Modularity, Atomicity and States in Archimedean Lattice Effect Algebras*

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Abstract. Effect algebras are a generalization of many structures which arise in quantum physics and in mathematical economics. We show that, in every modular Archimedean atomic lattice effect algebra E that is not an orthomodular lattice there exists an (o)-continuous state ω on E, which is subadditive. Moreover, we show properties of finite and compact elements of such lattice effect algebras.

Key words: effect algebra; state; modular lattice; finite element; compact element

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1 Introduction, basic definitions and some known facts

Effect algebras (introduced by D.J. Foulis and M.K. Bennett in [10] for modelling unsharp measurements in a Hilbert space) may be carriers of states or probabilities when events are noncompatible or unsharp resp. fuzzy. In this setting, the set $\mathcal{E}(\mathcal{H})$ of effects on a Hilbert space $\mathcal{H}$ is the set of all Hermitian operators on $\mathcal{H}$ between the null operator 0 and the identity operator 1, and the partial operation $\oplus$ is the restriction of the usual operator sum. D.J. Foulis and M.K. Bennett recognized that effect algebras are equivalent to D-posets introduced in general form by F. Kôpka and F. Chovanec (see [19]), firstly defined as axiomatic systems of fuzzy sets by F. Kôpka in [18].

Effect algebras are a generalization of many structures which arise in quantum physics (see [2]) and in mathematical economics (see [8, 9]). There are some basic ingredients in the study of the mathematical foundations of physics, typically the fundamental concepts are states, observables and symmetries. These concepts are tied together in [11] by employing effect algebras.

It is a remarkable fact that there are even finite effect algebras admitting no states, hence no probabilities. The smallest of them has only nine elements (see [28]). One possibility for eliminating this unfavourable situation is to consider modular complete lattice effect algebras (see [32]).

Having this in mind, we are going to show that, in every modular Archimedean atomic lattice effect algebra E that is not an orthomodular lattice there exists an (o)-continuous state ω on E, which is subadditive.

We show some further important properties of finite elements in modular lattice effect algebras. Namely, the set G of all finite elements in a modular lattice effect algebra is a lattice ideal of E. Moreover, any compact element in an Archimedean lattice effect algebra E is a finite join of finite elements of E.

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Definition 1 ([10]). A partial algebra $(E;\oplus,0,1)$ is called an *effect algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x,y,z\in E$:

(Ei) $x\oplus y=y\oplus x$ if $x\oplus y$ is defined,

(Eii) $(x\oplus y)\oplus z=x\oplus(y\oplus z)$ if one side is defined,

$(Eiii)$ for every $x\in E$ there exists a unique $y\in E$ such that $x\oplus y=1$ (we put $x'=y$),

(Eiv) if $1\oplus x$ is defined then $x=0$.

We put $\perp=\{(x,y)\in E\times E \mid x\oplus y \text{ is defined}\}$. We often denote the effect algebra $(E;\oplus,0,1)$ briefly by E. On every effect algebra E the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x\leq y \text{ and } y\ominus x=z \text{ iff } x\oplus z \text{ is defined and } x\oplus z=y.$$

Elements x and y of an effect algebra E are said to be (*Mackey*) *compatible* ($x\leftrightarrow y$ for short) iff there exist elements $x_1,y_1,d\in E$ with $x=x_1\oplus d$, $y=y_1\oplus d$ and $x_1\oplus y_1\oplus d\in E$.

If E with the defined partial order is a lattice (a complete lattice) then $(E;\oplus,0,1)$ is called a *lattice effect algebra* (*a complete lattice effect algebra*).

If, moreover, E is a modular or distributive lattice then E is called *modular* or *distributive* effect algebra.

Lattice effect algebras generalize two important structures: orthomodular lattices and MV-algebras. In fact a lattice effect algebra $(E;\oplus,0,1)$ is an orthomodular lattice [17] iff $x\wedge x'=0$ for every $x\in E$ (i.e., every $x\in E$ is a *sharp element*). A lattice effect algebra can be organized into an MV-algebra [4] (by extending $\oplus$ to a total binary operation on E) iff any two elements of E are compatible iff $(x\vee y)\ominus x=y\ominus(x\wedge y))$ for every pair of elements $x,y\in E$ [20, 5].

