Symmetry, Integrability and Geometry: Methods and Applications SIGMA **8** (2012), 038, 18 pages

On Free Pseudo-Product Fundamental Graded Lie Algebras

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Received December 16, 2011, in final form June 14, 2012; Published online June 27, 2012
http://dx.doi.org/10.3842/SIGMA.2012.038

Abstract. In this paper we first state the classification of the prolongations of complex free fundamental graded Lie algebras. Next we introduce the notion of free pseudo-product fundamental graded Lie algebras and study the prolongations of complex free pseudo-product fundamental graded Lie algebras. Furthermore we investigate the automorphism group of the prolongation of complex free pseudo-product fundamental graded Lie algebras.

Key words: fundamental graded Lie algebra; prolongation; pseudo-product graded Lie algebra

2010 Mathematics Subject Classification: 17B70

1 Introduction

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a graded Lie algebra over the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, and let μ be a positive integer. The graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called a fundamental graded Lie algebra if the following conditions hold: (i) $\mathfrak{m}$ is finite-dimensional; (ii) $\mathfrak{g}_{-1} \neq \{0\}$, and $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$. Moreover a fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is said to be of the μ-th kind if $\mathfrak{g}_{-\mu} \neq \{0\}$, and $\mathfrak{g}_p = \{0\}$ for all $p < -\mu$. It is shown that every fundamental graded algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is prolonged to a graded Lie algebra $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ satisfying the following conditions: (i) $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ for all $p < 0$; (ii) for $X \in \mathfrak{g}(\mathfrak{m})_p$ $(p \geqq 0)$, $[X, \mathfrak{m}] = \{0\}$ implies $X = 0$; (iii) $\mathfrak{g}(\mathfrak{m})$ is maximum among graded Lie algebras satisfying conditions (i) and (ii) above. The graded Lie algebra $\mathfrak{g}(\mathfrak{m})$ is called the prolongation of $\mathfrak{m}$. Note that $\mathfrak{g}(\mathfrak{m})_0$ is the Lie algebra of all the derivations of $\mathfrak{m}$ as a graded Lie algebra.

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra of the μ-th kind, where $\mu \geqq 2$. The fundamental graded Lie algebra $\mathfrak{m}$ is called a free fundamental graded Lie algebra of type (n, μ) if the following universal properties hold:

(i) $\dim \mathfrak{g}_{-1} = n$;

(ii) Let $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ be a fundamental graded Lie algebra of the μ-th kind and let φ be a surjective linear mapping of $\mathfrak{g}_{-1}$ onto $\mathfrak{g}'_{-1}$. Then φ can be extended uniquely to a graded Lie algebra epimorphism of $\mathfrak{m}$ onto $\mathfrak{m}'$.

In Section 3 we see that a universal fundamental graded Lie algebra $b(V, \mu)$ of the μ-th kind introduced by N. Tanaka [11] becomes a free fundamental graded Lie algebra of type (n, μ), where $\mu \geqq 2$, and V is a vector space such that $\dim V = n \geqq 2$.

In [13], B. Warhurst gave the complete list of the prolongations of real free fundamental graded Lie algebras by using a Hall basis of a free Lie algebra. The complex version of his theorem has the completely same form except for the ground number field as follows:

Theorem I. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free fundamental graded Lie algebra of type* (n,μ) *over* $\mathbb{C}$. *Then the prolongation* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ *of* $\mathfrak{m}$ *is one of the following types:*

(*a*) $(n,\mu) \neq (n,2)$ $(n \geqq 2)$, $(2,3)$. *In this case,* $\mathfrak{g}(\mathfrak{m})_1 = \{0\}$.

(*b*) $(n,\mu) = (n,2)$ $(n \geqq 3)$, $(2,3)$. *In this case,* $\dim \mathfrak{g}(\mathfrak{m}) < \infty$ *and* $\mathfrak{g}(\mathfrak{m})_1 \neq \{0\}$. *Furthermore* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to a finite-dimensional simple graded Lie algebra of type* $(B_n, \{\alpha_n\})$ $(n \geqq 3)$ *or* $(G_2, \{\alpha_1\})$ $(n = 2)$ (*see* [15] *or Section* 5 *for the gradations of finite-dimensional simple graded Lie algebras over* $\mathbb{C}$).

(*c*) $(n,\mu) = (2,2)$. *In this case,* $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. *Furthermore,* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to the contact algebra* $K(1)$ *as a graded Lie algebra.*

The first purpose of this paper is to give a proof of Theorem I by using the classification of complex irreducible transitive graded Lie algebras of finite depth (cf. [6]). Note that Warhurst's methods in [13] are available to the proof of Theorem I.

Next we introduce the notion of free pseudo-product fundamental graded Lie algebras. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra, and let $\mathfrak{e}$ and $\mathfrak{f}$ be nonzero subspaces of $\mathfrak{g}_{-1}$. Then $\mathfrak{m}$ is called a pseudo-product fundamental graded Lie algebra with pseudo-product structure $(\mathfrak{e},\mathfrak{f})$ if the following conditions hold: (*i*) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$; (*ii*) $[\mathfrak{e},\mathfrak{e}] = [\mathfrak{f},\mathfrak{f}] = \{0\}$ (cf. [10]).

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a pseudo-product fundamental graded Lie algebra with a pseudo-product structure $(\mathfrak{e},\mathfrak{f})$, and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of $\mathfrak{m}$. Moreover let $\mathfrak{g}_0$ be the Lie algebra of all the derivations of $\mathfrak{m}$ as a graded Lie algebra preserving $\mathfrak{e}$ and $\mathfrak{f}$. Also for $p \geqq 1$ we set $\mathfrak{g}_p = \{X \in \mathfrak{g}(\mathfrak{m})_p : [X,\mathfrak{g}_k] \subset \mathfrak{g}_{p+k}$ for all $k < 0\}$ inductively. Then the direct sum $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ becomes a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$, which is called the prolongation of $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$.

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a pseudo-product fundamental graded Lie algebra of the μ-th kind with pseudo-product structure $(\mathfrak{e},\mathfrak{f})$, where $\mu \geqq 2$. The pseudo-product fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called a free pseudo-product fundamental graded Lie algebra of type (m,n,μ) if the following conditions hold:

(*i*) $\dim \mathfrak{e} = m$ and $\dim \mathfrak{f} = n$;

(*ii*) Let $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ be a pseudo-product fundamental graded Lie algebra of the μ-th kind with pseudo-product structure $(\mathfrak{e}',\mathfrak{f}')$ and let φ be a surjective linear mapping of $\mathfrak{g}_{-1}$ onto $\mathfrak{g}'_{-1}$ such that $\varphi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\varphi(\mathfrak{f}) \subset \mathfrak{f}'$. Then φ can be extended uniquely to a graded Lie algebra epimorphism of $\mathfrak{m}$ onto $\mathfrak{m}'$.

The main purpose of this paper is to prove the following theorem.

Theorem II. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free pseudo-product fundamental graded Lie algebra of type* (m,n,μ) *with pseudo-product structure* $(\mathfrak{e},\mathfrak{f})$ *over* $\mathbb{C}$, *and let* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *be the prolongation of* $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$. *If* $\mathfrak{g}_1 \neq \{0\}$, *then* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is a finite-dimensional simple graded Lie algebra of type* $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be the prolongation of a free pseudo-product fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ over $\mathbb{C}$. We denote by $\mathrm{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0$ the group of all the automorphisms as a graded Lie algebra preserving $\mathfrak{e}$ and $\mathfrak{f}$, which is called the automorphism group of the pseudo-product graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. In Section 9, we show that $\mathrm{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0$ is isomorphic to $GL(\mathfrak{e}) \times GL(\mathfrak{f})$.

Notation and conventions

(1) From Section 2 to the last section, all vector spaces are considered over the field $\mathbb{C}$ of complex numbers.

(2) Let V be a vector space and let W_1 and W_2 be subspaces of V. We denote by $W_1 \wedge W_2$ the subspace of $\Lambda^2 V$ spanned by all the elements of the form $w_1 \wedge w_2$ $(w_1 \in W_1, w_2 \in W_2)$.

(3) Graded vector spaces are always $\mathbb{Z}$-graded. If we write $V = \bigoplus_{p<0} V_p$, then it is understood that $V_p = \{0\}$ for all $p \geqq 0$. Let $V = \bigoplus_{p\in\mathbb{Z}} V_p$ be a graded vector space. We denote by V_- the subspace $V = \bigoplus_{p<0} V_p$. Also for $k \in \mathbb{Z}$ we denote by $V_{\leqq k}$ the subspace $\bigoplus_{p\leqq k} V_p$.

Let $V = \bigoplus_{p\in\mathbb{Z}} V_p$ and $W = \bigoplus_{p\in\mathbb{Z}} W_p$ be graded vector spaces. For $r \in \mathbb{Z}$, we set

$$\mathrm{Hom}(V, W)_r = \{\varphi \in \mathrm{Hom}(V, W) : \varphi(V_p) \subset W_{p+r} \text{ for all } p \in \mathbb{Z}\}.$$

2 Free fundamental graded Lie algebras

First of all we give several definitions about graded Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. Assume that there is given a family of subspaces $(\mathfrak{g}_p)_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ satisfying the following conditions:

(i) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$;

(ii) $\dim \mathfrak{g}_p < \infty$ for all $p \in \mathbb{Z}$;

(iii) $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for all $p, q \in \mathbb{Z}$.

Under these conditions, we say that $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a graded Lie algebra (GLA). Moreover we define the notion of homomorphism, isomorphism, monomorphism, epimorphism, subalgebra and ideal for GLAs in an obvious manner.

A GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called transitive if for $X \in \mathfrak{g}_p$ $(p \geqq 0)$, $[X, \mathfrak{g}_-] = \{0\}$ implies $X = 0$, where $\mathfrak{g}_-$ is the negative part $\bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$. Furthermore a GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called irreducible if the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible.

Let μ be a positive integer. A GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is said to be of depth μ if $\mathfrak{g}_{-\mu} \neq \{0\}$ and $\mathfrak{g}_p = \{0\}$ for all $p < -\mu$.

Next we define fundamental GLAs. A GLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called a *fundamental graded Lie algebra* (FGLA) if the following conditions hold:

(i) $\dim \mathfrak{m} < \infty$;

(ii) $\mathfrak{g}_{-1} \neq \{0\}$, and $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$, or more precisely $\mathfrak{g}_{p-1} = [\mathfrak{g}_p, \mathfrak{g}_{-1}]$ for all $p < 0$.

If an FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is of depth μ, then $\mathfrak{m}$ is also said to be of the μ-th kind. Moreover an FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called non-degenerate if for $X \in \mathfrak{g}_{-1}$, $[X, \mathfrak{g}_{-1}] = \{0\}$ implies $X = 0$.

