{\infty}\lambda^k\zeta^{-k}=\frac{\zeta^n}{q^{(\zeta)}_\lambda}.$$

This implies by linearity that $\forall\psi\in H^-$,

$$(b\psi)_\lambda=\frac{\psi_\zeta}{q^{(\zeta)}_\lambda}=ev_\zeta(\psi)\frac{1}{q^{(\zeta)}_\lambda},$$

where

$$\begin{array}{cccc}ev_\zeta: & H & \longrightarrow & \mathbb{C}\\ & \psi & \longmapsto & \psi_\zeta\end{array}$$

is the evaluation map at $\lambda=\zeta$. Hence $b\psi=ev_\zeta(\psi)\frac{1}{q^{(\zeta)}}$. It means that b is a rank one operator, whose image is the line $\mathbb{C}\frac{1}{q^{(\zeta)}}$. This can also be pictured by the matrix product

$$b=\begin{pmatrix}\vdots\\ \zeta^{-2}\\ \zeta^{-1}\\ \underline{1}\end{pmatrix}\left(\!\mid\ \zeta^{-1}\quad\zeta^{-2}\quad\zeta^{-3}\quad\cdots\right).$$

Computation of a^{-1}. — Since $q^{(\zeta)} \in H^+$, we have that $\forall \psi \in H^+$,

$$a\psi = \left(\frac{\psi}{q^{(\zeta)}}\right)_+ = \frac{\psi}{q^{(\zeta)}}.$$

Hence a coincides with the restriction of $\left[q^{(\zeta)}\right]^{-1}$ on H^+, so a^{-1} is just the restriction of $\left[q^{(\zeta)}\right]$ on H^+, i.e., $\forall \psi \in H^+$,

$$\left(a^{-1}\psi\right)_\lambda = q^{(\zeta)}_\lambda \psi_\lambda = \left(1 - \frac{\lambda}{\zeta}\right)\psi_\lambda.$$

Hence as a matrix

$$a^{-1} = \left(\begin{array}{cccc|} \ddots & \ddots & & \vdots \\ \ddots & 1 & -\zeta^{-1} & 0 \\ & 0 & 1 & -\zeta^{-1} \\ \cdots & 0 & 0 & 1 \\ \hline \end{array}\right).$$

Computation of $a^{-1}b$. — $\forall \psi \in H^-$,

$$\left(a^{-1}b\psi\right)_\lambda = q^{(\zeta)}_\lambda \left(\frac{\psi_\zeta}{q^{(\zeta)}_\lambda}\right) = \psi_\zeta = ev_\zeta(\psi).$$

Hence $a^{-1}b = e_0 \otimes ev_\zeta$. As a matrix

$$a^{-1}b = \left(\begin{array}{cccc|} \ddots & \ddots & & \vdots \\ \ddots & 1 & -\zeta^{-1} & 0 \\ & 0 & 1 & -\zeta^{-1} \\ \cdots & 0 & 0 & 1 \\ \hline \end{array}\right) \left(\begin{array}{c} \vdots \\ \zeta^{-2} \\ \zeta^{-1} \\ 1 \\ \hline \end{array}\right) \left(\begin{array}{|cccc} \zeta^{-1} & \zeta^{-2} & \zeta^{-3} & \cdots \end{array}\right)$$

$$= \left(\begin{array}{c} \vdots \\ 0 \\ 0 \\ 1 \\ \hline \end{array}\right) \left(\begin{array}{|cccc} \zeta^{-1} & \zeta^{-2} & \zeta^{-3} & \cdots \end{array}\right) = \left(\begin{array}{|cccc} \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \zeta^{-1} & \zeta^{-2} & \zeta^{-3} & \cdots \\ \hline \end{array}\right).$$

Conclusion. — We now have

$$a^{-1}bA = \left(\begin{array}{|cccc} \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \zeta^{-1} & \zeta^{-2} & \zeta^{-3} & \cdots \\ \hline \end{array}\right) \left(\begin{array}{cccc|} \hline \cdots & A_2^{-1} & A_1^{-1} & A_0^{-1} \\ \cdots & A_2^{-2} & A_1^{-2} & A_0^{-2} \\ & \vdots & \vdots & \vdots \end{array}\right)$$

$$= \left(\begin{array}{cccc|} & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & \sum_{k=1}^\infty A_2^{-k}\zeta^{-k} & \sum_{k=1}^\infty A_1^{-k}\zeta^{-k} & \sum_{k=1}^\infty A_0^{-k}\zeta^{-k} \\ \hline \end{array}\right).$$

Hence

$$\tau_W\left(q^{(\zeta)}\right) = \det(Id_{H^+} + a^{-1}bA) = 1 + \sum_{k=1}^{\infty} A_0^{-k}\zeta^{-k} = 1 + \sum_{k=1}^{\infty} a_k\left(\frac{2i}{\zeta}\right)^k,$$

and the result follows. □

Corollary 1. — *Let φ_λ be the Baker–Akhiezer function, then*

$$\varphi_\zeta(\mathbf{t}) = e^{\ell_\zeta(\mathbf{t})}\frac{\tau_W\left(\left[e^{\ell(\mathbf{t})}\right]\left[q^{(\zeta)}\right]\right)}{\tau_W\left(\left[e^{\ell(\mathbf{t})}\right]\right)},$$

where $\mathbf{t} = (t_1, t_2, t_3, \cdots)$ *and* $e^{\ell(\mathbf{t})} : \lambda \longmapsto e^{\ell_\lambda(\mathbf{t})}$.

Proof. — We use the Proposition 2: the right hand side is equal to $e^{\ell_\zeta(\mathbf{t})}\tau_{\left[e^{-\ell(\mathbf{t})}\right]W}\left(\left[q^{(\zeta)}\right]\right)$, which is equal to $\varphi_\zeta(\mathbf{t})$ by the previous Lemma. □

The preceding relation can be further transformed, since we have $\left[e^{\ell(\mathbf{t})}\right]\left[q^{(\zeta)}\right] = \left[e^{\ell(\mathbf{t})}q^{(\zeta)}\right]$, with

$$q_\lambda^{(\zeta)} = e^{\log\left(1-\frac{\lambda}{\zeta}\right)} = e^{\left(-\sum_{k=1}^{\infty}\frac{\lambda^k}{k\zeta^k}\right)},$$

which implies

$$\left(e^{\ell(\mathbf{t})}q^{(\zeta)}\right)_\lambda = e^{\left(\frac{1}{2i}\sum_{k=1}^{\infty}\left(t_k - \frac{2i}{k\zeta^k}\right)\lambda^k\right)}.$$

So by introducing the notation $\tau_W(\mathbf{t}) := \tau_W\left(\left[e^{\ell(\mathbf{t})}\right]\right)$, we can write

$$e^{-\ell_\zeta(\mathbf{t})}\varphi_\zeta(\mathbf{t}) = \frac{\tau_W\left(t_1 - \frac{2i}{\zeta}, t_2 - \frac{2i}{2\zeta^2}, t_3 - \frac{2i}{3\zeta^3}, \cdots\right)}{\tau_W\left(t_1, t_2, t_3, \cdots\right)}.$$

By expanding the left and the right hand sides in powers of $\frac{2i}{\zeta}$ we obtain

$$1 + \sum_{k=1}^{\infty} a_k(\mathbf{t})\left(\frac{2i}{\zeta}\right)^k = 1 - \frac{1}{\tau_W(\mathbf{t})}\frac{\partial\tau_W(\mathbf{t})}{\partial t_1}\left(\frac{2i}{\zeta}\right) + \mathcal{O}\left(\left(\frac{2i}{\zeta}\right)^2\right),$$

so that we have the following expression of the solution of the KdV equation:

$$u = \frac{\partial a_1}{\partial x} = -\frac{\partial^2\log\tau_W}{\partial x^2}.$$

For more developments on the τ-function see for instance [**29**] and [**23**].

5. Constant mean curvature surfaces and minimal surfaces

A completely different problem concerns the study of immersed surfaces in Euclidean three-dimensional space. We know since Monge that at the infinitesimal scale the shape of such a surface near any point is characterized by two *principal curvature numbers* $k_1 \leq k_2$. The product $K = k_1k_2$ is called the *Gauss curvature* and the quantity $H = (k_1 + k_2)/2$ is the *mean curvature.* The surfaces which have an everywhere vanishing constant mean curvature are the critical points of the area functional and are called *minimal surfaces.* It is one of the oldest variational problem in several variables (the first results are due to Euler and Meusnier, a student of Monge). The surfaces which have a non-vanishing constant mean curvature are just called *constant mean curvature surfaces*: we shall abreviate them by CMC surfaces. We shall study these surfaces locally and view them as immersions of an open domain Ω of $\mathbb{R}^2$ in $\mathbb{R}^3$. A result that we will use from the beginning is that any simply connected smoothly immersed surface Σ can be parametrized by a conformal map

$$\begin{array}{rccc} X: & \Omega & \longrightarrow & \mathbb{R}^3 \\ & (x,y) & \longmapsto & X(x,y). \end{array}$$

The conformality assumption means here that dX is of rank 2 and that $\left|\frac{\partial X}{\partial x}\right|^2 - \left|\frac{\partial X}{\partial y}\right|^2 = 2\left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}\right\rangle = 0$ everywhere. This can be expressed by the fact that the first fundamental form of X

$$I := \begin{pmatrix} \left|\frac{\partial X}{\partial x}\right|^2 & \left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}\right\rangle \\ \left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}\right\rangle & \left|\frac{\partial X}{\partial y}\right|^2 \end{pmatrix}$$

satisfies everywhere

$$I = \begin{pmatrix} e^{2\omega} & 0 \\ 0 & e^{2\omega} \end{pmatrix}, \tag{5.53}$$

where $\omega : \Omega \longrightarrow \mathbb{R}$ is some function. Then for any $(x,y) \in \Omega$ the mean curvature at $X(x,y)$ is the unique real number H such that

$$\Delta X = 2H \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y}, \tag{5.54}$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\times$ denotes the vector product in $\mathbb{R}^3$.

5.1. Minimal surfaces. — We see immediately that the case $H = 0$ is much simpler, since X is then a harmonic map with values in $\mathbb{R}^3$. As a consequence it is possible to solve locally the system (5.53) and (5.54) by some elementary complex analysis: we set

$$f := \frac{\partial X}{\partial z} = \frac{1}{2}\left(\frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y}\right).$$

It defines a map $f : \Omega \longrightarrow \mathbb{C}^3$. Then the conformality assumption (5.53) means that $f \neq 0$ and f satisfies the constraint $(f)^2 := (f^1)^2 + (f^2)^2 + (f^3)^2 = 0$, or in a more geometrical language that f takes values into the pointed complex quadric $\mathcal{Q}^* := \{Z \in \mathbb{C}^3 \setminus \{0\} | (Z)^2 = 0\}$. And the Laplace equation (5.54) just means that f is holomorphic: $\frac{\partial f}{\partial \bar{z}} = \frac{1}{4}\Delta X = 0$. Hence f is a holomorphic curve into $\mathcal{Q}^*$ and can be constructed for instance by using the holomorphic parametrization $P : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathcal{Q}^*$ defined by

$$P(a,b) = \begin{pmatrix} \frac{1}{2}(a^2 - b^2) \\ \frac{i}{2}(a^2 + b^2) \\ ab \end{pmatrix}.$$

Note that P is a two-sheeted covering and hence f can be obtained by choosing an arbitrary holomorphic map $(\alpha,\beta):\Omega\longrightarrow\mathbb{C}^2\setminus\{0\}$ and by letting $f=P\circ(\alpha,\beta)$. The last step is to build the immersion X knowing f: we just set $X(z)=C+\mathrm{Re}\int_{z_0}^z f(\zeta)d\zeta$, where we integrate along a path joining z_0 to $z=x+iy$ in Ω and $C\in\mathbb{R}^3$. The final result is the famous Enneper–Weierstrass representation formula:

$$X(z)=C+\mathrm{Re}\int_{z_0}^z\begin{pmatrix}\frac{1}{2}(\alpha^2-\beta^2)\\ \frac{i}{2}(\alpha^2+\beta^2)\\ \alpha\beta\end{pmatrix}(\zeta)d\zeta.$$

In the following we shall ignore the constant C, it just reflects the invariance of the problem by translations. An interesting (and elementary) observation is that $\mathcal{Q}^*$ is invariant by transformations $Z\longmapsto\lambda^{-2}Z$, for $\lambda\in\mathbb{C}^*$ (we shall see the reason for the choice of λ^{-2} later on). As a consequence we can associate to X a family of minimal immersions $(X_\lambda)_{\lambda\in\mathbb{C}^*}$ by the relation

$$X_\lambda(z)=\mathrm{Re}\int_{z_0}^z\lambda^{-2}\begin{pmatrix}\frac{1}{2}(\alpha^2-\beta^2)\\ \frac{i}{2}(\alpha^2+\beta^2)\\ \alpha\beta\end{pmatrix}(\zeta)d\zeta.$$

If λ^{-2} is real then the resulting surface is just the image of the original one by some dilation in $\mathbb{R}^3$, which is not very interesting. So let us assume that $|\lambda|=1$; then if $\lambda^{-2}\neq\pm1$ the image X_λ is actually very different of the image of X. For instance if X is the parametrization of an helicoid, then $X_{e^{i\pi/4}}$ is the parametrization of a catenoid! We call $(X_\lambda)_{\lambda\in\mathbb{C}^*}$ the *associated family* of X.

