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Inversion of the Dual Dunkl–Sonine Transform on $\mathbb{R}$ Using Dunkl Wavelets

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Abstract. We prove a Calderón reproducing formula for the Dunkl continuous wavelet transform on $\mathbb{R}$. We apply this result to derive new inversion formulas for the dual Dunkl–Sonine integral transform.

Key words: Dunkl continuous wavelet transform; Calderón reproducing formula; dual Dunkl–Sonine integral transform

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1 Introduction

The one-dimensional Dunkl kernel e_γ, $\gamma > -1/2$, is defined by

$$e_\gamma(z) = j_\gamma(iz) + \frac{z}{2(\gamma+1)} j_{\gamma+1}(iz), \qquad z \in \mathbb{C},$$

where

$$j_\gamma(z) = \Gamma(\gamma+1) \sum_{n=0}^{\infty} \frac{(-1)^n \, (z/2)^{2n}}{n! \, \Gamma(n+\gamma+1)}$$

is the normalized spherical Bessel function of index γ. It is well-known (see [3]) that the functions $e_\gamma(\lambda\cdot)$, $\lambda \in \mathbb{C}$, are solutions of the differential-difference equation

$$\Lambda_\gamma u = \lambda u, \qquad u(0) = 1,$$

where

$$\Lambda_\gamma f(x) = f'(x) + \left(\gamma + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}$$

is the Dunkl operator with parameter $\gamma + 1/2$ associated with the reflection grour $\mathbb{Z}_2$ on $\mathbb{R}$. Those operators were introduced and studied by Dunkl [2, 3, 4] in connection with a generalization of the classical theory of spherical harmonics. Besides its mathematical interest, the Dunkl operator Λ_α has quantum-mechanical applications; it is naturally involved in the study of one-dimensional harmonic oscillators governed by Wigner's commutation rules [6, 11, 16].

It is known, see for example [14, 15], that the Dunkl kernels on $\mathbb{R}$ possess the following Sonine type integral representation

$$e_\beta(\lambda x) = \int_{-|x|}^{|x|} \mathcal{K}_{\alpha,\beta}(x,y) \, e_\alpha(\lambda y) \, |y|^{2\alpha+1} \, dy, \qquad \lambda \in \mathbb{C}, \qquad x \neq 0, \tag{1.1}$$

where

$$\mathcal{K}_{\alpha,\beta}(x,y) := \begin{cases} a_{\alpha,\beta}\,\mathrm{sgn}(x)\,(x+y)\,\dfrac{(x^2-y^2)^{\beta-\alpha-1}}{|x|^{2\beta+1}} & \text{if } |y|<|x|, \\ 0 & \text{if } |y|\geq |x|, \end{cases} \tag{1.2}$$

with $\beta > \alpha > -1/2$, and

$$a_{\alpha,\beta} := \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta-\alpha)}.$$

Define the Dunkl–Sonine integral transform $\mathcal{X}_{\alpha,\beta}$ and its dual ${}^t\mathcal{X}_{\alpha,\beta}$, respectively, by

$$\mathcal{X}_{\alpha,\beta}f(x) = \int_{-|x|}^{|x|} \mathcal{K}_{\alpha,\beta}(x,y)\, f(y)\, |y|^{2\alpha+1}\, dy,$$
$${}^t\mathcal{X}_{\alpha,\beta}f(y) = \int_{|x|\geq|y|} \mathcal{K}_{\alpha,\beta}(x,y)\, f(x)\, |x|^{2\beta+1}\, dx.$$

Soltani has showed in [14] that the dual Dunkl–Sonine integral transform ${}^t\mathcal{X}_{\alpha,\beta}$ is a transmutation operator between Λ_α and Λ_β on the Schwartz space $\mathcal{S}(\mathbb{R})$, i.e., it is an automorphism of $\mathcal{S}(\mathbb{R})$ satisfying the intertwining relation

$${}^t\mathcal{X}_{\alpha,\beta}\,\Lambda_\beta\, f = \Lambda_\alpha\, {}^t\mathcal{X}_{\alpha,\beta}\, f, \qquad f\in\mathcal{S}(\mathbb{R}).$$

The same author [14] has obtained inversion formulas for the transform ${}^t\mathcal{X}_{\alpha,\beta}$ involving pseudo-differential-difference operators and only valid on a restricted subspace of $\mathcal{S}(\mathbb{R})$.

The purpose of this paper is to investigate the use of Dunkl wavelets (see [5]) to derive a new inversion of the dual Dunkl–Sonine transform on some Lebesgue spaces. For other applications of wavelet type transforms to inverse problems we refer the reader to [7, 8] and the references therein.

The content of this article is as follows. In Section 2 we recall some basic harmonic analysis results related to the Dunkl operator. In Section 3 we list some basic properties of the Dunkl–Sonine integral trnsform and its dual. In Section 4 we give the definition of the Dunkl continuous wavelet transform and we establish for this transform a Calderón formula. By combining the results of the two previous sections, we obtain in Section 5 two new inversion formulas for the dual Dunkl–Sonine integral transform.

2 Preliminaries

Note 2.1. Throughout this section assume $\gamma > -1/2$. Define $L^p(\mathbb{R}, |x|^{2\gamma+1}dx)$, $1\leq p\leq\infty$, as the class of measurable functions f on $\mathbb{R}$ for which $||f||_{p,\gamma}<\infty$, where

$$||f||_{p,\gamma} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\gamma+1} dx\right)^{1/p}, \qquad \text{if} \quad p<\infty,$$

and $||f||_{\infty,\gamma} = ||f||_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}}|f(x)|$. $\mathcal{S}(\mathbb{R})$ stands for the usual Schwartz space.