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA of the μ-th kind, where $\mu \geqq 2$. $\mathfrak{m}$ is called a free fundamental graded Lie algebra of type (n, μ) if the following conditions hold:

(i) $\dim \mathfrak{g}_{-1} = n$;

(ii) Let $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ be an FGLA of the μ-th kind and let φ be a surjective linear mapping of $\mathfrak{g}_{-1}$ onto $\mathfrak{g}'_{-1}$. Then φ can be extended uniquely to a GLA epimorphism of $\mathfrak{m}$ onto $\mathfrak{m}'$.

Proposition 2.1. *Let n and μ be positive integers such that $n, \mu \geqq 2$.*

(1) *There exists a unique free FGLA of type (n, μ) up to isomorphism.*

(2) *Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a free FGLA of type (n, μ). We denote by $\mathrm{Der}(\mathfrak{m})_0$ the Lie algebra of all the derivations of $\mathfrak{m}$ preserving the gradation of $\mathfrak{m}$. Then the mapping $\Phi : \mathrm{Der}(\mathfrak{m})_0 \ni D \mapsto D|\mathfrak{g}_{-1} \in \mathfrak{gl}(\mathfrak{g}_{-1})$ is a Lie algebra isomorphism.*

Proof. (1) The uniqueness of a free FGLA of type (n, μ) follows from the definition. We set $X = \{1, \dots, n\}$. Let $L(X)$ be the free Lie algebra on X (see [1, Chapter II, § 2]) and let $i : X \to L(X)$ be the canonical injection. We define a mapping ϕ of X into $\mathbb{Z}$ by $\phi(k) = -1$ $(k \in X)$. The mapping ϕ defines the natural gradation $(L(X)_p)_{p<0}$ on $L(X)$ such that: (i) $L(X)$ is generated by $L(X)_{-1}$; (ii) $\{i(1), \dots, i(n)\}$ is a basis of $L(X)_{-1}$ (see [1, Chapter II, § 2, no. 6]). Note that if $n > 1$, then $L(X)_p \neq 0$ for all $p < 0$. We set $\mathfrak{a} = \bigoplus_{p<-\mu} L(X)_p$; then $\mathfrak{a}$ is a graded ideal of $L(X)$ and the factor GLA $\mathfrak{m} = L(X)/\mathfrak{a}$ becomes an FGLA of the μ-th kind. We put $\mathfrak{a}_p = \mathfrak{a} \cap L(X)_p$ and $\mathfrak{g}_p = L(X)_p/\mathfrak{a}_p$.

Now we prove that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a free FGLA of type (n, μ). Let $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ be an FGLA of the μ-th kind and let φ be a surjective linear mapping of $\mathfrak{g}_{-1}$ onto $\mathfrak{g}'_{-1}$. Let h be a mapping of X into $\mathfrak{m}'$ defined by $h(k) = \varphi(i(k))$ $(k \in X)$. Then there exists a Lie algebra homomorphism $\tilde{h}$ of $L(X)$ into $\mathfrak{m}'$ such that $\tilde{h} \circ i = h$. Since $L(X)$ (resp. $\mathfrak{m}'$) is generated by $L(X)_{-1}$ (resp. $\mathfrak{g}'_{-1}$), $\tilde{h}$ is surjective. Since $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ is of the μ-th kind, $\tilde{h}(\mathfrak{a}) = 0$, so $\tilde{h}$ induces a GLA epimorphism $L(\varphi)$ of $\mathfrak{m}$ onto $\mathfrak{m}'$ such that $L(\varphi)|\mathfrak{g}_{-1} = \varphi$. The homomorphism $L(\varphi)$ is unique, because $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$. Thus $\mathfrak{m}$ is a free FGLA of type (n, μ).

(2) Assume that $\mathfrak{m}$ is a free FGLA constructed in (1). Let ϕ be an endomorphism of $\mathfrak{g}_{-1}$. By Corollary to Proposition 8 of [1, Chapter II, § 2, no. 8], ϕ can be extended uniquely to a unique derivation D of $L(X)$. Since $D(L(X)_{-1}) = \phi(L(X)_{-1}) = \phi(\mathfrak{g}_{-1}) \subset L(X)_{-1}$, and since $L(X)$ is generated by $L(X)_{-1}$, we see that $D(L(X)_p) \subset L(X)_p$ and $D(\mathfrak{a}) \subset \mathfrak{a}$. Thus there is a derivation of D_ϕ of $\mathfrak{m}$ such that $\pi \circ D = D_\phi \circ \pi$, where π is the natural projection of $L(X)$ onto $\mathfrak{m}$. The correspondence $\mathfrak{gl}(\mathfrak{g}_{-1}) \ni \phi \mapsto D_\phi \in \mathrm{Der}(\mathfrak{m})_0$ is an injective linear mapping. Hence $\dim \mathfrak{gl}(\mathfrak{g}_{-1}) \leqq \dim \mathrm{Der}(\mathfrak{m})_0$. On the other hand, since $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$, the mapping Φ is a Lie algebra monomorphism. Therefore Φ is a Lie algebra isomorphism. ■

Remark 2.1. Let n and μ be positive integers with $n, \mu \geqq 2$, and let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a free FGLA of type (n, μ). Furthermore let $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ be an FGLA of the μ-th kind, and let φ be a linear mapping of $\mathfrak{g}_{-1}$ into $\mathfrak{g}'_{-1}$.

(1) From the proof of Proposition 2.1, there exists a unique GLA homomorphism $L(\varphi)$ of $\mathfrak{m}$ into $\mathfrak{m}'$ such that $L(\varphi)|\mathfrak{g}_{-1} = \varphi$.

(2) Let $\mathfrak{m}'' = \bigoplus_{p<0} \mathfrak{g}''_p$ be an FGLA of the μ-th kind, and let φ' be a linear mapping of $\mathfrak{g}'_{-1}$ into $\mathfrak{g}''_{-1}$. Assume that $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ is a free FGLA. By the uniqueness of $L(\varphi' \circ \varphi)$, we see that $L(\varphi' \circ \varphi) = L(\varphi') \circ L(\varphi)$.

(3) Assume that $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ is a free FGLA and φ is injective. By the result of (2), $L(\varphi)$ is a monomorphism.

(4) Let W be an m-dimensional subspace of $\mathfrak{g}_{-1}$ with $m \geqq 2$. By the result of (3), the subalgebra of $\mathfrak{m}$ generated by W is a free FGLA of type (m, μ).

By Remark 2.1 (4) and [1, Chapter II, § 2, Theorem 1], we get the following lemma.

Lemma 2.1. *Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a free FGLA of type (n, μ) with $\mu \geqq 3$. If X, Y are linearly independent elements of $\mathfrak{g}_{-1}$, then*

$$\operatorname{ad}(X)^{\mu}(Y) = 0, \qquad \operatorname{ad}(X)^{\mu-1}(Y) \neq 0,$$
$$\operatorname{ad}(Y)\operatorname{ad}(X)^{\mu-1}(Y) = 0, \qquad \operatorname{ad}(Y)\operatorname{ad}(X)^{\mu-2}(Y) \neq 0.$$

3 Universal fundamental graded Lie algebras

Following N. Tanaka [11], we introduce universal FGLAs of the μ-th kind.

Let V be an n-dimensional vector space. We define vector spaces $b(V)_p$ $(p < 0)$ and linear mappings B_p of $\sum_{r+s=p} b(V)_r \wedge b(V)_s$ into $b(V)_p$ $(p \leqq -2)$ as follows: First of all, we put $b(V)_{-1} = V$ and $b(V)_{-2} = \Lambda^2 V$. Further we define a mapping $B_{-2} : b(V)_{-1} \wedge b(V)_{-1} \to b(V)_{-2}$ to be the identity mapping. For $k \leqq -3$, we define $b(V)_k$ and B_k inductively as follows: We set $b(V)^{(k+1)} = \bigoplus_{p=-1}^{k+1} b(V)_p$ and we define a subspace $c(V)_k$ of $\Lambda^2(b(V)^{(k+1)})$ to be $\sum_{r+s=k} b(V)_r \wedge b(V)_s$. We denote by $A(V)_k$ the subspace of $c(V)_k$ spanned by the elements

$$\mathop{\mathfrak{S}}_{(X,Y,Z)} \sum_{r+s=k} \sum_{u+v=r} B_r(X_u \wedge Y_v) \wedge Z_s, \qquad X, Y, Z \in b(V)^{(k+1)},$$

where $\mathop{\mathfrak{S}}_{(X,Y,Z)}$ stands for the cyclic sum with respect to X, Y, Z, and X_u denotes the $b(V)_u$-component in the decomposition $b(V)^{(k+1)} = \bigoplus_{p=-1}^{k+1} b(V)_p$. Now we define $b(V)_k$ to be the factor space $c(V)_k / A(V)_k$, and B_k to be the projection of $c(V)_k$ onto $b(V)_k$. We put $b(V) = \bigoplus_{p<0} b(V)_p$ and define a bracket operation $[\ ,\ ]$ on $b(V)$ by

$$[X, Y] = \sum_{p \leqq -2} \sum_{r+s=p} B_p(X_r \wedge Y_s)$$

for all $X, Y \in b(V)$. Then $b(V) = \bigoplus_{p<0} b(V)_p$ becomes a GLA generated by $b(V)_{-1}$, and $b(V)_p \neq 0$ for all $p < 0$ if $\dim V > 1$.

Note that $b(V)_{-3}$ is isomorphic to $\Lambda^2(V) \otimes V / \Lambda^3 V$. Let μ be a positive integer. Assume that $\mu \geqq 2$ and $\dim V = n \geqq 2$. Since $\bigoplus_{p<-\mu} b(V)_p$ is a graded ideal of $b(V)$, we see that the factor space $b(V, \mu) = b(V) / \bigoplus_{p<-\mu} b(V)_p$ becomes an FGLA of μ-th kind, which is called a universal fundamental graded Lie algebra of the μ-th kind. By [11, Proposition 3.2], $b(V, \mu)$ is a free FGLA of type (n, μ).

4 The prolongations of fundamental graded Lie algebras

Following N. Tanaka [11], we introduce the prolongations of FGLAs. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA. A GLA $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ is called the prolongation of $\mathfrak{m}$ if the following conditions hold:

(*i*) $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ for all $p < 0$;

(*ii*) $\mathfrak{g}(\mathfrak{m})$ is a transitive GLA;

(*iii*) If $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ is a GLA satisfying conditions (i) and (ii) above, then $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ can be embedded in $\mathfrak{g}(\mathfrak{m})$ as a GLA.