5.2. Constant mean curvature surfaces. — We now look at the case where H is a constant different from 0. It turns out that for any CMC immersion X one can also construct an associated family: this result[(8)] was proved by O. Bonnet [**7**]. To see where it comes from, we need to introduce further the Gauss map $u:\Omega\longrightarrow S^2$ of the immersion X. Up to a sign this map is characterized by the fact that $\forall(x,y)\in\Omega$, $u(x,y)$ is orthogonal to $T_{X(x,y)}\Sigma$, the tangent space to the surface at $X(x,y)$. If an orientation is given on Σ then u is uniquely defined by requiring that $(e_1,e_2,u)(x,y)$ is an oriented basis of $\mathbb{R}^3$ if $(e_1,e_2)(x,y)$ is an oriented basis of $T_{X(x,y)}\Sigma$. Note that the parametrization X induces automatically an orientation to Σ for which

$$u=\frac{\frac{\partial X}{\partial x}\times\frac{\partial X}{\partial y}}{\left|\frac{\partial X}{\partial x}\times\frac{\partial X}{\partial y}\right|}.$$

We can now define the second fundamental form of the immersion X to be

$$II=\begin{pmatrix}\left\langle\frac{\partial^2X}{\partial x^2},u\right\rangle & \left\langle\frac{\partial^2X}{\partial x\partial y},u\right\rangle\\ \left\langle\frac{\partial^2X}{\partial x\partial y},u\right\rangle & \left\langle\frac{\partial^2X}{\partial x^2},u\right\rangle\end{pmatrix}.$$

(8) However we will see that the family will be parametrized by a variable λ in S^1 instead of $\mathbb{C}^*$.

So II can be understood as the orthogonal projection of the Hessian of X on the normal direction. It is easy to check that II is symmetric and that its trace is $2He^{2\omega}$. Hence there exist two real valued functions a, b on Ω such that

$$II := e^{2\omega} \begin{pmatrix} H+a & b \\ b & H-a \end{pmatrix}.$$

The key to understand Bonnet's result is to address the following **question**: we are given a constant $H \in \mathbb{R}$ and three real valued functions ω, a, b on Ω and we consider the two tensor fields on Ω

$$I = \begin{pmatrix} e^{2\omega} & 0 \\ 0 & e^{2\omega} \end{pmatrix} \quad \text{and} \quad II = e^{2\omega} \begin{pmatrix} H+a & b \\ b & H-a \end{pmatrix}.$$

Then we want to know whether there exists an immersion $X : \Omega \longrightarrow \mathbb{R}^3$ such that its first and second fundamental forms are respectively I and II. For simplicity we assume that Ω is simply connected.

The **answer** is that such an immersion exists if and only if the two following equations are satisfied on Ω:

$$\Delta\omega + (H^2 - a^2 - b^2)e^{2\omega} = 0, \tag{5.55}$$

$$\frac{\partial}{\partial \bar{z}}\left(e^{2\omega}(a - ib)\right) = 0. \tag{5.56}$$

The first equation is the (specialization of the) Gauss equation and the second one is the (specialization of the) Codazzi equation. If $(H, \omega, a - ib)$ satisfies these two conditions then X exists and is unique up to rigid motions in $\mathbb{R}^3$.

The next observation is then that these two equations are invariant by the transformation $(H, \omega, a - ib) \longmapsto (H, \omega, \lambda^{-2}(a - ib))$, where $\lambda \in S^1 \subset \mathbb{C}^*$. This has the following consequence: take any CMC conformal immersion X. Then its first and second fundamental forms provides us automatically with datas $(H, \omega, a - ib)$ which satisfies (5.55) and (5.56). But then we have a whole family of datas $(H, \omega, \lambda^{-2}(a - ib))_{\lambda \in S^1}$ which also satisfy (5.55) and (5.56) and hence for each $\lambda \in S^1$ there exists a CMC immersion X_λ whose first and second fundamental forms correspond to $(H, \omega, \lambda^{-2}(a - ib))$. This leads to the existence of the associated family $(X_\lambda)_{\lambda \in S^1}$. In the case where $H = 0$ we recover the family constructed through the Enneper–Weierstrass representation.

5.3. Introducing Darboux framings. — The fact that (5.55) and (5.56) are integrability conditions becomes more transparent if one uses a Darboux moving frame. We let $e_1, e_2 : \Omega \longrightarrow S^2$ be two smooth maps such that $\forall (x, y) \in \Omega$, $(e_1, e_2)(x, y)$ is an orthonormal oriented basis of $T_{X(x,y)}\Sigma$. Then (e_1, e_2) is called a *Darboux framing* of X. Alternatively, $\forall (x, y) \in \Omega$, $(e_1, e_2, u)(x, y)$ is an orthonormal oriented basis of $\mathbb{R}^3$. We can represent this moving frame by a map $F : \Omega \longrightarrow SO(3)$ whose columns are e_1, e_2 and u. Then all the informations contained in I and II can be encoded in the expression of the derivatives of e_1, e_2, u and X in the moving frame (e_1, e_2, u). These datas form the matrix valued 1-form

$$A := \begin{pmatrix} 0 & \langle de_2, e_1\rangle & \langle du, e_1\rangle & \langle dX, e_1\rangle \\ \langle de_1, e_2\rangle & 0 & \langle du, e_2\rangle & \langle dX, e_2\rangle \\ \langle de_1, u\rangle & \langle de_2, u\rangle & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\langle dX, u\rangle = 0$ by definition of u and $\langle du, u\rangle = 0$ because $|u|^2 = 1$. This matrix valued 1-form is also the *Maurer–Cartan* form

$$A := G^{-1}dG,$$

where

$$G := \begin{pmatrix} F & X \\ 0 & 1 \end{pmatrix}.$$

The nice point is that we have incorporated the symmetry group of the problem – which is here the group $SO(3) \ltimes \mathbb{R}^3$ of rigid motions of the Euclidean three space – in the formulation itself. Indeed G takes values in the Lie group

$$\left\{ \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} |R \in SO(3), T \in \mathbb{R}^3 \right\} \simeq SO(3) \ltimes \mathbb{R}^3,$$

and A is a 1-form with coefficients in the Lie algebra of $SO(3) \ltimes \mathbb{R}^3$. In our case we can compute that

$$A = \begin{pmatrix} 0 & -*d\omega & -\alpha & e^\omega dx \\ *d\omega & 0 & -\beta & e^\omega dy \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.57}$$

where

$$\alpha + i\beta = e^\omega \left(H dz + (a+ib)d\overline{z}\right).$$

Now the way to decide whether the datas $(H, \omega, a - ib)$ correspond to a conformal immersion is simply to ask whether given A as in (5.57) there exists a map $G : \Omega \longrightarrow SO(3) \ltimes \mathbb{R}^3$, such that $dG = GA$. But we know that this overdetermined equation has a solution if and only if A satisfies the zero curvature equation

$$dA + A \wedge A = 0.$$

And a straightforward computation shows that this condition is equivalent to (5.55) and (5.56).

It is interesting to look at the effect of the substitution $a - ib \longmapsto \lambda^{-2}(a - ib)$ in this framework: the Maurer–Cartan form A is then transformed into another form A_λ, which can be computed explicitly. Since the Gauss–Codazzi equations are still satisfied for the datas $(H, e^\omega, \lambda^{-2}(a-ib))$ and since these equations are equivalent to the zero curvature equation we know *a priori* that A_λ should be a solution of

$$dA_\lambda + A_\lambda \wedge A_\lambda = 0, \quad \forall \lambda \in S^1. \tag{5.58}$$

This of course indeed the case. And assuming that the domain Ω is simply connected, relation (5.58) is the necessary and sufficient condition for the existence for all $\lambda \in S^1$ of a map $G_\lambda : \Omega \longrightarrow SO(3) \ltimes \mathbb{R}^3$ which is a solution of $dG_\lambda = G_\lambda A_\lambda$ (it is unique if we fix the value of G_λ at one point in Ω). By extracting the fourth column of G_λ we obtain for all $\lambda \in S^1$ a map $X_\lambda : \Omega \longrightarrow \mathbb{R}^3$ which is the conformal parametrization of a new CMC surface. We get hence the associated family $(X_\lambda)_{\lambda\in S^1}$ of CMC conformal immersions.

The way A_λ depends on λ can be simplified if we do the gauge transformation

$$\alpha_\lambda := R_\lambda^{-1} A_\lambda R_\lambda,$$

where

$$R_\lambda := \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and } \lambda = e^{i\theta}.$$

Geometrically it just amounts to substitute $\Gamma_\lambda := G_\lambda R_\lambda$ to G_λ, since then $d\Gamma_\lambda = \Gamma_\lambda \alpha_\lambda$. This does not change X_λ but just rotates the Darboux framing of X_λ. Then the gain is that the holomorphic extension to $\mathbb{C}^*$ of the map $S^1 \ni \lambda \longmapsto \alpha_\lambda$ has the form

$$\alpha_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'', \tag{5.59}$$

where the entries of α_1', α_0 and α_1'' have the structure

$$\alpha_1', \alpha_1'' : \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_0 : \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover α_1' is a $(1,0)$-form (it means that $\alpha_1'(\frac{\partial}{\partial \overline{z}}) = 0$) and α_1'' is a $(0,1)$-form (it means that $\alpha_1(\frac{\partial}{\partial z}) = 0$). This provide us with a simple method to build up α_λ knowing $\alpha = A$: we split $\alpha = \alpha_0 + \alpha_1$ according to the entries structure, where α_0 is the block-diagonal part and α_1 is the off-block-diagonal part of α. Then we further decompose $\alpha_1 = \alpha_1' + \alpha_1''$, where $\alpha_1' := \alpha_1(\frac{\partial}{\partial z})dz$ and $\alpha_1'' := \alpha_1(\frac{\partial}{\partial \overline{z}})d\overline{z}$. We deduce α_λ by (5.59).

The last observation is actually more than just a trick: the splitting $\alpha = \alpha_0 + \alpha_1$ has a Lie algebra interpretation, it corresponds to the direct sum decomposition of the complexification $\mathfrak{g}^\mathbb{C}$ of the Lie algebra of $SO(3) \ltimes \mathbb{R}^3$

$$\mathfrak{g}^\mathbb{C} = \mathfrak{g}_0^\mathbb{C} \oplus \mathfrak{g}_1^\mathbb{C},$$

where $\forall a \in \{0,1\}$, $\mathfrak{g}_a^{\mathbb{C}}$ is the $(-1)^a$-eigenspace of a linear involution $\tau : \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathfrak{g}^{\mathbb{C}}$ called the *Cartan involution*. Actually τ is simply $M \longmapsto PMP^{-1}$, where

$$P := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P^{-1}.$$

The Cartan involution helps us to encapsulate the informations concerning the immersion X into $\mathbb{R}^3$ and its Gauss map u into S^2 inside the single map Γ into $SO(3) \ltimes \mathbb{R}^3$. And when we further formulate this theory using loop groups the Cartan involution plays again a crucial and similar role, by the introduction of *twisted loop groups* (see paragraph 5.5).

Lastly we can remark that if we denote $\alpha_\lambda = U_\lambda dx + V_\lambda dy$, then the relation $d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0$ is equivalent to

$$\frac{\partial V_\lambda}{\partial x} - \frac{\partial U_\lambda}{\partial y} + [U_\lambda, V_\lambda\ ] = 0, \quad \forall \lambda \in S^1,$$

which is more or less the same condition as (4.39).