The Dunkl transform of order $\gamma+1/2$ on $\mathbb{R}$ is defined for a function f in $L^1(\mathbb{R}, |x|^{2\gamma+1}dx)$ by

$$\mathcal{F}_\gamma f(\lambda) = \int_{\mathbb{R}} f(x)\, e_\gamma(-i\lambda x)\, |x|^{2\gamma+1} dx, \qquad \lambda\in\mathbb{R}. \tag{2.1}$$

Remark 2.2. It is known that the Dunkl transform $\mathcal{F}_\gamma$ maps continuously and injectively $L^1(\mathbb{R}, |x|^{2\gamma+1}dx)$ into the space $\mathcal{C}_0(\mathbb{R})$ (of continuous functions on $\mathbb{R}$ vanishing at infinity).

Two standard results about the Dunkl transform $\mathcal{F}_\gamma$ are as follows.

Theorem 2.3 (see [1]).

(*i*) *For every* $f \in L^1 \cap L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ *we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\gamma+1} dx = m_\gamma \int_{\mathbb{R}} |\mathcal{F}_\gamma f(\lambda)|^2 |\lambda|^{2\gamma+1} d\lambda,$$

where

$$m_\gamma = \frac{1}{2^{2\gamma+2}(\Gamma(\gamma+1))^2}. \tag{2.2}$$

(*ii*) *The Dunkl transform* $\mathcal{F}_\alpha$ *extends uniquely to an isometric isomorphism from* $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ *onto* $L^2(\mathbb{R}, m_\gamma|\lambda|^{2\gamma+1}d\lambda)$. *The inverse transform is given by*

$$\mathcal{F}_\gamma^{-1} g(x) = m_\gamma \int_{\mathbb{R}} g(\lambda) e_\gamma(i\lambda x) |\lambda|^{2\gamma+1} d\lambda,$$

where the integral converges in $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$.

Theorem 2.4 (see [1]). *The Dunkl transform* $\mathcal{F}_\alpha$ *is an automorphism of* $\mathcal{S}(\mathbb{R})$.

An outstanding result about Dunkl kernels on $\mathbb{R}$ (see [12]) is the product formula

$$e_\gamma(\lambda x) e_\gamma(\lambda y) = T_\gamma^x \left(e_\gamma(\lambda \cdot)\right)(y), \qquad \lambda \in \mathbb{C}, \qquad x, y \in \mathbb{R},$$

where T_γ^x stand for the Dunkl translation operators defined by

$$\begin{aligned} T_\gamma^x f(y) = {} & \frac{1}{2}\int_{-1}^{1} f\left(\sqrt{x^2+y^2-2xyt}\right)\left(1 + \frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right) W_\gamma(t) dt \\ & + \frac{1}{2}\int_{-1}^{1} f\left(-\sqrt{x^2+y^2-2xyt}\right)\left(1 - \frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right) W_\gamma(t) dt, \end{aligned} \tag{2.3}$$

with

$$W_\gamma(t) = \frac{\Gamma(\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+1/2)} (1+t)\left(1-t^2\right)^{\gamma-1/2}.$$

The Dunkl convolution of two functions f, g on $\mathbb{R}$ is defined by the relation

$$f *_\gamma g(x) = \int_{\mathbb{R}} T_\gamma^x f(-y) g(y) |y|^{2\gamma+1} dy. \tag{2.4}$$

Proposition 2.5 (see [13]).

(*i*) *Let* $p, q, r \in [1, \infty]$ *such that* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. *If* $f \in L^p(\mathbb{R}, |x|^{2\gamma+1}dx)$ *and* $g \in L^q(\mathbb{R}, |x|^{2\gamma+1}dx)$, *then* $f *_\gamma g \in L^r(\mathbb{R}, |x|^{2\gamma+1}dx)$ *and*

$$||f *_\gamma g||_{r,\gamma} \le 4 ||f||_{p,\gamma} ||g||_{q,\gamma}. \tag{2.5}$$

(*ii*) *For* $f \in L^1(\mathbb{R}, |x|^{2\gamma+1}dx)$ *and* $g \in L^p(\mathbb{R}, |x|^{2\gamma+1}dx)$, $p = 1$ *or* 2, *we have*

$$\mathcal{F}_\gamma(f *_\gamma g) = \mathcal{F}_\gamma f \mathcal{F}_\gamma g. \tag{2.6}$$

For more details about harmonic analysis related to the Dunkl operator on $\mathbb{R}$ the reader is referred, for example, to [9, 10].

3 The dual Dunkl–Sonine integral transform

Throughout this section assume $\beta > \alpha > -1/2$.

Definition 3.1 (see [14]). The dual Dunkl–Sonine integral transform ${}^t\mathcal{X}_{\alpha,\beta}$ is defined for smooth functions on $\mathbb{R}$ by

$$ {}^t\mathcal{X}_{\alpha,\beta} f(y) := \int_{|x|\geq|y|} \mathcal{K}_{\alpha,\beta}(x,y) f(x) |x|^{2\beta+1}\, dx, \qquad y \in \mathbb{R}, \tag{3.1}$$

where $\mathcal{K}_{\alpha,\beta}$ is the kernel given by (1.2).

Remark 3.2. Clearly, if $\operatorname{supp}(f) \subset [-a,a]$ then $\operatorname{supp}\left({}^t\mathcal{X}_{\alpha,\beta} f\right) \subset [-a,a]$.

The next statement provides formulas relating harmonic analysis tools tied to Λ_α with those tied to Λ_β, and involving the operator ${}^t\mathcal{X}_{\alpha,\beta}$.

Proposition 3.3.

