A Particular Solution of a Painlevé System in Terms of the Hypergeometric Function ${}_{n+1}F_n$ ⋆

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Abstract. In a recent work, we proposed the coupled Painlevé VI system with $A^{(1)}_{2n+1}$-symmetry, which is a higher order generalization of the sixth Painlevé equation (P_{VI}). In this article, we present its particular solution expressed in terms of the hypergeometric function ${}_{n+1}F_n$. We also discuss a degeneration structure of the Painlevé system derived from the confluence of ${}_{n+1}F_n$.

Key words: affine Weyl group; generalized hypergeometric functions; Painlevé equations

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1 Introduction

The main object in this article is the coupled Painlevé VI system with $A^{(1)}_{2n+1}$-symmetry given in [1, 4], or equivalently, the Hamiltonian system $\mathcal{H}_{n+1,1}$ given in [6]. It is expressed as a Hamiltonian system on $\mathbb{P}^1(\mathbb{C})$

$$t(t-1)\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad t(t-1)\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1,\dots,n, \tag{1}$$

with

$$H = \sum_{i=1}^{n} H_{\mathrm{VI}}\left[\sum_{j=0}^{n}\alpha_{2j+1} - \alpha_{2i-1} - \eta, \sum_{j=0}^{i-1}\alpha_{2j}, \sum_{j=i}^{n}\alpha_{2j}, \alpha_{2i-1}\eta; q_i, p_i\right] + \sum_{1\le i<j\le n}(q_i-1)(q_j-t)\{(q_ip_i+\alpha_{2i-1})p_j + p_i(q_jp_j+\alpha_{2j-1})\},$$

where H_{VI} is the Hamiltonian for P_{VI} defined as

$$H_{\mathrm{VI}}[\kappa_0,\kappa_1,\kappa_t,\kappa;q,p] = q(q-1)(q-t)p^2 - \kappa_0(q-1)(q-t)p - \kappa_1 q(q-t)p - (\kappa_t-1)q(q-1)p + \kappa q.$$

Here $\alpha_0,\dots,\alpha_{2n+1}$ and η are fixed parameters satisfying a relation $\sum_{i=0}^{2n+1}\alpha_i = 1$. We assume that the indices of α_i are congruent modulo $2n+2$. Note that the system (1) includes P_{VI} as the case $n = 1$. The aim of this article is to present a particular solution of the system (1) expressed in terms of the hypergeometric function ${}_{n+1}F_n$.

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The hypergeometric function ${}_{n+1}F_n$ is defined by the power series

$$ {}_{n+1}F_n \left[\begin{array}{c} a_0, \ldots, a_n \\ b_1, \ldots, b_n \end{array} ; t \right] = \sum_{i=0}^{\infty} \frac{(a_0)_i (a_1)_i \cdots (a_n)_i}{(1)_i (b_1)_i \cdots (b_n)_i} t^i, $$

where $(a)_i$ stands for the factorial function

$$ (a)_i = a(a+1)\cdots(a+i-1) = \frac{\Gamma(a+i)}{\Gamma(a)}. $$

Denoting by $\delta = td/dt$, we see that $x = {}_{n+1}F_n$ satisfies an $(n+1)$-th order linear differential equation

$$ [\delta(\delta + b_1 - 1)\cdots(\delta + b_n - 1) - t(\delta + a_0)\cdots(\delta + a_n)]x = 0, \tag{2} $$

which is called *the generalized hypergeometric equation* [3]. The equation (2) is of Fuchsian type with regular singular points at $t = 0, 1, \infty$ and the Riemann scheme

$$ \left[\begin{array}{ccc} t=0 & t=1 & t=\infty \\ 0 & 0 & a_0 \\ 1-b_1 & 1 & a_1 \\ \vdots & \vdots & \vdots \\ 1-b_{n-1} & n-1 & a_{n-1} \\ 1-b_n & -\sum_{i=1}^{n}(1-b_i) - \sum_{i=0}^{n} a_i & a_n \end{array} \right]. $$

Note that ${}_{n+1}F_n$ includes the Gauss hypergeometric function as the case $n = 1$.

