Dolbeault Complex on $S^4\backslash\{\cdot\}$ and $S^6\backslash\{\cdot\}$ through Supersymmetric Glasses

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Abstract. S^4 is not a complex manifold, but it is sufficient to remove one point to make it complex. Using supersymmetry methods, we show that the Dolbeault complex (involving the holomorphic exterior derivative ∂ and its Hermitian conjugate) can be perfectly well defined in this case. We calculate the spectrum of the Dolbeault Laplacian. It involves 3 bosonic zero modes such that the Dolbeault index on $S^4\backslash\{\cdot\}$ is equal to 3.

Key words: Dolbeault; supersymmetry

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1 Introduction

We start with reminding the standard definition for a complex manifold. Suppose a manifold of dimension $D = 2d$ is covered by several overlapping D-dimensional disks. Suppose that in each such map complex coordinates $w^{j=1,\dots,d}$, $\bar{w}^{\bar{j}=1,\dots,d}$ are introduced such that the metric has a Hermitian form

$$ds^2 = h_{j\bar{k}}(w,\bar{w})dw^j d\bar{w}^{\bar{k}}, \qquad h^*_{j\bar{k}} = h_{k\bar{j}}.$$

In the region where a couple of the maps with coordinates w, $\bar{w}$ and $\tilde{w}$, $\tilde{\bar{w}}$ overlap the latter are expressed into one another. The manifold is called *complex* if this relationship can be made holomorphic, $\tilde{w}^j = f^j(w^k)$.

For example, S^2 (actually, any 2-dimensional manifold) is complex. To see that, introduce the stereographic complex coordinates

$$w = \frac{x+iy}{\sqrt{2}}, \qquad \bar{w} = \frac{x-iy}{\sqrt{2}}$$

such that

$$ds^2 = \frac{2dwd\bar{w}}{(1+\bar{w}w)^2}.$$

This map covers the whole sphere except its north pole (corresponding to $w=\infty$). Introduce now another stereographic map that covers the whole sphere but its south pole. The metric is again

$$ds^2 = \frac{2d\tilde{w}d\tilde{\bar{w}}}{(1+\tilde{\bar{w}}\tilde{w})^2}.$$

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In the region where the maps overlap (the whole sphere but two points), the holomorphic relation $\tilde{w} = 1/w$ holds.

Let us try to do the same for S^4. Again, we can cover it by two stereographic maps with the coordinates w_j and $\tilde{w}_j$ [2] such that the metric is, on one hand,

$$ds^2 = \frac{2dw_j d\bar{w}_j}{(1+\bar{w}w)^2}. \tag{1}$$

($\bar{w}w \equiv \bar{w}_j w_j = (x^2+y^2+z^2+t^2)/2$) and, on the other hand,

$$ds^2 = \frac{2d\tilde{w}_j d\bar{\tilde{w}}_j}{(1+\bar{\tilde{w}}\tilde{w})^2}. \tag{2}$$

But the relationship $\tilde{w}_j = \bar{w}_j/(\bar{w}w)$ *is* not holomorphic anymore meaning that S^4 is not complex.

For complex compact manifolds, one can consider a set of holomorphic $(p,0)$-forms, introduce the operator of exterior holomorphic derivative ∂, its Hermitian conjugate $\partial^\dagger$ and define thereby the *Dolbeault* complex (see e.g. [1]). The operators ∂ and $\partial^\dagger$ are nilpotent and the Hermitian Dolbeault Laplacian $\partial\partial^\dagger + \partial^\dagger\partial$ commutes with both ∂ and $\partial^\dagger$. This algebra is isomorphic to the simplest supersymmetry algebra,

$$Q^2 = \bar{Q}^2 = 0, \qquad \{Q, \bar{Q}\} = H.$$

The supersymmetric description of the Dolbeault complex for any (not necessarily Kähler) complex manifold has been constructed in recent [2]. The superfield action (first written in [3]) is expressed in terms of $d+d$ chiral and antichiral superfields

$$W^j = w^j + \sqrt{2}\theta\psi^j - i\theta\bar{\theta}\dot{w}^j, \qquad \bar{W}^{\bar{j}} = \bar{w}^{\bar{j}} - \sqrt{2}\bar{\theta}\bar{\psi}^{\bar{j}} + i\theta\bar{\theta}\dot{\bar{w}}^{\bar{j}},$$
$$S = \int dt d^2\theta \left[-\frac{1}{4} h_{j\bar{k}}(W^l, \bar{W}^{\bar{l}}) DW^j \bar{D}\bar{W}^{\bar{k}} + G(\bar{W}, W)\right].$$

Deriving with this action the component Lagrangian, then classical and quantum Hamiltonian, using the Nöther theorem, and accurately resolving the ordering ambiguities [4], we arrive at the expressions for the quantum supercharges

$$Q = \psi^c e_c^k \left[\Pi_k - \frac{i}{4}\partial_k(\ln\det h) + i\psi^b \bar{\psi}^{\bar{a}} \Omega_{k,\bar{a}b}\right],$$
$$\bar{Q} = \bar{\psi}^{\bar{c}} e_{\bar{c}}^{\bar{k}} \left[\bar{\Pi}_{\bar{k}} - \frac{i}{4}\partial_{\bar{k}}(\ln\det h) + i\bar{\psi}^{\bar{b}} \psi^a \bar{\Omega}_{\bar{k},a\bar{b}}\right], \tag{3}$$

where e_j^c are the vielbeins, $e_j^c \bar{e}_{\bar{k}}^{\bar{c}} = h_{j\bar{k}}$, chosen such that $\det e = \det \bar{e} = \sqrt{\det h}$,

$$\Omega_{j,\bar{b}a} \equiv \Omega_{j,}{}^b{}_a = e_p^b\left(\partial_j e_a^p + \Gamma_{jk}^p e_a^k\right)$$

(and the complex conjugate $\bar{\Omega}_{\bar{j},b\bar{a}} \equiv \Omega_{\bar{j},}{}^{\bar{b}}{}_{\bar{a}}$) are the holomorphic and antiholomorphic components of the standard Levi-Civita spin connections[3], $\bar{\psi}^{\bar{a}} = \partial/\partial\psi^a$, and

$$\Pi_k = -i\left(\frac{\partial}{\partial z^k} - \partial_k G\right), \qquad \bar{\Pi}_{\bar{k}} = -i\left(\frac{\partial}{\partial \bar{z}^{\bar{k}}} + \partial_{\bar{k}} G\right).$$

[2] $j = 1, 2$ and, when going down to S^4 with its conformally flat metric, we will not bother to distingush between covariant and contravariant indices. Neither will we distiguish in this case the indices j and $\bar{j}$. The summation over the repeated indices in equations (1), (2) and in all the formulas in Sections 2–4 is assumed, as usual.