(i) *The dual Dunkl–Sonine integral transform ${}^t\mathcal{X}_{\alpha,\beta}$ maps $L^1(\mathbb{R}, |x|^{2\beta+1}dx)$ continuously into $L^1(\mathbb{R}, |x|^{2\alpha+1}dx)$.*

(ii) *For every $f \in L^1(\mathbb{R}, |x|^{2\beta+1}dx)$ we have the identity*

$$ \mathcal{F}_\beta(f) = \mathcal{F}_\alpha \circ {}^t\mathcal{X}_{\alpha,\beta}(f). \tag{3.2}$$

(iii) *Let $f, g \in L^1(\mathbb{R}, |x|^{2\beta+1}dx)$. Then*

$$ {}^t\mathcal{X}_{\alpha,\beta}(f *_\beta g) = {}^t\mathcal{X}_{\alpha,\beta} f *_\alpha {}^t\mathcal{X}_{\alpha,\beta} g. \tag{3.3}$$

Proof. Let $f \in L^1(\mathbb{R}, |x|^{2\beta+1}dx)$. By Fubini's theorem we have

$$ \begin{aligned} \int_{\mathbb{R}} {}^t\mathcal{X}_{\alpha,\beta}(|f|)(y)|y|^{2\alpha+1}dy &= \int_{\mathbb{R}} \left(\int_{|x|\geq|y|} \mathcal{K}_{\alpha,\beta}(x,y)|f(x)||x|^{2\beta+1}\, dx \right) |y|^{2\alpha+1}dy \\ &= \int_{\mathbb{R}} |f(x)| \left(\int_{-|x|}^{|x|} \mathcal{K}_{\alpha,\beta}(x,y)|y|^{2\alpha+1}dy \right) |x|^{2\beta+1}\, dx. \end{aligned} $$

But by (1.1),

$$ \int_{-|x|}^{|x|} \mathcal{K}_{\alpha,\beta}(x,y)|y|^{2\alpha+1}dy = e_\beta(0) = 1. \tag{3.4}$$

Hence, ${}^t\mathcal{X}_{\alpha,\beta} f$ is almost everywhere defined on $\mathbb{R}$, belongs to $L^1(\mathbb{R}, |x|^{2\alpha+1}dx)$ and $||{}^t\mathcal{X}_{\alpha,\beta} f||_{1,\alpha} \leq ||f||_{1,\beta}$, which proves (i). Identity (3.2) follows by using (1.1), (2.1), (3.1), and Fubini's theorem. Identity (3.3) follows by applying the Dunkl transform $\mathcal{F}_\alpha$ to both its sides and by using (2.6), (3.2) and Remark 2.2. ■

Remark 3.4. From (3.2) and Remark 2.2, we deduce that the transform ${}^t\mathcal{X}_{\alpha,\beta}$ maps $L^1(\mathbb{R}, |x|^{2\beta+1}dx)$ injectively into $L^1(\mathbb{R}, |x|^{2\alpha+1}dx)$.

From [14] we have the following result.

Theorem 3.5. *The dual Dunkl–Sonine integral transform ${}^t\mathcal{X}_{\alpha,\beta}$ is an automorphism of $\mathcal{S}(\mathbb{R})$ satisfying the intertwining relation*

$$ {}^t\mathcal{X}_{\alpha,\beta}\Lambda_\beta f = \Lambda_\alpha {}^t\mathcal{X}_{\alpha,\beta} f, \qquad f \in \mathcal{S}(\mathbb{R}). $$

Moreover ${}^t\mathcal{X}_{\alpha,\beta}$ admits the factorization

$$ {}^t\mathcal{X}_{\alpha,\beta} f = {}^tV_\alpha^{-1} \circ {}^tV_\beta f \qquad \text{for all} \ \ f \in \mathcal{S}(\mathbb{R}), $$

where for $\gamma > -1/2$, ${}^tV_\gamma$ denotes the dual Dunkl intertwining operator given by

$$ {}^tV_\gamma f(y) = \frac{\Gamma(\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+1/2)} \int_{|x|\geq|y|} \operatorname{sgn}(x)\,(x+y)\left(x^2-y^2\right)^{\gamma-1/2} f(x)\,dx. $$

Definition 3.6 (see [14]). The Dunkl–Sonine integral transform $\mathcal{X}_{\alpha,\beta}$ is defined for a locally bounded function f on $\mathbb{R}$ by

$$ \mathcal{X}_{\alpha,\beta} f(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} \mathcal{K}_{\alpha,\beta}(x,y)\, f(y)\, |y|^{2\alpha+1}\, dy & \text{if } x \neq 0, \\ f(0) & \text{if } x = 0. \end{cases} \tag{3.5} $$

Remark 3.7.

(i) Notice that by (3.4), $||\mathcal{X}_{\alpha,\beta} f||_\infty \leq ||f||_\infty$ if $f \in L^\infty(\mathbb{R})$.

(ii) It follows from (1.1) that

$$ e_\beta(\lambda x) = \mathcal{X}_{\alpha,\beta}(e_\alpha(\lambda\cdot)(x) \tag{3.6} $$

for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$.

Proposition 3.8.

(i) *For any $f \in L^\infty(\mathbb{R})$ and $g \in L^1(\mathbb{R}, |x|^{2\beta+1}dx)$ we have the duality relation*

$$ \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta} f(x) g(x) |x|^{2\beta+1} dx = \int_{\mathbb{R}} f(y)\, {}^t\mathcal{X}_{\alpha,\beta} g(y) |y|^{2\alpha+1} dy. \tag{3.7} $$

(ii) *Let $f \in L^1(\mathbb{R}, |x|^{2\beta+1}dx)$ and $g \in L^\infty(\mathbb{R})$. Then*

$$ \mathcal{X}_{\alpha,\beta}\left({}^t\mathcal{X}_{\alpha,\beta} f *_\alpha g\right) = f *_\beta \mathcal{X}_{\alpha,\beta} g. \tag{3.8} $$

