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Compact Simple Lie Groups and Their C-, S-, and E-Transforms

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Abstract. New continuous group transforms, together with their discretization over a lattice of any density and admissible symmetry, are defined for a general compact simple Lie groups of rank $2 \leq n < \infty$. Rank 1 transforms are known. Rank 2 exposition of C- and S-transforms is in the literature. The E-transforms appear here for the first time.

Key words: compact simple Lie groups; C-, S-, and E-transforms; discretization; fundamental region; Weyl group; weight lattice

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1 Introduction

In the talk [1] three types of transforms, called C-, S-, and E-transforms, are introduced for each compact simple Lie group G. Number of variables in the transforms equals the rank of G. Generalization from simple to semisimple Lie groups is straightforward but we disregard it here.

The transforms are defined on a finite region F of a real n-dimensional Euclidean space $\mathbb{R}^n$, more precisely F is a simplex often called the fundamental region of G. The transforms are introduced simultaneously as continuous ones on F, and also as discrete transforms on a lattice grid $F_M \subset F$ of any density fixed by a positive integer M. Respectively the three transforms are multidimensional generalizations of the cosine transform, the sine transform and the common Fourier transform using exponential functions.

A practical motivation for studying the group transforms introduced here, comes from the abundance of multidimensional digital data which need to be processed according to diverse criteria. In particular, continuous extension of discrete transforms, using C-functions, appears to be rather simple and advantageous way to interpolate digital data. It was noticed as a property of C-transforms based on $G = SU(2)$ and $SU(2) \times SU(2)$, in [2], although a crucial step toward that was made already in [3, 4]. There is every reason to expect that the same is true about the S- and E-transforms. Moreover, the general approach described here allows one to treat not only the data in 2 or 3 dimensions, but in any dimension $n < \infty$ using any of the semisimple Lie groups of rank n.

General discrete C-transforms are the content of [5]. Their applications are in [6, 7, 8, 9]. In [10, 11] one finds an explicit description of the continuous and discrete C-transforms for the four semisimple Lie groups of rank 2. For other applications see [2, 12] and [13, 14, 15]. Let us also point out a forthcoming review of a C-functions [16].

The S-transform for rank 2 groups are the content of [17].

The E-transforms are not found in the literature sofar [18].

2 General discrete and continuous transforms in n dimensions

There is an underlying compact semisimple Lie group G implied in the considerations here. Let Φ denote the new expansion functions of either type, C or S, or E. Within each family the functions are orthogonal when integrated over F. The following relations are called the continuous transform and its inversion:

$$f(x) = \sum_{\lambda} f_\lambda \Phi_\lambda(x), \qquad f_\lambda = \int_F f(x)\overline{\Phi_\lambda(x)}\,dx \tag{1}$$

assuming a suitable normalization of the expansion functions. Here $x \in \mathbb{R}^n$ stands for n continuous variables, $\lambda \in P^+$ is a point of an n dimensional lattice P, which is taken to be in the 'positive chamber' P^+ of P whenever it is convenient. The finite region $F \subset \mathbb{R}^n$ is the fundamental region of the appropriate Weyl group for C- and S-transform, and it is a pair of adjacent copies of F in case of the E-transform. Overbar denotes complex conjugation.

Note that Φ in (1) could be the irreducible character of G. Practical advantage of any of the three families of functions here is in that they are much simpler than the irreducible characters.

Discretization of the transform (1) and its inversion, namely

$$f(x_k) = \sum_{\lambda \in S_M} f_\lambda \Phi_\lambda(x_k), \qquad f_\lambda = \sum_{x_k \in F_M} f(x_k)\overline{\Phi_\lambda(x_k)}, \tag{2}$$

require two modifications of the continuous case. Firstly the continuous variables x need to be replaced by a suitable grid F_M of discrete points in F. Secondly, over the finite set F_M of points only a finite number of functions Φ_λ, given by $\lambda \in S_M$, can be pairwise orthogonal and thus participate in the expansion (2). Density of the points in F_M is fixed by a positive integer M.

In order to define the transform (1), one needs to choose the group G (for simplicity of formulation here, G is supposed to be simple), and to define the appropriate set of orthogonal and distinct expansion functions.

In order to define the discrete transform (2), one needs as well to choose the sets F_M and S_M. For a fixed value of M, the set F_M is unique, while S_M is not. However there is always a unique lowest set S_M; we assume that one to be always chosen.