In this article, we clarify a relationship between the system (1) and the function ${}_{n+1}F_n$. For $n = 1$ among them, the relationship between P_{VI} and the Gauss hypergeometric function is well known. Under the system (1) of the case $n = 1$, we consider a specialization $p = \eta = 0$. Then we obtain a Riccati equation

$$ t(t-1)\frac{dq}{dt} = \alpha_1 q^2 + \{(\alpha_3 + \alpha_0)t - (\alpha_0 + \alpha_1)\} q - \alpha_3 t. $$

Via a transformation of a dependent variable

$$ q = \frac{t(1-t)}{\alpha_1} \frac{d}{dt} \log\{(t-1)^{\alpha_3} x(t)\}, $$

we obtain the Gauss hypergeometric equation

$$ [\delta(\delta + \alpha_2 + \alpha_3 - 1) - t(\delta + \alpha_1 + \alpha_2 + \alpha_3)(\delta + \alpha_3)]x = 0. $$

The result of this article gives a natural extension of the above fact. For general n, we consider a specialization $p_1 = \cdots = p_n = \eta = 0$. Then we obtain the generalized hypergeometric equation by a certain transformation of dependent variables.

We also discuss a degeneration structure of the system (1) derived from the confluence of ${}_{n+1}F_n$. The confluent hypergeometric functions ${}_{n-r+1}F_n$ $(r = 1, \ldots, n+1)$ are defined by the power series

$$ {}_{n-r+1}F_n \left[\begin{array}{c} a_r, \ldots, a_n \\ b_1, \ldots, b_n \end{array} ; t \right] = \sum_{i=0}^{\infty} \frac{(a_r)_i \cdots (a_n)_i}{(b_1)_i \cdots (b_n)_i} t^i. $$

The process of confluence ${}_{n-r+2}F_n \to {}_{n-r+1}F_n$ is given by a replacement

$$t \to \varepsilon t, \qquad a_{r-1} \to \varepsilon^{-1},$$

and taking a limit $\varepsilon \to 0$. We see that $x = {}_{n-r+1}F_n$ satisfy the confluent hypergeometric differential equations

$$[\delta(\delta + b_1 - 1) \cdots (\delta + b_n - 1) - t(\delta + a_r) \cdots (\delta + a_n)]x = 0. \tag{3}$$

In this article, we propose a class of higher order Painlevé systems which admit particular solutions expressed in terms of ${}_{n-r+1}F_n$.

Remark 1. In this article, we study a higher order generalization of $P_{\rm VI}$. On the other hand, for a multi-time generalization, it is known that the Garnier system admits a particular solution in terms of the Appell–Lauricella hypergeometric function F_D [2].

This article is organized as follows. In Section 2, we derive a system of linear differential equations from the system (1) by a specialization $p_1 = \cdots = p_n = \eta = 0$. In Section 3, we give its fundamental solutions expressed in terms of the hypergeometric function ${}_{n+1}F_n$ in a neighborhood of the singular point $t = 0$. In Section 4, we discuss a degeneration structure of the system (1) derived from the confluence of ${}_{n+1}F_n$.

2 Linear differential equations

In this section, we derive a system of linear differential equations from the system (1) by a specialization $p_1 = \cdots = p_n = \eta = 0$.

We first consider a *symmetric form* of (1) in order to derive a system of linear differential equations. Let x_i, y_i $(i = 0, \ldots, n)$ be dependent variables such that

$$t(1-t)\frac{d}{dt}\log x_n = \sum_{i=1}^{n} \{(q_i - 1)(q_i - t)p_i + \alpha_{2i-1}q_i\} + t\alpha_{2n+1} - (t+1)\eta,$$

and

$$x_{i-1} = \frac{x_n q_i}{t}, \qquad y_{i-1} = \frac{tp_i}{x_n}, \qquad i = 1, \ldots, n, \qquad y_n = -\frac{1}{x_n}\left(\sum_{j=1}^{n} q_j p_j + \eta\right).$$

Then we obtain a Hamiltonian system of $(2n+2)$-th order

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \qquad i = 0, \ldots, n, \tag{4}$$

with a Hamiltonian

$$\begin{aligned} H = \frac{1}{t}\sum_{i=0}^{n}\left\{\frac{1}{2}x_i^2 y_i^2 - \alpha_{2i+2}^{2n-2i-1}x_i y_i + \sum_{j=0}^{i-1} x_i(x_i y_i + \alpha_{2i+1})y_j\right\} \\ + \frac{1}{1-t}\sum_{i=0}^{n}\sum_{j=0}^{n} x_i(x_i y_i + \alpha_{2i+1})y_j, \end{aligned}$$

where

$$\alpha_k^l = \begin{cases} 0, & l \in \mathbb{Z}_{<0}, \\ \displaystyle\sum_{i=k}^{k+l} \alpha_i, & l \in \mathbb{Z}_{\geq 0}. \end{cases}$$