[3] In the Kähler case, nonholomorphic components like $\Omega_j{}^a{}_{\bar{b}}$ vanish. For generic complex manifold, they do not vanish (though they vanish again for some special torsionful connections (10) below) but *do* not enter the supercharges (3). We refer the reader to [2] and to recent [5] for further pedagogical explanations.

are the covariant derivatives involving the gauge field

$$A_{j,\bar{k}} = (-i\partial_j G, i\bar{\partial}_{\bar{k}} G). \tag{4}$$

The quantum supercharges (3) act on the wave functions

$$\Psi\big(w^j, \bar{w}^{\bar{k}}; \psi^a\big) = A^{(0)}\big(w^j, \bar{w}^{\bar{k}}\big) + \psi^a A_a^{(1)}\big(w^j, \bar{w}^{\bar{k}}\big) + \cdots + \psi^{a_1}\cdots\psi^{a_d} A^{(d)}_{[a_1\cdots a_d]}\big(w^j, \bar{w}^{\bar{k}}\big).$$

The components of this wave function $A^{(0)}$, $A_a^{(1)}$, etc. can be mapped onto the space of the holomorphic forms $A^{(0)}$, $e^a_j A^{(1)}_a dw^j$, etc. A $(p,0)$-form corresponds to the wave function with the eigenvalue $p \equiv F$ of the fermion charge operator, $\bar{F} = \psi^a\bar{\psi}^a$. Each component is normalized with the covariant measure,

$$\mu d^D x = \sqrt{\det g}\, d^D x = \det h d^d w d^d \bar{w}. \tag{5}$$

The supercharges (3) are conjugate to each other with respect to this measure, $\bar{Q} = \mu^{-1} Q^\dagger \mu$, where $Q^\dagger$ is a "naive" Hermitian conjugate.

It was shown in [2] that the supercharge Q in (3) is isomorphic in this setting to the exterior derivative operator ∂ and the Dolbeault complex is reproduced, if choosing the function G in a special way,

$$G = \frac{1}{4}\ln\det h. \tag{6}$$

Another distinguished choice is $G = -(1/4)\ln\det h$ when the operator $\bar{Q}$ is isomorphic to the antiholomorphic exterior derivative $\bar{\partial}$, and we arrive at the anti-Dolbeault complex. For an arbitrary G, we are dealing with a *twisted* Dolbeault complex.

The Hamiltonian is given by the expression

$$\begin{aligned} H = &-\frac{1}{2}\triangle^{\text{cov}} + \frac{1}{8}\left(R - \frac{1}{2} h^{\bar{k}j} h^{\bar{l}t} h^{\bar{i}n} C_{j t\bar{i}} C_{\bar{k}\bar{l}n}\right) \\ &- 2\langle\psi^a\bar{\psi}^{\bar{b}}\rangle\, e^k_a e^{\bar{l}}_{\bar{b}} \partial_k\partial_{\bar{l}} G - \langle\psi^a\psi^c\bar{\psi}^{\bar{b}}\bar{\psi}^{\bar{d}}\rangle e^t_a e^j_c e^{\bar{l}}_{\bar{b}} e^{\bar{k}}_{\bar{d}} (\partial_t\partial_{\bar{l}} h_{j\bar{k}}). \end{aligned} \tag{7}$$

Here, $\langle\ldots\rangle$ denotes the Weyl-ordered products of fermions, $\langle\psi^a\bar{\psi}^{\bar{b}}\rangle = (\psi^a\bar{\psi}^{\bar{b}} - \bar{\psi}^{\bar{b}}\psi^a)/2$, etc. R is the standard scalar curvature of the metric $h_{j\bar{k}}$, while

$$C_{jk\bar{l}} = \partial_k h_{j\bar{l}} - \partial_j h_{k\bar{l}}, \qquad C_{\bar{j}\bar{k}l} = (C_{jk\bar{l}})^* = \partial_{\bar{k}} h_{l\bar{j}} - \partial_{\bar{j}} h_{l\bar{k}} \tag{8}$$

is the metric-dependent torsion tensor. The covariant Laplacian $\triangle^{\text{cov}}$ is defined with taking into account the torsion,

$$-\triangle^{\text{cov}} = h^{\bar{k}j}\big(\mathcal{P}_j\bar{\mathcal{P}}_{\bar{k}} + i\hat{\Gamma}^{\bar{q}}_{j\bar{k}}\bar{\mathcal{P}}_{\bar{q}} + \bar{\mathcal{P}}_{\bar{k}}\mathcal{P}_j + i\hat{\Gamma}^{s}_{\bar{k}j}\mathcal{P}_s\big),$$

where $\mathcal{P}_j = \Pi_j + i\hat{\Omega}_{j,\bar{b}a}\langle\psi^a\bar{\psi}^{\bar{b}}\rangle$ and $\bar{\mathcal{P}}_{\bar{k}} = \bar{\Pi}_{\bar{k}} - i\hat{\bar{\Omega}}_{\bar{k},a\bar{b}}\langle\psi^a\bar{\psi}^{\bar{b}}\rangle$ with some particular torsionfull affine and spin connections (the so called Bismut connections [6]),

$$\hat{\Gamma}^M_{NK} = \Gamma^M_{NK} + \frac{1}{2} g^{ML} C_{LNK}, \tag{9}$$

$$\hat{\Omega}_{M,AB} = \Omega_{M,AB} + \frac{1}{2} e^K_A e^L_B C_{KML}, \qquad M \equiv \{m, \bar{m}\}. \tag{10}$$