A minimal nonzero element of an effect algebra E is called an *atom* and E is called *atomic* if under every nonzero element of E there is an atom.

We say that a finite system $F=(x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra $(E;\oplus,0,1)$ is $\oplus$-*orthogonal* if $x_1\oplus x_2\oplus\cdots\oplus x_n$ (written $\bigoplus\limits_{k=1}^{n} x_k$ or $\bigoplus F$) exists in E. Here we define $x_1\oplus x_2\oplus\cdots\oplus x_n=(x_1\oplus x_2\oplus\cdots\oplus x_{n-1})\oplus x_n$ supposing that $\bigoplus\limits_{k=1}^{n-1} x_k$ is defined and $\bigoplus\limits_{k=1}^{n-1} x_k\leq x_n'$. We also define $\bigoplus\varnothing=0$. An arbitrary system $G=(x_\kappa)_{\kappa\in H}$ of not necessarily different elements of E is called $\oplus$-orthogonal if $\bigoplus K$ exists for every finite $K\subseteq G$. We say that for a $\oplus$-orthogonal system $G=(x_\kappa)_{\kappa\in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K\subseteq G$ is finite$\}$ exists in E and then we put $\bigoplus G=\bigvee\{\bigoplus K\mid K\subseteq G$ is finite$\}$. (Here we write $G_1\subseteq G$ iff there is $H_1\subseteq H$ such that $G_1=(x_\kappa)_{\kappa\in H_1}$).

An element $u\in E$ is called *finite* if either $u=0$ or there is a finite sequence $\{a_1,a_2,\ldots,a_n\}$ of not necessarily different atoms of E such that $u=a_1\oplus a_2\oplus\cdots\oplus a_n$. Note that any atom of E is evidently finite.

For an element x of an effect algebra E we write $\mathrm{ord}(x)=\infty$ if $nx=x\oplus x\oplus\cdots\oplus x$ (n-times) exists for every positive integer n and we write $\mathrm{ord}(x)=n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\mathrm{ord}(x)<\infty$ for all $x\in E$.

Definition 2. Let E be an effect algebra. Then $Q\subseteq E$ is called a *sub-effect algebra* of E if

(i) $0,1\in Q$,

(ii) if $x,y\in Q$ then $x'\in Q$ and $x\perp y\Longrightarrow x\oplus y\in Q$.

If E is a lattice effect algebra and Q is a sub-lattice and a sub-effect algebra of E then Q is called a *sub-lattice effect algebra* of E.

Note that a sub-effect algebra Q (sub-lattice effect algebra Q) of an effect algebra E (of a lattice effect algebra E) with inherited operation $\oplus$ is an effect algebra (lattice effect algebra) in its own right.

Let E be an effect algebra and let $(E_\kappa)_{\kappa\in H}$ be a family of sub-effect algebras of E such that:

(i) $E = \bigcup_{\kappa\in H} E_\kappa$.

(ii) If $x \in E_{\kappa_1} \setminus \{0,1\}$, $y \in E_{\kappa_2} \setminus \{0,1\}$ and $\kappa_1 \neq \kappa_2$, $\kappa_1, \kappa_2 \in H$, then $x \wedge y = 0$ and $x \vee y = 1$.

Then E is called the *horizontal sum* of effect algebras $(E_\kappa)_{\kappa\in H}$.

Important sub-lattice effect algebras of a lattice effect algebra E are

(i) $S(E) = \{x \in E \mid x \wedge x' = 0\}$ the *set of all sharp elements of* E (see [13, 14]), which is an orthomodular lattice (see [16]).

(ii) Maximal subsets of pairwise compatible elements of E called *blocks* of E (see [26]), which are in fact maximal sub-MV-algebras of E.