Proof. Identity (3.7) follows by using (3.1), (3.5) and Fubini's theorem. Let us check (3.8). Let $\psi \in \mathcal{S}(\mathbb{R})$. By using (3.3), (3.7) and Fubini's theorem, we have

$$ \begin{aligned} \int_{\mathbb{R}} f *_\beta \mathcal{X}_{\alpha,\beta} g(x) \psi(x) |x|^{2\beta+1} dx &= \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta} g(x) \psi *_\beta f^-(x)\, |x|^{2\beta+1} dx \\ &= \int_{\mathbb{R}} g(y)\, {}^t\mathcal{X}_{\alpha,\beta}(\psi *_\beta f^-)(y) |y|^{2\alpha+1} dy = \int_{\mathbb{R}} g(y) \left({}^t\mathcal{X}_{\alpha,\beta}\psi *_\alpha {}^t\mathcal{X}_{\alpha,\beta} f^-\right)(y) |y|^{2\alpha+1} dy, \end{aligned} $$

where $f^-(x) = f(-x)$, $x \in \mathbb{R}$. But an easy computation shows that ${}^t\mathcal{X}_{\alpha,\beta} f^- = \left({}^t\mathcal{X}_{\alpha,\beta} f\right)^-$. Hence,

$$ \begin{aligned} \int_{\mathbb{R}} f *_\beta \mathcal{X}_{\alpha,\beta} g(x) \psi(x) |x|^{2\beta+1} dx &= \int_{\mathbb{R}} g(y)\, {}^t\mathcal{X}_{\alpha,\beta}\psi *_\alpha \left({}^t\mathcal{X}_{\alpha,\beta} f\right)^-(y) |y|^{2\alpha+1} dy \\ &= \int_{\mathbb{R}} {}^t\mathcal{X}_{\alpha,\beta} f *_\alpha g(y)\, {}^t\mathcal{X}_{\alpha,\beta}\psi(y) |y|^{2\alpha+1} dy = \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta}\left({}^t\mathcal{X}_{\alpha,\beta} f *_\alpha g\right)(x) \psi(x) |x|^{2\beta+1} dx. \end{aligned} $$

This clearly yields the result. ∎

4 Calderón's formula for the Dunkl continuous wavelet transform

Throughout this section assume $\gamma > -1/2$.

Definition 4.1. We say that a function $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ is a Dunkl wavelet of order γ, if it satisfies the admissibility condition

$$0 < C_g^\gamma := \int_0^\infty |\mathcal{F}_\gamma g(\lambda)|^2 \frac{d\lambda}{\lambda} = \int_0^\infty |\mathcal{F}_\gamma g(-\lambda)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{4.1}$$

Remark 4.2.

(i) If g is real-valued we have $\mathcal{F}_\gamma g(-\lambda) = \overline{\mathcal{F}_\gamma g(\lambda)}$, so (4.1) reduces to

$$0 < C_g^\gamma := \int_0^\infty |\mathcal{F}_\gamma g(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

(ii) If $0 \neq g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ is real-valued and satisfies

$$\exists\, \eta > 0 \qquad \text{such that} \quad \mathcal{F}_\gamma g(\lambda) - \mathcal{F}_\gamma g(0) = \mathcal{O}(\lambda^\eta) \quad \text{as} \quad \lambda \to 0^+$$

then (4.1) is equivalent to $\mathcal{F}_\gamma g(0) = 0$.

Note 4.3. For a function g in $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and for $(a, b) \in (0, \infty) \times \mathbb{R}$ we write

$$g_{a,b}^\gamma(x) := \frac{1}{a^{2\gamma+2}} T_\gamma^{-b} g_a(x),$$

where T_γ^{-b} are the generalized translation operators given by (2.3), and $g_a(x) := g(x/a)$, $x \in \mathbb{R}$.

Remark 4.4. Let $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and $a > 0$. Then it is easily checked that $g_a \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$, $||g_a||_{2,\gamma} = a^{\gamma+1}\, ||g||_{2,\gamma}$, and $\mathcal{F}_\gamma(g_a)(\lambda) = a^{2\gamma+2}\mathcal{F}_\gamma(g)(a\lambda)$.

Definition 4.5. Let $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ be a Dunkl wavelet of order γ. We define for regular functions f on $\mathbb{R}$, the Dunkl continuous wavelet transform by

$$\Phi_g^\gamma(f)(a, b) := \int_{\mathbb{R}} f(x)\overline{g_{a,b}^\gamma(x)}|x|^{2\gamma+1}dx$$

which can also be written in the form

$$\Phi_g^\gamma(f)(a, b) = \frac{1}{a^{2\gamma+2}}\, f *_\gamma \widetilde{g}_a(b),$$

where $*_\gamma$ is the generalized convolution product given by (2.4), and $\widetilde{g}_a(x) := \overline{g(-x/a)}$, $x \in \mathbb{R}$.

The Dunkl continuous wavelet transform has been investigated in depth in [5] in which precise definitions, examples, and a more complete discussion of its properties can be found. We look here for a Calderón formula for this transform. We start with some technical lemmas.

Lemma 4.6. *For all $f, g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and all $\psi \in \mathcal{S}(\mathbb{R})$ we have the identity*

$$\int_{\mathbb{R}} f *_\gamma g(x)\mathcal{F}_\gamma^{-1}\psi(x)|x|^{2\gamma+1}dx = m_\gamma \int_{\mathbb{R}} \mathcal{F}_\gamma f(\lambda)\mathcal{F}_\gamma g(\lambda)\psi^-(\lambda)|\lambda|^{2\gamma+1}d\lambda,$$

where m_γ is given by (2.2).