Note that this rather complicated expression for the Hamiltonian is greatly simplified in the Kähler case. Then the torsion (8) vanishes, the 4-fermion term in (7) vanishes too, and

$$H_{\text{Kähl}} = -\frac{1}{2}\triangle^{\text{cov}} + \frac{R}{8} - 2\langle\psi^a\bar{\psi}^{\bar{b}}\rangle e^k_a e^{\bar{l}}_{\bar{b}}\partial_k\partial_{\bar{l}} G,$$

where now $-\triangle^{\text{cov}} = h^{\bar{k}j}\left(\mathcal{P}_j\bar{\mathcal{P}}_{\bar{k}} + \bar{\mathcal{P}}_{\bar{k}}\mathcal{P}_j\right)$ with $\hat{\Omega}_{j,\bar{b}a} = \Omega_{j,\bar{b}a} = e^{\bar{k}}_{\bar{b}}\partial_j e^{\bar{a}}_{\bar{k}}$.

In this paper, we are interested, however, with S^4, which is not Kähler and, as was mentioned, not even globally complex. This notwithstanding, one can write the supercharges (3) and the Hamiltonian (7) with the metric (1), which is well defined everywhere on S^4 except the north pole and study the spectrum.

For sure, to determine the spectrum, we have to define first the *spectral problem* and to specify the boundary conditions for the wave functions. There are two different reasonable choices: (i) We can consider the functions that are regular on S^4. (ii) We can allow the singularity at the pole, but require that the functions are square integrable with the measure (5),

$$\int \frac{|\Psi|^2 d^2w d^2\bar{w}}{(1+\bar{w}w)^4} < \infty.$$

It turns out that, for the *first* spectral problem, the Hamiltonian is well defined and Hermitian. However, the Hilbert space of all nonsingular on S^4 functions does not constitute the domain of the supercharges: there exist nonsingular functions Ψ such that $Q\Psi$ are singular. In physical language, this means that the supersymmetry is broken – some states do not have superpartners. In mathematical language, this means that the Dolbeault complex is not well defined on the manifolds that are not complex, of which S^4 is an example.

What is, however, rather nontrivial and somewhat surprising is that, in the Hilbert space of square integrable functions, everything works fine. In the main body of the paper, we will show that all excited square integrable states of the Hamiltonian are doubly degenerate (i.e. supersymmetry *is* there) and that there are 3 bosonic zero modes such that the Witten index of this system is $I_W = 3$. In other words, even though the Dolbeault complex is not well defined on S^4, there is a nontrivial self-consistent way to define it on $S^4\backslash\{\cdot\}$.

There is a kinship between the problem under consideration and a problem of the Dirac complex on S^2 with noninteger magnetic flux [7]. In both cases, the requirement for the spectrum to be supersymmetric brings about restrictions on the Hilbert space (see also [8]). However, for a noninteger flux, these restrictions are extremely stringent: they simply leave the Hilbert space empty, a Dirac complex with noninteger flux is not defined. And, for S^4, the Hilbert space of regular wave functions is not supersymmetric, while its *extension* – the space of square integrable functions is.

2 The Dolbeault Hamiltonian and its spectrum

On S^4 with the metric (1) and with $G(\bar{W}, W)$ given by (6), the supercharges (3) acquire the following simple form

$$\begin{aligned} Q &= i(1+\bar{w}w)\psi_j\partial_j + i\psi_j\psi_k\bar{\psi}_j\bar{w}_k, \\ \bar{Q} &= i\bar{\psi}_j\left[(1+\bar{w}w)\bar{\partial}_j - 2w_j\right] + i\bar{\psi}_j\bar{\psi}_k\psi_j w_k. \end{aligned} \tag{11}$$

The complicated expression (7) for the Hamiltonian also simplifies a lot. There are three sectors: $F=0$, $F=1$, and $F=2$. Consider first the sector $F=0$. We obtain

$$H^{F=0} = -(1+\bar{w}w)^2\partial_j\bar{\partial}_j + 2(1+\bar{w}w)w_j\partial_j. \tag{12}$$

It is instructive to compare this Dolbeault Laplacian with the standard covariant Laplacian on S^4,

$$-\triangle_{S^4} = -(1+\bar{w}w)^2\bar{\partial}_j\partial_j + (1+\bar{w}w)\big(w_j\partial_j + \bar{w}_j\bar{\partial}_j\big). \tag{13}$$

Note that both (12) and (13) commute with the angular momentum operator $m = w_j\partial_j - \bar{w}_j\bar{\partial}_j$.[4] The eigenvalues of m are integer.

[4] The Hamiltonian (12) commutes also with two other generators of $SU(2)$ such that the states represent $SU(2)$-multiplets. The standard Laplacian (13) has $O(5)$ symmetry.

The supercharges (11) admit 3 normalizable zero modes satisfying $Q\Psi^{(0)} = \bar{Q}\Psi^{(0)} = 0$ in the sector $F = 0$,

$$\Psi^{(0)} = 1, \bar{w}_1, \bar{w}_2. \tag{14}$$

They represent the ground states of the Hamiltonian (12).

Consider now excited states. The eigenfunctions of the Hamiltonian can be sought for in the form

$$\Psi_{ms} = S_{ms} F_{ms}(\bar{w}w), \tag{15}$$

where S_{ms} ($m = 0, \pm 1, \ldots$; $s = 0, 1, \ldots$) are mutually orthogonal tensor structures that vanish under the action of the "naive Laplacian" $\bar{\partial}_j \partial_j$. Each structure S_{ms} has $2s+|m|+1$ independent components (and the corresponding energy level has degeneracy $2s+|m|+1$). The explicit form of first few such structures is

$$S_{00} = 1, \qquad S_{01} = w_j \bar{w}_k - \frac{\bar{w}w}{2}\delta_{jk},$$
$$S_{02} = w_i w_j \bar{w}_k \bar{w}_l - \frac{\bar{w}w}{4}\left(w_i \bar{w}_k \delta_{jl} + w_i \bar{w}_l \delta_{jk} + w_j \bar{w}_k \delta_{il} + w_j \bar{w}_l \delta_{ik}\right) + \frac{(\bar{w}w)^2}{12}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$
$$S_{10} = w_j, \qquad S_{11} = w_i w_j \bar{w}_k - \frac{\bar{w}w}{3}(w_i \delta_{jk} + w_j \delta_{ik}), \qquad S_{-1,0} = \bar{w}_j.$$

It is straightforward to see that the action of the Hamiltonian on the Ansatz (15) preserves its tensor form. The radial dependence is then determined from the solution of scalar spectral equations for F_{ms}. It is convenient to introduce the variable

$$z = \frac{1 - \bar{w}w}{1 + \bar{w}w}$$

(it is nothing but $\cos\theta$, θ being the polar angle on S^4).