5.4. A reduction to the harmonic map problem. — A map u from a domain Ω of $\mathbb{R}^2$ into the unit sphere $S^2 \subset \mathbb{R}^3$ is called *harmonic* if it is a solution of the system

$$\Delta u + u|\nabla u|^2 = 0 \quad \text{in } \mathbb{R}^3,$$

where $|\nabla u|^2 := |\frac{\partial u}{\partial x}|^2 + |\frac{\partial u}{\partial y}|^2$. Harmonic maps are actually the critical points of the Dirichlet energy functional $E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 dxdy$ with the *constraint* that $u(x,y) \in S^2 \subset \mathbb{R}^3$, $\forall (x,y) \in \Omega$. There are related to the CMC surfaces by the following

Theorem 3. — *Let $X : \Omega \longrightarrow \mathbb{R}^3$ be a conformal immersion and $u : \Omega \longrightarrow S^2$ its Gauss map. Then the image of X is a CMC surface if and only if u is harmonic.*

If furthermore Ω is simply connected we can also construct a weakly conformal CMC immersion from any harmonic map $u : \Omega \longrightarrow S^2$. For that one observes that the harmonic map equation implies that the $\mathbb{R}^3$-valued 1-form $\psi := (u \times \frac{\partial u}{\partial y})dx - (u \times \frac{\partial u}{\partial x})dy$ (where $\times$ is the vector product in $\mathbb{R}^3$) is closed and hence we can integrate it: $\psi = dB$, where B is unique up to a constant in $\mathbb{R}^3$. Then the two maps $B \pm u$ are weakly conformal CMC immersions.

Now harmonic maps into S^2 can be characterized by a similar construction as for CMC surfaces. For any map $u : \Omega \longrightarrow S^2$ we build a moving frame (e_1, e_2), i.e., such that $(e_1(x,y), e_2(x,y))$ is an oriented orthonormal basis of $T_{u(x,y)}S^2$, $\forall (x,y) \in S^2$. Then we get a map $F = (e_1, e_2, u)$ from Ω to $SO(3)$ and its Maurer–Cartan form

$\alpha := F^{-1}dF$. We split $\alpha = \alpha_0 + \alpha_1$, where α_0 and α_1 have the structure

$$\alpha_1 : \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}, \quad \alpha_0 : \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to the eigenspace decomposition of the Lie algebra $\mathfrak{so}(3)$ for the automorphism $\tau : M \longmapsto PMP^{-1}$ where $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. And we further decompose $\alpha_1 = \alpha_1' + \alpha_1''$, where α_1' is the $(1,0)$-part of α_1 and α_1'' is its $(0,1)$-part. Then u is harmonic if and only if $\alpha_\lambda := \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1''$ satisfies the zero curvature condition $d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0$, $\forall \lambda \in \mathbb{C}^*$. As a consequence the overdetermined equation $d\varphi_\lambda = \alpha_\lambda\varphi_\lambda$ has a solution (unique if we prescribe the value of φ_λ at some point).

5.5. Construction of all harmonic maps into S^2. — As an application we describe here an algorithm for constructing all harmonic maps $\Omega \longrightarrow S^2$ (where Ω is simply connected) starting with holomorphic data. This construction is due to J. Dorfmeister, F. Pedit and H.Y. Wu [**11**]. For that purpose we need to introduce the twisted loop group $LSO(3)_\tau := \{g : S^1 \ni \lambda \longmapsto g_\lambda \in SO(3) \mid \tau(g_\lambda) = g_{-\lambda}\}$ and its complexification[9] $LSO(3)^{\mathbb{C}}_\tau := \{g : S^1 \ni \lambda \longmapsto g_\lambda \in SO(3)^{\mathbb{C}} \mid \tau(g_\lambda) = g_{-\lambda}\}$.

Step 1: choosing a potential

Let $a, b : \Omega \longrightarrow \mathbb{C}$ be holomorphic maps, and define a matrix-valued (actually loop algebra-valued) holomorphic 1-form

$$\mu_\lambda = \lambda^{-1} \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix} dz$$

which we call the *potential*. Observe that this 1-form has its coefficients in $L\mathfrak{so}(3)^{\mathbb{C}}_\tau$, the Lie algebra of $LSO(3)^{\mathbb{C}}_\tau$.

Step 2: integrating μ_λ

The potential trivially satisfies $d\mu_\lambda + \mu_\lambda \wedge \mu_\lambda = 0$, which is the necessary and sufficient condition for the existence of $g_\lambda : \Omega \longrightarrow LSO(3)^{\mathbb{C}}_\tau$ such that

$$dg_\lambda = g_\lambda\mu_\lambda.$$

Step 3: splitting

We write g_λ as the product $\varphi_\lambda b_\lambda$, with $\varphi_\lambda : \Omega \longrightarrow LSO(3)_\tau$ and $b_\lambda : \Omega \longrightarrow L^+_{\mathfrak{B}}SO(3)^{\mathbb{C}}_\tau$, where $L^+_{\mathfrak{B}}SO(3)^{\mathbb{C}}_\tau$ is the subgroup[10] of the loops $b_\lambda \in LSO(3)^{\mathbb{C}}_\tau$ which have a holomorphic extension (in λ) from the closed unit disk to $SO(3)^{\mathbb{C}}$. This step

[9] Here $SO(3)^{\mathbb{C}} := \{M \in M(3,\mathbb{C}) | \, {}^tMM = 1_3 \text{ and } \det M = 1\}$.

[10] An extra condition in the definition of $L^+_{\mathfrak{B}}SO(3)^{\mathbb{C}}_\tau$ is that $\forall b_\lambda \in L^+_{\mathfrak{B}}SO(3)^{\mathbb{C}}_\tau$, $b_0 \in \mathfrak{B}$, where $\mathfrak{B}$ is a Borel subgroup of $SO(3)^{\mathbb{C}}$ and the Iwasawa decomposition $SO(3)^{\mathbb{C}} = SO(3) \cdot \mathfrak{B}$ holds.

rests on an Iwasawa decomposition result similar to Theorem 2, which states that any loop $g_\lambda \in LSO(3)^{\mathbb{C}}_\tau$ can be written uniquely as the product of $\varphi_\lambda \in LSO(3)_\tau$ and $b_\lambda \in L^+_{\mathfrak{B}}SO(3)^{\mathbb{C}}_\tau$ (hence the decomposition of maps is done pointwise in z).

Then the map φ_λ produced in this way is a lift of a harmonic map into the sphere, i.e., the third column of φ_λ gives us the components of a harmonic map into S^2 for all $\lambda \in S^1$.

Note that this algorithm accounts for the construction of *almost* all harmonic maps. Actually J. Dorfmeister, F. Pedit and H.-Y. Wu show how to associate to any harmonic map a unique such potential μ_λ where the data (a, b) is meromorphic, albeit with non accumulating poles. This is based on solving the Riemann–Hilbert problem of splitting $\varphi_\lambda(z) = \varphi^-_\lambda(z)\varphi^+_\lambda(z)$ for each fixed $z \in \Omega$, where φ^-_λ takes values in $L^-_*SO(3)^{\mathbb{C}}_\tau$, the sub loop group of loops $g \in LSO(3)^{\mathbb{C}}_\tau$ which admit a holomorphic extension in λ outside the unit disk in $P\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$ and such that $g_\infty = 1_3$, and where φ^+_λ takes values in $L^+SO(3)^{\mathbb{C}}_\tau$. Again this decomposition follows from results in [**28**]. Then the potential is given by $\mu_\lambda = \left(\varphi^-_\lambda\right)^{-1} d\varphi^-_\lambda$. There are other constructions along the same lines which avoid using meromorphic data (but μ_λ may be more complicated).

Lastly one can remark that the algorithm parallels the Enneper–Weierstrass representation formula (hence its name). Indeed μ_λ is the analog of

$$f_\lambda dz := \lambda^{-2} \begin{pmatrix} \frac{1}{2}(\alpha^2 - \beta^2) \\ \frac{i}{2}(\alpha^2 + \beta^2) \\ \alpha\beta \end{pmatrix} (z)dz,$$

The map g_λ obtained from μ_λ corresponds to the (standard) integral $\int_{z_0}^z f_\lambda dz$. Finally Iwasawa decomposition reduces to taking the real part. Notice that the analogy is not only in spirit, but that under some conditions, the Dorfmeister–Pedit–Wu algorithm deforms into the Enneper–Weierstrass representation formula.

A construction of harmonic maps by using the Adler–Kostant–Symes theory is possible. One starts from the Lie algebra decomposition $L\mathfrak{so}(3)^{\mathbb{C}}_\tau = L\mathfrak{so}(3)_\tau \oplus L^+_{\mathfrak{b}}\mathfrak{so}(3)^{\mathbb{C}}_\tau$, which is in fact the linearization at the identity of the Iwasawa decomposition $LSO(3)^{\mathbb{C}}_\tau = LSO(3)_\tau \cdot L^+_{\mathfrak{B}}SO(3)^{\mathbb{C}}_\tau$. Then for each odd positive integer d one can construct a pair of ad*-invariant functions on the dual space of $L\mathfrak{so}(3)^{\mathbb{C}}_\tau$ which induces a pair of commuting Hamiltonian vector fields on a suitable (finite dimensional) subspace V^d of the dual space of $L^+_{\mathfrak{b}}\mathfrak{so}(3)^{\mathbb{C}}_\tau$. Their flow equations read as Lax equations and admit solutions which stay in V^d. By integrating this pair of vector fields one obtains harmonic maps, called *finite type* solutions (see [**8**]). Actually in the Dorfmeister–Pedit–Wu description this finite type solution arises from a potential $\mu_\lambda = \lambda^{d-1}\eta_\lambda$, where η_λ is a *constant* loop in $L\mathfrak{so}(3)_\tau$ with a Fourier expansion $\eta_\lambda = \sum_{k=-d}^{d} \widehat{\eta}_k \lambda^k$, see [**8, 17, 18, 14**] for more details.

6. Anti-self-dual curvature two-forms

6.1. The Hodge operator on forms. — Let $(\mathcal{M}, g)$ be a Riemannian manifold of dimension n. We denote by $(x^1, \cdots, x^n)$ local coordinates on $\mathcal{M}$ and by $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ the expression of the metric g in these coordinates. This allows us to define a scalar product on each cotangent space: if $a, b \in T^*_m\mathcal{M}$ and $a = a_i dx^i$ and $b = b_j dx^j$, then $\langle a, b\rangle := g^{ij} a_i b_j$, where (g^{ij}) is the inverse matrix of (g_{ij}). We can actually extend this scalar product to $\Lambda^p T^*_m\mathcal{M}$ for all $1 \leq p \leq n$ to be the unique bilinear form such that if $\alpha = a^1 \wedge \cdots \wedge a^p$ and $\beta = b^1 \wedge \cdots \wedge b^p$ (where the a^i's and the b^j's are in $T^*_m\mathcal{M}$), then

$$\langle \alpha, \beta\rangle := \det\left(\langle a^i, b^j\rangle\right).$$

Now let us assume that our manifold $\mathcal{M}$ and the coordinates $(x^1, \cdots, x^n)$ are oriented. Then there is a uniquely defined volume form σ on $\mathcal{M}$ which has the local expression

$$\sigma = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n.$$

All these ingredients allows us to define an operator which transforms any p-form on $\mathcal{M}$ onto an $(n-p)$-form: it is the *Hodge operator*

$$\begin{array}{cccc} * : & \Lambda^p T^*_m\mathcal{M} & \longrightarrow & \Lambda^{n-p} T^*_m\mathcal{M} \\ & \alpha & \longmapsto & *\alpha \end{array}$$

which is characterized by the following property

$$\forall \alpha \in \Lambda^p T^*_m\mathcal{M}, \forall \beta \in \Lambda^{n-p} T^*_m\mathcal{M}, \quad \alpha \wedge \beta = \langle *\alpha, \beta\rangle \sigma.$$

It is easy to observe that if we reverse the orientation, then $*$ is changed into $-*$. Let us study how $*$ is changed by a conformal deformation of the metric. Let $\widetilde{g}_{ij} = e^{2f} g_{ij}$ be another metric (where $f : \mathcal{M} \longrightarrow \mathbb{R}$ is any smooth function). Then the corresponding scalar product on $(n-p)$-forms is transformed into:

$$\langle \alpha, \beta\rangle_\sim = e^{-2(n-p)f}\langle \alpha, \beta\rangle$$

and the volume form is $\widetilde{\sigma} = e^{nf}\sigma$. So the corresponding Hodge operator $\widetilde{*}$ is characterized by

$$\alpha \wedge \beta = \langle \widetilde{*}\alpha, \beta\rangle_\sim\, \widetilde{\sigma} = e^{-2(n-p)f}\langle \widetilde{*}\alpha, \beta\rangle\, e^{nf}\sigma = e^{(2p-n)f}\langle \widetilde{*}\alpha, \beta\rangle.$$

Hence we deduce that $\forall \alpha \in \Gamma(\mathcal{M}, \Lambda^p T^*_m\mathcal{M})$,

$$\widetilde{*}\alpha = e^{(n-2p)f} * \alpha.$$

It is interesting to observe that if n is even and if $2p = n$ (which will be the case in the following), then the Hodge operator does not depend on the metric, but only on the conformal class of the metric and on the orientation.