The dependent variables x_i, y_i and the fixed parameter η satisfy a relation

$$\sum_{i=0}^{n} x_i y_i + \eta = 0.$$

Remark 2. The symmetric form (4) is suggested by the Hamiltonian system given in Theorem 3.2 of [4]. Their relationship is given by

$$t = \frac{1}{t_1^{n+1}}, \qquad x_i = \frac{w_{2i+1}}{t_1^{i-n+\rho_1+\kappa_{2n+1}-\kappa_0}}, \qquad y_i = \frac{t_1^{i-n+\rho_1+\kappa_{2n+1}-\kappa_0}\varphi_{2i+1}}{n+1},$$

and

$$\eta = \sum_{j=0}^{n} \frac{\rho_1 + \kappa_{2i} - \kappa_{2i+1}}{n+1}, \qquad \alpha_{2i} = \frac{1 + \kappa_{2i-1} - 2\kappa_{2i} + \kappa_{2i+1}}{n+1},$$
$$\alpha_{2i+1} = \frac{\kappa_{2i} - 2\kappa_{2i+1} + \kappa_{2i+2}}{n+1},$$

for $i = 0, \dots, n$.

Remark 3. The system (4), or equivalently the system (1), admits the affine Weyl group symmetry of type $A_{2n+1}^{(1)}$; see Appendix B.

We can derive easily a system of linear differential equations from the symmetric form by the specialization $y_0 = \dots = y_n = \eta = 0$, which is equivalent to $p_1 = \dots = p_n = \eta = 0$. Let $E_{i,j}$ be the matrix unit defined by

$$E_{i,j} = (\delta_{i,k}\delta_{j,l})_{k,l=0}^{n}.$$

For example, in the case $n = 2$, it is explicitly given as

$$E_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad E_{2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad E_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we obtain

Proposition 1. *The system* (4) *admits a specialization*

$$y_i = 0, \qquad i = 0, \dots, n, \qquad \eta = 0.$$

Then a vector of the variables $\mathbf{x} = {}^t(x_0, \dots, x_n)$ *satisfies a system of linear differential equations on* $\mathbb{P}^1(\mathbb{C})$

$$\frac{d\mathbf{x}}{dt} = \left(\frac{A_0}{t} + \frac{A_1}{1-t} \right) \mathbf{x}, \tag{5}$$

with

$$A_0 = \sum_{i=0}^{n-1} \left(-\alpha_{2i+2}^{2n-2i-1} \right) E_{i,i} + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \alpha_{2j+1} E_{i,j}, \qquad A_1 = \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{2j+1} E_{i,j}.$$

Furthermore, the system (5) *is of Fuchsian type with regular singular points at* $t = 0, 1, \infty$. *The data of eigenvalues of its residue matrices is given as*

$$-\alpha_2^{2n-1}, \dots, -\alpha_{2n}^{1}, 0 \quad \textit{at} \quad t = 0,$$
$$0, \dots, 0, -\sum_{i=0}^{n} \alpha_{2i+1} \quad \textit{at} \quad t = 1,$$
$$\alpha_1^{2n}, \dots, \alpha_{2n-1}^{2}, \alpha_{2n+1} \quad \textit{at} \quad t = \infty.$$

Remark 4. The system (4) also admits a specialization

$$x_i = 0, \qquad i = 0, \dots, n-1, \qquad x_n y_n + \eta = 0, \qquad \eta - \alpha_{2n+1} = 0,$$

which is equivalent to $q_1 = \cdots = q_n = \eta - \alpha_{2n+1} = 0$. Then a vector of the variables $\mathbf{y} = {}^t(y_0, \dots, y_n)$ satisfies a system of linear differential equations

$$\frac{d\mathbf{y}}{dt} = \left(\frac{A_0}{t} + \frac{A_1}{1-t} \right) \mathbf{y},$$

with

$$A_0 = \sum_{i=0}^{n-1} \alpha_{2i+2}^{2n-2i-1} E_{i,i} + \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} (-\alpha_{2i+1}) E_{i,j} + \sum_{j=0}^{n} \alpha_{2n+1} E_{n,j},$$

$$A_1 = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-\alpha_{2i+1}) E_{i,j} + \sum_{j=0}^{n} \alpha_{2n+1} E_{n,j}.$$

We always assume that

$$\alpha_{2i}^{2j-1} \notin \mathbb{Z}, \qquad \sum_{i=0}^{n} \alpha_{2i+1} \notin \mathbb{Z}, \qquad \alpha_{2i-1}^{2j-1} \notin \mathbb{Z}, \qquad i = 1, \dots, n, \quad j = 1, \dots, n-i+1.$$

In the next section, we describe fundamental solutions of the system (5) in a neighborhood of the singular point $t = 0$ explicitly.