The spectral equations acquire then the form

$$(z^2 - 1)F''(z) + 2(2z + m + 2s)F'(z) + \frac{4(m+s)}{1+z}F(z) = \lambda F(z), \qquad m \geq 0,$$
$$(z^2 - 1)F''(z) + 2(2z + |m| + 2s)F'(z) + \frac{4s}{1+z}F(z) = \lambda F(z), \qquad m \leq 0.$$

Their formal solutions are

$$F(z) = (1+z)^{\gamma_{ms}} P_n^{|m|+2s+1, \pm\Delta_{ms}}(z),$$
$$\lambda_{msn} = \gamma_{ms}^2 + 3\gamma_{ms} + n(n + |m| + 2s + 2 \pm \Delta_{ms}), \tag{16}$$

with

$$\gamma_{ms} = \frac{|m| + 2s - 1 \pm \Delta_{ms}}{2} \tag{17}$$

and

$$\Delta_{ms} = \sqrt{(1 - m - 2s)^2 + 8(m+s)}, \qquad m \geq 0,$$
$$\Delta_{ms} = \sqrt{(1 - |m| - 2s)^2 + 8s}, \qquad m \leq 0.$$

$P_n^{\alpha,\beta}$ ($n = 0, 1, \ldots$) are the Jacobi polynomials,

$$P_n^{\alpha,\beta}(z) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1+z)^k(z-1)^{n-k}.$$

For $\alpha > -1$, $\beta > -1$, the Jacobi polynomials are mutually orthogonal on the interval $z \in (-1, 1)$ with the weight $\mu = (1-z)^\alpha(1+z)^\beta$.

Not all the solutions in (16) are admissible, however. One can observe the following:

- First of all, all the solutions with $s > 0$ and/or $m > 0$ and the negative sign of Δ_{ms} in (16), (17) are not square integrable and should not be included in the spectrum. If $s = 0$ and $m \leq 0$, the solution with negative sign of Δ_{ms} are not independent being expressed into the solutions with positive sign in virtue of the identity

$$P_{n+\beta}^{\alpha,-\beta}(z) = 2^{-\beta}(z+1)^{\beta}\frac{n!(n+\alpha+\beta)!}{(n+\alpha)!(n+\beta)!}P_n^{\alpha,\beta}(z), \tag{18}$$

which holds for integer α [9].

- On the other hand, the solutions with positive sign of Δ_{ms} and with nonnegative m are all not only square integrable, but also nonsingular on S^4. In addition, they belong to the domain of Q: $Q\Psi_{m\geq 0,s}$ is never singular.

- Most of the solutions with $m < 0$ also have this property. However, there are three distinguished families of solutions: the solutions

$$\Psi_{-1,0,n} = \bar{w}_j P_n^{2,0}(z), \tag{19}$$

the solutions

$$\Psi_{-2,0,n} = \frac{\bar{w}_j \bar{w}_k}{1+\bar{w}w} P_n^{3,1}(z), \tag{20}$$

and the solutions

$$\Psi_{-3,0,n} = \frac{\bar{w}_j \bar{w}_k \bar{w}_l}{(1+\bar{w}w)^2} P_n^{4,2}(z). \tag{21}$$

(i) The functions (19) are all singular at infinity, but integrable. Two lowest such functions $\Psi = \bar{w}_j$ are zero modes of the Hamiltonian (12). The functions $Q\Psi_{-1,0,n}$ are less singular: they do not grow at infinity (though do not have a definite value there when $n > 0$).

(ii) The functions (20) are bounded at infinity. The supercharge action produces growing functions, $Q\Psi_{-2,0,n}(w = \infty) = \infty$. Still, $Q\Psi_{-2,0,n}$ is square integrable.

(iii) The functions (21) are regular at infinity. The supercharge action produces singular bounded functions.

- Note that if Ψ is a *non-normalizable* eigenfunction in the sector $F = 0$, the function $Q\Psi$ is also not normalizable. Indeed, the action of Q brings about generically an extra power of $|w|$, which makes the divergence still stronger. An exception would only be provided by the functions with the asymptotics $\propto \bar{w}_{j_1} \cdots \bar{w}_{j_k}$ at infinity. But the only *eigenfunctions* with such asymptotics are written in equation (19). They are normalizable.

Consider now the sector $F = 2$. The Hamiltonian is

$$H^{F=2} \;=\; -(1+\bar{w}w)^2 \bar{\partial}_j \partial_j + 2(1+\bar{w}w) w_j \partial_j + 2(2+\bar{w}w).$$

The eigenfunctions have the same form as in (15), (16), (17) with modified

$$\Delta_{ms}^{F=2} = \sqrt{(1-m-2s)^2 + 8(m+s+1)}, \qquad m \geq 0,$$
$$\Delta_{ms}^{F=2} = \sqrt{(1-|m|-2s)^2 + 8(s+1)}, \qquad m \leq 0.$$

Again, almost all functions with negative sign in (17) are not normalizable. The exceptions are the sectors $s = 0$, $m = 0, -2$, where the functions with the negative sign are expressed into the functions with positive sign. Speaking of the latter, they are not only normalizable, but also nonsingular in this case. All these functions are annihilated by Q and belong to the domain of $\bar{Q}$, $\bar{Q}\Psi^{F=2}$ being regular on S^4.