Proof. Fix $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and $\psi \in \mathcal{S}(\mathbb{R})$. For $f \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ put

$$S_1(f) := \int_{\mathbb{R}} f *_\gamma g(x)\mathcal{F}_\gamma^{-1}\psi(x)|x|^{2\gamma+1}dx$$

and

$$S_2(f) := m_\gamma \int_{\mathbb{R}} \mathcal{F}_\gamma f(\lambda)\mathcal{F}_\gamma g(\lambda)\psi^-(\lambda)|\lambda|^{2\gamma+1}d\lambda.$$

By (2.5), (2.6) and Theorem 2.3, we see that $S_1(f) = S_2(f)$ for each $f \in L^1 \cap L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. Moreover, by using (2.5), Hölder's inequality and Theorem 2.3 we have

$$|S_1(f)| \le ||f *_\gamma g||_\infty ||\mathcal{F}_\gamma^{-1}\psi||_{1,\gamma} \le 4||f||_{2,\gamma}||g||_{2,\gamma}||\mathcal{F}_\gamma^{-1}\psi||_{1,\gamma}$$

and

$$|S_2(f)| \le m_\gamma\, ||\mathcal{F}_\gamma f \mathcal{F}_\gamma g||_{1,\gamma}||\psi||_\infty \le m_\gamma\, ||\mathcal{F}_\gamma f||_{2,\gamma}||\mathcal{F}_\gamma g||_{2,\gamma}||\psi||_\infty = ||f||_{2,\gamma}||g||_{2,\gamma}||\psi||_\infty,$$

which shows that the linear functionals S_1 and S_2 are bounded on $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. Therefore $S_1 \equiv S_2$, and the lemma is proved. ∎

Lemma 4.7. *Let $f_1, f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. Then $f_1 *_\gamma f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ if and only if $\mathcal{F}_\gamma f_1 \mathcal{F}_\gamma f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and we have*

$$\mathcal{F}_\gamma(f_1 *_\gamma f_2) = \mathcal{F}_\gamma f_1 \mathcal{F}_\gamma f_2$$

in the L^2-case.

Proof. Suppose $f_1 *_\gamma f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. By Lemma 4.6 and Theorem 2.3, we have for any $\psi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} m_\gamma \int_{\mathbb{R}} \mathcal{F}_\gamma f_1(\lambda)\mathcal{F}_\gamma f_2(\lambda)\psi(\lambda)|\lambda|^{2\gamma+1}d\lambda &= \int_{\mathbb{R}} f_1 *_\gamma f_2(x)\mathcal{F}_\gamma^{-1}\psi^-(x)|x|^{2\gamma+1}dx \\ &= \int_{\mathbb{R}} f_1 *_\gamma f_2(x)\overline{\mathcal{F}_\gamma^{-1}\overline{\psi}(x)}|x|^{2\gamma+1}dx = m_\gamma \int_{\mathbb{R}} \mathcal{F}_\gamma(f_1 *_\gamma f_2)(\lambda)\psi(\lambda)|\lambda|^{2\gamma+1}d\lambda, \end{aligned}$$

which shows that $\mathcal{F}_\gamma f_1 \mathcal{F}_\gamma f_2 = \mathcal{F}_\gamma(f_1 *_\gamma f_2)$. Conversely, if $\mathcal{F}_\gamma f_1 \mathcal{F}_\gamma f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$, then by Lemma 4.6 and Theorem 2.3, we have for any $\psi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}} f_1 *_\gamma f_2(x)\mathcal{F}_\gamma^{-1}\psi(x)|x|^{2\gamma+1}dx &= m_\gamma \int_{\mathbb{R}} \mathcal{F}_\gamma f_1(\lambda)\mathcal{F}_\gamma f_2(\lambda)\overline{\widetilde{\psi}(\lambda)}|\lambda|^{2\gamma+1}d\lambda \\ &= \int_{\mathbb{R}} \mathcal{F}_\gamma^{-1}(\mathcal{F}_\gamma f_1 \mathcal{F}_\gamma f_2)(x)\mathcal{F}_\gamma^{-1}\psi(x)|x|^{2\gamma+1}dx, \end{aligned}$$

which shows, in view of Theorem 2.4, that $f_1 *_\gamma f_2 = \mathcal{F}_\gamma^{-1}(\mathcal{F}_\gamma f_1 \mathcal{F}_\gamma f_2)$. This achieves the proof of Lemma 4.7. ∎

A combination of Lemma 4.7 and Theorem 2.3 gives us the following.

Lemma 4.8. *Let $f_1, f_2 \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$. Then*

$$\int_{\mathbb{R}} |f_1 *_\gamma f_2(x)|^2|x|^{2\gamma+1}dx = m_\gamma \int_{\mathbb{R}} |\mathcal{F}_\gamma f_1(\lambda)|^2|\mathcal{F}_\gamma f_2(\lambda)|^2|\lambda|^{2\gamma+1}d\lambda,$$

where both sides are finite or infinite.

Lemma 4.9. *Let $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ be a Dunkl wavelet of order γ such that $\mathcal{F}_\gamma g \in L^\infty(\mathbb{R})$. For $0 < \varepsilon < \delta < \infty$ define*

$$G_{\varepsilon,\delta}(x) := \frac{1}{C_g^\gamma} \int_\varepsilon^\delta g_a *_\gamma \widetilde{g}_a(x) \frac{da}{a^{4\gamma+5}} \tag{4.2}$$

and

$$K_{\varepsilon,\delta}(\lambda) := \frac{1}{C_g^\gamma} \int_\varepsilon^\delta |\mathcal{F}_\gamma g(a\lambda)|^2 \frac{da}{a}. \tag{4.3}$$