We construct the prolongation $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ of $\mathfrak{m}$. We set $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ $(p < 0)$. We define subspaces $\mathfrak{g}(\mathfrak{m})_k$ $(k \geqq 0)$ of $\mathrm{Hom}(\mathfrak{m}, \bigoplus_{p\leqq k-1} \mathfrak{g}(\mathfrak{m})_p)_k$ and a bracket operation on $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ inductively. First $\mathfrak{g}(\mathfrak{m})_0$ is defined to be $\mathrm{Der}(\mathfrak{m})_0$ and a bracket operation $[\ ,\ ] : \bigoplus_{p\leqq 0} \mathfrak{g}(\mathfrak{m})_p \times \bigoplus_{p\leqq 0} \mathfrak{g}(\mathfrak{m})_p \to \bigoplus_{p\leqq 0} \mathfrak{g}(\mathfrak{m})_p$ is defined by

$$[X, Y] = -[Y, X] = X(Y), \qquad X \in \mathfrak{g}(\mathfrak{m})_0, \quad Y \in \mathfrak{m},$$
$$[X, Y] = XY - YX, \qquad X, Y \in \mathfrak{g}(\mathfrak{m})_0.$$

Next for $k > 0$ we define $\mathfrak{g}(\mathfrak{m})_k$ $(k \geqq 1)$ inductively as follows:

$$\mathfrak{g}(\mathfrak{m})_k = \left\{ X \in \mathrm{Hom}\left(\mathfrak{m}, \bigoplus_{p\leqq k-1} \mathfrak{g}(\mathfrak{m})_p\right)_k : X([u, v]) = [X(u), v] + [u, X(v)] \text{ for all } u, v \in \mathfrak{m} \right\},$$

where for $X \in \mathfrak{g}(\mathfrak{m})_r$, $u \in \mathfrak{m}$, we set $[X, u] = -[u, X] = X(u)$. Further for $X \in \mathfrak{g}(\mathfrak{m})_k$, $Y \in \mathfrak{g}(\mathfrak{m})_l$ $(k, l \geqq 0)$, by induction on $k + l \geqq 0$, we define $[X, Y] \in \mathrm{Hom}(\mathfrak{m}, \mathfrak{g}(\mathfrak{m}))_{k+l}$ by

$$[X, Y](u) = [X, [Y, u]] - [Y, [X, u]], \qquad u \in \mathfrak{m}.$$

It follows easily that $[X, Y] \in \mathfrak{g}(\mathfrak{m})_{k+l}$. With this bracket operation, $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ becomes a graded Lie algebra satisfying conditions (*i*), (*ii*) and (*iii*) above.

Let $\mathfrak{m}$ and $\mathfrak{g}(\mathfrak{m})$ be as above. Assume that we are given a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}(\mathfrak{m})_0$. We define subspaces $\mathfrak{g}_k$ $(k \geqq 1)$ of $\mathfrak{g}(\mathfrak{m})_k$ inductively as follows:

$$\mathfrak{g}_k = \{X \in \mathfrak{g}(\mathfrak{m})_k : [X, \mathfrak{g}_p] \subset \mathfrak{g}_{p+k} \text{ for all } p < 0\}.$$

If we put $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ becomes a transitive graded Lie subalgebra of $\mathfrak{g}(\mathfrak{m})$, which is called the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

By Proposition 2.1 (2) we get the following proposition.

Proposition 4.1. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free FGLA and let* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ *be the prolongation of* $\mathfrak{m}$*. Then the mapping* $\mathfrak{g}(\mathfrak{m})_0 \ni D \mapsto D|\mathfrak{g}_{-1} \in \mathfrak{gl}(\mathfrak{g}_{-1})$ *is an isomorphism.*

Conversely we obtain the following proposition.

Proposition 4.2. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be an FGLA of the* μ*-th kind and let* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ *be the prolongation of* $\mathfrak{m}$*. Assume that* $\mathfrak{g}(\mathfrak{m})_0$ *is isomorphic to* $\mathfrak{gl}(\mathfrak{g}_{-1})$*. If* $\mu = 2$ *or* $\mu = 3$*, then* $\mathfrak{m}$ *is a free FGLA.*

Proof. We put $n = \dim \mathfrak{g}_{-1}$. We consider a universal FGLA $b(\mathfrak{g}_{-1},\mu) = \bigoplus_{p<0} b(\mathfrak{g}_{-1},\mu)_p$ of the μ-th kind. Since $b(\mathfrak{g}_{-1},\mu)$ is a free FGLA of type (n,μ), there exists a GLA epimorphism φ of $b(\mathfrak{g}_{-1},\mu)$ onto $\mathfrak{m}$ such that the restriction $\varphi|b(\mathfrak{g}_{-1},\mu)_{-1}$ is the identity mapping. Let $\check{b}(\mathfrak{g}_{-1},\mu) = \bigoplus_{p\in\mathbb{Z}} \check{b}(\mathfrak{g}_{-1},\mu)_p$ be the prolongation of $b(\mathfrak{g}_{-1},\mu)$. Since the mapping $\mathfrak{g}(\mathfrak{m})_0 \ni D \mapsto D|\mathfrak{g}_{-1} \in \mathfrak{gl}(\mathfrak{g}_{-1})$ is an isomorphism, φ can be extended to be a homomorphism $\check{\varphi}$ of $\bigoplus_{p\leqq 0} \check{b}(\mathfrak{g}_{-1},\mu)_p$ onto $\bigoplus_{p\leqq 0} \mathfrak{g}(\mathfrak{m})_p$. Let $\mathfrak{a}$ be the kernel of $\check{\varphi}$; then $\mathfrak{a}$ is a graded ideal of $\bigoplus_{p\leqq 0} \check{b}(\mathfrak{g}_{-1},\mu)_p$. We set $\mathfrak{a}_p = \mathfrak{a}\cap\check{b}(\mathfrak{g}_{-1},\mu)_p$; then $\mathfrak{a} = \bigoplus_{p\leqq 0} \mathfrak{a}_p$. Since the restriction of $\check{\varphi}$ to $\check{b}(\mathfrak{g}_{-1},\mu)_{-1}\oplus\check{b}(\mathfrak{g}_{-1},\mu)_0$ is injective, $\mathfrak{a}_p = \{0\}$ for $p \geqq -1$. Also each $\mathfrak{a}_p$ is a $\check{b}(\mathfrak{g}_{-1},\mu)_0$-submodule of $\check{b}(\mathfrak{g}_{-1},\mu)_p$. From the construction of $b(\mathfrak{g}_{-1},\mu)$, we see that $b(\mathfrak{g}_{-1},\mu)_{-2}$ (resp. $b(\mathfrak{g}_{-1},\mu)_{-3}$) is isomorphic to $\Lambda^2(\mathfrak{g}_{-1})$ (resp. $\Lambda^2(\mathfrak{g}_{-1})\otimes\mathfrak{g}_{-1}/\Lambda^3(\mathfrak{g}_{-1})$) as a $\check{b}(\mathfrak{g}_{-1},\mu)_0$-module. By the table of [8], $\Lambda^2(\mathfrak{g}_{-1})$ and $\Lambda^2(\mathfrak{g}_{-1})\otimes\mathfrak{g}_{-1}/\Lambda^3(\mathfrak{g}_{-1})$ are irreducible $\mathfrak{gl}(\mathfrak{g}_{-1})$-modules. Thus we see that $\mathfrak{a}_{-2} = \mathfrak{a}_{-3} = \{0\}$. From $\mu \leqq 3$ it follows that φ is an isomorphism. ∎

5 Finite-dimensional simple graded Lie algebras

Following [15], we first state the classification of finite-dimensional simple GLAs.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA of the μ-th kind over $\mathbb{C}$ such that the negative part $\mathfrak{g}_-$ is an FGLA. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$; then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $E \in \mathfrak{h}$, where E is the element of $\mathfrak{g}_0$ such that $[E,x] = px$ for all $x \in \mathfrak{g}_p$ and p. Let Δ be a root system of $(\mathfrak{g},\mathfrak{h})$. For $\alpha\in\Delta$, we denote by $\mathfrak{g}^\alpha$ the root space corresponding to α. We set $\mathfrak{h}_\mathbb{R} = \{h\in\mathfrak{h} : \alpha(h)\in\mathbb{R} \text{ for all } \alpha\in\Delta\}$ and let $(h_1,\dots,h_l)$ be a basis of $\mathfrak{h}_\mathbb{R}$ such that $h_1 = E$. We define the set of positive roots Δ^+ as the set of roots which are positive with respect to the lexicographical ordering in $\mathfrak{h}_\mathbb{R}^*$ determined by the basis $(h_1,\dots,h_l)$ of $\mathfrak{h}_\mathbb{R}$. Let $\Pi\subset\Delta^+$ be the corresponding simple root system. We denote by $\{m_1,\dots,m_l\}$ the coordinate functions corresponding to Π, i.e., for $\alpha\in\Delta$, we can write $\alpha = \sum_{i=1}^{l} m_i(\alpha)\alpha_i$.

We set $\alpha_i(E) = s_i$ and $\boldsymbol{s} = (s_1,\dots,s_l)$; then each s_i is a non-negative integer. For $\alpha\in\Delta$, we call the integer $\ell_{\boldsymbol{s}}(\alpha) = \sum_{i=1}^{l} m_i(\alpha)s_i$ the $\boldsymbol{s}$-length of α. We put $\Delta_p = \{\alpha\in\Delta : \ell_{\boldsymbol{s}}(\alpha) = p\}$, $\Pi_p = \Delta_p\cap\Pi$ and $I = \{i\in\{1,\dots,l\} : s_i = 1\}$. Let θ be the highest root of $\mathfrak{g}$; then $\ell_{\boldsymbol{s}}(\theta) = \mu$. Also since the $\mathfrak{g}_0$-module $\mathfrak{g}_{-\mu}$ is irreducible, $\dim\mathfrak{g}_{-\mu} = 1$ if and only if $\langle\theta,\alpha_i^\vee\rangle = 0$ for all $i\in\{1,\dots,l\}\setminus I$, where $\{\alpha_i^\vee\}$ is the simple root system of the dual root system $\Delta^\vee$ of Δ corresponding to $\{\alpha_i\}$. In our situation, since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$, we have $s_i = 0$ or 1 for all i. The l-tuple $\boldsymbol{s} = (s_1,\dots,s_l)$ of non-negative integers is determined only by the ordering of $(\alpha_1,\dots,\alpha_l)$. In what follows, we assume that the ordering of $(\alpha_1,\dots,\alpha_l)$ is as in the table of [2]. If $\mathfrak{g}$ has the Dynkin diagram of type X_l $(X = A,\dots,G)$, then the simple GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is said to be of type (X_l,Π_1). Here we remark that for an automorphism $\bar{\mu}$ of the Dynkin diagram, a simple GLA of type (X_l,Π_1) is isomorphic to that of type $(X_l,\bar{\mu}(\Pi_1))$. We will identify a simple GLA of type (X_l,Π_1) with that of type $(X_l,\bar{\mu}(\Pi_1))$.