In the sector $F = 1$, the wave functions have two components, $\Psi^{F=1} = \psi_j C_j(\bar{w}_k, w_k)$. No new zero modes appear. Indeed, if the Hamiltonian *had* zero modes in this sector, they would satisfy the conditions $Q\Psi = \bar{Q}\Psi = 0$ giving

$$(1 + \bar{w}w)\partial_{[j}C_{k]} - \bar{w}_{[j}C_{k]} = 0,$$
$$(1 + \bar{w}w)\bar{\partial}_k C_k - 3w_k C_k = 0.$$

The first equation can be rewritten as

$$\partial_{[j}\left[\frac{C_{k]}}{1 + \bar{w}w}\right] = 0$$

with a generic solution $C_k = (1 + \bar{w}w)\partial_k\Phi$. Then the second equation gives $H^{F=0}\Phi = 0$. If C_k is normalizable, Φ must also be normalizable (modulo a pure antiholomorphic part). But we have seen that the only normalizable zero modes of $H^{F=0}$ are 1 and $\bar{w}_j$ annihilated by holomorphic derivatives and giving $C_k = 0$.[5]

To find the nonzero modes in the sector $F = 1$, one needs not to solve the Schrödinger equation again. All such normalizable functions are obtained by the action of Q or $\bar{Q}$ onto the normalizable functions in the sectors $F = 0$ or $F = 2$, correspondingly. This follows from the last itemized statement above, which is valid also in the sector $F = 2$. By construction, these functions are annihilated by Q or $\bar{Q}$ and belong to the domain of $\bar{Q}$ or Q, correspondingly.

2.1 Twisted Dolbeault complex

The Hamiltonian (7) is supersymmetric not only under the condition (6) that distinguishes the pure Dolbeault complex, but also with other choices of G describing twisted Dolbeault complexes.

First of all, we can set $G = 0$. As was shown in [2], the Hamiltonian (7) with $G = 0$ coincides with the extended $N = 4$ supersymmetric Hamiltonian written in [10],

$$H = -\frac{1}{2}f^3\partial_M^2\frac{1}{f} - \frac{1}{2}\psi\sigma^\dagger_{[M}\sigma_{N]}\bar{\psi}f(\partial_M f)\partial_N + f(\partial^2 f)\left(\psi\bar{\psi} - \frac{1}{2}(\psi\bar{\psi})^2\right), \qquad (22)$$

with $f = 1 + x_M^2/2$.

This model belongs to the class of the so called "hyperkähler with torsion" (HKT) models [11], which were classified using the harmonic superspace formalism in recent [12]. The Hamiltonian (22) does not admit normalizable zero-energy solutions and its index is zero.

Consider now a model with

$$G = \frac{q}{4}\ln\det h = -q\ln(1 + \bar{w}w)$$

with an integer $q > 1$. The supercharges are then

$$Q = i(1 + \bar{w}w)\psi_j\partial_j + i(q - 1)\bar{w}w\psi_j\bar{w}_j + i\psi_j\psi_k\bar{\psi}_j\bar{w}_k,$$
$$\bar{Q} = i\bar{\psi}_j\left[(1 + \bar{w}w)\bar{\partial}_j - (q + 1)w_j\right] + i\bar{\psi}_j\bar{\psi}_k\psi_j w_k.$$

The zero modes all dwell in the sector $F = 0$. They have the form

$$\Psi^{(0)} = \frac{P(\bar{w})}{(1 + \bar{w}w)^{q-1}},$$

[5]Note that if one lifts the normalizability condition, a nontrivial solution of the equation $H^{F=0}\Phi = 0$ exists: $\Phi = \bar{w}w + 2\ln(\bar{w}w) - \frac{1}{\bar{w}w}$ giving $C_k = \bar{w}_k(1 + \bar{w}w)^3/(\bar{w}w)^2$.

where $P(\bar{w})$ is an antiholomorphic polynomial of degree $2q-1$. It has $2q^2+q$ independent coefficients, which gives $2q^2+q$ independent zero modes.

When q is negative, the analysis is similar. It gives $2q^2-q$ zero modes in the sector $F=2$. The same consideration as in the pure Dolbeault case displays the absence of the normalized zero modes in the sector $F=1$. The final result for the index of the twisted Dolbeault complex is

$$I(q) = 2q^2 + |q|. \tag{23}$$

3 The index and the functional integral

As was mentioned, for pure Dolbeault complex, there are 3 bosonic zero modes (14) in the sector $F=0$ and no zero modes in the other sectors. This means that the Witten index of this system,

$$I_W = \mathrm{Tr}\big\{(-1)^F e^{-\beta H}\big\}$$

is equal to 3.

For *compact* complex manifolds, the Witten index of the supersymmetric Hamiltonian (7) under the condition (6) is known to mathematicians by the name of *arithmetic genus* of the manifold. This invariant admits an integral representation known as the Hirzebruch–Riemann–Roch theorem [13],[6]

$$I \;=\; \int \mathrm{Td}(TM),$$

where the symbol $\mathrm{Td}(TM)$ (*Todd class of a complex tangent bundle* associated with the manifold M) is spelled out as

$$\mathrm{Td}(TM) = \prod_{\alpha=1}^{n} \frac{\lambda_\alpha/2\pi}{1-e^{-\lambda_\alpha/2\pi}}, \tag{24}$$

where λ_α are eigenvalues of the curvature matrix corresponding to this bundle[7].

The representation (24) can be derived by using the fact that the sum of the supercharges (11) can be interpreted as a Dirac operator involving an Abelian gauge field and torsions. (The presence of torsions is a complication that distinguishes the HRR theorem from a version of the Atiyah–Singer theorem discussed usually by physicists.) It is important in this derivation that the gauge field represents a regular fiber bundle on the manifold, while torsions are regular tensors.