(iii) The *center of compatibility* $B(E)$ of E, $B(E) = \bigcap\{M \subseteq E \mid M$ is a block of $E\} = \{x \in E \mid x \leftrightarrow y$ for every $y \in E\}$ which is in fact an MV-algebra (MV-effect algebra).

(iv) The *center* $C(E) = \{x \in E \mid y = (y \wedge x) \vee (y \wedge x')$ for all $y \in E\}$ of E which is a Boolean algebra (see [12]). In every lattice effect algebra it holds $C(E) = B(E) \cap S(E)$ (see [24, 25]).

For a poset P and its subposet $Q \subseteq P$ we denote, for all $X \subseteq Q$, by $\bigvee_Q X$ the join of the subset X in the poset Q whenever it exists.

For a study of effect algebras, we refer to [7].

2 Finite elements, modularity and atomicity in lattice effect algebras

It is quite natural, for a lattice effect algebra, to investigate whether the join of two finite elements is again finite and whether each element below a finite element is again finite. The following example shows that generally it is not the case.

Example 1. Let B be an infinite complete atomic Boolean algebra, C a finite chain MV-algebra. Then

1. The set F of finite elements of the horizontal sum of B and C is not closed under order (namely, the top element 1 is finite but the coatoms from B are not finite).
2. The set F of finite elements of the horizontal sum of two copies of B is not closed under join (namely, the join of two atoms in different copies of B is the top element which is not finite).

Theorem 1. *Let E be a modular lattice effect algebra, $x \in E$. Then*

(i) *If x is finite, then, for every $y \in E$, $(x \vee y) \ominus y$ is finite* [1, Proposition 2.16].

(ii) *If x is finite, then every $z \in E$, $z \leq x$ is finite.*

(iii) *If x is finite, then every chain in the interval $[0, x]$ is finite.*

(iv) *If x is finite, then $[0, x]$ is a complete lattice.*

(v) *If x and y are finite, then $x \vee y$ is finite.*

(vi) *The set F of all finite elements of E is a lattice ideal of E.*

Proof. (i) See [1, Proposition 2.16].

(ii) This follows at once from [1, Proposition 2.16] by putting $y = x \ominus z$.

(iii) This is an immediate consequence of [1, Proposition 2.15] that gives a characterization of finite elements in modular lattice effect algebras using the height function and of general and well-known facts about modular lattices (which can be found, for instance, in [15, § VII.4] and are also recalled in [1, § 2.2, page 5]).

(iv) Since the interval $[0, x]$ has no infinite chains it is complete by [6, Theorem 2.41].

(v) Indeed $x \vee y = ((x \vee y) \ominus y) \oplus y$ and, clearly, the sum of finite elements is finite.

(vi) It follows immediately from the above facts. ■

Special types of effect algebras called sharply dominating and S-dominating have been introduced by S. Gudder in [13, 14]. Important example is the standard Hilbert spaces effect algebra $\mathcal{E}(\mathcal{H})$.

Definition 3 ([13, 14]). An effect algebra $(E, \oplus, 0, 1)$ is called *sharply dominating* if for every $a \in E$ there exists a smallest sharp element $\hat{a}$ such that $a \leq \hat{a}$. That is $\hat{a} \in S(E)$ and if $b \in S(E)$ satisfies $a \leq b$ then $\hat{a} \leq b$.

Similarly to [33, Theorem 2.7] we have the following.

Theorem 2. *Let E be a modular Archimedean lattice effect algebra and let $E_1 = \{x \in E \mid x$ is finite or x' is finite$\}$. Then*

- (i) *E_1 is a sub-lattice effect algebra of E.*
- (ii) *For every finite $x \in E$, there exist a smallest sharp element $\widehat{x}$ over x and a greatest sharp element $\widetilde{x}$ under x.*
- (iii) *E_1 is sharply dominating.*

Proof. (i): Clearly, $x \in E_1$ iff $x' \in E_1$ by definition of E_1. Further for any finite $x, y \in E_1$ we have by Theorem 1 that $x \vee y \in E_1$ and $x \oplus y \in E_1$ whenever $x \oplus y$ exists. The rest follows by de Morgan laws and the fact that $v \leq u$, u is finite implies v is finite (Theorem 1).