For $i\in I$, we put $\Delta_p^{(i)} = \{\alpha\in\Delta : m_i(\alpha) = p \text{ and } m_j(\alpha) = 0 \text{ for } j\in I\setminus\{i\}\}$ and $\mathfrak{g}_p^{(i)} = \sum_{\alpha\in\Delta_p^{(i)}} \mathfrak{g}^\alpha$; then $\mathfrak{g}_{-1}^{(i)}$ is an irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$ with highest weight $-\alpha_i$. In particular, if the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, then $\#(I) = 1$.

For $i \in I$, we denote by $\mathfrak{g}^{(i)}$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^{(i)}_{-1} \oplus \mathfrak{g}^{(i)}_1$; then $\mathfrak{g}^{(i)}$ is a simple GLA whose Dynkin diagram is the connected component containing the vertex i of the subdiagram of X_l corresponding to vertices $(\{1, \ldots, l\} \setminus I) \cup \{i\}$. We denote by $\theta^{(i)}$ the highest root of $\mathfrak{g}^{(i)}$. Then $[\mathfrak{g}^{(i)}_{-1}, \mathfrak{g}^{(i)}_{-1}] = \{0\}$ if and only if $m_i(\theta^{(i)}) = 1$.

From Theorem 5.2 of [15], we obtain the following theorem:

Theorem 5.1. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA over $\mathbb{C}$ such that $\mathfrak{g}_-$ is an FGLA and the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{g}_-$ except for the following cases:*

- (a) *$\mathfrak{g}_-$ is of the first kind;*
- (b) *$\mathfrak{g}_-$ is of the second kind and* $\dim \mathfrak{g}_{-2} = 1$.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA. Now we assume that $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(\mathfrak{g}_{-1})$; then the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible. The derived subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ of $\mathfrak{g}_0$ is a semisimple Lie algebra whose Dynkin diagram is the subdiagram of X_l consisting of the vertices $\{1, \ldots, l\} \setminus I$. Since $[\mathfrak{g}_0, \mathfrak{g}_0]$ is of type A_{l-1} and since the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is elementary, (X_l, Δ_1) is one of the following cases:

$$(A_l, \{\alpha_1\}), \qquad (B_l, \{\alpha_l\}), \quad l \geqq 2, \qquad (G_2, \{\alpha_1\}).$$

From this result and Propositions 4.1 and 4.2, we get the following theorem:

Theorem 5.2. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA of type (X_l, Π_1) over $\mathbb{C}$ satisfying the following conditions:*

- (i) *$\mathfrak{g}_-$ is an FGLA of the μ-th kind;*
- (ii) *The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible;*
- (iii) *$\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(\mathfrak{g}_{-1})$;*
- (iv) *$\mathfrak{g}$ is the prolongation of $\mathfrak{g}_-$.*

Then $\mathfrak{g}_-$ is a free FGLA of type (l, μ), and $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is one of the following types:

- (a) *$l \geqq 3$, $\mu = 2$, $(X_l, \Pi_1) = (B_l, \{\alpha_l\})$.*
- (b) *$l = 2$, $\mu = 3$, $(X_l, \Pi_1) = (G_2, \{\alpha_1\})$.*

6 Graded Lie algebras $W(n)$, $K(n)$ of Cartan type

In this section, following V.G. Kac [3], we describe Lie algebras $W(n)$, $K(n)$ of Cartan type and their standard gradations.

Let $A(m)$ denote the monoid (under addition) of all m-tuples of non-negative integers. For an m-tuple $\boldsymbol{s} = (s_1, \ldots, s_m)$ of positive integers and $\alpha = (\alpha_1, \ldots, \alpha_m) \in A(m)$ we set $\|\alpha\|_{\boldsymbol{s}} = \sum_{i=1}^{m} s_i\alpha_i$. Also we denote the m-tuple $(1, \ldots, 1)$ by $\mathbf{1}_m$ and we denote the $(m+1)$-tuple $(1, \ldots, 1, 2)$ by $(\mathbf{1}_m, 2)$. Let $\mathfrak{A}(m) = \mathbb{C}[x_1, \ldots, x_m]$. For any m-tuple $\boldsymbol{s}$ of positive integers, we denote by $\mathfrak{A}(m; \boldsymbol{s})_p$ the subspace of $\mathfrak{A}(m)$ spanned by polynomials

$$x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}, \qquad \alpha = (\alpha_1, \ldots, \alpha_m) \in A(m), \quad \|\alpha\|_{\boldsymbol{s}} = p.$$

Let $W(m)$ be the Lie algebra consisting of all the polynomial vector fields

$$\sum_{i=1}^{m} P_i \frac{\partial}{\partial x_i}, \qquad P_i \in \mathfrak{A}(m). \tag{6.1}$$

For an m-tuple $\boldsymbol{s} = (s_1, \dots, s_m)$ of positive integers, we denote by $W(m; \boldsymbol{s})_p$ the subspaces of $W(m)$ consisting of those polynomial vector fields (6.1) such that the polynomials P_i are contained in $\mathfrak{A}(m; \boldsymbol{s})_{p+s_i}$; then $W(m; \boldsymbol{s}) = \bigoplus_{p\in\mathbb{Z}} W(m; \boldsymbol{s})_p$ is a transitive GLA. In particular, $W(m; \mathbf{1}_m) = \bigoplus_{p\geqq -1} W(m; \mathbf{1}_m)_p$ is a transitive irreducible GLA such that: (i) $W(m; \mathbf{1}_m)_0$ is isomorphic to $\mathfrak{gl}(m, \mathbb{C})$; (ii) the $W(m; \mathbf{1}_m)_0$-module $W(m; \mathbf{1}_m)_{-1}$ is elementary; (iii) $W(m; \mathbf{1}_m)$ is the prolongation of $W(m; \mathbf{1}_m)_-$.

We now consider the following differential form

$$\omega_K = dx_{2n+1} - \sum_{i=1}^{n} x_{i+n} dx_i.$$

Define

$$K(n) = \{D \in W(2n+1) : D\omega_K \in \mathfrak{A}(2n+1)\omega_K\}.$$

(Here the action of D on the differential forms is extended from its action $\mathfrak{A}(2n+1)$ by requiring that D be derivation of the exterior algebra satisfying $D(df) = d(Df)$, where $df = \sum \frac{\partial f}{\partial x_i} dx_i$, $f \in \mathfrak{A}(m)$.) We set $K(n)_p = W(2n+1; (\mathbf{1}_{2n}, 2))_p \cap K(n)$. Then $K(n) = \bigoplus_{p\geqq -2} K(n)_p$ is a transitive irreducible GLA such that: (i) $K(n)_0$ is isomorphic to $\mathfrak{csp}(n, \mathbb{C})$; (ii) the $K(n)_0$-module $K(n)_{-1}$ is elementary; (iii) $K(n)$ is the prolongation of $K(n)_-$ (cf. [3, 5]).

From Proposition 2.2 of [6], we get

Theorem 6.1. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a transitive GLA over $\mathbb{C}$ satisfying the following conditions:*

- (i) *$\mathfrak{g}_-$ is an FGLA of the μ-th kind;*
- (ii) *$\mathfrak{g}$ is infinite-dimensional;*
- (iii) *The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible;*
- (iv) *$\mathfrak{g}$ is the prolongation of $\mathfrak{g}_-$.*

Then $\mu \leqq 2$ and $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic to $W(m; \mathbf{1}_m)$ or $K(n)$.

7 Classification of the prolongations of free fundamental graded Lie algebras

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a free FGLA of type (n, μ) over $\mathbb{C}$, and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of $\mathfrak{m}$. First of all, we assume that $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. By Theorem 6.1, $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(m)$ as a GLA, where $n = 2m$. Since $K(m)_0$ is isomorphic to $\mathfrak{csp}(m, \mathbb{C})$ and since $\mathfrak{g}(\mathfrak{m})_0$ is isomorphic to $\mathfrak{gl}(n, \mathbb{C})$, we see that $m = 1$. Therefore $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(1)$ as a GLA.

Next we assume that $\dim \mathfrak{g}(\mathfrak{m}) < \infty$ and $\mathfrak{g}(\mathfrak{m})_1 \neq 0$. Since the $\mathfrak{g}(\mathfrak{m})_0$-module $\mathfrak{g}(\mathfrak{m})_{-1}$ is irreducible, $\mathfrak{g}(\mathfrak{m})$ is a finite-dimensional simple GLA (see [4, 7]). By Theorem 5.2, $\mathfrak{g}(\mathfrak{m})$ is isomorphic to one of the following types:

$$(B_l, \{\alpha_l\}) \quad l \geqq 3, \qquad (G_2, \{\alpha_1\}).$$

Thus we get a proof of the following theorem:

Theorem 7.1. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free FGLA of type* (n, μ) *over* $\mathbb{C}$*, and let* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ *be the prolongation of* $\mathfrak{m}$*. Then one of the following cases occurs:*

(*a*) $(n, \mu) \neq (n, 2)$ $(n \geqq 2)$, $(2, 3)$*. In this case,* $\mathfrak{g}(\mathfrak{m})_1 = \{0\}$*.*

(*b*) $(n, \mu) = (n, 2)$ $(n \geqq 3)$, $(2, 3)$*. In this case,* $\dim \mathfrak{g}(\mathfrak{m}) < \infty$ *and* $\mathfrak{g}(\mathfrak{m})_1 \neq \{0\}$*. Furthermore* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to a finite-dimensional simple GLA of type* $(B_n, \{\alpha_n\})$ $(n \geqq 3)$ *or* $(G_2, \{\alpha_1\})$ $(n = 2)$*.*

(*c*) $(n, \mu) = (2, 2)$*. In this case,* $\dim \mathfrak{g}(\mathfrak{m}) = \infty$*. Furthermore,* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to* $K(1)$ *as a GLA.*

8 Free pseudo-product fundamental graded Lie algebras

An FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ equipped with nonzero subspaces $\mathfrak{e}$, $\mathfrak{f}$ of $\mathfrak{g}_{-1}$ is called a pseudo-product FGLA if the following conditions hold:

(*i*) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$;

(*ii*) $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = \{0\}$.