In our S^4 case, however, these conditions are not fulfilled: the gauge field and the torsion are singular at $w=\infty$. Indeed, the torsion (8) with the metric (1) behaves at infinity as $\sim |x|^{-5}$. Then $g^{MN}g^{PQ}g^{ST}C_{MPS}C_{NQT} \sim (x^4)^3 \cdot (x^{-5})^2 \sim x^2$ and diverges. The gauge field (4), (6) is rather peculiar. It is *disguised* as a benign fiber bundle having an integer Chern class

$$\mathrm{Ch}_2 = \frac{1}{8\pi^2}\int F\wedge F = 2. \tag{25}$$

However, a topologically nontrivial $U(1)$ bundle on S^4 *does* not exist because $\pi_3[U(1)] = 1$.[8] Indeed, the field strength tensor is singular in this case, which manifests itself in the fact that the "action integral" $\sim \int d^4x \sqrt{g} F_{MN}F^{MN}$ diverges logarithmically.

[6] Following the ideas of [14], it has been recently derived also in physical way by studying the path integral for the supersymmetric partition function [5].

[7] For (24) to be correct and simply to make sense, the connection and its curvature should respect the complex structure. For example, the Bismut connection (9) is appropriate for this purpose, while the usual torsionless Levi-Civita connection is not, if the manifold is not Kähler.

[8] On the other hand, topologically nontrivial bundles on S^4 with non-Abelian gauge groups (*instantons*), of course, exist.

For such a singular Dirac operator, one cannot get rid of torsions by a smooth deformation (a key step in the derivation of (24)), because the index integral may in this case acquire contributions from total derivatives of singular expressions. Moreover, with all probability, the Dirac operator on S^4 with the singular field (4) and *without* torsions does not describe a benign supersymmetric system – the situation must be the same as for the gauge field on S^2 with non-integer magnetic flux [7].

Having no further mathematical methods at our disposal (at least, we are not aware of such methods), we can try to calculate the Witten index in a physical way by evaluating directly the corresponding path integral,

$$I = \int d\mu \exp\left\{-\int_0^\beta L_E(\tau)d\tau\right\},$$

where L_E is the Euclidean Lagrangian of our supersymmetric quantum system, $d\mu$ is the appropriate functional integral measure, and the periodic boundary conditions are imposed onto all variables.

For most supersymmetric systems, this integral is reduced for small β to an ordinary phase space integral [15]. This is true e.g. for a supersymmetric Hamiltonian describing the de Rham complex on a compact manifold, where the Witten index is given by its Euler characteristics. For the Dirac complex on compact manifolds, the situation is more complicated, a naive semiclassical reduction is not justified and one has to perform a honest calculation of the path integral in the one-loop approximation [14], which is not so trivial (see [2] for detailed pedagogical explanations). For the Dolbeault complex on compact non-Kähler complex manifolds, the life is still more difficult. Generically, one has to perform a two-loop calculation for $4d$ and $6d$ manifolds, a three-loop calculation for $8d$ and $10d$ manifolds, etc. This complication is due to the appearance of the new 4-fermionic term in the Lagrangian,

$$L_E = \frac{1}{2}\left[g_{MN}\dot{x}^M\dot{x}^N + g_{MN}\,\psi^M\hat{\nabla}\psi^N + \frac{1}{6}\partial_P C_{MNT}\psi^P\psi^M\psi^N\psi^T\right] - iA_M\dot{x}^M + \frac{i}{2}F_{MN}\psi^M\psi^N \tag{26}$$

($\hat{\nabla}\psi^M = \dot{\psi}^M + \hat{\Gamma}^M_{NK}\dot{x}^N\psi^K$ is the Bismut covariant derivative). For example, for a 4-dimensional manifold, the leading (at small β) contribution to the index is

$$I \sim \frac{1}{\beta}\int d^4x\,\epsilon^{MNPQ}\partial_M C_{NPQ}. \tag{27}$$

For sure, the integrand is a total derivative here and the integral vanishes, but the appearance of the large factor $1/\beta$ does not allow one to ignore 2-loop corrections anymore. They are essential. For $8d$ manifolds, the leading contribution in the integrand is of order $\sim 1/\beta^2$, and this makes essential 3 loop contributions, etc.

As was mentioned above, for compact manifolds, one needs not actually to come to grips with these complicated multiloop contributions. One can, instead, perform a smooth deformation that kills the torsion and makes the problem and the corresponding path integral tractable. For a particular class of manifolds where the fermion term in (26) vanishes (the so called SKT manifolds), this program was in fact carried out in [6]. (In this mathematical paper, path integrals and supersymmetry were not mentioned and the author described the results in the language of *heat kernel* technique, which is equivalent, however, to the path integral approach.) The generic case is discussed in [5].

For S^4, we have no other choice than to try to evaluate the path integral directly. In 4 dimensions, this is difficult, but feasible and, in the case when the Dirac operator involves only torsions, but no extra gauge field ($G = 0$ in our language), has been performed in [16]. The index integral has been represented in these papers as

$$I = -\frac{1}{4\pi^2}\int d^4x\,\sqrt{g}\bigg\{\frac{1}{2\beta}\nabla_M B^M + \frac{1}{192}\frac{1}{\sqrt{g}}\epsilon^{RSKL}\left[R_{MNRS}R^{MN}{}_{KL} + \frac{1}{2}B_{RS}B_{KL}\right] + \frac{1}{24}\nabla_M \mathcal{K}^M\bigg\} + \mathcal{O}(\beta) \tag{28}$$

with

$$\mathcal{K}^M = \left(\nabla^N\nabla_N + \frac{1}{4}B^N B_N + \frac{1}{2}R\right)B^M, \qquad B_{MN} = \nabla_M B_N - \nabla_N B_M. \tag{29}$$

Here ∇_M is the standard Levi-Civita covariant derivative and R_{MNRS} is the standard Riemann tensor. B_M is the axial vector dual to the torsion tensor,

$$B^M = \frac{1}{6\sqrt{g}}\epsilon^{MNPQ}C_{NPQ} = 2x^M\left(1 + x^2/2\right).$$

The first term in equation (28) is a singular ($\sim 1/\beta$) but vanishing integral of a total derivative. It was discussed before. The last term also represents a total derivative and also vanishes[9]. The "field strength" B_{MN} is zero in our case. The quadratic in the Riemann tensor integral is proportional to a certain topological invariant called Hirzebruch signature. For S^4, it vanishes.

As a result, the index of the corresponding supersymmetric Hamiltonian vanishes. This agrees with the direct analysis of its spectrum (see the remark after equation (22)).