6.1.1. The Hodge operator on surfaces. — Let us consider the case where $n=2$ and $(\mathcal{M},g)$ is $\mathbb{R}^2$ with the Euclidean metric[11]. Then

$$\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 \quad \Longrightarrow \quad *\alpha = -\alpha_2 dx^1 + \alpha_1 dx^2.$$

So in particular $*\circ *$ is minus the identity, i.e., $*$ is a complex structure. At any point x the action of $*$ on $T^*_x\mathbb{R}^2$ can be diagonalized over $\mathbb{C}$ with eigenvalues $-i$ and i and the eigenvectors are just $dz = dx^1 + idx^2$ and $d\overline{z} = dx^1 - idx^2$, i.e., we have $*dz = -idz$ and $*d\overline{z} = id\overline{z}$. If we denote by $T^{*,(1,0)}_x\mathbb{R}^2 := \{\alpha \in T^*_x\mathbb{R}^2 |\ *\alpha = -i\alpha\}$ and $T^{*,(0,1)}_x\mathbb{R}^2 := \{\alpha \in T^*_x\mathbb{R}^2 |\ *\alpha = i\alpha\}$ then we have the eigenspace decomposition $\left(T^*_x\mathbb{R}^2\right)^{\mathbb{C}} = T^{*,(1,0)}_x\mathbb{R}^2 \oplus T^{*,(0,1)}_x\mathbb{R}^2$. This has the following consequence: we say that a smooth 1-form $\alpha \in \Gamma(\mathbb{R}^2, T^*\mathbb{R}^2)$ is *harmonic* if and only if

$$d\alpha = 0 \quad \text{and} \quad d(*\alpha) = 0.$$

This definition is quite natural since, in particular one can check that the two components of α should be harmonic functions. Now any 1-form α can be splitted according to the eigenspace decomposition of $\left(T^*_x\mathbb{R}^2\right)^{\mathbb{C}}$: $\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}$, where

$$\alpha^{(1,0)} = \frac{1}{2}\left(\alpha + i*\alpha\right) \quad \text{and} \quad \alpha^{(0,1)} = \frac{1}{2}\left(\alpha - i*\alpha\right),$$

and α is harmonic if and only if $\alpha^{(1,0)}$ or $\alpha^{(0,1)}$ is closed. If so it means that $\alpha^{(1,0)}$ is a holomorphic form, i.e., if we write $\alpha^{(1,0)} = f(z,\overline{z})dz$ and $\alpha^{(1,0)} = g(z,\overline{z})d\overline{z}$ then $0 = d\alpha^{(1,0)} = \frac{\partial f}{\partial \overline{z}} d\overline{z}\wedge dz$ so that f is holomorphic and similarly g is anti-holomorphic (note that $g = \overline{f}$).

Eventually we recover the following simple fact: let $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function. Then φ is harmonic if and only if $\alpha := d\varphi$ is harmonic[12]. If so we have

$$d\varphi = f(z)dz + g(\overline{z})d\overline{z} = d\left(F(z) + G(\overline{z})\right),$$

where $F' = f$ and $G' = g$. Hence up to some constant we have $\varphi = F + G$, where F is holomorphic and G is antiholomorphic.

6.2. The dimension 4. — Let us now look at 2-forms on $\mathbb{R}^4$ with its standard orientation and metric. Then any 2-form $F \in \Lambda^2 T^*_x\mathbb{R}^4$ can be decomposed as $F = \sum_{1\leq i<j\leq 4} F_{ij} dx^i \wedge dx^j$ and its image by $*$ is

$$\begin{aligned} *F \quad = \quad & F_{34}dx^1\wedge dx^2 - F_{24}dx^1\wedge dx^3 + F_{23}dx^1\wedge dx^4 \\ & + F_{12}dx^3\wedge dx^4 - F_{13}dx^2\wedge dx^4 + F_{14}dx^2\wedge dx^3. \end{aligned}$$

In particular we see that $*$ is an involution (its square is the identity) so it has eigenvalues ± 1. Hence $\Lambda^2 T^*_x\mathbb{R}^4$ splits as the sum of two eigenspaces: $\Lambda^2 T^*_x\mathbb{R}^4 = \Lambda^+ T^*_x\mathbb{R}^4 \oplus \Lambda^- T^*_x\mathbb{R}^4$, where $\Lambda^+ T^*_x\mathbb{R}^4$ (resp. $\Lambda^- T^*_x\mathbb{R}^4$) is the space of self-dual (resp. anti-self-dual) 2-forms.

(11) This situation is locally very general since any Riemannian surface is conformally flat.
(12) Since $d(*\alpha) = \Delta\varphi\, dx^1\wedge dx^2$.

As for 1-forms on surfaces, we can consider harmonic 2-forms, which are sections F of $\Gamma(\mathbb{R}^4, \Lambda^2 T^*\mathbb{R}^4)$ which satisfy

$$dF = 0 \quad \text{and} \quad d(*F) = 0.$$

This system may be seen as an Euclidean version of Maxwell equations in empty space. Now we can decompose such an harmonic form into its self-dual and anti-self-dual parts: $F = F^+ + F^-$, where(13)

$$F^+ = \frac{1}{2}(F + *F), \quad F^- = \frac{1}{2}(F - *F).$$

And we see that F is harmonic if and only F^+ and F^- are closed. Hence we can reduce the study of harmonic 2-forms to the study of closed self-dual or anti-self-dual 2-forms. In the following we will focus on anti-self-dual 2-forms (this is just a matter of orientation). The key for understanding closed ASD (anti-self-dual) 2-forms is to interpret the ASD condition in terms of complex structures.

6.3. Introducing complex structures on $\mathbb{R}^4$. — For any $2n$-dimensional oriented Euclidean space V, we call a *compatible complex structure* an oriented isometry J of V such that $J^2 = -\mathrm{Id}_V$. The set $\mathcal{J}_{2n}$ of compatible complex structures on a $2n$-dimensional oriented Euclidean space is a homogeneous space ($\mathcal{J}_{2n} \simeq SO(2n)/U(n)$). For $n = 1$, $\mathcal{J}_2$ is reduced to points. For $n = 2$, $\mathcal{J}_4 \simeq S^2 \cup S^2$: this can be seen precisely by using the identification (that we systematically use in the following)

$$\begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{H} \\ (x^1, x^2, x^3, x^4) & \longmapsto & x^1 + ix^2 + jx^3 + kx^4 = x \end{array}$$

and setting $S^2_{\mathbb{H}} := \{u \in \mathrm{Im}\mathbb{H} | \, |u| = 1\}$. Then $\forall u \in S^2_{\mathbb{H}}$ the linear maps $L_u : \mathbb{H} \longrightarrow \mathbb{H}$, $x \longmapsto ux$ and $R_u : \mathbb{H} \longrightarrow \mathbb{H}$, $x \longmapsto xu$ are compatible complex structures and $\mathcal{J}_4 = \{L_u | u \in S^2_{\mathbb{H}}\} \cup \{R_u | u \in S^2_{\mathbb{H}}\} =: \mathcal{J}_L \cup \mathcal{J}_R$. In the following, beside the identification of

(13)Note that F^+ and F^- can also be expressed by using quaternion numbers $\mathbb{H}$. For that purpose we use the identification $\mathbb{R}^4 \longrightarrow \mathbb{H}$, $(x^1, x^2, x^3, x^4) \longmapsto x^1 + ix^2 + jx^3 + kx^4$ and we set

$$dx = dx^1 + idx^2 + jdx^3 + kdx^4 \quad \text{and} \quad d\overline{x} = dx^1 - idx^2 - jdx^3 - kdx^4.$$

Then a basis of $\Lambda^+ T^x_x \mathbb{R}^4$ is made of the three components of

$$dx \wedge d\overline{x} = -2\left[i\left(dx^1 \wedge dx^2 + dx^3 \wedge dx^4\right) + j\left(dx^1 \wedge dx^3 - dx^2 \wedge dx^4\right) + k\left(dx^1 \wedge dx^4 + dx^2 \wedge dx^3\right)\right]$$

and similarly a basis of $\Lambda^- T^x_x \mathbb{R}^4$ is made of the three components of

$$d\overline{x} \wedge dx = 2\left[i\left(dx^1 \wedge dx^2 - dx^3 \wedge dx^4\right) + j\left(dx^1 \wedge dx^3 + dx^2 \wedge dx^4\right) + k\left(dx^1 \wedge dx^4 - dx^2 \wedge dx^3\right)\right]$$

In particular if we denote

$$\begin{array}{l} f^+ := \frac{1}{4}\left[i\left(F_{12} + F_{34}\right) + j\left(F_{13} - F_{24}\right) + k\left(F_{14} + F_{23}\right)\right] \\ f^- := -\frac{1}{4}\left[i\left(F_{12} - F_{34}\right) + j\left(F_{13} + F_{24}\right) + k\left(F_{14} - F_{23}\right)\right] \end{array}$$

then $F^+ = \mathrm{Re}(f^+ dx \wedge d\overline{x})$ and $F^- = \mathrm{Re}(f^- d\overline{x} \wedge dx)$.

$\mathbb{R}^4$ with $\mathbb{H}$, we also sometime use the complex coordinates (X^1, X^2) on $\mathbb{H}$ defined by

$$\begin{array}{ccccc} \mathbb{R}^4 & \longrightarrow & \mathbb{C}^2 & \longrightarrow & \mathbb{H} \\ (x^1, x^2, x^3, x^4) & \longmapsto & (x^1 + ix^2, x^3 + ix^4) = (X^1, X^2) & \longmapsto & X^1 + X^2 j = x \end{array}$$

But we will not privilege the particular complex structure associated with (X^1, X^2). Instead we want to explore the ASD condition from the point of view of all complex structures in $\mathcal{J}_L$. For that purpose consider the complex manifold $\mathbb{T}^* := \mathbb{C}^2 \times (\mathbb{C}^2 \setminus \{0\}) \subset \mathbb{C}^4$ with coordinates $Z = (z^1, z^2, \alpha, \beta)$ and the smooth map

$$\begin{array}{cccc} P: & \mathbb{T}^* & \longrightarrow & \mathbb{H} \\ & (z^1, z^2, \alpha, \beta) & \longmapsto & x = (\alpha + \beta j)^{-1}(z^1 + z^2 j). \end{array}$$

In terms of the coordinates (X^1, X^2), $X^1 + X^2 j = P(Z)$ reads[14]

$$X^1 + X^2 j = (\alpha + \beta j)^{-1}(z^1 + z^2 j) \quad \Longleftrightarrow \quad (X^1, X^2) = \left(\frac{\overline{\alpha} z^1 + \beta \overline{z^2}}{|\alpha|^2 + |\beta|^2}, \frac{-\beta \overline{z^1} + \overline{\alpha} z^2}{|\alpha|^2 + |\beta|^2} \right).$$

Note that for any fixed $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$ the map $P|_{(\alpha,\beta)} : \mathbb{C}^2 \longrightarrow \mathbb{H}$, $(z^1, z^2) \longmapsto P(z^1, z^2, \alpha, \beta)$ is an isomorphism of real vector space, whose inverse is given by the relation

$$z^1 + z^2 j = (\alpha + \beta j)(X^1 + X^2 j) \quad \Longleftrightarrow \quad (z^1, z^2) = (\alpha X^1 - \beta \overline{X^2}, \beta \overline{X^1} + \alpha X^2). \tag{6.60}$$

What is the meaning of the map P? answer: to each (α, β) it corresponds a complex structure $L_{u(\alpha,\beta)} \in \mathcal{J}_L$ and then (z^1, z^2) are complex coordinates on $\mathbb{H}$ for the complex structure $L_{u(\alpha,\beta)}$. It means that (abbreviating $u = u(\alpha, \beta)$), $\forall (\zeta^1, \zeta^2) \in T_{(z^1,z^2)}\mathbb{C}^2$,

$$L_u \circ dP_{(z^1,z^2)|(\alpha,\beta)}(\zeta^1, \zeta^2) = dP_{(z^1,z^2)|(\alpha,\beta)}(i\zeta^1, i\zeta^2).$$

This can be proved by the following: for any fixed (α, β),

$$dz^1 + dz^2 j = (\alpha + \beta j)(dX^1 + dX^2 j)$$

implies that

$$idz^1 + idz^2 j = i(\alpha + \beta j)(dX^1 + dX^2 j) = (\alpha + \beta j) u(\alpha, \beta)(dX^1 + dX^2 j),$$

where

$$u(\alpha, \beta) = (\alpha + \beta j)^{-1} i (\alpha + \beta j) = i \frac{|\alpha|^2 - |\beta|^2 + 2\overline{\alpha}\beta j}{|\alpha|^2 + |\beta|^2}.$$

In particular we see that $u(t\alpha, t\beta) = u(\alpha, \beta)$, $\forall t \in \mathbb{C}^*$, so that u induces a map from $P\mathbb{C}^1$ (with homogeneous coordinates $[\alpha : \beta]$) to $S^2_{\mathbb{H}}$, which is just the stereographic projection.