3 Preliminaries

Standard Lie theory provides description of the following objects for any compact simple Lie group G of rank $1 \le n < \infty$:

> maximal torus T; the root system and its subset of simple roots $\{\alpha_1, \dots, \alpha_n\}$; finite Weyl group W of G; the weight lattice P and its dual $\hat{P}$ in the real Euclidean space $\mathbb{R}^n$ of dimension n; the bases $\{\omega_1, \dots, \omega_n\}$ and $\{\hat{\omega}_1, \dots, \hat{\omega}_n\}$ of P and $\hat{P}$ respectively; the Weyl group orbit W_λ of $\lambda \in P$; the size $|W_\lambda|$ of W_λ; the fundamental region F for W action on T; finite W-invariant subgroup $A_M \subset T$ generated by the elements of order M in T.

A description of the points $x_s \in F_M = A_M \cap F$ is given next

$$F_M = \left\{ x_s = \sum_{k=1}^{n} \frac{s_k}{M}\hat{\omega}_k \,\middle|\, s_k \in \mathbb{Z}^{\ge 0};\ M = s_0 + \sum_{m=1}^{n} q_m s_m \right\}, \tag{3}$$

where the coefficients q_m are given in the highest root ξ of G:

$$\xi = \sum_{m=1}^{n} q_m \alpha_m.$$

To find the points of F_M, it suffices to find all the non-negative integers $\{s_0, s_1, \dots, s_n\}$ that add up to M according to (3). It is an easy computing task. In the case of E-functions one has to take into account that F^e consists of two copies of F.

More theoretically, points of F_M are representatives of conjugacy classes of elements of an Abelian subgroup $A \subset T$ of the maximal torus T, which is generated by all elements of order M in T. Thus one has $F_M = A \cap F$.

4 Definition of C-functions

A review of properties of C-functions denoted here $C_\lambda(t)$ is in the forthcoming paper [16]

$$C_\lambda(t) := \sum_{\lambda' \in W_\lambda} e^{2\pi i \langle \lambda' | t \rangle}, \qquad \lambda \in P^+ \subset \mathbb{R}^n, \quad t \in \mathbb{R}^n. \tag{4}$$

Here $\langle \lambda' \mid t \rangle$ denotes the scalar product in $\mathbb{R}^n$. The number of summands is finite, it is equal to the size of the Weyl group orbit of λ.

Among the useful properties of C-functions, note the complete decomposition of products

$$C_\lambda(t) C_{\lambda'}(t) = C_{\lambda+\lambda'}(t) + C_\mu(t) + \cdots .$$

The functions are continuous and have all derivatives continuous in $\mathbb{R}^n$. They are W-invariant and have interesting symmetry properties [10, 11] with respect to affine W. In particular, they are symmetric with respect to reflection in the sides of F of maximal dimension, i.e. $n-1$. Hence their normal derivative at the boundary is zero.

One has the continuous orthogonality of C-functions,

$$(C_\lambda, C_\mu) = \int_F C_\lambda(t) \overline{C_\mu(t)} dt \sim \delta_{\lambda\mu}, \tag{5}$$

and the discrete orthogonality of C-functions,

$$\sum_{t \in A \cap F} |Wt| C_\lambda(t) \overline{C_\mu(t)} \sim \delta_{\lambda\mu},$$

where the integer $|Wt|$ is the size of the W-orbit of t. The lattice points $\lambda, \mu \in P^+$ are subjects to additional restriction assuring that $C_\lambda(t)$, $C_\mu(t)$ belong to a finite set of the functions denoted S_M. The Abelian W-invariant group A can be built as the group generated by representatives of the conjugacy classes of elements of given order $M < \infty$ in the Lie group. For the rank 2 cases, see [10, 11].

5 Definition of S-functions

Comparing with (4), we have

$$S_\lambda(t) := \sum_{\lambda' \in W_\lambda} (-1)^{l(\lambda,\lambda')} e^{2\pi i \langle \lambda' | t \rangle}, \qquad \lambda \in P^{++} \subset \mathbb{R}^n, \quad t \in \mathbb{R}^n. \tag{6}$$

Here $l(\lambda, \lambda')$ is the minimal number of elementary reflections from W needed to transform λ into λ'; P^{++} denotes the interior of the positive chamber P^+ of the weight lattice P. The S-functions are continuous and antisymmetric with respect to reflection in the sides of F of maximal dimension, i.e. $n-1$. Hence their value at the boundary is zero.

A product of an even number of S-functions decomposes into the sum of C-functions where coefficients are positive and negative integers. A product of C- and S-functions decomposes into the sum of S-functions:

$$S_\lambda(t)S_{\lambda'}(t) = C_{\lambda+\lambda'}(t) - C_\mu(t) + \cdots , \tag{7}$$

$$C_\lambda(t)S_{\lambda'}(t) = S_{\lambda+\lambda'}(t) + S_\mu(t) + \cdots . \tag{8}$$

See some examples in [17].