(ii), (iii): Let $x = \bigoplus_{i=1}^{n} k_i a_i$ for some set $\{a_1, \ldots, a_n\}$ of atoms of E. Clearly, for any index $j, 1 \leq j \leq n$, $k_j a_j \wedge \bigoplus_{i=1, i\neq j}^{n} k_i a_i = 0$ and $\bigoplus_{i=1, i\neq j}^{n} k_i a_i \leq (k_j a_j)'$. Hence by [22, Lemma 3.3] $n_{a_j} a_j \wedge \bigoplus_{i=1, i\neq j}^{n} k_i a_i = 0$ and $\bigoplus_{i=1, i\neq j}^{n} k_i a_i \leq (n_{a_j} a_j)'$. By a successive application of the above argument this yields the existence of the sum $\bigoplus_{i=1}^{n} n_{a_i} a_i$. Then by Theorem 1 the interval $[0, \widehat{x}]$, $\widehat{x} = \bigoplus_{i=1}^{n} n_{a_i} a_i$ is a complete lattice effect algebra, hence it is sharply dominating. Moreover by [34, Theorem 3.5], $\bigoplus_{i=1}^{n} n_{a_i} a_i$ is the smallest sharp element $\widehat{x}$ over x. It follows by [13] that there exists a greatest sharp element $\widetilde{x}$ under x in $[0, \widehat{x}]$ and so in E and E_1 as well.

If $x' = \bigoplus_{i=1}^{m} l_i b_i$ for some set $\{b_1, \ldots, b_m\}$ of atoms of E then $w = \bigoplus_{i=1}^{m} n_{b_i} b_i$ is the smallest sharp element over x'. Hence w' is the greatest sharp element under x both in E and E_1. ■

Note that, in any effect algebra E, the following infinite distributive law holds (see [7, Proposition 1.8.7]):

$$\Big(\bigvee_{\alpha} c_\alpha\Big) \oplus b = \bigvee_{\alpha} (c_\alpha \oplus b)$$

provided that $\bigvee_\alpha c_\alpha$ and $(\bigvee_\alpha c_\alpha) \oplus b$ exist.

Proposition 1. *Let $\{b_\alpha \mid \alpha \in \Lambda\}$ be a family of elements in a lattice effect algebra E and let $a \in E$ with $a \leq b_\alpha$ for all $\alpha \in \Lambda$. Then*

$$\Big(\bigvee \{b_\alpha \mid \alpha \in \Lambda\}\Big) \ominus a = \bigvee \{b_\alpha \ominus a \mid \alpha \in \Lambda\}$$

if one side is defined.

Proof. Assume first that $(\bigvee\{b_\alpha \mid \alpha \in \Lambda\}) \ominus a$ is defined. Then $\bigvee\{b_\alpha \mid \alpha \in \Lambda\}$ exists. Clearly, $b_\alpha \ominus a \leq (\bigvee\{b_\alpha \mid \alpha \in \Lambda\}) \ominus a$ for all $\alpha \in \Lambda$. Let $b_\alpha \ominus a \leq c$ for all $\alpha \in \Lambda$. Let us put $d = c \wedge ((\bigvee\{b_\alpha \mid \alpha \in \Lambda\}) \ominus a)$. Then $b_\alpha \ominus a \leq d$ for all $\alpha \in \Lambda$ and $d \leq (\bigvee\{b_\alpha \mid \alpha \in \Lambda\}) \ominus a$. Hence $d \oplus a$ exists and $b_\alpha \leq d \oplus a$ for all $\alpha \in \Lambda$. This yields $\bigvee\{b_\alpha \mid \alpha \in \Lambda\} \leq d \oplus a$. Consequently, $\bigvee\{b_\alpha \mid \alpha \in \Lambda\} \ominus a \leq d$, so $d = \bigvee\{b_\alpha \mid \alpha \in \Lambda\} \ominus a \leq c$.