The pair $(\mathfrak{e}, \mathfrak{f})$ is called the pseudo-product structure of the pseudo-product FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$. We will also denote by the triplet $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ the pseudo-product FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ (resp. $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$) be a pseudo-product FGLA with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ (resp. $(\mathfrak{e}', \mathfrak{f}')$). We say that two pseudo-product FGLAs $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ and $(\mathfrak{m}'; \mathfrak{e}', \mathfrak{f}')$ are isomorphic if there exists a GLA isomorphism φ of $\mathfrak{m}$ onto $\mathfrak{m}'$ such that $\varphi(\mathfrak{e}) = \mathfrak{e}'$ and $\varphi(\mathfrak{f}) = \mathfrak{f}'$.

Proposition 8.1. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a pseudo-product FGLA of the* μ*-th kind with pseudo-product structure* $(\mathfrak{e}, \mathfrak{f})$*. If* $\mathfrak{m}$ *is a free FGLA of type* (n, μ)*, then* $n = 2$*.*

Proof. Let $(e_1, \dots, e_m)$ (resp. $(f_1, \dots, f_l)$) be a basis of $\mathfrak{e}$ (resp. $\mathfrak{f}$). Since $[\mathfrak{e}, \mathfrak{f}] = \mathfrak{g}_{-2}$, the space $\mathfrak{g}_{-2}$ is generated by $\{[e_i, f_j] : i = 1, \dots, m, j = 1, \dots, l\}$ as a vector space, so $\dim \mathfrak{g}_{-2} \leqq ml$. On the other hand, since $\mathfrak{m}$ is a free FGLA,

$$\dim \mathfrak{g}_{-2} = \dim b(\mathfrak{g}_{-1}, \mu)_{-2} = \dim \Lambda^2(\mathfrak{g}_{-1}) = \frac{(m+l)(m+l-1)}{2},$$

so $ml \geqq \frac{(m+l)(m+l-1)}{2}$. From this fact it follows that $m = l = 1$. ■

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a pseudo-product FGLA of the μ-th kind with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$, where $\mu \geqq 2$. $\mathfrak{m}$ is called a free pseudo-product FGLA of type (m, n, μ) if the following conditions hold:

(*i*) $\dim \mathfrak{e} = m$ and $\dim \mathfrak{f} = n$;

(*ii*) Let $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ be a pseudo-product FGLA of the μ-th kind with pseudo-product structure $(\mathfrak{e}', \mathfrak{f}')$ and let φ be a surjective linear mapping of $\mathfrak{g}_{-1}$ onto $\mathfrak{g}'_{-1}$ such that $\varphi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\varphi(\mathfrak{f}) \subset \mathfrak{f}'$. Then φ can be extended uniquely to a GLA epimorphism of $\mathfrak{m}$ onto $\mathfrak{m}'$.

Proposition 8.2. *Let* m*,* n *and* μ *be positive integers such that* $\mu \geqq 2$*.*

(1) *There exists a unique free pseudo-product FGLA of type* (m, n, μ) *up to isomorphism.*

(2) *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free pseudo-product FGLA of type* (m, n, μ) *with pseudo-product structure* $(\mathfrak{e}, \mathfrak{f})$*. We denote by* $\mathrm{Der}(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})_0$ *the Lie algebra of all the derivations of* $\mathfrak{m}$ *preserving the gradation of* $\mathfrak{m}$*,* $\mathfrak{e}$ *and* $\mathfrak{f}$*. Then the mapping* $\Phi : \mathrm{Der}(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})_0 \ni D \mapsto (D|\mathfrak{e}, D|\mathfrak{f}) \in \mathfrak{gl}(\mathfrak{e}) \times \mathfrak{gl}(\mathfrak{f})$ *is a Lie algebra isomorphism.*

Proof. (1) The uniqueness of a free pseudo-product FGLA of type (m, n, μ) follows from the definition. Let V be an $(m+n)$-dimensional vector space and let $\mathfrak{e}$, $\mathfrak{f}$ be subspaces of V such that $V = \mathfrak{e} \oplus \mathfrak{f}$, $\dim \mathfrak{e} = m$ and $\dim \mathfrak{f} = n$. Let $\mathfrak{a} = \bigoplus_{p<0} \mathfrak{a}_p$ be the graded ideal of $b(V, \mu)$ generated by $[\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]$. We set $\mathfrak{m} = b(V, \mu)/\mathfrak{a}$, $\mathfrak{g}_p = b(V, \mu)_p/\mathfrak{a}_p$. Clearly $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a pseudo-product FGLA. We show that the factor algebra $\mathfrak{m}$ is a free pseudo-product FGLA of type (m, n, μ). First we prove that $\mathfrak{m}$ is of the μ-th kind. Let $\mathfrak{n} = \bigoplus_{p<0} \mathfrak{g}''_p$ be a free FGLA of type $(2, \mu)$ and let $\mathfrak{e}''$ and $\mathfrak{f}''$ be one-dimensional subspaces of $\mathfrak{g}''_{-1}$ such that $\mathfrak{g}''_{-1} = \mathfrak{e}'' \oplus \mathfrak{f}''$. Let φ_1 be an injective linear mapping of $\mathfrak{g}''_{-1}$ into V such that $\varphi_1(\mathfrak{e}'') \subset \mathfrak{e}$ and $\varphi_1(\mathfrak{f}'') \subset \mathfrak{f}$. Let φ_2 be a linear mapping of V into $\mathfrak{g}''_{-1}$ such that $\varphi_2 \circ \varphi_1 = 1_{\mathfrak{g}''_{-1}}$, $\varphi_2(\mathfrak{e}) = \mathfrak{e}''$ and $\varphi_2(\mathfrak{f}) = \mathfrak{f}''$. There exists a homomorphism $L(\varphi_1)$ (resp. $L(\varphi_2)$) of $\mathfrak{n}$ (resp. $b(V, \mu)$) into $b(V, \mu)$ (resp. $\mathfrak{n}$) such that $L(\varphi_1)|\mathfrak{g}''_{-1} = \varphi_1$ (resp. $L(\varphi_2)|V = \varphi_2$). Since $L(\varphi_2)([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) = \{0\}$, $L(\varphi_2)$ induces a homomorphism $\hat{L}(\varphi_2)$ of $\mathfrak{m}$ into $\mathfrak{n}$ such that $L(\varphi_2) = \hat{L}(\varphi_2) \circ \pi$, where π is the natural projection of $b(V, \mu)$ onto $\mathfrak{m}$. Since

$$1_{\mathfrak{n}} = L(\varphi_2) \circ L(\varphi_1) = \hat{L}(\varphi_2) \circ \pi \circ L(\varphi_1),$$

$\pi \circ L(\varphi_1)$ is a monomorphism of $\mathfrak{n}$ into $\mathfrak{m}$, so $\mathfrak{g}_{-\mu} \neq \{0\}$. Thus $\mathfrak{m}$ is of the μ-th kind. Let $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$ be a pseudo-product FGLA of the μ-th kind with pseudo-product structure $(\mathfrak{e}', \mathfrak{f}')$ and let ϕ be a surjective linear mapping of $b(V, \mu)_{-1}$ onto $\mathfrak{g}'_{-1}$ such that $\phi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\phi(\mathfrak{f}) \subset \mathfrak{f}'$. By the definition of a free FGLA, there exists a GLA epimorphism $L(\phi)$ of $b(V, \mu)$ onto $\mathfrak{m}'$ such that $L(\phi)|b(V, \mu)_{-1} = \phi$. Since $L(\phi)([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) \subset [\mathfrak{e}', \mathfrak{e}'] + [\mathfrak{f}', \mathfrak{f}'] = \{0\}$, we see that $L(\phi)(\mathfrak{a}) = \{0\}$, so the epimorphism $L(\phi)$ induces a GLA epimorphism $\hat{L}(\phi)$ of $\mathfrak{m}$ onto $\mathfrak{m}'$ such that $\hat{L}(\phi)|\mathfrak{g}_{-1} = \phi$.

(2) We may prove the fact that the mapping Φ is surjective. Let ϕ be an endomorphism of $\mathfrak{g}_{-1}$ such that $\phi(\mathfrak{e}) \subset \mathfrak{e}$ and $\phi(\mathfrak{f}) \subset \mathfrak{f}$. By Proposition 2.1 (2), there exists a $D \in \mathrm{Der}(b(V, \mu))_0$ such that $D|b(V, \mu)_{-1} = \phi$. Since $D([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) \subset [\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]$, D induces a derivation $\hat{D}$ of $\mathfrak{m}$ such that $\hat{D}|\mathfrak{g}_{-1} = \phi$. ∎

Remark 8.1. Let m, n, m', n' and μ be positive integers with $\mu \geqq 2$, and let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ (resp. $\mathfrak{m}' = \bigoplus_{p<0} \mathfrak{g}'_p$) be a free pseudo-product FGLA of type (m, n, μ) (resp. (m', n', μ)) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ (resp. $(\mathfrak{e}', \mathfrak{f}')$). Furthermore let φ be a linear mapping of $\mathfrak{g}_{-1}$ into $\mathfrak{g}'_{-1}$ such that $\varphi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\varphi(\mathfrak{f}) \subset \mathfrak{f}'$.

(1) From the proof of Proposition 8.2, there exists a unique GLA homomorphism $\hat{L}(\varphi)$ of $\mathfrak{m}$ into $\mathfrak{m}'$ such that $\hat{L}(\varphi)|\mathfrak{g}_{-1} = \varphi$. If φ is injective, then $\hat{L}(\varphi)$ is a monomorphism.

(2) Assume that $m = n = 1$ and φ is injective. Then $\hat{L}(\varphi)(\mathfrak{m})$ is a graded subalgebra of $\mathfrak{m}'$ isomorphic to a free FGLA of type $(2, \mu)$. From this result, the subalgebra of $\mathfrak{m}'$ generated by a nonzero element X of $\mathfrak{e}'$ and a nonzero element Y of $\mathfrak{f}'$ is a free FGLA of type $(2, \mu)$.

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a pseudo-product FGLA of the μ-th kind with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$. We denote by $\mathfrak{g}_0$ the Lie algebra of all the derivations of $\mathfrak{m}$ preserving the gradation of $\mathfrak{m}$, $\mathfrak{e}$ and $\mathfrak{f}$:

$$\mathfrak{g}_0 = \{D \in \mathrm{Der}(\mathfrak{g})_0 : D(\mathfrak{e}) \subset \mathfrak{e}, D(\mathfrak{f}) \subset \mathfrak{f}\}.$$

The prolongation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $(\mathfrak{m}, \mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$.

A transitive GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called a pseudo-product GLA if there are given nonzero subspaces $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}_{-1}$ satisfying the following conditions:

(*i*) The negative part $\mathfrak{g}_-$ is a pseudo-product FGLA with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$;

(*ii*) $[\mathfrak{g}_0, \mathfrak{e}] \subset \mathfrak{e}$ and $[\mathfrak{g}_0, \mathfrak{f}] \subset \mathfrak{f}$.