When $G \neq 0$, the Lagrangian and the Hamiltonian involve an extra gauge field. The functional integral for the index acquires the (tree-level) contribution (25).

On top of this, there might have been a 1-loop contribution associated with the 4-fermion term. To evaluate it, it is convenient to expand the periodic fields $x^M(\tau)$ and $\psi^M(\tau)$ into the Fourrier modes,

$$x^M(\tau) = x_0^M + \sum_{m\neq 0} x_m^M e^{2\pi i m\tau/\beta}, \qquad \psi^M(\tau) = \psi_0^M + \sum_{m\neq 0}\psi_m^M e^{2\pi i m\tau/\beta},$$

with integer m ($\bar{x}_m^M = x_{-m}^M$, $\bar{\psi}_m^M = \psi_{-m}^M$). We obtain instead of (27)

$$I \sim \frac{1}{\beta}\int d^4x_0\,\epsilon^{MNPQ}\partial_M C_{NPQ}\mu \times \prod_{M,m\neq 0} dx_m^M d\psi_m^M \exp\left\{-\frac{1}{2\beta}\sum_m (2\pi m)^2 x_m^M x_{-m}^N\left(g_{MN} - \frac{\beta F_{MN}}{2\pi m}\right) + i\sum_m (2\pi m)\psi_m^M\psi_{-m}^N\left(g_{MN} - \frac{\beta F_{MN}}{2\pi m}\right)\right\}, \tag{30}$$

where ∂C, g, F, and the infinite factor μ [10] depend on the zero coordinate modes x_0^M.

[9]This vanishing is due to a certain cancellation. One can easily check that *individual* terms $\sim \nabla^N\nabla_N$ and $\sim B^N B_N$ in (29) are expressed into the integral $\sim \int d^4x\,\partial_M(x^M/x^4)$ that does not vanish. These contributions cancel, however, in the sum.

[10]It can be made finite when imposing an ultraviolet cutoff, see equation (6.27) of [2]. When $F = 0$,

$$\mu\int \prod_{M,m\neq 0} dx_m^M d\psi_m^M \exp\{\cdots\} = 1.$$

The individual contributions due to bosonic and fermionic Gaussian integrals are nontrivial. For example, the fermionic integral gives

$$\text{fermion factor} = \prod_{m=1}^{\infty} \det \left\| \delta_M^N - \frac{\beta F_M{}^N}{2\pi m} \right\| = \det{}^{1/2} \frac{2\sin(\beta F/2)}{\beta F}.$$

Formally, the correction to unity is proportional to β^2, but it multiplies $\mathrm{Tr}\{F^2\} = -F_{MN}F^{MN}$, which grows at infinity $\sim x^4$. The integral is then saturated by large x values, and, as a result, the correction is of order β. When multiplied by the overall factor $1/\beta$ in front of the integral, this gives a correction of order 1 to the index.

Anyway, as is clear from (30), this fermionic correction exactly cancels the bosonic one (obviously, this cancelation is due to supersymmetry), and we have to conclude that the functional integral calculation gives the value (25) for the index, which contradicts the direct analysis above giving $I = 3$. In addition, for the twisted Dolbeault complex, we obtain

$$I_{\text{funct. int.}} = 2q^2,$$

which contradicts the estimate (23) above.

Certainly, this mismatch is disappointing and paradoxical. We want to emphasize, however, that there is no *logical* contradiction here. We calculated the functional integral by semiclassical methods expanding it in β. In particular, we studied only one-loop corrections to the index associated with gauge field, because two- and higher-loop corrections are suppressed by the naive β counting. We have seen, however, that this expansion breaks down near the singularity where β is multiplied by a large factor $\sim x^2$. In this situation, one cannot reliably justify ignoring higher-loop contributions. They can give something (though we do not see at the moment how this can come about).

Note that there are some other examples where the presence of singularities invalidates the semiclassical calculation of the path integral. In particular, in [17], we constructed SQM systems associated with chiral supersymmetric gauge theories in finite volume. The Hamiltonian of these systems is singular near the origin, $H \sim 1/x^2$. And, though this singularity is of repulsive benign nature, unitarity is not broken, and the spectrum of the Hamiltonian is discrete, the semiclassical approximation for the path integral breaks down near the origin. This manifests itself in the senseless fractional values of the path integral for the index evaluated at the leading order.

Definitely, more studies of this very interesting question are necessary.

4 S^6

A similar analysis can be done for S^6 and also for higher even-dimensional spheres. The metric of S^6 is still given by equation (1) where now $j = 1, 2, 3$. The supercharges of the SQM system describing the pure Dolbeault complex have almost the same form as for S^4,

$$Q = i(1+\bar{w}w)\psi_j\partial_j + i\psi_j\psi_k\bar{\psi}_j\bar{w}_k,$$
$$\bar{Q} = i\bar{\psi}_j\left[(1+\bar{w}w)\bar{\partial}_j - 3w_j\right] + i\bar{\psi}_j\bar{\psi}_k\psi_j w_k.$$

The Hamiltonian in the sector $F = 0$ is

$$H^{F=0} = -(1+\bar{w}w)^2\partial_j\bar{\partial}_j + 4(1+\bar{w}w)w_j\partial_j, \qquad j = 1, 2, 3,$$

to be compared with the standard covariant Laplacian on S^6,

$$-\triangle_{S^6} = -(1+\bar{w}w)^2\bar{\partial}_j\partial_j + 2(1+\bar{w}w)(w_j\partial_j + \bar{w}_j\bar{\partial}_j).$$

We choose the basis

$$\Psi_{pq} = T_{pq} F(\bar{w}w),$$

where a tensor structure T_{pq} having p factors w and q factors $\bar{w}$ and annihilated by $\partial_j \bar{\partial}_j$ represents a $\binom{p}{q}$ multiplet of $SU(3)$ and has $(p+1)(q+1)(p+q+2)/2$ independent components. For example,

$$T_{11} = w_j \bar{w}_k - \frac{\bar{w}w}{3}\delta_{jk}$$

is an octet[11]. The spectral equations for the coefficients $F(z)$ are

$$\left(z^2-1\right)F''(z) + 2(3z+p+q)F'(z) + \frac{4p}{1+z}F(z) = \lambda F(z).$$