Now, assume that $\bigvee\{b_\alpha \ominus a \mid \alpha \in \Lambda\}$ is defined. Then $\bigvee\{b_\alpha \ominus a \mid \alpha \in \Lambda\} \leq 1 \ominus a$, which gives $\bigvee\{b_\alpha \ominus a \mid \alpha \in \Lambda\} \oplus a$ exists. Hence by the above infinite distributive law

$$\bigvee\{b_\alpha \ominus a \mid \alpha \in \Lambda\} \oplus a = \bigvee\{(b_\alpha \ominus a) \oplus a \mid \alpha \in \Lambda\} = \bigvee\{b_\alpha \mid \alpha \in \Lambda\}. \qquad \blacksquare$$

Now we are ready for the next proposition that was motivated by [35, § 6, Theorem 20] for complete modular lattices.

Proposition 2. *Let E be a modular lattice effect algebra, $z \in E$ and let $F_z = \{x \in E \mid x \text{ is finite}, x \leq z\}$ and suppose that $\bigvee F_z = z$. Then the interval $[0, z]$ is atomic.*

Proof. Let $0 \neq y \in E$, $y \leq z$. We shall show that there exists an atom $a \leq y$. We have (by the same argument as in [1, Lemma 3.1 (a)] for complete lattices) that

$$\bigvee\{(x \vee (z \ominus y)) \ominus (z \ominus y) \mid x \in F_z\}$$
$$= \bigvee\{x \vee (z \ominus y) \mid x \in F_z\} \ominus (z \ominus y) = z \ominus (z \ominus y) = y.$$

This yields $(x \vee (z \ominus y)) \ominus (z \ominus y) \neq 0$ for some $x \in F_z$. By Theorem 1 *(i)* $(x \vee (z \ominus y)) \ominus (z \ominus y) \in F_z$, $(x \vee (z \ominus y)) \ominus (z \ominus y) \leq y$. Hence, there exists an atom $a \leq (x \vee (z \ominus y)) \ominus (z \ominus y) \leq y$. ■

Corollary 1. *Let E be a modular lattice effect algebra and let $F = \{x \in E \mid x \text{ is finite}\}$ and suppose that $\bigvee F = 1$. Then E is atomic.*

Corollary 2. *Let E be a modular lattice effect algebra. Let at least one block M of E be Archimedean and atomic. Then E is atomic.*

Proof. Let us put $F = \{x \in E \mid x \text{ is finite}\}$. Clearly, F contains all finite elements of the block M. Hence by [30, Theorem 3.3] we have $1 = \bigvee_M (F \cap M)$. From [23, Lemma 2.7] we obtain that the joins in E and M coincide. Therefore $1 = \bigvee_M (F \cap M) = \bigvee_E (F \cap M) \leq \bigvee_E F$. By Corollary 1 we get that E is atomic. ■

Further recall that an element u of a lattice L is called a *compact element* if, for any $D \subseteq L$ with $\bigvee D \in L$, $u \leq \bigvee D$ implies $u \leq \bigvee F$ for some finite $F \subseteq D$.

Moreover, the lattice L is called *compactly generated* if every element of L is a join of compact elements.

It was proved in [21, Theorem 6] that every compactly generated lattice effect algebra is atomic. If moreover E is Archimedean then every compact element $u \in E$ is finite [21, Lemma 4] and conversely [23, Lemma 2.5].

Example 2 ([33, Example 2.9]). If a is an atom of a compactly generated Archimedean lattice effect algebra E (hence atomic) then $n_a a$ need not be an atom of $S(E)$.

Indeed, let E be a horizontal sum of a Boolean algebra $B = \{0, a, a', 1 = a \oplus a'\}$ and a chain $M = \{0, b, 1 = 2b\}$. Then $S(E) = B$ and $1 = 2b$ is not an atom of $S(E)$.

Remark 1. The atomicity of the set of sharp elements $S(E)$ is not completely solved till now. For example, if E is a complete modular Archimedean atomic lattice effect algebra then $S(E)$ is an atomic orthomodular lattice (see [23]).