The pair $(\mathfrak{e}, \mathfrak{f})$ is called the pseudo-product structure of the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. If the $\mathfrak{g}_0$-modules $\mathfrak{e}$ and $\mathfrak{f}$ are irreducible, then the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is said to be of irreducible type.

The following lemma is due to N. Tanaka (cf. [9]). Here we give a proof for the convenience of the readers.

Lemma 8.1. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a pseudo-product GLA of depth μ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$.*

(1) *If $\mathfrak{g}_-$ is non-degenerate, then $\mathfrak{g}$ is finite-dimensional.*

(2) *If $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is of irreducible type and $\mu \geqq 2$, then $\mathfrak{g}$ is finite-dimensional.*

Proof. (1) The proof is essentially due to the proof of [11, Corollary 3 to Theorem 11.1]. For $p \in \mathbb{Z}$, we set $\mathfrak{h}_p = \{X \in \mathfrak{g}_p : [X, \mathfrak{g}_{\leqq -2}] = \{0\}\}$. We define $I \in \mathfrak{gl}(\mathfrak{g}_{-1})$ as follows: $I(x) = -\sqrt{-1}x$ for $x \in \mathfrak{e}$, $I(x) = \sqrt{-1}x$ for $x \in \mathfrak{f}$. Then $I^2 = -1$, $I([a, x]) = [a, I(x)]$ and $[I(x), I(y)] = [x, y]$ for $a \in \mathfrak{g}_0$ and $x, y \in \mathfrak{g}_{-1}$. We put $\langle x, y\rangle = [I(x), y]$ for $x, y \in \mathfrak{g}_{-1}$. Then $\langle x, y\rangle = \langle y, x\rangle$, and for $x \in \mathfrak{g}_{-1}$, $\langle x, \mathfrak{g}_{-1}\rangle = \{0\}$ implies $x = 0$, since $\mathfrak{g}_-$ is non-degenerate. Also $\langle [a, x], y\rangle + \langle x, [a, y]\rangle = 0$ and $[[b, x], y] = [[b, y], x]$ for $a \in \mathfrak{h}_0$, $b \in \mathfrak{h}_1$ and $x, y \in \mathfrak{g}_{-1}$. Then, for $b \in \mathfrak{h}_1$, $x, y, z \in \mathfrak{g}_{-1}$, we have $\langle [[b, x], y], z\rangle = -\langle y, [[b, x], z]\rangle = -\langle y, [[b, z], x]\rangle = \langle [[b, z], y], x\rangle = \langle [[b, y], z], x\rangle = -\langle z, [[b, y], x]\rangle = -\langle [[b, x], y], z\rangle$, so $\langle [[b, x], y], z\rangle = 0$. By transitivity of $\mathfrak{g}$, $\mathfrak{h}_1 = \{0\}$. Therefore by [11, Corollary 1 to Theorem 11.1], $\mathfrak{g}$ is finite-dimensional.

(2) We may assume that $\mathfrak{g}_1 \neq \{0\}$. By [16, Lemma 2.4], the $\mathfrak{g}_0$-modules $\mathfrak{e}$, $\mathfrak{f}$ are not isomorphic to each other. We put $\mathfrak{d} = \{X \in \mathfrak{g}_{-1} : [X, \mathfrak{g}_{-1}] = \{0\}\}$; then $\mathfrak{d}$ is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$. Hence $\mathfrak{d} = \{0\}$, $\mathfrak{d} = \mathfrak{e}$, $\mathfrak{d} = \mathfrak{f}$ or $\mathfrak{d} = \mathfrak{g}_{-1}$. If $\mathfrak{d} \neq \{0\}$, then $\mathfrak{g}_{-2} = [\mathfrak{e}, \mathfrak{f}] = \{0\}$, which is a contradiction. Thus $\mathfrak{g}_-$ is non-degenerate. By (1), $\mathfrak{g}$ is finite-dimensional. ∎

The prolongation of a pseudo-product FGLA becomes a pseudo-product GLA. By Proposition 8.2 (2), the prolongation of a free pseudo-product FGLA is a pseudo-product GLA of irreducible type. By Lemma 8.1 (2), the prolongation of a free pseudo-product FGLA is finite-dimensional.

Proposition 8.3. *Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a free pseudo-product FGLA of type (m, n, μ) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ and let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$.*

(1) *$\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(\mathfrak{e}) \oplus \mathfrak{gl}(\mathfrak{f})$ as a Lie algebra.*

(2) *$\mathfrak{g}_{-2}$ is isomorphic to $\mathfrak{e} \otimes \mathfrak{f}$ as a $\mathfrak{g}_0$-module. In particular,* $\dim \mathfrak{g}_{-2} = mn$.

(3) $\mathfrak{g}_{-3}$ *is isomorphic to* $S^2(\mathfrak{e}) \otimes \mathfrak{f} \oplus S^2(\mathfrak{f}) \otimes \mathfrak{e}$ *as a* $\mathfrak{g}_0$*-module. In particular,* $\dim \mathfrak{g}_{-3} = \frac{mn(m+n+2)}{2}$.

Proof. (1) This follows from Proposition 8.2 (2).

(2) Let $\mathfrak{a} = \bigoplus_{p<0} \mathfrak{a}_p$ be the graded ideal of $b(\mathfrak{g}_{-1}, \mu)$ generated by $[\mathfrak{e}, \mathfrak{e}]+[\mathfrak{f}, \mathfrak{f}]$. By the construction of $b(\mathfrak{g}_{-1}, \mu)_{-2}$, $\mathfrak{a}_{-2}$ is isomorphic to $\Lambda^2(\mathfrak{e}) \oplus \Lambda^2(\mathfrak{f})$, so $\mathfrak{g}_{-2} = b(\mathfrak{g}_{-1}, \mu)_{-2}/\mathfrak{a}_{-2}$ is isomorphic to $\mathfrak{e} \otimes \mathfrak{f}$.

(3) By the construction of $b(\mathfrak{g}_{-1}, \mu)_{-3}$, $b(\mathfrak{g}_{-1}, \mu)_{-3}$ is isomorphic to

$$(\mathfrak{e} \oplus \mathfrak{f}) \otimes \Lambda^2(\mathfrak{e} \oplus \mathfrak{f})/\Lambda^3(\mathfrak{e} \oplus \mathfrak{f}) \cong (\mathfrak{e} \otimes \mathfrak{e} \otimes \mathfrak{f}) \oplus (\mathfrak{e} \otimes \mathfrak{f} \otimes \mathfrak{f}).$$

Moreover, $\mathfrak{a}_{-3}$ is isomorphic to

$$(\mathfrak{e} \oplus \mathfrak{f}) \otimes \Lambda^2(\mathfrak{e}) \oplus (\mathfrak{e} \oplus \mathfrak{f}) \otimes \Lambda^2(\mathfrak{f})/\Lambda^3(\mathfrak{e} \oplus \mathfrak{f}) \cong \mathfrak{e} \otimes \Lambda^2(\mathfrak{e}) \oplus \mathfrak{f} \otimes \Lambda^2(\mathfrak{f}).$$

Hence $\mathfrak{g}_{-3} = b(\mathfrak{g}_{-1}, \mu)_{-3}/\mathfrak{a}_{-3}$ is isomorphic to

$$(\mathfrak{e} \otimes \mathfrak{e} \otimes \mathfrak{f})/\Lambda^2(\mathfrak{e}) \otimes \mathfrak{f} \oplus (\mathfrak{e} \otimes \mathfrak{f} \otimes \mathfrak{f})/\mathfrak{e} \otimes \Lambda^2(\mathfrak{f}) \cong S^2(\mathfrak{e}) \otimes \mathfrak{f} \oplus S^2(\mathfrak{f}) \otimes \mathfrak{e}.$$

This completes the proof. ■

Proposition 8.4. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a pseudo-product FGLA of the* μ*-th kind with pseudo-product structure* $(\mathfrak{e}, \mathfrak{f})$*, where* $\mu \geqq 2$*. We denote by* $\mathfrak{c}$ *the centralizer of* $\mathfrak{g}_{-2}$ *in* $\mathfrak{g}_{-1}$*. Let* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *be the prolongation of* $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$*. Assume that* $\mathfrak{g}_0$ *is isomorphic to* $\mathfrak{gl}(\mathfrak{e}) \oplus \mathfrak{gl}(\mathfrak{f})$ *as a Lie algebra.*

(1) *If* $\mu = 2$*, then* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free pseudo-product FGLA.*

(2) *If* $\mu \geqq 3$ *and* $\mathfrak{c} \neq \{0\}$*, then* $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ *is not a free pseudo-product FGLA.*

(3) *If* $\mu = 3$ *and* $\mathfrak{c} = \{0\}$*, then* $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ *is a free pseudo-product FGLA.*

Proof. Let $\check{\mathfrak{m}} = \bigoplus_{p<0} \check{\mathfrak{g}}_p$ be the free pseudo-product FGLA of type (m, n, μ) with pseudo-product structure $(\check{\mathfrak{e}}, \check{\mathfrak{f}})$ such that $\check{\mathfrak{g}}_{-1} = \mathfrak{g}_{-1}$, $\check{\mathfrak{e}} = \mathfrak{e}$ and $\check{\mathfrak{f}} = \mathfrak{f}$. Let $\check{\mathfrak{g}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{g}}_p$ be the prolongation of $(\check{\mathfrak{m}}; \check{\mathfrak{e}}, \check{\mathfrak{f}})$. There exists a GLA epimorphism φ of $\check{\mathfrak{m}}$ onto $\mathfrak{m}$ such that the restriction $\varphi|\check{\mathfrak{g}}_{-1}$ is the identity mapping. Since the mapping $\check{\mathfrak{g}}_0 \ni D \mapsto (D|\mathfrak{e}, D|\mathfrak{f}) \in \mathfrak{gl}(\mathfrak{e}) \times \mathfrak{gl}(\mathfrak{f})$ is an isomorphism, φ can be extended to be a homomorphism $\check{\varphi}$ of $\bigoplus_{p\leqq 0} \check{\mathfrak{g}}_p$ onto $\bigoplus_{p\leqq 0} \mathfrak{g}_p$. Let $\mathfrak{a}$ be the kernel of $\check{\varphi}$; then $\mathfrak{a}$ is a graded ideal of $\bigoplus_{p\leqq 0} \check{\mathfrak{g}}_p$. We set $\mathfrak{a}_p = \mathfrak{a} \cap \check{\mathfrak{g}}_p$; then $\mathfrak{a} = \bigoplus_{p\leqq 0} \mathfrak{a}_p$. Since the restriction of $\check{\varphi}$ to $\check{\mathfrak{g}}_{-1} \oplus \check{\mathfrak{g}}_0$ is injective, $\mathfrak{a}_p = \{0\}$ for $p \geqq -1$. Also each $\mathfrak{a}_p$ is a $\check{\mathfrak{g}}_0$-submodule of $\check{\mathfrak{g}}_p$. Since the $\check{\mathfrak{g}}_0$-module $\check{\mathfrak{g}}_{-2}$ is irreducible (Proposition 8.3 (2)), $\varphi|\mathfrak{g}_{-2}$ is injective. If $\mu = 2$, then φ is an isomorphism. This proves the assertion (1). Now we assume that $\mu \geqq 3$. Then

$$\check{\mathfrak{g}}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}] \oplus [[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}].$$