[14] We use here constantly the basic property that for any complex quantity f, $jf = \overline{f} j$.

Now let us exploit the canonical complex structure on $\mathbb{T}^*$. In general for any complex manifold $\mathcal{M}$, for any point $Z \in \mathcal{M}$ and for any integer $p \leq 2\dim_{\mathbb{C}}\mathcal{M}$ we have the decomposition

$$(\Lambda^p T_Z\mathcal{M})^{\mathbb{C}} = \bigoplus_{q=0}^{p} \Lambda^{(q,p-q)} T_Z\mathcal{M},$$

where for any $\psi \in \Lambda^{(q,p-q)}T_Z\mathcal{M}$, $\forall t \in \mathbb{C}$, $\forall \zeta^1, \cdots, \zeta^p \in T_Z\mathcal{M}$, $\psi(t\zeta^1, \cdots, t\zeta^p) = t^q\overline{t}^{p-q}\psi(\zeta^1, \cdots, \zeta^p)$. In other words if $z^1, \cdots, z^n$ are complex coordinates on $\mathcal{M}$, then any ψ in $\Lambda^{(q,p-q)}T_Z\mathcal{M}$ writes $\sum \psi_{i_1\cdots i_q\overline{i}_{q+1}\cdots\overline{i}_p} dz^{i_1} \wedge \cdots dz^{i_q} \wedge \overline{dz^{i_{q+1}}} \wedge \cdots \wedge \overline{dz^{i_p}}$. Thus for instance any 2-form ψ on $\mathbb{T}^*$ can be splitted according to the decomposition of $\left(\Lambda^2 T\mathbb{T}^*\right)^{\mathbb{C}}$:

$$\psi = \psi^{(2,0)} + \psi^{(1,1)} + \psi^{(0,2)},$$

where, if ψ is real valued, $\psi^{(0,2)} = \overline{\psi^{(2,0)}}$. The key result is:

Theorem 4. — *Let $F \in \Gamma(\mathbb{H}, \Lambda^2 T^*\mathbb{H})$ be a 2-form on $\mathbb{H}$. Then F is ASD if and only if $\left(P^*F\right)^{(0,2)} = 0$.*

Proof. — We write the equation of the graph of P in $\mathbb{T}^* \times \mathbb{H}$: $z^1 + z^2 j = (\alpha + \beta j)(X^1 + X^2 j)$ and derivate it:

$$dz^1 + dz^2 j = (d\alpha + d\beta j)(X^1 + X^2 j) + (\alpha + \beta j)(dX^1 + dX^2 j).$$

This implies (by using $(\alpha + \beta j)^{-1} = (\overline{\alpha} - \beta j)|\alpha + \beta j|^{-2}$)

$$\begin{aligned} dX^1 + dX^2 j &= (\alpha + \beta j)^{-1}\left[dz^1 + dz^2 j - (d\alpha + d\beta j)(X^1 + X^2 j)\right] \\ &= (\overline{\alpha} - \beta j)(\theta^1 + \theta^2 j) \\ &= (\overline{\alpha}\theta^1 + \beta\overline{\theta^2}) + (\overline{\alpha}\theta^2 - \beta\overline{\theta^1})j, \end{aligned} \tag{6.61}$$

where

$$\theta^1 := \frac{dz^1 - X^1 d\alpha + \overline{X^2}d\beta}{|\alpha|^2 + |\beta|^2} = \frac{dz^1}{|\alpha|^2 + |\beta|^2} - \frac{\overline{\alpha}z^1 + \beta\overline{z^2}}{(|\alpha|^2 + |\beta|^2)^2}d\alpha + \frac{-\overline{\beta}z^1 + \alpha\overline{z^2}}{(|\alpha|^2 + |\beta|^2)^2}d\beta,$$

$$\theta^2 := \frac{dz^2 - X^2 d\alpha - \overline{X^1}d\beta}{|\alpha|^2 + |\beta|^2} = \frac{dz^2}{|\alpha|^2 + |\beta|^2} - \frac{-\beta\overline{z^1} + \overline{\alpha}z^2}{(|\alpha|^2 + |\beta|^2)^2}d\alpha - \frac{\alpha\overline{z^1} + \overline{\beta}z^2}{(|\alpha|^2 + |\beta|^2)^2}d\beta.$$

If we pull-back the relation (6.61) by the parametrization $\mathrm{Id}_{\mathbb{T}^*} \times P : (z^1, z^2, \alpha, \beta) \longmapsto (z^1, z^2, \alpha, \beta, X^1 + X^2 j)$ we obtain

$$P^*dX^1 = \overline{\alpha}\theta^1 + \beta\overline{\theta^2}\,, \quad P^*dX^2 = \overline{\alpha}\theta^2 - \beta\overline{\theta^1}$$

and by complex conjugation

$$P^*d\overline{X^1} = \alpha\overline{\theta^1} + \overline{\beta}\theta^2\,, \quad P^*d\overline{X^2} = \alpha\overline{\theta^2} - \overline{\beta}\theta^1.$$

The point here is that θ^1 and θ^2 are $(1,0)$-forms whereas $\overline{\theta^1}$ and $\overline{\theta^2}$ are $(0,1)$-forms. Hence,

$$
\begin{aligned}
\left(P^*dX^1\right)^{(0,1)} &= \beta\overline{\theta^2}, & \left(P^*d\overline{X^1}\right)^{(0,1)} &= \alpha\overline{\theta^1}\\
\left(P^*dX^2\right)^{(0,1)} &= -\beta\overline{\theta^1}, & \left(P^*d\overline{X^2}\right)^{(0,1)} &= \alpha\overline{\theta^2}.
\end{aligned}
\tag{6.62}
$$

We now write F^+ and F^- by using coordinates[15] X^1 and X^2:

$$
\begin{aligned}
F^+ = \; & \tfrac{i}{4}(F_{12}+F_{34}) && (dX^1\wedge d\overline{X^1}+dX^2\wedge d\overline{X^2})\\
&+\tfrac{1}{4}\left[(F_{13}-F_{24})-i(F_{14}+F_{23})\right] && dX^1\wedge dX^2\\
&+\tfrac{1}{4}\left[(F_{13}-F_{24})+i(F_{14}+F_{23})\right] && d\overline{X^1}\wedge d\overline{X^2},\\
F^- = \; & \tfrac{i}{4}(F_{12}-F_{34}) && (dX^1\wedge d\overline{X^1}-dX^2\wedge d\overline{X^2})\\
&+\tfrac{1}{4}\left[(F_{13}+F_{24})+i(F_{14}-F_{23})\right] && dX^1\wedge d\overline{X^2}\\
&+\tfrac{1}{4}\left[(F_{13}+F_{24})-i(F_{14}-F_{23})\right] && d\overline{X^1}\wedge dX^2.
\end{aligned}
$$

And we express $(P^*F^{\pm})^{(0,2)}$: this accounts to compute terms like

$$\left(P^*dX^1\wedge dX^2\right)^{(0,2)} = \left(P^*dX^1\right)^{(0,1)}\wedge\left(P^*dX^2\right)^{(0,1)} = \beta\overline{\theta^2}\wedge(-\beta\overline{\theta^1}) = \beta^2\overline{\theta^1}\wedge\overline{\theta^2}.$$

Proceeding this way we find

$$
\begin{aligned}
(P^*F^+)^{(0,2)} = \; & \tfrac{1}{4}\left[i(F_{12}+F_{34})(-2\alpha\beta)+\left[(F_{13}-F_{24})-i(F_{14}+F_{23}\right]\beta^2\right.\\
&\left.+\left[(F_{13}-F_{24})+i(F_{14}+F_{23})\right]\alpha^2\right]\overline{\theta^1}\wedge\overline{\theta^2}.
\end{aligned}
$$

and $(P^*F^-)^{(0,2)} = 0$. Hence we have $(P^*F)^{(0,2)} = (P^*F^+)^{(0,2)}$ and this quantity vanishes if and only if $F^+ = 0$. □

6.4. The holomorphic twistor function. — Let A be a smooth 1-form on $\mathbb{H}$ such that dA is ASD. Then the preceding result implies

$$(dP^*A)^{(0,2)} = (P^*dA)^{(0,2)} = 0.$$

This means somehow that P^*A is closed with respect to half of the variables, $(\overline{z^1}, \overline{z^2}, \overline{\alpha}, \overline{\beta})$, so that we could think that P^*A is also exact with respect to the same variables. This is indeed the case except that we need not only to take care of the topology of the domain, but also of its shape (polyconvexity assumption). For instance this is true on a complex vector space:

Lemma 6. — *For any smooth complex valued* 1*-form* b *on* $\mathbb{C}^n$ *such that*

$$(db)^{(0,2)} = 0,$$

there exists a 1*-form* a *on* $\mathbb{C}^n$ *such that*

$$(da)^{(0,1)} = b^{(0,1)} \quad \text{or} \quad d''a = b''.$$

[15] Alternatively we could compute $P^*dx\wedge d\overline{x}$ (self-dual part) and $P^*d\overline{x}\wedge dx$ (anti-self-dual part), where $dx = (\overline{\alpha}-\beta j)(\theta^1+\theta^2 j)$ and $d\overline{x} = (\overline{\theta^1}-\theta^2 j)(\alpha+\beta j)$.

Remark 1. — Here we introduce the notations $d''a$ for $(da)^{(0,1)}$ and b'' for $b^{(0,1)}$. Several proofs[16] of this results exist, see for instance [**30**].

We can use this result on any complex hyperplane $H \subset \mathbb{T}^*$. Indeed let $j_H : H \longrightarrow \mathbb{T}^*$ be the inclusion map, then $((P \circ j_h)^* dA)^{(0,2)} = 0$, so that by the preceding lemma there exists a function $f : H \longrightarrow \mathbb{C}$ such that

$$d''f + ((P \circ j_h)^* A)'' = 0.$$

But we can then extend this solution on the cone $\mathbb{C}^*H := \{tZ |\ t \in \mathbb{C}^*, Z \in H\}$ (which fills almost all of $\mathbb{T}^*$) by homogeneity. Indeed first remark that P is complex homogeneous of degree 0: $\forall t \in \mathbb{C}^*$, $\forall Z \in \mathbb{T}^*$, $P(tZ) = P(Z)$. Hence in particular, if $\pi_H : \mathbb{C}^*H \longrightarrow H$ denotes the radial projection onto H, then $P = P \circ j_H \circ \pi_H$ on $\mathbb{C}^*H$, so that[17]

$$(P^*A)'' = (\pi_H^* ((P \circ j_H)^* A))'' = \pi_H^* ((P \circ j_H)^* A)'' = -\pi_H^*(df)'' = -d''(f \circ \pi_H).$$

Hence $f := f \circ \pi_H$ is a solution of

$$f(tZ) = f(Z) \quad \text{and} \quad d''f + (P^*A)'' = 0 \quad \text{on } \mathbb{C}^*H. \tag{6.63}$$

Note that f is not unique, since for any complex homogeneous holomorphic function g on $\mathbb{C}^*H$, $f + g$ is also a solution of (6.63). We use this construction for the hyperplane H_1 of equation $\beta = 1$ and the hyperplane H_2 of equation $\alpha = 1$. Let f_1 be a solution of (6.63) on $\mathbb{C}^*H_1$ (the cone of equation $\beta \neq 0$), and f_2 be a solution of (6.63) on $\mathbb{C}^*H_2$ (the cone of equation $\alpha \neq 0$). We then observe that, on $\mathbb{C}^*H_1 \cap \mathbb{C}^*H_2$,

$$d''f_1 = -(P^*A)'' = d''f_2,$$

so that $d''(f_2 - f_1) = 0$, i.e., $h := f_2 - f_1$ is holomorphic on $\mathbb{C}^*H_1 \cap \mathbb{C}^*H_2$. The complex homogeneous function h is called the **twistor function**: it encodes the 1-form A, up to gauge transformations $A \longmapsto A + dV$.