3 Hypergeometric function ${}_{n+1}F_n$

In this section, we give fundamental solutions of the system (5) expressed in terms of the hypergeometric function ${}_{n+1}F_n$ in a neighborhood of the singular point $t = 0$.

For each $k = 0, \dots, n$, we consider a gauge transformation

$$\mathbf{x}^k = t^{\alpha_{2k+2}^{2n-2k-1}} \left(\sum_{i=0}^{n-k-1} t^{-1} E_{i,i+k+1} + \sum_{i=n-k}^{n} E_{i,i-n+k} \right) \mathbf{x}.$$

Then the system (5) is transformed into

$$\frac{d\mathbf{x}^k}{dt} = \left(\frac{A_0^k}{t} + \frac{A_1^k}{1-t} \right) \mathbf{x}^k, \tag{6}$$

with

$$A_0^k = \sum_{i=0}^{n-1} (-\alpha_{2k+2i+4}^{2n-2i-1}) E_{i,i} + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \alpha_{2j+2k+3} E_{i,j},$$

$$A_1^k = \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{2j+2k+3} E_{i,j}.$$

Recall that indices of the fixed parameters α_i are congruent modulo $2n+2$, from which we have $\alpha_{2k+2n+2}^1 = \alpha_{2k}^1$. We also consider a formal power series of $\mathbf{x}^k$ at $t = 0$

$$\mathbf{x}^k = \sum_{i=0}^{\infty} \mathbf{x}_i^k t^i.$$

Then the system (6) implies

$$\begin{aligned} &A_0^k \mathbf{x}_0^k = \mathbf{0}, \\ &\{A_0^k - (i+1)I\}\mathbf{x}_{i+1}^k = \left(A_0^k - A_1^k - iI\right)\mathbf{x}_i^k, \qquad i \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{7}$$

where I stands for the identity matrix. The matrices A_0^k and $A_0^k - (i+1)I$ are of rank n and $n+1$, respectively. It follows that the recurrence formula (7) admits one parameter family of solutions.

For each $k = 0, \ldots, n$, we can show that a sequence of vectors

$$\mathbf{x}_i^k = \begin{bmatrix} \prod_{j=0}^{n-1} \frac{(\alpha_{2k-2j+1}^{2j})_{i+1}}{(\alpha_{2k-2j}^{2j+1})_{i+1}} \cdot \frac{(\alpha_{2k+3}^{2n})_i}{(\alpha_{2k+2}^{2n+1})_i} \\ \prod_{j=0}^{n-2} \frac{(\alpha_{2k-2j+1}^{2j})_{i+1}}{(\alpha_{2k-2j}^{2j+1})_{i+1}} \cdot \frac{(\alpha_{2k+5}^{2n-2})_i(\alpha_{2k+3}^{2n})_i}{(\alpha_{2k+4}^{2n-1})_i(\alpha_{2k+2}^{2n+1})_i} \\ \vdots \\ \frac{(\alpha_{2k+1})_{i+1}}{(\alpha_{2k}^{1})_{i+1}} \prod_{j=0}^{n-1} \frac{(\alpha_{2k+2j+3}^{2n-2j})_i}{(\alpha_{2k+2j+2}^{2n-2j+1})_i} \\ \prod_{j=0}^{n} \frac{(\alpha_{2k+2j+3}^{2n-2j})_i}{(\alpha_{2k+2j+2}^{2n-2j+1})_i} \end{bmatrix}, \qquad i \in \mathbb{Z}_{\geq 0},$$

satisfies the recurrence formula (7) by a direct computation. Note that $\alpha_{2k+2}^{2n+1} = 1$. Therefore we arrive at