Continuous orthogonality of S-functions formally coincides with that for C-functions. The only difference is in the fact that $\lambda \in P^{++}$ rather than in P^+.

$$(S_\lambda, S_\mu) = \int_F S_\lambda(t)\overline{S_\mu(t)}dt \sim \delta_{\lambda\mu}. \tag{9}$$

Discrete orthogonality of S-functions happens to be on the same grids in F as in the case of C-functions. Since the W-orbit of all the points in the interior P^{++} of the positive chamber P^+ is the same,

$$\sum_{t\in A\cap F} S_\lambda(t)\overline{S_\mu(t)} \sim \delta_{\lambda\mu}.$$

As before, one needs to assume that λ and μ belong to the lowest finite set, denoted S_M, of dominant weights of pairwise orthogonal functions over the grid fixed by the positive integer M. For that some additional restrictions on λ and μ need to be imposed.

6 Definition of E-functions

In order to define the E-functions in a way analogous to the C- and S-functions, one needs to replace the Weyl group W by its even subgroup $W^e \subset W$ and correspondingly enlarge its fundamental region F^e.

Let r be any simple reflection from the Weyl group. Then

$$F^e := F \cup rF,$$

where F^e is the fundamental region for W^e.

For $\lambda \in P$,

$$E_\lambda(t) := \sum_{\lambda' \in W^e_\lambda} e^{2\pi i\langle\lambda', t\rangle}. \tag{10}$$

Since $E_\lambda(t)$ depends on $W^e\lambda$, not on λ, we can suppose $\lambda \in P^{+e} = P^+ \cup rP^+$. One verifies directly that

$$C_\lambda = \begin{cases} E_\lambda + E_{r\lambda} & \text{if } \lambda \neq r\lambda, \\ E_\lambda & \text{if } \lambda = r\lambda, \end{cases} \qquad S_\lambda = \begin{cases} E_\lambda - E_{r\lambda} & \text{if } \lambda \neq r\lambda, \\ 0 & \text{if } \lambda = r\lambda. \end{cases} \tag{11}$$

Continuous orthogonality of E-functions involves integration over F^e:

$$(E_\lambda, E_\mu) = \int_{F^e} E_\lambda(t)\overline{E_\mu(t)}dt \sim \delta_{\lambda\mu}. \tag{12}$$

Discrete orthogonality of E-functions can be introduced in a similar way as for the C- and S-functions, using the same grid of points on F^e.

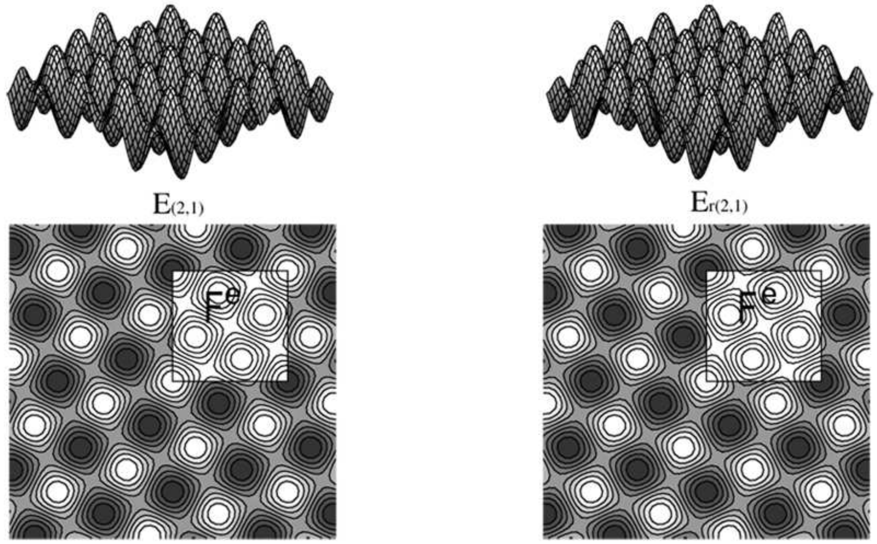

Figure 1. 3-D and contour graphs of $E_{(2,1)}$ and $E_{r(2,1)}$ for Lie group C_2. The light color square is the fundamental region F^e of C_2.