Then

$$G_{\varepsilon,\delta} \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx), \qquad K_{\varepsilon,\delta} \in \left(L^1 \cap L^2\right)(\mathbb{R}, |x|^{2\gamma+1}dx), \tag{4.4}$$

and

$$\mathcal{F}_\gamma(G_{\varepsilon,\delta}) = K_{\varepsilon,\delta}.$$

Proof. Using Schwarz inequality for the measure $\dfrac{da}{a^{4\gamma+5}}$ we obtain

$$|G_{\varepsilon,\delta}(x)|^2 \leq \frac{1}{(C_g^\gamma)^2} \left(\int_\varepsilon^\delta \frac{da}{a^{4\gamma+5}}\right) \int_\varepsilon^\delta |g_a *_\gamma \widetilde{g}_a(x)|^2 \frac{da}{a^{4\gamma+5}},$$

so

$$\int_\mathbb{R} |G_{\varepsilon,\delta}(x)|^2 |x|^{2\gamma+1} dx \leq \frac{1}{(C_g^\gamma)^2} \left(\int_\varepsilon^\delta \frac{da}{a^{4\gamma+5}}\right) \int_\varepsilon^\delta \int_\mathbb{R} |g_a *_\gamma \widetilde{g}_a(x)|^2 |x|^{2\gamma+1} dx \frac{da}{a^{4\gamma+5}}.$$

By Theorem 2.3, Lemma 4.8, and Remark 4.4, we have

$$\begin{aligned}
\int_\mathbb{R} |g_a *_\gamma \widetilde{g}_a(x)|^2 |x|^{2\gamma+1} dx &= m_\gamma \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)(\lambda)|^4 |\lambda|^{2\gamma+1} d\lambda \\
&\leq m_\gamma \, ||\mathcal{F}_\gamma(g_a)||_\infty^2 \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)(\lambda)|^2 |\lambda|^{2\gamma+1} d\lambda \\
&= ||\mathcal{F}_\gamma(g_a)||_\infty^2 \, ||g_a||_{2,\gamma}^2 = a^{6\gamma+6} \, ||\mathcal{F}_\gamma g||_\infty^2 \, ||g||_{2,\gamma}^2 .
\end{aligned}$$

Hence

$$\int_\mathbb{R} |G_{\varepsilon,\delta}(x)|^2 |x|^{2\gamma+1} dx \leq \frac{||\mathcal{F}_\gamma g||_\infty^2 \, ||g||_{2,\gamma}^2}{(C_g^\gamma)^2} \left(\int_\varepsilon^\delta a^{2\gamma+1} da\right) \left(\int_\varepsilon^\delta \frac{da}{a^{4\gamma+5}}\right) < \infty.$$

The second assertion in (4.4) is easily checked. Let us calculate $\mathcal{F}_\gamma(G_{\varepsilon,\delta})$. Fix $x \in \mathbb{R}$. From Theorem 2.3 and Lemma 4.7 we get

$$g_a *_\gamma \widetilde{g}_a(x) = m_\gamma \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)(\lambda)|^2 e_\gamma(i\lambda x) |\lambda|^{2\gamma+1} d\lambda,$$

so

$$G_{\varepsilon,\delta}(x) = \frac{m_\gamma}{C_g^\gamma} \int_\varepsilon^\delta \left(\int_\mathbb{R} |\mathcal{F}_\gamma(g_a)(\lambda)|^2 e_\gamma(i\lambda x) |\lambda|^{2\gamma+1} d\lambda\right) \frac{da}{a^{4\gamma+5}}.$$

As $|e_\gamma(iz)| \leq 1$ for all $z \in \mathbb{R}$ (see [12]), we deduce by Theorem 2.3 that

$$m_\gamma \int_\varepsilon^\delta \int_\mathbb{R} |\mathcal{F}_\gamma(g_a)(\lambda)|^2 |e_\gamma(i\lambda x)| |\lambda|^{2\gamma+1} d\lambda \frac{da}{a^{4\gamma+5}}$$

$$\leq \int_\varepsilon^\delta ||g_a||_{2,\gamma}^2 \frac{da}{a^{4\gamma+5}} = ||g||_{2,\gamma}^2 \int_\varepsilon^\delta \frac{da}{a^{2\gamma+3}} < \infty.$$

Hence, applying Fubini's theorem, we find that

$$\begin{aligned} G_{\varepsilon,\delta}(x) &= m_\gamma \int_{\mathbb{R}} \left(\frac{1}{C_g^\gamma} \int_\varepsilon^\delta |\mathcal{F}_\gamma g(a\lambda)|^2 \frac{da}{a} \right) e_\gamma(i\lambda x)|\lambda|^{2\gamma+1} d\lambda \\ &= m_\gamma \int_{\mathbb{R}} K_{\varepsilon,\delta}(\lambda) e_\gamma(i\lambda x)|\lambda|^{2\gamma+1} d\lambda \end{aligned}$$

which completes the proof. ■

We can now state the main result of this section.

Theorem 4.10 (Calderón's formula). *Let $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ be a Dunkl wavelet of order γ such that $\mathcal{F}_\gamma g \in L^\infty(\mathbb{R})$. Then for $f \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and $0 < \varepsilon < \delta < \infty$, the function*

$$f^{\varepsilon,\delta}(x) := \frac{1}{C_g^\gamma} \int_\varepsilon^\delta \int_{\mathbb{R}} \Phi_g^\gamma(f)(a,b) g_{a,b}(x)|b|^{2\gamma+1} db \frac{da}{a}$$

belongs to $L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and satisfies

$$\lim_{\varepsilon\to 0,\, \delta\to\infty} \left\| f^{\varepsilon,\delta} - f \right\|_{2,\gamma} = 0. \tag{4.5}$$