Theorem 1. *On a domain $|t| < 1$, the system (5) admits fundamental solutions*

$$\mathbf{x} = t^{-\alpha_{2k+2}^{2n-2k-1}} \begin{bmatrix} f^{k,k} \\ \vdots \\ f^{k,0} \\ tf^{k,n} \\ \vdots \\ tf^{k,k+1} \end{bmatrix}, \qquad k = 0, \ldots, n,$$

where

$$f^{k,l} = \prod_{i=1}^{l} \frac{\alpha_{2k-2i+3}^{2i-2}}{\alpha_{2k-2i+2}^{2i-1}} \cdot {}_{n+1}F_n \left[\begin{matrix} a_0, \ldots, a_n \\ b_1, \ldots, b_n \end{matrix} ; t \right],$$

with

$$\begin{aligned} &a_0 = \alpha_{2k-2n+1}^{2n}, \\ &a_i = 1 + \alpha_{2k-2i+3}^{2i-2}, \qquad b_i = 1 + \alpha_{2k-2i+2}^{2i-1}, \qquad i = 1, \ldots, l, \\ &a_i = \alpha_{2k-2i+3}^{2i-2}, \qquad b_i = \alpha_{2k-2i+2}^{2i-1}, \qquad i = l+1, \ldots, n. \end{aligned}$$

Corollary 1. *If the vector $\mathbf{x} = {}^t(x_0, \ldots, x_n)$ satisfies the system (5), each component x_i satisfies the generalized hypergeometric equation (2) with*

$$\begin{aligned} &a_0 = \alpha_1^{2n}, \\ &a_j = 1 + \alpha_{2n-2j+3}^{2j-2}, \qquad b_j = 1 + \alpha_{2n-2j+2}^{2j-1}, \qquad j = 1, \ldots, n-i, \\ &a_j = \alpha_{2n-2j+3}^{2j-2}, \qquad b_j = \alpha_{2n-2j+2}^{2j-1}, \qquad j = n-i+1, \ldots, n. \end{aligned}$$

Remark 5. The system (5) have been already studied by Okubo–Takano–Yoshida [3]. They considered the Fuchsian differential equation of Okubo type and obtained its fundamental solutions at singular points $t = 0, 1$.

4 Degeneration structure

In this section, we discuss a degeneration structure of the system (1) derived from the confluence of ${}_{n+1}F_n$.

For each $r = 1, \dots, n+1$, we consider a Hamiltonian system

$$
{}_{n-r+1}\mathcal{H}_n : \quad \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \qquad i = 0, \dots, n,
$$

with a Hamiltonian

$$
tH = \sum_{i=0}^{n} \frac{1}{2} x_i y_i \big(x_i y_i - 2\alpha_{2i+2}^{2n-2i-1}\big) + \sum_{i=0}^{r-2} x_{i+1} y_i + \sum_{i=r-1}^{n} \left\{ t x_0 + \sum_{j=i+1}^{n} x_j (x_j y_j + \alpha_{2j+1}) \right\} y_i.
$$

Here α_i $(i = 0, \dots, 2n+1)$ and η are fixed parameters satisfying

$$
\alpha_{2i} = 0, \quad i = 0, \dots, r-1, \qquad \sum_{j=0}^{n} \alpha_{2j+1} + \sum_{j=r}^{n} \alpha_{2j} = 1,
$$

and

$$
\sum_{j=0}^{n} x_j y_j + \eta = 0.
$$

Note that

$$
\alpha_{2i+2}^{2n-2i-1} = \sum_{j=2i+2}^{2n+1} \alpha_j = \sum_{j=i+1}^{n} \alpha_{2j+1} + \sum_{j=\max(r,i+1)}^{n} \alpha_{2j}.
$$

The system ${}_{n-r+1}\mathcal{H}_n$ is obtained from ${}_{n-r+2}\mathcal{H}_n$ by a replacement

$$
\begin{aligned}
&t \to \varepsilon t, \qquad \alpha_{2r-2} \to -\varepsilon^{-1}, \qquad \alpha_{2r-1} \to \alpha_{2r-1} + \varepsilon^{-1}, \\
&x_i \to \varepsilon^{-1} x_i, \qquad y_i \to \varepsilon y_i, \qquad i = 0, \dots, r-2.
\end{aligned}
$$

and taking a limit $\varepsilon \to 0$, where ${}_{n+1}\mathcal{H}_n$ stands for the system (4).