This remark leads us to

Proposition 3. *Let E be a modular Archimedean atomic lattice effect algebra. Then $S(E)$ is an atomic orthomodular lattice.*

Proof. Let $x \in S(E)$, $x \neq 0$. From [30, Theorem 3.3] we get that there is an atom a of E such that $n_a a \leq x$. Then by Theorem 1 the interval $[0, n_a a]$ is a complete modular atomic lattice effect algebra. This yields that $[0, n_a a]$ is a compactly generated complete modular lattice effect algebra and all elements of $[0, n_a a]$ are compact in $[0, n_a a]$. Hence also $S([0, n_a a])$ is a compactly generated complete modular lattice effect algebra i.e. it is atomic. Clearly, any atom p of $S([0, n_a a])$ is an atom of $S(E)$ and $p \leq x$. ∎

A *basic algebra* [3] (*lattice with sectional antitone involutions*) is a system $L = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice such that every principal order-filter $[a, 1]$ (which is called a *section*) possesses an antitone involution $x \mapsto x^a$.

Clearly, any principal ideal $[0, x]$, $x \in L$ of a basic algebra L is again a basic algebra. Moreover, any lattice effect algebra is a basic algebra. Note that every interval $[a, b]$, for $a < b$ in an effect algebra E can be organized (in a natural way) into an effect algebra (see [36, Theorem 1]), hence every interval $[a, b]$ in E possesses an antitone involution.

The following Lemma is in fact implicitly contained in the proof of [21, Theorem 5].

Lemma 1 ([21, Theorem 5]). *Let L be a basic algebra, $u \in L$, $u \neq 0$ a compact element. Then there is an atom $a \in L$ such that $a \leq u$.*

Lemma 2. *Let E be an Archimedean lattice effect algebra, $u \in E$ a compact element. Then u is a finite join of finite elements of E.*

Proof. If $u = 0$ we are finished. Let $u \neq 0$. Let $\mathcal{Q}$ be a maximal pairwise compatible subset of finite elements of E under u. Clearly, $0 \in \mathcal{Q} \neq \emptyset$. Assume that u is not the smallest upper bound of $\mathcal{Q}$ in E. Hence there is an element $c \in E$ such that c is an upper bound of $\mathcal{Q}$, $c \not\geq u$. Let us put $d = c \wedge u$. Then $d < u$. Clearly, the interval $[d, 1]$ is a basic algebra and u is compact in $[d, 1]$. Hence by Lemma 1 there is an atom $b \in [d, 1]$ such that $b \leq u$. Let us put $a = b \ominus d$. Then a is an atom of E. Let M be a block of E containing the compatible set $\mathcal{Q} \cup \{d, b, u\}$. Evidently $a \in M$ and $\mathcal{Q} \cup \{q \oplus a \mid q \in \mathcal{Q}\} \subseteq M$ is a compatible subset of finite elements of E under u. From the maximality of $\mathcal{Q}$ we get that $\{q \oplus a \mid q \in \mathcal{Q}\} \subseteq \mathcal{Q}$. Hence, for all $n \in \mathbb{N}$, $na \in \mathcal{Q}$, a contradiction with the assuption that E is Archimedean. Therefore $u = \bigvee \mathcal{Q}$. Since u is compact there are finitely many finite elements $q_1, \ldots, q_n$ of $\mathcal{Q}$ such that $u = \bigvee_{i=1}^{n} q_i$. ∎

Remark 2. The condition that E is Archimedean in Lemma 2 cannot be omitted (e.g., the Chang MV-effect algebra $E = \{0, a, 2a, 3a, \ldots, (3a)', (2a)', a', 1\}$ is not Archimedean, every $x \in E$ is compact and the top element 1 is not a finite join of finite elements of E).

Corollary 3. *Let E be a modular Archimedean lattice effect algebra, $u \in E$ a compact element. Then u is finite.*

Proof. Since u is a finite join of finite elements of E and in a modular lattice effect algebra a finite join of finite elements is finite by Theorem 1 we are done. ∎

Thus we obtain the following common corollary of Proposition 2 and Corollary 3.