Proof. It is easily seen that

$$f^{\varepsilon,\delta} = f *_\gamma G_{\varepsilon,\delta},$$

where $G_{\varepsilon,\delta}$ is given by (4.2). It follows by Lemmas 4.7 and 4.9 that $f^{\varepsilon,\delta} \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ and $\mathcal{F}_\gamma(f^{\varepsilon,\delta}) = \mathcal{F}_\gamma(f)\, K_{\varepsilon,\delta}$, where $K_{\varepsilon,\delta}$ is as in (4.3). From this and Theorem 2.3 we obtain

$$\begin{aligned} \left\| f^{\varepsilon,\delta} - f \right\|_{2,\gamma}^2 &= m_\gamma \int_{\mathbb{R}} |\mathcal{F}_\gamma(f^{\varepsilon,\delta} - f)(\lambda)|^2 |\lambda|^{2\gamma+1} d\lambda \\ &= m_\gamma \int_{\mathbb{R}} |\mathcal{F}_\gamma f(\lambda)|^2 (1 - K_{\varepsilon,\delta}(\lambda))^2 |\lambda|^{2\gamma+1} d\lambda. \end{aligned}$$

But by (4.1) we have

$$\lim_{\varepsilon\to 0,\, \delta\to\infty} K_{\varepsilon,\delta}(\lambda) = 1, \qquad \text{for almost all} \quad \lambda \in \mathbb{R}.$$

So (4.5) follows from the dominated convergence theorem. ■

Another pointwise inversion formula for the Dunkl wavelet transform, proved in [5], is as follows.

Theorem 4.11. *Let $g \in L^2(\mathbb{R}, |x|^{2\gamma+1}dx)$ be a Dunkl wavelet of order γ. If both f and $\mathcal{F}_\gamma f$ are in $L^1(\mathbb{R}, |x|^{2\gamma+1}dx)$ then we have*

$$f(x) = \frac{1}{C_g^\gamma} \int_0^\infty \left(\int_{\mathbb{R}} \Phi_g^\gamma(f)(a,b) g_{a,b}^\gamma(x)|b|^{2\gamma+1} db \right) \frac{da}{a}, \quad a.e.,$$

where, for each $x \in \mathbb{R}$, both the inner integral and the outer integral are absolutely convergent, but possibly not the double integral.

5 Inversion of the dual Dunkl–Sonine transform using Dunkl wavelets

From now on assume $\beta > \alpha > -1/2$. In order to invert the dual Dunkl–Sonine transform, we need the following two technical lemmas.

Lemma 5.1. *Let* $0 \neq g \in L^1 \cap L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$ *such that* $\mathcal{F}_\alpha g \in L^1(\mathbb{R}, |x|^{2\alpha+1}dx)$ *and satisfying*

$$\exists\, \eta > \beta - 2\alpha - 1 \qquad \text{such that} \quad \mathcal{F}_\alpha g(\lambda) = \mathcal{O}\left(|\lambda|^\eta\right) \quad \text{as} \quad \lambda \to 0. \tag{5.1}$$

Then $\mathcal{X}_{\alpha,\beta} g \in L^2(\mathbb{R}, |x|^{2\beta+1}dx)$ *and*

$$\mathcal{F}_\beta(\mathcal{X}_{\alpha,\beta} g)(\lambda) = \frac{m_\alpha}{m_\beta} \frac{\mathcal{F}_\alpha g(\lambda)}{|\lambda|^{2(\beta-\alpha)}}.$$

Proof. By Theorem 2.3 we have

$$g(x) = m_\alpha \int_{\mathbb{R}} \mathcal{F}_\alpha g(\lambda) e_\alpha(i\lambda x) |\lambda|^{2\alpha+1} d\lambda, \qquad \text{a.e.}$$

So using (3.6), we find that

$$\mathcal{X}_{\alpha,\beta} g(x) = m_\beta \int_{\mathbb{R}} h_{\alpha,\beta}(\lambda) e_\beta(i\lambda x) |\lambda|^{2\beta+1} d\lambda, \qquad \text{a.e.} \tag{5.2}$$

with

$$h_{\alpha,\beta}(\lambda) := \frac{m_\alpha}{m_\beta} \frac{\mathcal{F}_\alpha g(\lambda)}{|\lambda|^{2(\beta-\alpha)}}.$$

Clearly, $h_{\alpha,\beta} \in L^1(\mathbb{R}, |x|^{2\beta+1}dx)$. So it suffices, in view of (5.2) and Theorem 2.3, to prove that $h_{\alpha,\beta}$ belongs to $L^2(\mathbb{R}, |x|^{2\beta+1}dx)$. We have

$$\begin{aligned} \int_{\mathbb{R}} |h_{\alpha,\beta}(\lambda)|^2 |\lambda|^{2\beta+1} d\lambda &= \left(\frac{m_\alpha}{m_\beta}\right)^2 \int_{\mathbb{R}} |\lambda|^{4\alpha-2\beta+1} |\mathcal{F}_\alpha g(\lambda)|^2 d\lambda \\ &= \left(\frac{m_\alpha}{m_\beta}\right)^2 \left(\int_{|\lambda|\leq 1} + \int_{|\lambda|\geq 1}\right) |\lambda|^{4\alpha-2\beta+1} |\mathcal{F}_\alpha g(\lambda)|^2 d\lambda := I_1 + I_2. \end{aligned}$$

By (5.1) there is a positive constant k such that

$$I_1 \leq k \int_{|\lambda|\leq 1} |\lambda|^{2\eta+4\alpha-2\beta+1} d\lambda = \frac{k}{\eta + 2\alpha - \beta + 1} < \infty.$$