Remark 6. Such degenerate systems also can be rewritten into the Hamiltonian systems in terms of the canonical coordinates. We give their explicit formulas for $n = 1$ and $n = 2$ in Appendix A.

The system ${}_{n-r+1}\mathcal{H}_n$ admits a specialization

$$
y_i = 0, \quad i = 0, \dots, n, \qquad \eta = 0.
$$

Then a vector of the variables $\mathbf{x} = {}^t(x_0, \dots, x_n)$ satisfies a system of linear differential equations

$$
{}_{n-r+1}\mathcal{L}_n : \quad \frac{d\mathbf{x}}{dt} = \left(\frac{A_0}{t} + A_1 \right) \mathbf{x},
$$

with

$$
\begin{aligned}
A_0 &= \sum_{i=0}^{n-1} \big(-\alpha_{2i+2}^{2n-2i-1} \big) E_{i,i} + \sum_{i=0}^{r-2} E_{i,i+1} + \sum_{i=r-1}^{n-1} \sum_{j=i+1}^{n} \alpha_{2j+1} E_{i,j}, \\
A_1 &= \sum_{i=r-1}^{n} E_{i,0}.
\end{aligned}
$$

Note that ${}_{n-r+1}\mathcal{L}_n$ is obtained from ${}_{n-r+2}\mathcal{L}_n$ through the above process of confluence.

In a similar manner as Section 3, we arrive at

Theorem 2. *On a domain $|t| < 1$, the system ${}_{n-r+1}\mathcal{L}_n$ admits fundamental solutions*

$$\mathbf{x} = t^{-\alpha_{2k+2}^{2n-2k-1}} \begin{bmatrix} f_r^{k,k} \\ \vdots \\ f_r^{k,0} \\ t f_r^{k,n} \\ \vdots \\ t f_r^{k,k+1} \end{bmatrix}, \qquad k = 0, \ldots, n,$$

where

$$f_r^{k,l} = \prod_{\substack{1 \le i \le l \\ \mathrm{mod}[k-i+1, n+1] \ge r}} \alpha_{2k-2i+3}^{2i-2} \prod_{1 \le i \le l} \frac{1}{\alpha_{2k-2i+2}^{2i-1}} \cdot {}_{n-r+1}F_n \left[\begin{matrix} a_r, \ldots, a_n \\ b_1, \ldots, b_n \end{matrix} ; t \right],$$

and

$$\mathrm{mod}[i, n+1] = i - m(n+1) \qquad \text{for} \quad m(n+1) \le i < (m+1)(n+1).$$

Here the parameters $a_r, \ldots, a_n$ are given by

$$a_i = \alpha_{2r-2i-1}^{2k-2r+2i+2}, \qquad i = r, \ldots, n,$$

for $k + 1 \le r$ and $l < k + 2$;

$$\begin{aligned} a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r, \ldots, r-k+l-2, \\ a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r-k+l-1, \ldots, n, \end{aligned}$$

for $k + 1 \le r$ and $k + 2 \le l$;

$$\begin{aligned} a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r, \ldots, n+r-k-1, \\ a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= n+r-k, \ldots, n+r-k+l-1, \\ a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= n+r-k+l, \ldots, n, \end{aligned}$$

for $r < k + 1$ and $l < k - r + 1$;

$$\begin{aligned} a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r, \ldots, n+r-k-1, \\ a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= n+r-k, \ldots, n, \end{aligned}$$

for $r < k + 1$ and $k - r + 1 \le l < k + 2$;

$$\begin{aligned} a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r, \ldots, r-k+l-2, \\ a_i &= \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= r-k+l-1, \ldots, n+r-k-1, \\ a_i &= 1 + \alpha_{2r-2i-1}^{2k-2r+2i+2}, & i &= n+r-k, \ldots, n, \end{aligned}$$

for $r < k + 1$ and $k + 2 \le l$. The parameters $b_1, \ldots, b_n$ are given by

$$\begin{aligned} b_i &= 1 + \alpha_{2k-2i+2}^{2i-1}, & i &= 1, \ldots, l, \\ b_i &= \alpha_{2k-2i+2}^{2i-1}, & i &= l+1, \ldots, n. \end{aligned}$$