Since $\check{\mathfrak{g}}_0$-modules $[[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}]$ and $[[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}]$ are irreducible and not isomorphic to each other (Proposition 8.3 (3)), one of the following cases occurs: (*i*) $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}]$; (*ii*) $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}]$; (*iii*) $\mathfrak{a}_{-3} = \{0\}$. If $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}]$ (resp. $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}]$), then $\mathfrak{c} = \mathfrak{f}$ (resp. $\mathfrak{c} = \mathfrak{e}$). Also since $\mathfrak{g}_0$-modules $\mathfrak{e}$, $\mathfrak{f}$ are irreducible and not isomorphic to each other, one of the following cases occurs: (*i*) $\mathfrak{c} = \mathfrak{e}$; (*ii*) $\mathfrak{c} = \mathfrak{f}$; (*iii*) $\mathfrak{c} = \{0\}$. If $\mathfrak{c} = \mathfrak{e}$ (resp. $\mathfrak{c} = \mathfrak{f}$), then $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}]$ (resp. $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}]$). In this case, φ is not injective. Hence $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is not free. If $\mathfrak{c} = \{0\}$, then $\mathfrak{a}_{-3} = \{0\}$. Hence $\varphi|\check{\mathfrak{g}}_{-3}$ is an isomorphism. In particular, if $\mu = 3$, then $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is free. ■

Example 8.1. Let V and W be finite-dimensional vector spaces and $k \geqq 1$. We set

$$\begin{aligned} &\mathfrak{C}^k(V,W) = \bigoplus_{p=-k-1}^{-1} \mathfrak{C}^k(V,W)_p, \\ &\mathfrak{C}^k(V,W)_p = W \otimes S^{k+p+1}(V^*), \qquad -k-1 \leqq p \leqq -2, \\ &\mathfrak{C}^k(V,W)_{-1} = V \oplus (W \otimes S^k(V^*)). \end{aligned}$$

The bracket operation of $\mathfrak{C}^k(V,W)$ is defined as follows:

$$\begin{aligned} &[W,V] = \{0\}, \qquad [V,V] = \{0\}, \qquad [W \otimes S^r(V^*), W \otimes S^s(V^*)] = \{0\}, \\ &[w \otimes s_r, v] = w \otimes (v \lrcorner\, s_r) \qquad \text{for } v \in V,\ w \in W,\ s_r \in S^r(V^*). \end{aligned}$$

Equipped with this bracket operation, $\mathfrak{C}^k(V,W)$ becomes a pseudo-product FGLA of the $(k+1)$-th kind with pseudo-product structure $(V, W \otimes S^k(V^*))$, which is called *the contact algebra of order k of bidegree (n,m)*, where $n = \dim V$ and $m = \dim W$ (cf. [14, p. 133]). We assume that $\mathfrak{C}^k(V,W)$ is a free pseudo-product FGLA. Since

$$\dim \mathfrak{C}^k(V,W)_{-2} = m\binom{n+k-2}{k-1}, \qquad \dim V \dim(W \otimes S^k(V^*)) = nm\binom{n+k-1}{k},$$

we get $n = 1$. Since $W \otimes S^k(V^*)$ is contained in the centralizer of $\mathfrak{C}^k(V,W)_{-2}$ in $\mathfrak{C}^k(V,W)_{-1}$, we get $k = 1$. Thus we obtain that $\mathfrak{C}^k(V,W)$ is a free pseudo-product FGLA if and only if $k = 1$, $n = 1$.

Example 8.2. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$. We set $\mathfrak{e} = \mathfrak{g}_{-1}^{(m)}$, $\mathfrak{f} = \mathfrak{g}_{-1}^{(m+1)}$. Then $(\mathfrak{g}_-; \mathfrak{e}, \mathfrak{f})$ is a pseudo-product FGLA. Since $\dim \mathfrak{e} = m$, $\dim \mathfrak{f} = n$ and $\dim \mathfrak{g}_{-2} = mn$, the pseudo-product FGLA $(\mathfrak{g}_-; \mathfrak{e}, \mathfrak{f})$ is a free pseudo-product FGLA of type $(m, n, 2)$ (Proposition 8.3 (2)). Also $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{g}_-$ except for the following cases (see [15]):

(1) $m = n = 1$. In this case, the prolongation of $\mathfrak{g}_-$ is isomorphic to $K(1)$.

(2) $m = 1$ or $n = 1$ and $l = \max\{m, n\} \geqq 2$. In this case, the prolongation of $\mathfrak{g}_-$ is isomorphic to $W(l+1; \boldsymbol{s})$, where $\boldsymbol{s} = (1, 2, \dots, 2)$.

Example 8.3. Let V and W be finite-dimensional vector spaces such that $\dim V = m \geqq 1$ and $\dim W = n \geqq 1$. We set

$$\begin{aligned} &\mathfrak{g}_{-1} = V \oplus W, \qquad \mathfrak{g}_{-2} = V \otimes W, \\ &\mathfrak{g}_{-3} = V \otimes S^2(W) \oplus S^2(V) \otimes W, \qquad \mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}. \end{aligned}$$

The bracket operation of $\mathfrak{m}$ is defined as follows:

$$\begin{aligned} &[\mathfrak{g}_{-3}, \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}] = [\mathfrak{g}_{-2}, \mathfrak{g}_{-2}] = \{0\}, \qquad [V,V] = [W,W] = \{0\}, \\ &[v,w] = -[w,v] = v \otimes w, \qquad [v, v' \otimes w] = -[v' \otimes w, v] = v \odot v' \otimes w, \\ &[v \otimes w, w'] = -[w', v \otimes w] = v \otimes w \odot w', \end{aligned}$$

where $v, v' \in V$ and $w, w' \in W$. Equipped with this bracket operation, $\mathfrak{m}$ becomes a free pseudo-product FGLA of type $(m, n, 3)$ with pseudo-product structure (V, W) (Proposition 8.3).

Theorem 8.1. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free pseudo-product FGLA of type* (m, n, μ) *with pseudo-product structure* $(\mathfrak{e}, \mathfrak{f})$ *over* $\mathbb{C}$. *Furthermore let* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *(resp.* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$*) be the prolongation of* $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ *(resp.* $\mathfrak{m}$*).*

(1) *Assume that* $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. *Then* $m = 1$ *or* $n = 1$, *and* $\mu = 2$. *Furthermore* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic to a finite-dimensional simple GLA of type* $(A_{l+1}, \{\alpha_1, \alpha_2\})$, *where* $l = \max\{m, n\}$. *If* $l = 1$, *then* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to* $K(1)$. *If* $l \geqq 2$, *then* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to* $W(l + 1; \boldsymbol{s})$, *where* $\boldsymbol{s} = (1, 2, \ldots, 2)$.

(2) *If* $\mathfrak{g}_1 \neq \{0\}$, *then* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is a finite-dimensional simple GLA of type* $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$.

Proof. (1) For $p \geqq -1$, we put $\mathfrak{h}_p = \{X \in \mathfrak{g}(\mathfrak{m})_p : [X, \mathfrak{g}_{\leqq -2}] = \{0\}\}$. Assume that $\dim \mathfrak{g}(\mathfrak{m}) = \infty$ and $\mu \geqq 3$. By Proposition 8.4 (2), $\mathfrak{h}_{-1} = \{0\}$. Since $[\mathfrak{h}_0, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{-1} = \{0\}$, we see that $\mathfrak{h}_0 = \{0\}$. By [11, Corollary 1 to Theorem 11.1], we obtain that $\dim \mathfrak{g}(\mathfrak{m}) < \infty$, which is a contradiction. Thus we see that $\mu = 2$ if $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. The remaining assertion follows from Example 8.2.

(2) Assume that $\mathfrak{g}_1 \neq \{0\}$ and $\mu \geqq 3$. By transitivity of $\mathfrak{g}$, $[\mathfrak{g}_1, \mathfrak{e}] \neq \{0\}$ or $[\mathfrak{g}_1, \mathfrak{f}] \neq \{0\}$. We may assume that $[\mathfrak{g}_1, \mathfrak{e}] \neq \{0\}$. Then there exists an irreducible component $\mathfrak{g}'_1$ of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ such that $[\mathfrak{g}'_1, \mathfrak{e}] \neq \{0\}$ and $[\mathfrak{g}'_1, \mathfrak{f}] = \{0\}$. The subalgebra $\mathfrak{e} \oplus [\mathfrak{e}, \mathfrak{g}'_1] \oplus \mathfrak{g}'_1$ is a simple GLA of the first kind. Since $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(\mathfrak{e}) \oplus \mathfrak{gl}(\mathfrak{f})$, $\mathfrak{e} \oplus [\mathfrak{e}, \mathfrak{g}'_1] \oplus \mathfrak{g}'_1$ is of type $(A_m, \{\alpha_1\})$. Let D be a nonzero element of $\mathfrak{g}'_1$. There exist $\lambda \in \mathfrak{e}^*$ and $\eta \in \mathfrak{f}^*$ such that

$$[[D, Z], U] = \lambda(U)Z + \lambda(Z)U, \qquad [[D, Z], W] = \eta(Z)W,$$

where $Z, U \in \mathfrak{e}$ and $W \in \mathfrak{f}$ (cf. [12, p. 4]). Let X (resp. Y) be a nonzero element of $\mathfrak{e}$ (resp. $\mathfrak{f}$). Since the subalgebra generated by X, Y is a free FGLA of type $(2, \mu)$ (Remark 8.1 (2)),

$$\begin{aligned} &\mathrm{ad}(X)^{\mu}(Y) = 0, \qquad \mathrm{ad}(X)^{\mu-1}(Y) \neq 0, \\ &\mathrm{ad}(Y)\,\mathrm{ad}(X)^{\mu-1}(Y) = 0, \qquad \mathrm{ad}(Y)\,\mathrm{ad}(X)^{\mu-2}(Y) \neq 0 \end{aligned}$$