Let $A \subset \hat{T}$ be a finite subgroup which is W^e-invariant. Again a convenient and versatile set up would be to take the group generated by representatives of conjugacy classes of elements of a fixed order M in the Lie group. The intersection of A with F^e is a finite set of points

$$A \cap F^e = \{z_1^e, z_2^e, \dots, z_m^e\}, \qquad o_j^e = |W^e z_j^e|.$$

Since E_λ, E_μ are W^e-invariant, we have

$$\sum_{j=1}^{m} o_j^e E_\lambda(z_j^e)\overline{E_\mu(z_j^e)} \sim \delta_{\lambda\mu}.$$

As in the previous cases, only a finite set of E-functions can be pairwise orthogonal and distinct on a finite grid of points given by fixed value of M. Therefore some additional restrictions on λ, μ need to be imposed.

Some two dimensional examples can be found in the forthcoming [18]. Fig. 1 shows an example of $E_{(2,1)}$ and $E_{r(2,1)}$ for Lie group C_2.

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[1] Patera J., Invited talk at the Sixth International Conference "Symmetry in Nonlinear Mathematical Physics" (June 20–26, 2005, Kyiv).

[2] Atoyan A., Patera J., Properties of continuous Fourier extension of the discrete cosine transform and its multidimensional generalization, *J. Math. Phys.*, 2004, V.45, 2468–2491.

[3] Wang Zhongde, Interpolation using type I discrete cosine transform, *Electron. Lett.*, 1990, V.26, 1170–1171.

[4] Agbinya J.I., Two dimensional interpolation of real sequences using the DCT, *Electron. Lett.*, 1993, V.29, 204–205.

[5] Moody R.V., Patera J., Computation of character decompositions of class functions on compact semisimple Lie groups, *Math. Comp.*, 1987, V.48, 799–827.

[6] Moody R.V., Patera J., Elements of finite order in Lie groups and their applications, XIII Int. Colloq. on Group Theoretical Methods in Physics, Editor W. Zachary, Singapore, World Scientific Publishers, 1984, 308–318.

[7] McKay W.G., Moody R.V., Patera J., Decomposition of tensor products of E_8 representations, *Algebras Groups Geom.*, 1986, V.3, 286–328.

[8] McKay W.G., Moody R.V., Patera J., Tables of E_8 characters and decomposition of plethysms, in Lie Algebras and Related Topics, Editors D.J. Britten, F.W. Lemire and R.V. Moody, Providence R.I., Amer. Math. Society, 1985, 227–264.

[9] Grimm S., Patera J., Decomposition of tensor products of the fundamental representations of E_8, in Advances in Mathematical Sciences: CRM's 25 Years, Editor L. Vinet, *CRM Proc. Lecture Notes*, Providence RI, Amer. Math. Soc., 1997, V.11, 329–355.

[10] Patera J., Zaratsyan A., Cosine transform generalized to Lie groups $SU(2) \times SU(2)$ and $O(5)$, *J. Math. Phys.*, 2005, V.46, 053514, 25 pages.

[11] Patera J., Zaratsyan A., Cosine transform generalized to Lie groups $SU(3)$ and $G(2)$, *J. Math. Phys.*, 2005, V.46, 113506, 17 pages.

[12] Atoyan A., Patera J., Sahakian V., Akhperjanian A., Fourier transform method for imaging atmospheric Cherenkov telescopes, *Astroparticle Phys.*, 2005, V.23, 79–95.

[13] Patera J., Orbit functions of compact semisimple Lie groups as special functions, in Proceedinds of Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" (June 23–29, 2003, Kyiv), Editors A.G. Nikitin, V.M. Boyko, R.O. Popovych and I.A. Yehorchenko, *Proceedings of Institute of Mathematics*, Kyiv, 2004, V.50, Part 3, 1152–1160.

[14] Patera J., Algebraic solutions of the Neumann boundary value problems on fundamental region of a compact semisimple Lie group, talk given at the Workshop on Group Theory and Numerical Methods (May 26–31, 2003, Montreal).

[15] Atoyan A., Patera J., Continuous extension of the discrete cosine transform, and its applications to data processing, Proceedings of the Workshop on Group Theory and Numerical Methods (May 26–31, 2003, Montreal).

[16] Klimyk A., Patera J., Orbit functions, in preparation.

[17] Patera J., Zaratsyan A., The sine transform generalized to semisimple Lie groups rank 2, Preprint, 2005.

[18] Kashuba I., Patera J., Zaratsyan A., The E-functions of compact semisimple Lie groups and their discretization, in preparation.

[19] Bremner M.R., Moody R.V., Patera J., Tables of dominant weight multiplicities for representations of simple Lie algebras, New York, Marcel Dekker, 1985, 340 pages.

[20] Moody R.V., Patera J., Discrete and continuous orthogonality of C-, S-, and E-functions of a compact semisimple Lie group, in preparation.