6.4.1. The reality condition. — The fact that A is a real valued 1-form can be encoded in the construction of h. The complex antilinear map

$$\begin{array}{cccc} \tau : & \mathbb{T}^* & \longrightarrow & \mathbb{T}^* \\ & (z^1, z^2, \alpha, \beta) & \longmapsto & (-\overline{z^2}, \overline{z^1}, -\overline{\beta}, \overline{\alpha}) \end{array}$$

(16) If we assume that b is $\mathcal{C}^1$ and that there exists $s > 1$ and $C > 0$ such that $|z|^s |b(z, \overline{z})| + |z|^{s+1} |db(z, \overline{z})| < C$ then a solution is

$$a(z, \overline{z}) = \int_{\mathbb{C}} \frac{1}{\pi(1 - \zeta)} \sum_{k=1}^{n} b_{\overline{k}}(\zeta z, \overline{\zeta z})\ \overline{z^k}\ \frac{d\overline{\zeta} \wedge d\zeta}{2i}.$$

(17) Here we use the fact that π_h^* and j_H^* commute with the $(0, 1)$-projection because π_h and j_H are complex maps.

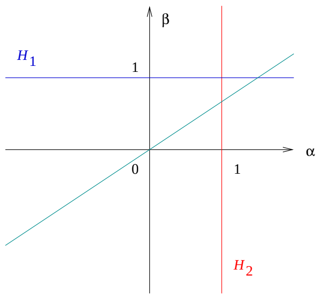

FIGURE 6. The hyperplanes H_1 and H_2

respects[(18)] the fibers of $P : \mathbb{T}^* \longrightarrow \mathbb{H}$, i.e., $P \circ \tau = P$: this can be easily seen if one observes that τ acts on $(z^1 + z^2 j, \alpha + \beta j)$ by left multiplication by j and using $P(z^1+z^2 j, \alpha+\beta j) = (\alpha+\beta j)^{-1}(z^1+z^2 j)$. This has the consequence that if f satisfies $d'' f = (P^* A)''$ then

$$d''\left(\overline{f\circ\tau}\right) = \overline{d'(f\circ\tau)} = \overline{\tau^* d'' f} = \overline{\tau^*(P^*A)''} = \overline{((P\circ\tau)^*A)'} = \overline{(P^*A)'} = (P^*A)'',$$

where we have used the fact that A is real in the last equality. Hence $\overline{f\circ\tau}$ is also a solution of (6.63). Note also that if we apply this to f_1, which is defined on $\mathbb{C}^* H_1$, then $\overline{f_1\circ\tau}$ is defined on $\mathbb{C}^* H_2$, so that we may choose $f_2 = \overline{f_1\circ\tau}$ in the preceding construction. Then the twistor function h satisfies the condition $\overline{h\circ\tau} = -h$.

6.5. An alternative description of the twistor function. — We now translate the previous construction on the original space $\mathbb{H}$ plus a further variable $\lambda \simeq [\lambda : 1] \in P\mathbb{C}$, the complex projective line. Consider the map

$$\begin{array}{cccc} Q: & \mathbb{T}^* & \longrightarrow & \mathbb{H}\times P\mathbb{C} \\ & (z^1,z^2,\alpha,\beta) & \longmapsto & (P(z^1,z^2,\alpha,\beta),[\alpha:\beta]) = ((\alpha+\beta j)^{-1}(z^1+z^2 j),\alpha/\beta). \end{array}$$

The inverse image of $(X^1 + X^2 j, \lambda)$ by Q is the pointed complex line $\{(\alpha X^1 - \beta\overline{X^2}, \alpha X^2 + \beta\overline{X^1}, \alpha,\beta) | (\alpha,\beta) \in \mathbb{C}^2\setminus\{0\}, \alpha = \lambda\beta\}$. Let H be an affine hyperplane of $\mathbb{T}^*$ and $f : \mathbb{C}^* H \longrightarrow \mathbb{C}$ be a complex homogeneous map. Then there exists a unique map $\varphi : Q(\mathbb{C}^* H) \longrightarrow \mathbb{C}$ (note that $Q(\mathbb{C}^* H_1) = \mathbb{H}\times(P\mathbb{C}\setminus\{\infty\})$ and $Q(\mathbb{C}^* H_2) = \mathbb{H}\times(P\mathbb{C}\setminus\{0\}))$ such that

$$\varphi\circ Q = f \quad \text{on } \mathbb{C}^* H.$$

In coordinates:

$$\begin{aligned} \varphi(X^1+X^2 j,\lambda) &= f(\alpha X^1 - \beta\overline{X^2}, \alpha X^2+\beta\overline{X^1},\alpha,\beta), \quad \text{with } \alpha=\lambda\beta \\ &= f(\lambda X^1 - \overline{X^2}, \lambda X^2 + \overline{X^1}, \lambda, 1) \quad \text{if } \lambda\neq\infty. \end{aligned}$$

(18) One also remarks that τ maps the complex structure on $\mathbb{H}$ to the opposite one, i.e., τ maps $u(\alpha,\beta)$ to $-u(\alpha,\beta)$.

We first translate the relation $d''f + (P^*A)'' = 0$ in terms of φ. For that purpose write

$$d\varphi = \frac{\partial\varphi}{\partial X^1}dX^1 + \frac{\partial\varphi}{\partial X^2}dX^2 + \frac{\partial\varphi}{\partial\overline{X^1}}d\overline{X^1} + \frac{\partial\varphi}{\partial\overline{X^2}}d\overline{X^2} + \frac{\partial\varphi}{\partial\lambda}d\lambda + \frac{\partial\varphi}{\partial\overline{\lambda}}d\overline{\lambda}.$$

Hence by using (6.62) (note that $P^*dX^a = Q^*dX^a$)

$$\begin{aligned} d''f = (Q^*d\varphi)'' \quad = \quad & \beta\left(\frac{\partial\varphi}{\partial X^1}\circ Q\right)\overline{\theta^2} - \beta\left(\frac{\partial\varphi}{\partial X^2}\circ Q\right)\overline{\theta^1} \\ & +\alpha\left(\frac{\partial\varphi}{\partial\overline{X^1}}\circ Q\right)\overline{\theta^1} + \alpha\left(\frac{\partial\varphi}{\partial\overline{X^2}}\circ Q\right)\overline{\theta^2} \\ & +\left(\frac{\partial\varphi}{\partial\overline{\lambda}}\circ Q\right)\frac{\overline{\beta}d\overline{\alpha} - \overline{\alpha}d\overline{\beta}}{\overline{\beta}^2}. \end{aligned}$$

Similarly we write $A = A_1dX^1 + A_2dX^2 + A_{\overline{1}}d\overline{X^1} + A_{\overline{2}}d\overline{X^2}$, then the preceding quantity should be opposite to

$$(P^*A)'' = \beta(A_1\circ Q)\overline{\theta^2} - \beta(A_2\circ Q)\overline{\theta^1} + \alpha(A_{\overline{1}}\circ Q)\overline{\theta^1} + \alpha(A_{\overline{2}}\circ Q)\overline{\theta^2}.$$

Hence using the fact that $(\overline{\theta^1},\overline{\theta^2},\overline{\alpha},\overline{\beta})$ is a basis of $T_Z^{*,(0,1)}\mathbb{T}^*$,

$$\left\{\begin{array}{rcl} \alpha\frac{\partial\varphi}{\partial\overline{X^1}}\circ Q - \beta\frac{\partial\varphi}{\partial X^2}\circ Q & + & \alpha A_{\overline{1}}\circ Q - \beta A_2\circ Q \quad = \quad 0 \\ \alpha\frac{\partial\varphi}{\partial\overline{X^2}}\circ Q + \beta\frac{\partial\varphi}{\partial X^1}\circ Q & + & \alpha A_{\overline{2}}\circ Q + \beta A_1\circ Q \quad = \quad 0 \\ \frac{\partial\varphi}{\partial\overline{\lambda}}\circ Q & & \quad = \quad 0. \end{array}\right.$$

Thus dividing by β

$$\left\{\begin{array}{rcl} \lambda\frac{\partial\varphi}{\partial\overline{X^1}} - \frac{\partial\varphi}{\partial X^2} & + & \lambda A_{\overline{1}} - A_2 \quad = \quad 0 \\ \lambda\frac{\partial\varphi}{\partial\overline{X^2}} + \frac{\partial\varphi}{\partial X^1} & + & \lambda A_{\overline{2}} + A_1 \quad = \quad 0 \\ \frac{\partial\varphi}{\partial\overline{\lambda}} & & \quad = \quad 0. \end{array}\right. \tag{6.64}$$

We observe in particular that, for any fixed (X^1, X^2), φ is holomorphic in $\lambda\in\mathbb{C}P$. We apply this to $\varphi_1 : \mathbb{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ and $\varphi_2 : \mathbb{H}\times(\mathbb{C}^*\cup\{\infty\}) \longrightarrow \mathbb{C}$ such that $\varphi_1\circ Q = f_1$ and $\varphi_2\circ Q = f_2$. Note also that the reality condition $f_2 = \overline{f_1\circ\tau}$ implies that

$$\forall x\in\mathbb{H}, \forall\lambda\in\mathbb{C}^* \quad \varphi_2(x,\lambda) = \overline{\varphi_1(x, -\overline{\lambda}^{-1})}.$$

Hence φ_1 and φ_2 can be expanded as

$$\varphi_1(x,\lambda) = \sum_{n=0}^{\infty} a_n(x)\lambda^n \quad \text{and} \quad \varphi_2(x,\lambda) = \sum_{n=0}^{\infty}(-1)^n\overline{a_n(x)}\lambda^{-n}.$$

Now let $\psi:\mathbb{H}\times\mathbb{C}^*\longrightarrow\mathbb{C}$ such that $\psi\circ Q=h$, i.e., $\psi=\varphi_2-\varphi_1$ on $\mathbb{H}\times\mathbb{C}^*$: it is the twistor function in the variables (x,λ). It has the expansion

$$\psi(x,\lambda)=\sum_{n=0}^{\infty}(-1)^n\overline{a_n(x)}\lambda^{-n}-\sum_{n=0}^{\infty}a_n(x)\lambda^n.$$

Hence we can extract φ_1 and φ_2 by splitting the Laurent expansion of ψ into respectively nonnegative and nonpositive powers of λ. The only ambiguity is that the λ^0 coefficient of ψ gives us only $\overline{a_0(x)}-a_0(x)=-2i\mathrm{Im}(a_0(x))$ and we cannot recover the real part of a_0. For this reason we will only be able to recover A modulo an exact form, as we shall see later. This situation is actually characteristic of gauge theories.