Theorem 3. *Let E be a modular Archimedean lattice effect algebra, $z \in E$ and let $C_z = \{x \in E \mid x \text{ is compact}, x \leq z\}$ and suppose that $\bigvee C_z = z$. Then the interval $[0, z]$ is atomic. Moreover, if $z = 1$ and $\bigvee C_1 = 1$ then E is atomic.*

3 States on modular Archimedean atomic lattice effect algebras

The aim of this section is to apply results of previous section in order to study (o)-continuous states on modular Archimedean atomic lattice effect algebras.

Definition 4. Let E be an effect algebra. A map $\omega : E \to [0,1]$ is called a *state* on E if $\omega(0) = 0$, $\omega(1) = 1$ and $\omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \oplus y$ exists in E. If, moreover, E is lattice ordered then ω is called *subadditive* if $\omega(x \vee y) \leq \omega(x) + \omega(y)$, for all $x, y \in E$.

It is easy to check that the notion of a state ω on an orthomodular lattice L coincides with the notion of a state on its derived effect algebra L. It is because $x \leq y'$ iff $x \oplus y$ exists in L, hence $\omega(x \vee y) = \omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \leq y'$ (see [17]).

It is easy to verify that, if ω is a subadditive state on a lattice effect algebra E, then in fact $\omega(x) + \omega(y) = \omega(x \vee y) + \omega(x \wedge y)$ for all $x, y \in E$ (see [29, Theorem 2.5]), so that ω is a *modular measure*, as defined for example in [1, § 5, page 13].

Assume that $(\mathcal{E}; \prec)$ is a directed set and E is an effect algebra. A net of elements of E is denoted by $(x_\alpha)_{\alpha \in \mathcal{E}}$. Then $x_\alpha \uparrow x$ means that $x_{\alpha_1} \leq x_{\alpha_2}$ for every $\alpha_1 \prec \alpha_2$, $\alpha_1, \alpha_2 \in \mathcal{E}$ and $x = \bigvee\{x_\alpha \mid \alpha \in \mathcal{E}\}$. The meaning of $x_\alpha \downarrow x$ is dual. A net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of an effect algebra E *order converges to a point* $x \in E$ if there are nets $(u_\alpha)_{\alpha \in \mathcal{E}}$ and $(v_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E such that

$$u_\alpha \uparrow x,\ v_\alpha \downarrow x,\ \text{and } u_\alpha \leq x_\alpha \leq v_\alpha \text{ for all } \alpha \in \mathcal{E}.$$

We write $x_\alpha \xrightarrow{(o)} x$, $\alpha \in \mathcal{E}$ *in* E (or briefly $x_\alpha \xrightarrow{(o)} x$).

A state ω is called (o)-*continuous* (*order-continuous*) if, for every net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E, $x_\alpha \xrightarrow{(o)} x \implies \omega(x_\alpha) \to \omega(x)$ (equivalently $x_\alpha \uparrow x \Rightarrow \omega(x_\alpha) \uparrow \omega(x)$).

We are going to prove statements about the existence of (o)-continuous states which are subadditive.

Theorem 4. *Let E be a Archimedean atomic lattice effect algebra, $c \in C(E)$, c finite in E, $c \neq 0$, $[0,c]$ a modular lattice. Then there exists an (o)-continuous state ω on E, which is subadditive.*

Proof. Note that, for every central element z of a lattice effect algebra E, the interval $[0,z]$ with the $\oplus$ operation inherited from E and the new unit z is a lattice effect algebra in its own right.

Since c is central we have have the direct product decomposition $E \cong [0,c] \times [0,c']$. Hence $E = \{y \oplus z \mid y \in [0,c], z \in [0,c']\}$.

Since c is finite in E and hence in $[0,c]$ we have by Theorem 1 that the interval $[0,c]$ is a complete modular atomic lattice effect algebra. From [32, Theorem 4.2] we get a subadditive (o)-continuous state ω_c on $[0,c]$.