Their solutions are

$$F(z) = (1+z)^{\gamma_{pq}} P_n^{p+q+2,\pm\Delta_{pq}}(z), \qquad \lambda_{pqn} = \gamma_{pq}^2 + 5\gamma_{pq} + n(n+p+q+3\pm\Delta_{pq}), \tag{31}$$

with

$$\gamma_{pq} = \frac{p+q-2\pm\Delta_{pq}}{2} \tag{32}$$

and

$$\Delta_{pq} = \sqrt{(2-p-q)^2+16p}.$$

The observations to be made are exactly parallel to the observations in the S^4 case. In particular,

- The solutions with $p > 0$ and the negative sign of Δ_{pq} are not square integrable and should not be included in the spectrum. If $p = 0$ or $p = q = 1$, the solutions with negative sign of Δ_{ms} are not independent being expressed into the solutions with positive sign in virtue of (18).
- On the other hand, the solutions with positive sign of Δ_{pq} and $p > 0$ are all not only square integrable, but also nonsingular on S^6. In addition, they belong to the domain of Q (meaning here that $Q\Psi$ are all normalizable).
- The solutions with $p = 0$ and $q > 4$ also have this property.
- The normalizable[12] families of solutions with $p = 0$ and $q = 0, 1, 2, 3, 4, 5$ are

$$\Psi_{00n} = P_n^{22}(z), \qquad \Psi_{01n} = \bar{w}_j P_n^{31}(z),$$
$$\Psi_{02n} = \bar{w}_j \bar{w}_k\, P_n^{40}(z), \qquad \Psi_{03n} = \frac{\bar{w}_j \bar{w}_k \bar{w}_l}{1+\bar{w}w}\, P_n^{51}(z),$$
$$\Psi_{04n} = \frac{\bar{w}_j \bar{w}_k \bar{w}_l \bar{w}_p}{(1+\bar{w}w)^2} P_n^{62}(z), \qquad \Psi_{05n} = \frac{\bar{w}_j \bar{w}_k \bar{w}_l \bar{w}_p \bar{w}_s}{(1+\bar{w}w)^3} P_n^{73}(z).$$

[11] The (p,q) notation can also be used for S^4, in which case $m = p - q$ and $s = \min\{p,q\}$.

[12] This means here that

$$\int \frac{|\Psi|^2\, d^3w d^3\bar{w}}{(1+\bar{w}w)^6} < \infty,$$

such that the singularity $\Psi \sim |w|^2$ is still allowed, while $\Psi \sim |w|^3$ is already not.

The families with $q = 1, 2, 3$ grow at infinity. The functions Ψ_{04n} are bounded, but still singular ($\Psi(\infty)$ is not defined). However, one cannot restrict oneself with the regular functions. The last family in the list above is regular on S^6, but would not belong to the domain of Q in this case: $Q\Psi_{05n}$ is not regular at infinity. In addition, by the same token as for S^4, the family $Q\Psi_{01n}$ is regular on S^6, but does not belong to the domain of $\bar{Q}$ (because Ψ_{01n} are singular).

For normalizable functions, we probably have a nice complex. As we have just shown, all normalizable functions in the sector $F = 0$ have normalizable superpartners.

The Hamiltonian in the sector $F = 3$ is

$$H = -(1 + \bar{w}w)^2\partial_j\bar{\partial}_j + (1 + \bar{w}w)\left(\bar{w}_j\bar{\partial}_j + 3w_j\partial_j\right) + 3(3 + \bar{w}w).$$

The spectral equations for the coefficients $F(z)$ of the structures T_{pq} are

$$\left(z^2 - 1\right)F''(z) + 2(3z + p + q)F'(z) + \frac{3(p + 1) + q}{1 + z}F(z) = (\lambda - 6)F(z),$$

and the solutions are also given by (31), (32) with

$$\Delta_{pq}^{F=3} = \sqrt{(2 - p - q)^2 + 4[3(p + 1) + q]}.$$

The eigenfunctions have better convergence here than in the sector $F = 0$. Actually, all normalizable eigenfunctions as well as their superpartners (they have fermion charge $F = 2$) are regular on S^6.

To *prove* that the Dolbeault complex is well defined in this case in the space of square integrable functions, we have also to solve the Schrödinger equation in the sectors $F = 1$ and $F = 2$. In this case, it is more difficult than for S^4 because *some* states in the sector $F = 1$ are annihilated by $\bar{Q}$ and cannot be found as superpartners of the states in the sector $F = 0$. Likewise, there are states in the sector $F = 2$ that are not superpartners to the states with $F = 3$. (These new states are superpartners to each other.) A special analysis of the matrix Schrödinger equation is thus required. We do not think, however, that such an analysis would unravel unpleasant surprises and believe that the Dolbeault complex is well defined on $S^6\backslash\{\cdot\}$.

The Witten index of this system is equal to

$$I^{S^6\backslash\{\cdot\}} = 1 + 3 + 6 = 10.$$

(There is one zero mode, $\Psi = 1$, in the sector $F = 0$, $p = q = 0$, three zero modes, $\Psi = \bar{w}_j$, in the sector $F = 0$, $p = 0$, $q = 1$ and six zero modes $\Psi = \bar{w}_j\bar{w}_k$, in the sector $F = 0$, $p = 0$, $q = 2$. No zero modes in the other sectors are present.) Generalizing this analysis to higher spheres, we obtain the result

$$I^{S^{2d}\backslash\{\cdot\}} = C_{2d-1}^{d-1}$$

for the index of the pure Dolbeault complex.

Again, we can try to make contact of this result with functional integral calculations. Unfortunately, this does not work better here than in the S^4 case. At the tree level, one obtains a *fractional* contribution to the path integral,

$$\frac{1}{48\pi^3}\int F \wedge F \wedge F = \frac{9}{2}.$$

The one-loop contribution associated with the gauge field vanishes by the same token as for S^4 (see equation (30) and the discussion thereabout). Higher loops seem to be suppressed for small β, but the presence of singularity does not allow one to make a definite statement ....

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