6.5.1. How to recover A from its twistor function. — Assume now that we are given a holomorphic complex homogeneous function h on $\mathbb{C}^*H_1\cap\mathbb{C}^*H_2\subset\mathbb{T}^*$ which satisfies $\overline{h\circ\tau}=-h$. Let ψ be defined on $\mathbb{H}\times\mathbb{C}^*$ by $\psi\circ Q=h$. By computations similar to the previous one on φ_1 and φ_2, the condition $d''h=0$ translates to

$$\left(\lambda\frac{\partial}{\partial\overline{X^1}}-\frac{\partial}{\partial X^2}\right)\psi(x,\lambda)=\left(\lambda\frac{\partial}{\partial\overline{X^2}}+\frac{\partial}{\partial X^1}\right)\psi(x,\lambda)=\frac{\partial}{\partial\overline{\lambda}}\psi(x,\lambda)=0.\tag{6.65}$$

Hence in particular

$$\psi(x,\lambda)=-\sum_{n\in\mathbb{Z}}a_n(x)\lambda^n\quad\text{and}\quad\overline{\psi(x,-\overline{\lambda}^{-1})}+\psi(x,\lambda)=0.$$

(Note that this implies that a_0 is imaginary). Let us choose an arbitrary smooth function $g:\mathbb{H}\longrightarrow\mathbb{R}$ and let

$$\varphi_1(x,\lambda):=g(x)+\frac{1}{2}a_0(x)+\sum_{n=1}^{\infty}a_n(x)\lambda^n$$

and

$$\varphi_2(x,\lambda):=\overline{\varphi_2(x,-\overline{\lambda}^{-1})}=g(x)-\frac{1}{2}a_0(x)+\sum_{n=1}^{\infty}(-1)^n\overline{a_n(x)}\lambda^{-n}.$$

Then $\psi=\varphi_2-\varphi_1$. By substitution of $\psi=\varphi_2-\varphi_1$ in (6.65) we obtain that

$$\left(\lambda\frac{\partial}{\partial\overline{X^1}}-\frac{\partial}{\partial X^2}\right)\varphi_1(x,\lambda)=\left(\lambda\frac{\partial}{\partial\overline{X^1}}-\frac{\partial}{\partial X^2}\right)\varphi_2(x,\lambda),\tag{6.66}$$

$$\left(\lambda\frac{\partial}{\partial\overline{X^2}}+\frac{\partial}{\partial X^1}\right)\varphi_1(x,\lambda)=\left(\lambda\frac{\partial}{\partial\overline{X^2}}+\frac{\partial}{\partial X^1}\right)\varphi_2(x,\lambda).\tag{6.67}$$

The left hand side of (6.66) has the expansion $\sum_{n=0}^{\infty}(\cdot)_n\lambda^n$, whereas the right hand side of (6.66) has the expansion $\sum_{n=-\infty}^{1}(\cdot)_n\lambda^n$. Hence both terms should be of the form

$$\left(\lambda\frac{\partial}{\partial\overline{X^1}}-\frac{\partial}{\partial X^2}\right)\varphi_1(x,\lambda)=\left(\lambda\frac{\partial}{\partial\overline{X^1}}-\frac{\partial}{\partial X^2}\right)\varphi_2(x,\lambda)=-\lambda A_{\overline{1}}(x)+A_2(x).\tag{6.68}$$

A similar reasoning with (6.67) shows that

$$(6.69)\quad \left(\lambda\frac{\partial}{\partial\overline{X^2}}+\frac{\partial}{\partial X^1}\right)\varphi_1(x,\lambda)=\left(\lambda\frac{\partial}{\partial\overline{X^2}}+\frac{\partial}{\partial X^1}\right)\varphi_2(x,\lambda)=-\lambda A_{\overline{2}}(x)-A_1(x).$$

We hence construct the 1-form $A=A_1dX^1+A_2dX^2+A_{\overline{1}}d\overline{X^1}+A_{\overline{2}}d\overline{X^2}$ and we can check that it has the desired properties:

- First A is real valued: for instance the relation involving φ_2 in (6.68) implies that $A_{\overline{1}}=-\frac{\partial}{\partial\overline{X^1}}(g-\frac{1}{2}a_0)$, whereas the relation involving φ_1 in (6.69) implies that $A_1=-\frac{\partial}{\partial X^1}(g+\frac{1}{2}a_0)$ and hence $A_{\overline{1}}=\overline{A_1}$, since a_0 is imaginary
- Second if we had choosed another value for g, say $g+\tilde{g}$, this would change A into $A-d\tilde{g}$
- Lastly dA is ASD precisely because (6.68) and (6.69) mean that $(P^*A)''=-d''(\varphi_1\circ P)=-d''(\varphi_2\circ P)$ and thus that $(P^*dA)^{(0,2)}=0$.

6.6. The projective twistor space. — The reader certainly remarked that functions used on $\mathbb{T}^*$ were complex homogeneous of degree zero. Hence they can be alternatively described by functions on open subsets of the complex projective space

$$P\mathbb{T}:=\{[z^1:z^2:\alpha:\beta]|\ (z^1,z^2,\alpha,\beta)\in\mathbb{C}^4\setminus\{0\}\ \}\simeq P\mathbb{C}^3.$$

This space is called the **projective twistor space** (it is the projective version of the twistor space $\mathbb{T}\simeq\mathbb{C}^4$). By the canonical projection $\pi:\mathbb{T}\longrightarrow P\mathbb{T}$ the image of the cone $\mathbb{T}^*$ is $P\mathbb{T}\setminus P\mathbb{C}$, where here $P\mathbb{C}=\{[z^1:z^2:0:0]\}$. And the map P induces

$$\begin{array}{cccc} P: & P\mathbb{T}\setminus P\mathbb{C} & \longrightarrow & \mathbb{H}\\ & [z^1:z^2:\alpha:\beta] & \longmapsto & (\alpha+\beta j)^{-1}(z^1+z^2j).\end{array}$$

We remark also that this map can be extended to $P\mathbb{T}$: we obtain

$$\begin{array}{cccc} P: & P\mathbb{T} & \longrightarrow & P\mathbb{H}\\ & [z^1:z^2:\alpha:\beta] & \longmapsto & [z^1+z^2j:\alpha+\beta j],\end{array}$$

a map onto the projective quaternionic line $P\mathbb{H}\simeq S^4$, which is the compactification of $\mathbb{R}^4\simeq\mathbb{H}$. Moreover τ induces an involution of $P\mathbb{T}$ without fixed points. It has however fixed complex projective lines which are exactly the fibers of the fibration $P\mathbb{T}\longrightarrow P\mathbb{H}$.

The homogeneous complex functions f_1, f_2, h on $\mathbb{T}^*$ correspond to complex functions on respectively $U_1:=\{[z^1:z^2:\alpha:1]\}$, $U_2:=\{[z^1:z^2:1:\beta]\}$ and $U_{12}:=U_1\cap U_2$. So we can reformulate the previous result by saying that we have a one to one correspondence between:

- 1-forms A on $\mathbb{H}$ such that dA is ASD, up to exact 1-forms, i.e., the cohomology group corresponding to the sequence:

$$\{\text{functions on }\mathbb{H}\}\xrightarrow{d}\{\text{1-forms on }\mathbb{H}\}\xrightarrow{d}\{\text{ASD 2-forms on }\mathbb{H}\}.$$

– holomorphic functions $h : U_{12} \longrightarrow \mathbb{C}$ up to the addition of restrictions on U_{12} of holomorphic functions $f_1 : U_1 \longrightarrow \mathbb{C}$ and $f_2 : U_2 \longrightarrow \mathbb{C}$ (with the reality condition $\overline{h \circ \tau} = -h$).

The latter data can be reformulated in terms of sheaf theory. We do not go into details and refer to [**19**] for an introduction to this topic and to [**34**] for more details. Roughly the idea is to think the set of holomorphic functions on open subsets of $P\mathbb{T}$ as a collection of commutative groups $\mathcal{O}|_V$, where V runs over all possible open subsets of $P\mathbb{T}$: each $\mathcal{O}|_V$ is just the set of holomorphic functions, called *sections*, on V. These groups are related together by restriction morphisms $\rho_{VV'} : \mathcal{O}|_V \longrightarrow \mathcal{O}|_{V'}$ (each time we have $V' \subset V$) satisfying some natural axioms. Hence we see that (ignoring reality conditions) the datas on $P\mathbb{T}$ consists in the quotient

$$\mathcal{O}|_{U_{12}} / \left(\rho_{U_1U_{12}}\left(\mathcal{O}|_{U_1}\right) + \rho_{U_2U_{12}}\left(\mathcal{O}|_{U_2}\right)\right).$$

One can then prove that this set is the Čech cohomology of sheaves group $H^1(P\mathbb{T} \setminus P\mathbb{C}, \mathcal{O})$.

The previous construction admits generalizations if we replace the sheaf $\mathcal{O}$ by other sheaves, namely the sheaves of sections of complex line bundles over $P\mathbb{T}$. Interesting examples of bundles over $P\mathbb{T}$ are the bundles L^m, for $m \in \mathbb{Z}$: L^0 is the trivial bundle $P\mathbb{T} \times \mathbb{C}$, L^1 is the canonical bundle(19), for which the fiber at each point $[z^1 : z^2 : \alpha : \beta]$ is the complex line in $\mathbb{C}^4$ spanned by $(z^1, z^2, \alpha, \beta)$, and for $m \geq 1$, L^m is the m-th tensorial product $L \otimes \cdots \otimes L$; lastly for $m < 0$ L^{-m} is the dual bundle of L^m. The sheaf of holomorphic sections of L^m is denoted by $\mathcal{O}(-m)$ because its sections can be identified with complex homogeneous functions on open cones in $\mathbb{T}$ of degree $-m$. An example is for $m = 2$: then one can show that the cohomology group $H^1(P\mathbb{T}^*, \mathcal{O}(-2))$ corresponds to harmonic functions on $\mathbb{H} \simeq \mathbb{R}^4$. This results in the following representation formula: every harmonic function $\varphi : \mathbb{H} \longrightarrow \mathbb{C}$ can be written

$$\varphi(X^1 + X^2 j) = \int_{S^1} f(\lambda X^1 - \overline{X^2}, \lambda X^2 + \overline{X^1}, \lambda) d\lambda,$$

where f is holomorphic in the three variables $(z^1, z^2, \lambda) \in \mathbb{C}^2 \times \mathbb{C}^*$ and S^1 is a path around 0 in $\mathbb{C}^*$. This formula was proved by H. Bateman in 1904 [**6**] (previously an

(19)The restriction of L on $P\mathbb{T} \setminus P\mathbb{C}$ can be described as follows: $P\mathbb{T} \setminus P\mathbb{C}$ is covered by the two open subsets U_1 (for which $\beta \neq 0$) and U_1 (for which $\alpha \neq 0$), with local charts $g_a : U_a \longrightarrow \mathbb{C}^3$ (for $a = 1, 2$) given by $g_1([z^1 : z^2 : \alpha : \beta]) = (\frac{z^1}{\beta}, \frac{z^2}{\beta}, \frac{\alpha}{\beta})$ and $g_1([z^1 : z^2 : \alpha : \beta]) = (\frac{z^1}{\alpha}, \frac{z^2}{\alpha}, \frac{\beta}{\alpha})$. On U_1 we have the canonically defined section of L: $\sigma_1([z^1 : z^2 : \alpha : \beta]) = (\frac{z^1}{\beta}, \frac{z^2}{\beta}, \frac{\alpha}{\beta}, 1) \in \mathbb{C}^4$ and on U_2: $\sigma_2([z^1 : z^2 : \alpha : \beta]) = (\frac{z^1}{\alpha}, \frac{z^2}{\alpha}, 1, \frac{\beta}{\alpha}) \in \mathbb{C}^4$. These sections allows us to trivialize the inverse images by the canonical fibration $\pi : L \longrightarrow P\mathbb{T}$ of U_1 and U_2 by $Pg_a : \pi^* U_a \longrightarrow \mathbb{C}^3 \times \mathbb{C}$ such that $Pg_a([Z], Z) = (g_a(Z), Z/\sigma_a(Z))$, where $Z/\sigma_a(Z)$ is the complex number k_a such that $Z = k_a \sigma_a(Z)$. Then on $Pg_1(U_{12})$ we have the transition map $\varphi = Pg_2 \circ Pg_1^{-1}$ given by $\varphi((\zeta^1, \zeta^2, t), k) = ((\frac{\zeta^1}{t}, \frac{\zeta^2}{t}, \frac{1}{t}), tk)$. For L^m the transition function becomes $\varphi((\zeta^1, \zeta^2, t), k) = ((\frac{\zeta^1}{t}, \frac{\zeta^2}{t}, \frac{1}{t}), t^m k)$.

analogous formula in dimension 3 was obtained by E.T. Whitakker [**35**] in 1902) and was rediscovered by R. Penrose [**26**] in 1969. Here f is not unique but represents a unique cohomology class in $H^1(P\mathbb{T}^*, \mathcal{O}(-2))$. Note that this formula may be proven by hand by checking that it works for any harmonic homogeneous polynomial φ on $\mathbb{R}^4$ by using some complex homogeneous function f.

All that is a part of a whole theory (named *twistor theory*) developped by R. Penrose and its collaborators. An important geometrical construction is the Grassmannian manifold $\mathbb{M}$ of complex projective lines in $P\mathbb{T}$ (or equivalentely of complex planes in $\mathbb{T}$): it is a 4-dimensional complex manifold which can be embedded in a natural way in $P\mathbb{C}^5$ (Plücker embedding), its image being the Klein quadric. Now $\mathbb{M}$ is just the complexification of $S^4 \simeq P\mathbb{H}$ and the map $P : P\mathbb{T} \longrightarrow P\mathbb{H}$ can be interpreted in this context. But $\mathbb{M}$ is also the complexification of the compactification of the Minkowski space-time and an analogous theory exists for relativistic equations. We refer to [**4**], [**19**], [**34**] for more details.