Corollary 2. *If the vector* $\mathbf{x} = {}^t(x_0, \ldots, x_n)$ *satisfies the system* ${}_{n-r+1}\mathcal{L}_n$*, each component* x_i *satisfies the confluent hypergeometric equation* (3) *with*

$$a_j = 1 + \alpha_{2r-2j-1}^{2n-2r+2j+2}, \qquad j = r, \ldots, n,$$

for $i \leq r-1$*,*

$$a_j = 1 + \alpha_{2r-2j-1}^{2n-2r+2j+2}, \qquad j = r, \ldots, n+r-i-1,$$
$$a_j = \alpha_{2r-2j-1}^{2n-2r+2j+2}, \qquad j = n+r-i, \ldots, n,$$

for $r-1 < i$ *and*

$$b_j = 1 + \alpha_{2n-2j+2}^{2j-1}, \qquad j = 1, \ldots, n-i,$$
$$b_j = \alpha_{2n-2j+2}^{2j-1}, \qquad j = n-i+1, \ldots, n,$$

for any i*.*

A Canonical Hamiltonian system

The systems ${}_{n-r+1}\mathcal{H}_n$ can be rewritten into the Hamiltonian systems in terms of canonical coordinates. In this section, we give their explicit formulas for $n = 1$ and $n = 2$. Note that ${}_{3-r}\mathcal{H}_2$ appear in the classification of the fourth order Painlevé type differential equations [5].

A.1 Case $n = 1$, $r = 1$

Under the system ${}_1\mathcal{H}_1$, we take canonical coordinates

$$q = \frac{x_0}{x_1}, \qquad p = -\frac{x_1(x_1y_1 + \alpha_3)}{x_0}.$$

Via a transformation of the independent variable $t \to -t$, we obtain a Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

with a Hamiltonian

$$tH = q(q-1)p(p+t) - qp(\eta + \alpha_2 - \alpha_3) + (\eta - \alpha_3)p + t\alpha_3 q.$$

It is equivalent to the fifth Painlevé equation.

A.2 Case $n = 1$, $r = 2$

Under the system ${}_0\mathcal{H}_1$, we take canonical coordinates

$$q = \frac{x_1}{x_0}, \qquad p = x_0y_1.$$

Then we obtain a Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

with a Hamiltonian

$$tH = q^2p(p-1) + (\eta + \alpha_3)qp + tp - \eta q.$$

It is equivalent to the third Painlevé equation.

A.3 Case $n = 2$, $r = 1$

Under the system ${}_2\mathcal{H}_2$, we take canonical coordinates

$$q_1 = \frac{x_0}{x_1}, \qquad p_1 = -\frac{x_1(x_1y_1 + \alpha_3)}{x_0}, \qquad q_2 = \frac{x_0}{x_2}, \qquad p_2 = -\frac{x_2(x_2y_2 + \alpha_5)}{x_0}.$$

Via a transformation of the independent variable $t \to -t$, we obtain a Hamiltonian system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2,$$

with a Hamiltonian

$$\begin{aligned} tH = {} & q_1(q_1 - 1)p_1(p_1 + t) - (\eta + \alpha_2 - \alpha_3 - \alpha_5)q_1p_1 + (\eta - \alpha_3 - \alpha_5)p_1 \\ & + \alpha_3 t q_1 + (q_1 - 1)p_1q_2p_2 + (q_1 - 1)(q_1p_1 + \alpha_3)p_2 \\ & + q_2(q_2 - 1)p_2(p_2 + t) - (\eta + \alpha_2 + \alpha_4 - \alpha_5)q_2p_2 + (\eta - \alpha_5)p_2 + \alpha_5 t q_2. \end{aligned}$$

A.4 Case $n = 2$, $r = 2$

Under the system ${}_1\mathcal{H}_2$, we take canonical coordinates

$$q_1 = -\frac{x_1}{x_0}, \qquad p_1 = 1 - x_0y_1, \qquad q_2 = -\frac{x_2}{x_0}, \qquad p_2 = -x_0y_2.$$

Via a transformation of the independent variable $t \to -t$, we obtain a Hamiltonian system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2,$$

with a Hamiltonian

$$\begin{aligned} tH = {} & q_1^2p_1(p_1 - 1) + (\eta + \alpha_3)q_1p_1 + tp_1 - \alpha_3q_1 + q_1p_1q_2p_2 + p_1q_2(q_2p_2 + \alpha_5) \\ & + q_2^2p_2(p_2 - 1) + (\eta + \alpha_3 + \alpha_4 + \alpha_5)q_2p_2 + tp_2 - \alpha_5q_2. \end{aligned}$$