Let us define $\omega : E \to [0,1] \subseteq \mathbb{R}$ by setting $\omega(x) = \omega_c(y)$, for every $x = y \oplus z$, $y \in [0,c], z \in [0,c']$. It is easy to check that ω is an (o)-continuous state on E, which is subadditive. These properties follow by the fact that the effect algebra operations as well as the lattice operations on the direct product $[0,c] \times [0,c']$ are defined coordinatewise and ω_c is a state on the complete modular atomic lattice effect algebra $[0,c]$ with all enumerated properties. ■

Corollary 4. *Let E be a modular Archimedean atomic lattice effect algebra, $c \in C(E)$, c finite in E, $c \neq 0$. Then there exists an (o)-continuous state ω on E, which is subadditive.*

In [16] it was proved that for every lattice effect algebra E the subset $S(E)$ is an orthomodular lattice. It follows that E is an orthomodular lattice iff $E = S(E)$. If E is atomic then E is an orthomodular lattice iff $a \in S(E)$ for every atom a of E. This is because if $x \in E$ with $x \wedge x' \neq 0$ exists then there exists an atom a of E with $a \leq x \wedge x'$, which gives $a \leq x' \leq a'$ and hence $a \wedge a' = a \neq 0$, a contradiction.

Theorem 5. *Let E be an Archimedean atomic lattice effect algebra with $S(E) \neq E$. Let $F = \{x \in E \mid x \text{ is finite}\}$ be an ideal of E such that F is a modular lattice. Then there exists an (o)-continuous state ω on E, which is subadditive.*

Proof. Let $x \in E \setminus S(E)$. From Theorem [30, Theorem 3.3] we have that there are mutually distinct atoms $a_\alpha \in E$ and positive integers k_α, $\alpha \in \mathcal{E}$ such that $x = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\}$, and $x \in S(E)$ iff $k_\alpha = n_{a_\alpha} = \mathrm{ord}(a_\alpha)$ for all $\alpha \in \mathcal{E}$. Hence there is an atom $a \in E$ such that $a \notin S(E)$ i.e., $a \leq a'$.

We shall proceed similarly as in [32, Theorem 3.1].

(i): Assume that $a \in B(E)$. Then also $n_a a \in B(E)$ (by [26]) and, by [31, Theorem 2.4], $n_a a \in S(E)$. Thus $n_a a \in B(E) \cap S(E) = C(E)$.

(ii): Assume now that $a \notin B(E)$. Then there exists an atom $b \in E$ with $b \not\leftrightarrow a$. As F is a modular lattice we have $[0, b] = [a \wedge b, b] \cong [a, a \vee b]$ which yields that $a \vee b$ covers a both in F and E. Hence there exists an atom $c \in E$ such that $a \oplus c = a \vee b$, which gives $c \leq a'$. Evidently, $c \neq b$ as $b \not\leq a'$. If $c \neq a$ then $a \vee c = a \oplus c = a \vee b$, which implies $b \leq a \vee c \leq a'$, a contradiction. Thus $c = a$ and $a \vee b = 2a$.

Let $p \in E$ be an atom. Then either $p \not\leftrightarrow a$ which, as we have just shown, implies that $p \leq p \vee a = 2a \leq n_a a$, or $p \leftrightarrow a$ and hence $p \leftrightarrow n_a a$ for every atom $p \in E$. By [27], for every $x \in E$ we have $x = \bigvee\{u \in E | u \leq x,\ u \text{ is a sum of finite sequence of atoms}\}$. Since $n_a a \leftrightarrow p$ for every atom p and hence $n_a a \leftrightarrow u$ for every finite sum u of atoms, we conclude that $n_a a \leftrightarrow x$ for every $x \in E$. Thus again $n_a a \in B(E) \cap S(E) = C(E)$.

Since $n_a a$ is finite and the interval $[0, n_a a]$ is modular we can apply Theorem 4. ■

Corollary 5. *Let E be a modular Archimedean atomic lattice effect algebra with $S(E) \neq E$. Then there exists an (o)-continuous state ω on E, which is subadditive.*

Proof. It follows immediately from Theorem 1 (vi) and Theorem 5. ■

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