6.7. Ward theorem. — The construction of a twistor function for 1-forms A on $\mathbb{R}^4$ such that dA is ASD has a beautiful generalization for connections. Consider for instance a complex Hermitian vector bundle E over $\mathbb{H}$ (or on an open ball of $\mathbb{H}$). By choosing a global section of E we can identify (*trivialization*) it with the product $\mathbb{H} \times \mathbb{C}^k$, where k is the dimension of the fiber and $\mathbb{C}^k$ has the standard Hermitian metric $(\cdot,\cdot)$. A **connection** ∇ on this bundle is an object which allows us to make sense of the derivative of a section φ of E with respect to some vector $\xi \in T_x\mathbb{H}$ by $\nabla_\xi\varphi = d_\xi\varphi + A(\xi)\varphi$, where $d_\xi\varphi = \xi^j \frac{\partial\varphi}{\partial x^j}$ and A is a 1-form with coefficients in the Lie algebra $\mathfrak{u}(k)$ of $U(k)$ (this then means that $(\nabla_\xi\varphi,\psi) + (\varphi,\nabla_\xi\psi) = d_\xi(\varphi,\psi)$). Basic properties of connections follow:

- for any section φ of E, if ξ and ζ are commuting vector fields on $\mathbb{H}$, i.e., if $[\xi,\zeta] = 0$, then
$$[\nabla_\xi, \nabla_\zeta] := \nabla_\xi(\nabla_\zeta\varphi) - \nabla_\zeta(\nabla_\xi\varphi) = F_\nabla(\xi,\zeta)\varphi,$$
i.e., the right hand side is the product of the *curvature* $F_\nabla(\xi,\zeta)$ by the value of φ (in other words there are no derivatives of φ). The expression of the curvature is
$$F_\nabla(\xi,\zeta) = d_\xi(A(\zeta)) - d_\zeta(A(\xi)) + [A(\xi), A(\zeta)].$$
In particular F_∇ is a 2-form with coefficients in $\mathfrak{u}(k)$. If $A = \sum_{\mu=1}^4 A_\mu dx^\mu$, then $F = \sum_{1\leq\mu,\nu\leq 4} F_{\mu\nu}dx^\mu \wedge dx^\nu$ with $F_{\mu\nu} := \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu]$.
- The curvature F_∇ satisfies the *Bianchi identity* $\nabla F = 0$, or equivalentely
$$\forall \mu,\nu,\lambda, \quad \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} + [A_\mu, F_{\nu\lambda}] + [A_\lambda, F_{\mu\nu}] + [A_\nu, F_{\lambda\mu}] = 0,$$
which generalizes the identity $dd\varphi = 0$ for forms.

– if we had chosen another trivialization of the bundle E, then the connection ∇ would have another expression, still of the form $\nabla = d + \widetilde{A}_\mu$, but with

$$\widetilde{A}_\mu = g^{-1}A_\mu g + g^{-1}dg,$$

where g is a smooth map into $U(k)$ (a *gauge transformation*). However the expression of the curvature in this trivialization is $\widetilde{F}_{\mu\nu} = g^{-1}F_{\mu\nu}g$.

Now by using the Hermitian form $\langle A, B\rangle := tr(A^\dagger B)$ on $\mathfrak{u}(k)$ we define the Yang–Mills functional on the set of connections

$$\mathcal{YM}[A] := -\frac{1}{4}\int_{\mathbb{H}} |F_\nabla|^2 d^4x.$$

Critical points of $\mathcal{YM}$ satisfy the Yang–Mills equations

$$\nabla(*F) = 0 \quad \Longleftrightarrow \quad \forall\mu, \quad \sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x^\nu} + [A_\nu, F_{\mu\nu}] = 0. \tag{6.70}$$

Note that these equations are in general non linear because of the commutator $[A_\nu, F_{\mu\nu}]$, unless the group is Abelian, i.e., for $U(1)$. In this case we precisely recover the Maxwell equations. Also the Yang–Mills functional and its Euler–Lagrange equations are invariant by gauge transformations.

We observe that if the curvature of ∇ is anti-self-dual, i.e., if $*F = -F$, then the Bianchi identity immediately implies that ∇ is a solution of the Yang–Mills system of equations. Hence a first order condition on ∇ (that F_∇ is ASD) implies the second order equation (6.70). Similarly connections with a self-dual curvature 2-form are also Yang–Mills connections. We now have the following.

***Theorem 5* (R.S. Ward, [33]).** — *The gauge classes of connections on $\mathbb{H}$ (or on an open ball in $\mathbb{H}$) for the gauge group $U(k)$ whose curvature is ASD are in one to one correspondence with holomorphic complex vector bundles on $P\mathbb{T} \setminus P\mathbb{C}$ (or in some open subset of $P\mathbb{T}$).*

***Remark 2*.** — The preceding statement is relatively vague, but we shall precise it at the end of the proof. Also the result holds for connections on $S^4 \simeq P\mathbb{H}$: this case is more interesting than in the linear case because there are nontrivial ASD Yang–Mills fields on S^4, called *instantons*, see [**4**] (whereas there are no nontrivial ASD Maxwell fields on S^4).

Sketch of the proof. — The idea of the proof is very similar to the preceding construction for 1-forms with an ASD differential. Let $\nabla = d + A$ be a connection such that F_∇ is ASD. Consider the pull-back $P^*\nabla$ of ∇ by $P : \mathbb{T}^* \longrightarrow \mathbb{H}$ which acts on the pull-back bundle P^*E. Then the curvature of $P^*\nabla$ is P^*F_∇, so it satisfies $(P^*F_\nabla)'' = 0$. This implies that the over-determined system

$$(P^*\nabla)''f = d''f + (P^*A)''f = 0 \tag{6.71}$$

has nontrivial solutions. This step is however harder than in the linear case: we may either assume that the connection is analytic and use a result of A. Weil, as in [**33**], or use the Newlander–Nirenberg theorem [**25**] valid locally for $\mathcal{C}^{2n}$ connections forms[20], and then glue together the local solutions by using results in [**16**] or [**5**] as in [**5**]. Hence on U_1 and U_2, which are respectively the images by the canonical fibration $\pi : \mathbb{T} \longrightarrow P\mathbb{T}$ of the hyperplanes H_1 and H_2, we can construct respectively the maps f_1 and f_2 into $GL(k,\mathbb{C})$, which are solutions of (6.71). We can moreover impose the reality condition that $f_2 = \left((f_1\circ\tau)^\dagger\right)^{-1}$. Then $h := (f_2)^{-1}f_1$ is holomorphic on $U_{12} = U_1 \cap U_2$: denoting $A \simeq P^*A$,

$$d''h = -(f_2)^{-1}(d''f_2)(f_2)^{-1}f_1 + (f_2)^{-1}(d''f_1) = (f_2)^{-1}Af_1 - (f_2)^{-1}Af_1 = 0.$$

So we obtain holomorphic datas, a twistor function. Again f_1 and f_2 (and hence h) are not uniquely defined but only up to right multiplication by holomorphic maps on respectively U_1 and U_2 into $GL(k,\mathbb{C})$. (And the reality condition writes $(h\circ\tau)^\dagger = h$.) So the right interpretation is that h is a transition function defining a holomorphic rank k complex vector bundle over $P\mathbb{T}\setminus P\mathbb{C}$. But because of the definition $h := (f_2)^{-1}f_1$ the restriction of this bundle on any complex projective line of the type $P^{-1}(x)$, for $x\in\mathbb{H}$, is trivial.

We can also represent f_1 and f_2 as respectively functions φ_1 and φ_2 of $x\in\mathbb{H}$ and of $\lambda\in P\mathbb{C}$ by setting $\varphi_a\circ Q = f_a$. Then relation (6.71) reads

$$\left\{\begin{array}{rcl} \left(\lambda\dfrac{\partial}{\partial\overline{X^1}} - \dfrac{\partial}{\partial X^2}\right)\varphi_a + \left(\lambda A_{\overline{1}} - A_2\right)\varphi_a & = & 0 \\ \left(\lambda\dfrac{\partial}{\partial\overline{X^2}} + \dfrac{\partial}{\partial X^1}\right)\varphi_a + \left(\lambda A_{\overline{2}} + A_1\right)\varphi_a & = & 0 \\ \dfrac{\partial\varphi_a}{\partial\overline{\lambda}} & = & 0. \end{array}\right.$$

and φ_1 and φ_2 have the same expansions in powers of λ as before: φ_1 involves non-negative powers of λ whereas φ_2 involves nonpositive powers of λ.

The construction of A starting from the twistor function h follows also the same lines as for the linear case. We define ψ such that $\psi\circ Q = h$, then we obtain that ψ satisfies equation (6.65). Now in order to deduce φ_1 and φ_2 from ψ we need a more sophisticated argument than just a Fourier splitting, namely the solution of the following Riemann–Hilbert problem: for each fixed $x\in\mathbb{H}$, find $[\lambda\mapsto\varphi_1(x,\lambda)]\in L^+GL(k,\mathbb{C})$ and $[\lambda\mapsto\varphi_2(x,\lambda)]\in L^-GL(k,\mathbb{C})$ such that

$$\psi(x,\lambda) = \varphi_2(x,\lambda)^{-1}\varphi_1(x,\lambda).$$

The fact that this problem has a solution precisely means that the restriction of the holomorphic bundle to $P^{-1}(x)$ is trivial. This implies in particular that

[20] Here $n = \dim P\mathbb{T} = 3$.

$d\psi = (\varphi_2)^{-1}\left(-d\varphi_2(\varphi_2)^{-1} + d\varphi_1(\varphi_1)^{-1}\right)\varphi_1$. Hence by substituting this identity in (6.65) we obtain

$$\left[\left(\lambda\frac{\partial}{\partial\overline{X^1}} - \frac{\partial}{\partial X^2}\right)\varphi_1\right](\varphi_1)^{-1} = \left[\left(\lambda\frac{\partial}{\partial\overline{X^1}} - \frac{\partial}{\partial X^2}\right)\varphi_2\right](\varphi_2)^{-1}, \tag{6.72}$$

$$\left[\left(\lambda\frac{\partial}{\partial\overline{X^2}} + \frac{\partial}{\partial X^1}\right)\varphi_1\right](\varphi_1)^{-1} = \left[\left(\lambda\frac{\partial}{\partial\overline{X^2}} + \frac{\partial}{\partial X^1}\right)\varphi_2\right](\varphi_2)^{-1}. \tag{6.73}$$

A similar reasoning as for the linear case then allows us to reconstruct the connection by identifying both sides of (6.72) with $-\lambda A_{\overline{1}} + A_2$ and both sides of (6.73) with $-\lambda A_{\overline{2}} - A_1$. Again the ambiguity in the Riemann–Hilbert decomposition corresponds to gauge invariance, the reality conditions ensures us that A is a $\mathfrak{u}(k)$-valued 1-form and the ASD condition on the curvature of $d + A$ follows by the construction. □

So we can now complete the statement of Theorem 5: a complex vector bundle which corresponds to a connection with an ASD curvature has the further properties that its restriction to any projective line of the form $P^{-1}(x)$, where $x \in \mathbb{H}$, is trivial and that the transition function h satisfies the reality condition $h = (h \circ \tau)^\dagger$.

As a conclusion note that anti-self-dual connections can be considered on the 4-dimensional *Minkowski* space (i.e., $\mathbb{R}^4$ with a metric of signature $+ - --$) and on the *ultrahyperbolic* space (i.e., $\mathbb{R}^4$ with a metric of signature $+ + --$). Real ASD connections exist on the Euclidean and the ultrahyperbolic spaces whereas ASD connections on the Minkowski space must be complex. This is due to the fact that the Hodge operator has eigenvalues ± 1 on the Euclidean and the ultrahyperbolic spaces, whereas its eigenvalues are $\pm i$ on the Minkowski space. Moreover given a subgroup H of the conformal group acting on the space, one can look at ASD connections which are invariant under the action of this subgroup. If the quotient space $\mathbb{R}^4/H$ is a submanifold (which is the case if, for instance, H is composed of translations) the coefficients of the connection descend to fields on $\mathbb{R}^4/H$ which, in good cases, are solutions of some interesting completely integrable system. This process, called a *reduction*, was studied extensively by L. Mason and N. Woodhouse [**22**]. On the ultrahyperbolic space and if H is a 2-dimensional group spanned by two commuting translation vector fields X and Y, it gives particularly interesting examples whose nature depends on the signature of the metric on the plane spanned by (X, Y). If this signature is $++$, then one recovers 2-dimensional harmonic maps to a Lie group, if however this metric is degenerate of rank 1, then one can obtain the KdV equation or the non-linear Schrödinger equation by setting the gauge group of the ASD connection to be $SL(2, \mathbb{C})$ (and also by choosing a suitable gauge).

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