A.5 Case $n = 2$, $r = 3$

Under the system ${}_0\mathcal{H}_2$, we take canonical coordinates

$$q_1 = -\frac{x_1}{x_0}, \qquad p_1 = 1 - x_0y_1, \qquad q_2 = -\frac{x_2}{x_0}, \qquad p_2 = -x_0y_2.$$

Via a transformation of the independent variable $t \to -t$, we obtain a Hamiltonian system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2,$$

with a Hamiltonian

$$\begin{aligned} tH = {} & q_1^2p_1(p_1 - 1) + (\eta + \alpha_3)q_1p_1 - \alpha_3q_1 + q_1p_1q_2p_2 + p_1q_2 \\ & + q_2^2p_2^2 + (\eta + \alpha_3 + \alpha_5)q_2p_2 + tp_2 - q_2. \end{aligned}$$

B Affine Weyl group symmetry

The system (4) admits the affine Weyl group symmetry of type $A^{(1)}_{2n+1}$. In this section, we describe its action on the dependent variables and parameters.

Recall that the affine Weyl group of type $A^{(1)}_{2n+1}$ is generated by the transformations r_i $(i = 0,\dots,2n+1)$ with the fundamental relations

$$r_i^2 = 1, \qquad i = 0,\dots,2n+1,$$
$$(r_i r_j)^{2-a_{i,j}} = 0, \qquad i,j = 0,\dots,2n+1, \quad i \neq j,$$

where

$$a_{i,i} = 2, \qquad i = 0,\dots,2n+1,$$
$$a_{i,i+1} = a_{2n+1,0} = a_{i+1,i} = a_{0,2n+1} = -1, \qquad i = 0,\dots,2n,$$
$$a_{i,j} = 0, \qquad \text{otherwise}.$$

We define the Poisson structure by

$$\{x_i, y_j\} = -\delta_{i,j}, \qquad i,j = 0,\dots,n.$$

Then the Hamiltonian system (4) is invariant under the birational transformations $r_0,\dots,r_{2n+1}$ defined by

$$r_0(x_j) = t^{-\alpha_0} x_j, \qquad r_0(y_j) = t^{\alpha_0}\left(y_j + \frac{\alpha_0}{x_n - t x_0}\{x_n - t x_0, y_j\}\right),$$
$$r_{2i+1}(x_j) = x_j + \frac{\alpha_{2i+1}}{y_i}\{y_i, x_j\}, \qquad r_{2i+1}(y_j) = y_j, \qquad i = 0,\dots,n-1,$$
$$r_{2i}(x_j) = x_j, \qquad r_{2i}(y_j) = y_j + \frac{\alpha_{2i}}{x_{i-1} - x_i}\{x_{i-1} - x_i, y_j\}, \qquad i = 1,\dots,n,$$
$$r_{2n+1}(x_j) = t^{\alpha_{2n+1}}\left(x_j + \frac{\alpha_{2n+1}}{y_n}\{y_n, x_j\}\right), \qquad r_{2n+1}(y_j) = t^{-\alpha_{2n+1}} y_j,$$

for $j = 0,\dots,n$ and

$$r_i(\alpha_j) = \alpha_j - a_{i,j}\alpha_i, \qquad r_i(\eta) = \eta + (-1)^i \alpha_i, \qquad i,j = 0,\dots,2n+1.$$

The group of symmetries $\langle r_0,\dots,r_{2n+1}\rangle$ is isomorphic to the affine Weyl group of type $A^{(1)}_{2n+1}$.

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References

[1] Fuji K., Suzuki T., Drinfeld–Sokolov hierarchies of type A and fourth order Painlevé systems, *Funkcial. Ekvac.* **53** (2010), 143–167, arXiv:0904.3434.

[2] Iwasaki K., Kimura H., Shimomura S., Yoshida M., From Gauss to Painlevé. A modern theory of special functions, *Aspects of Mathematics*, Vol. E16, Friedr. Vieweg & Sohn, Braunschweig, 1991.

[3] Okubo K., Takano K., Yoshida S., A connection problem for the generalized hypergeometric equation, *Funkcial. Ekvac.* **31** (1988), 483–495.

[4] Suzuki T., A class of higher order Painlevé systems arising from integrable hierarchies of type A, arXiv:1002.2685.

[5] Sakai H., Private communication.

[6] Tsuda T., UC hierarchy and monodromy preserving deformation, arXiv:1007.3450.