From Theorem 2.3, it follows that

$$\begin{aligned} I_2 &= \left(\frac{m_\alpha}{m_\beta}\right)^2 \int_{|\lambda|\geq 1} |\lambda|^{2(\alpha-\beta)} |\mathcal{F}_\alpha g(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \\ &\leq \left(\frac{m_\alpha}{m_\beta}\right)^2 \int_{|\lambda|\geq 1} |\mathcal{F}_\alpha g(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \leq \left(\frac{m_\alpha}{m_\beta}\right)^2 ||\mathcal{F}_\alpha g||_{2,\alpha}^2 = \frac{m_\alpha}{(m_\beta)^2} ||g||_{2,\alpha}^2 < \infty \end{aligned}$$

which ends the proof. ■

Lemma 5.2. *Let* $0 \neq g \in L^1 \cap L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$ *be real-valued such that* $\mathcal{F}_\alpha g \in L^1(\mathbb{R}, |x|^{2\alpha+1}dx)$ *and satisfying*

$$\exists\, \eta > 2(\beta - \alpha) \qquad \text{such that} \quad \mathcal{F}_\alpha g(\lambda) = \mathcal{O}(\lambda^\eta) \quad \text{as} \quad \lambda \to 0^+. \tag{5.3}$$

Then $\mathcal{X}_{\alpha,\beta} g \in L^2(\mathbb{R}, |x|^{2\beta+1}dx)$ *is a Dunkl wavelet of order* β *and* $\mathcal{F}_\beta(\mathcal{X}_{\alpha,\beta} g) \in L^\infty(\mathbb{R})$.

Proof. By combining (5.3) and Lemma 5.1 we see that $\mathcal{X}_{\alpha,\beta} g \in L^2(\mathbb{R}, |x|^{2\beta+1}dx)$, $\mathcal{F}_\beta(\mathcal{X}_{\alpha,\beta} g)$ is bounded and

$$\mathcal{F}_\beta(\mathcal{X}_{\alpha,\beta} g)(\lambda) = \mathcal{O}\big(\lambda^{\eta - 2(\beta-\alpha)}\big) \qquad \text{as} \quad \lambda \to 0^+.$$

Thus, in view of Remark 4.2, $\mathcal{X}_{\alpha,\beta} g$ satisfies the admissibility condition (4.1) for $\gamma = \beta$. ■

Remark 5.3. In view of Remark 4.2, each function satisfying the conditions of Lemma 5.1 is a Dunkl wavelet of order α.

Lemma 5.4. *Let* g *be as in Lemma* 5.2*. Then for all* $f \in L^1(\mathbb{R}, |x|^{2\beta+1}dx)$ *we have*

$$\Phi^\beta_{\mathcal{X}_{\alpha,\beta} g}(f)(a,b) = \frac{1}{a^{2(\beta-\alpha)}}\, \mathcal{X}_{\alpha,\beta}\big[\Phi^\alpha_g\left({}^t\mathcal{X}_{\alpha,\beta} f\right)(a,\cdot)\big](b).$$

Proof. By Definition 4.5 we have

$$\Phi^\beta_{\mathcal{X}_{\alpha,\beta} g}(f)(a,b) = \frac{1}{a^{2\beta+2}}\, f *_\beta \big(\widetilde{\mathcal{X}_{\alpha,\beta} g}\big)_a(b).$$

But $\big(\widetilde{\mathcal{X}_{\alpha,\beta} g}\big)_a = \mathcal{X}_{\alpha,\beta}\left(\widetilde{g}_a\right)$ by virtue of (1.2) and (3.5). So using (3.8) we find that

$$\begin{aligned}\Phi^\beta_{\mathcal{X}_{\alpha,\beta} g}(f)(a,b) &= \frac{1}{a^{2\beta+2}} f *_\beta \left[\mathcal{X}_{\alpha,\beta}\left(\widetilde{g}_a\right)\right](b)\\ &= \frac{1}{a^{2\beta+2}} \mathcal{X}_{\alpha,\beta}\left[{}^t\mathcal{X}_{\alpha,\beta} f *_\alpha \widetilde{g}_a\right](b) = \frac{1}{a^{2(\beta-\alpha)}} \mathcal{X}_{\alpha,\beta}\big[\Phi^\alpha_g\left({}^t\mathcal{X}_{\alpha,\beta} f\right)(a,\cdot)\big](b),\end{aligned}$$

which gives the desired result. ■

Combining Theorems 4.10, 4.11 with Lemmas 5.2, 5.4 we get

Theorem 5.5. *Let* g *be as in Lemma* 5.2*. Then we have the following inversion formulas for the dual Dunkl–Sonine transform:*

(i) If both f *and* $\mathcal{F}_\beta f$ *are in* $L^1(\mathbb{R}, |x|^{2\beta+1}dx)$ *then for almost all* $x \in \mathbb{R}$ *we have*

$$f(x) = \frac{1}{C^\beta_{\mathcal{X}_{\alpha,\beta} g}} \int_0^\infty \left(\int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta}\left[\Phi^\alpha_g\left({}^t\mathcal{X}_{\alpha,\beta} f\right)(a,\cdot)\right](b) \big(\mathcal{X}_{\alpha,\beta} g\big)^\beta_{a,b}(x) |b|^{2\beta+1} db\right) \frac{da}{a^{2(\beta-\alpha)+1}}.$$

(ii) For $f \in L^1 \cap L^2(\mathbb{R}, |x|^{2\beta+1}dx)$ *and* $0 < \varepsilon < \delta < \infty$*, the function*

$$f^{\varepsilon,\delta}(x) := \frac{1}{C^\beta_{\mathcal{X}_{\alpha,\beta} g}} \int_\varepsilon^\delta \int_{\mathbb{R}} \mathcal{X}_{\alpha,\beta}\left[\Phi^\alpha_g\left({}^t\mathcal{X}_{\alpha,\beta} f\right)(a,\cdot)\right](b) \big(\mathcal{X}_{\alpha,\beta} g\big)^\beta_{a,b}(x) |b|^{2\beta+1} db\, \frac{da}{a^{2(\beta-\alpha)+1}}$$

satisfies

$$\lim_{\varepsilon\to 0,\, \delta\to\infty} \big\| f^{\varepsilon,\delta} - f \big\|_{2,\beta} = 0.$$

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