(Lemma 2.1). By induction on μ, we see that

$$\begin{aligned} 0 &= \mathrm{ad}(D)\,\mathrm{ad}(X)^{\mu}(Y) = (\mu(\mu-1)\lambda(X) + \mu\eta(X))\,\mathrm{ad}(X)^{\mu-1}(Y), \\ 0 &= \mathrm{ad}(D)\,\mathrm{ad}(Y)\,\mathrm{ad}(X)^{\mu-1}(Y) \\ &= ((\mu-1)(\mu-2)\lambda(X) + (\mu-1)\eta(X))\,\mathrm{ad}(Y)\,\mathrm{ad}(X)^{\mu-2}(Y). \end{aligned}$$

Since

$$\det \begin{bmatrix} \mu(\mu-1) & \mu \\ (\mu-1)(\mu-2) & \mu-1 \end{bmatrix} = \mu(\mu-1) \neq 0,$$

we see that $\lambda(X) = \eta(X) = 0$, which is a contradiction. Thus we obtain that $\mu = 2$ if $\dim \mathfrak{g}_1 \neq \{0\}$. From Example 8.2, it follows that $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a simple GLA of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$ if $\dim \mathfrak{g}_1 \neq \{0\}$. ■

9 Automorphism groups of the prolongations of free pseudo-product fundamental graded Lie algebras

For a GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ we denote by $\mathrm{Aut}(\mathfrak{g})_0$ the group of all the automorphisms of $\mathfrak{g}$ preserving the gradation of $\mathfrak{g}$:

$$\mathrm{Aut}(\mathfrak{g})_0 = \{\varphi \in \mathrm{Aut}(\mathfrak{g}) : \varphi(\mathfrak{g}_p) = \mathfrak{g}_p \text{ for all } p \in \mathbb{Z}\}.$$

Proposition 9.1. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be an FGLA and let* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ *be the prolongation of* $\mathfrak{m}$*. The mapping* $\Phi : \mathrm{Aut}(\mathfrak{g}(\mathfrak{m}))_0 \ni \phi \mapsto \phi|\mathfrak{m} \in \mathrm{Aut}(\mathfrak{m})_0$ *is an isomorphism.*

Proof. It is clear that Φ is a group homomorphism. We prove that Φ is injective. Let ϕ be an element of $\mathrm{Ker}\,\Phi$. Assume that $\phi(X) = X$ for all $X \in \mathfrak{g}(\mathfrak{m})_p$ $(p < k)$. For $X \in \mathfrak{g}(\mathfrak{m})_k$, $Y \in \mathfrak{g}_{-1}$,

$$[\phi(X) - X, Y] = \phi([X,Y]) - [X,Y].$$

Since $[X,Y] \in \mathfrak{g}(\mathfrak{m})_{k-1}$, we have $[\phi(X) - X, Y] = 0$. By transitivity, $\phi(X) = X$. By induction, we have proved ϕ to be the identity mapping. Hence Φ is a monomorphism.

We prove that Φ is surjective. Let $\varphi \in \mathrm{Aut}(\mathfrak{m})_0$. We construct the mapping $\varphi_p : \mathfrak{g}(\mathfrak{m})_p \to \mathfrak{g}(\mathfrak{m})_p$ inductively as follows: First for $X \in \mathfrak{g}(\mathfrak{m})_0$, we set $\varphi_0(X) = \varphi X \varphi^{-1}$. Then for $Y, Z \in \mathfrak{m}$

$$\varphi_0(X)([Y,Z]) = [\varphi(X(\varphi^{-1}(Y))), Z] + [Y, \varphi(X(\varphi^{-1}(Z)))],$$

so $\varphi_0(X) \in \mathfrak{g}(\mathfrak{m})_0$. Furthermore we can prove easily that $[\varphi_0(X), \varphi_p(Y)] = \varphi_p([X,Y])$ for $X \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_p$ $(p \leqq 0)$. Here for $p < 0$ we set $\varphi_p = \varphi|\mathfrak{g}(\mathfrak{m})_p$. Assume that we have defined linear isomorphisms φ_p of $\mathfrak{g}(\mathfrak{m})_p$ onto itself $(0 \leqq p < k)$ such that

$$\varphi_{r+s}([X,Y]) = [\varphi_r(X), \varphi_s(Y)]$$

for $X \in \mathfrak{g}(\mathfrak{m})_r$, $Y \in \mathfrak{g}(\mathfrak{m})_s$ $(r + s < k,\ r < k,\ s < k)$. For $X \in \mathfrak{g}(\mathfrak{m})_k$ we define $\varphi_k(X) \in \mathrm{Hom}(\mathfrak{m}, \bigoplus_{p\leqq k-1} \mathfrak{g}(\mathfrak{m})_p)_k$ as follows:

$$\varphi_k(X)(Y) = \varphi_{k+s}([X, \varphi^{-1}(Y)]), \qquad Y \in \mathfrak{g}_s,\ s < 0.$$

For $Y \in \mathfrak{g}_s$, $Z \in \mathfrak{g}_t$ $(s,t < 0)$,

$$\begin{aligned}\varphi_k(X)([Y,Z]) &= \varphi_{k+t+s}([X, \varphi^{-1}([Y,Z]])\\ &= \varphi_{k+s+t}([[X,\varphi^{-1}(Y)], \varphi^{-1}(Z)] + [\varphi^{-1}(Y), [X, \varphi^{-1}(Z)]])\\ &= [\varphi_{k+s}([X,\varphi^{-1}(Y)]), Z] + [Y, \varphi_{k+t}([X, \varphi^{-1}(Z)])]\\ &= [\varphi_k(X)(Y), Z] + [Y, \varphi_k(X)(Z)],\end{aligned}$$

so $\varphi_k(X) \in \mathfrak{g}(\mathfrak{m})_k$. Next we prove that for $X \in \mathfrak{g}_p$, $Y \in \mathfrak{g}_q$ $(p + q = k,\ 0 \leqq p \leqq k,\ 0 \leqq q \leqq k)$,

$$\varphi_k([X,Y]) = [\varphi_p(X), \varphi_q(Y)].$$

For $Z \in \mathfrak{g}_s$ $(s < 0)$,

$$\begin{aligned}[[\varphi_p(X), \varphi_q(Y)], Z] &= [\varphi_p(X), [\varphi_q(Y), Z]] - [\varphi_q(Y), [\varphi_p(X), Z]]\\ &= \varphi_{p+q+s}([X, [Y, \varphi^{-1}(Z)]] - [Y, [X, \varphi^{-1}(Z)]])\\ &= \varphi_{p+q+s}([[X,Y], \varphi^{-1}(Z)]) = [\varphi_k([X,Y]), Z].\end{aligned}$$

By transitivity, we see that $\varphi_k([X,Y]) = [\varphi_p(X), \varphi_q(Y)]$. We define a mapping $\check{\varphi}$ of $\mathfrak{g}(\mathfrak{m})$ into itself as follows:

$$\check{\varphi}(X) = \begin{cases}\varphi(X), & X \in \mathfrak{m},\\ \varphi_k(X), & k \geqq 0,\ X \in \mathfrak{g}(\mathfrak{m})_k.\end{cases}$$

From the above results and the definition of φ_k $(k \geqq 0)$, we see that $\check{\varphi}$ is a GLA homomorphism.

Assume that φ_{k-1} $(k \geqq 0)$ is a linear isomorphism. For $X \in \mathfrak{g}(\mathfrak{m})_k$, if $\varphi_k(X) = 0$, then $0 = [\varphi_k(X), Y] = \varphi_{k-1}([X, \varphi^{-1}(Y)])$ for all $Y \in \mathfrak{g}_{-1}$. By transitivity, we see that $X = 0$, so φ_k is a linear isomorphism. Therefore $\check{\varphi}$ is an automorphism of $\mathfrak{g}(\mathfrak{m})$. ■

Theorem 9.1. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free FGLA over* $\mathbb{C}$*, and let* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ *be the prolongation of* $\mathfrak{m}$*. The mapping* $\Phi : \mathrm{Aut}(\mathfrak{g}(\mathfrak{m}))_0 \ni \phi \mapsto \phi|\mathfrak{g}_{-1} \in GL(\mathfrak{g}_{-1})$ *is an isomorphism.*

Proof. We may assume that $\mathfrak{m}$ is a universal FGLA $b(\mathfrak{g}_{-1}, \mu)$ of the μ-th kind. By Corollary 1 to Proposition 3.2 of [11], the mapping $\mathrm{Aut}(\mathfrak{m})_0 \ni a \mapsto a|\mathfrak{g}_{-1} \in GL(\mathfrak{g}_{-1})$ is an isomorphism. By Proposition 9.1, we see that the mapping $\Phi : \mathrm{Aut}(\mathfrak{g}(\mathfrak{m}))_0 \ni \phi \mapsto \phi|\mathfrak{g}_{-1} \in GL(\mathfrak{g}_{-1})$ is an isomorphism. ■

For a pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$, we denote by $\mathrm{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0$ the group of all the automorphisms of $\mathfrak{g}$ preserving the gradation of $\mathfrak{g}$, $\mathfrak{e}$ and $\mathfrak{f}$:

$$\mathrm{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0 = \{\varphi \in \mathrm{Aut}(\mathfrak{g})_0 : \varphi(\mathfrak{e}) = \mathfrak{e}, \varphi(\mathfrak{f}) = \mathfrak{f}\}.$$

Theorem 9.2. *Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *be a free pseudo-product FGLA of type* (m, n, μ) *with pseudo-product structure* $(\mathfrak{e}, \mathfrak{f})$ *over* $\mathbb{C}$*, and let* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *be the prolongation of* $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$*. The mapping* $\Phi : \mathrm{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0 \ni \phi \mapsto (\phi|\mathfrak{e}, \phi|\mathfrak{f}) \in GL(\mathfrak{e}) \times GL(\mathfrak{f})$ *is an isomorphism. Furthermore if* $\dim \mathfrak{e} \neq \dim \mathfrak{f}$*, then* $\mathrm{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0 = \mathrm{Aut}(\mathfrak{g})_0$*.*

Proof. Clearly Φ is a monomorphism. We show that Φ is surjective. Let (ϕ_1, ϕ_2) be an element of $GL(\mathfrak{e}) \times GL(\mathfrak{f})$. We set $\phi = \phi_1 \oplus \phi_2 \in GL(\mathfrak{g}_{-1})$. By Corollary 1 to Proposition 3.2 of [11], there exists an element $\varphi_1 \in \mathrm{Aut}(b(\mathfrak{g}_{-1}, \mu))_0$ such that $\varphi_1|\mathfrak{g}_{-1} = \phi$. Since $\varphi_1([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) = [\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]$, φ_1 induces an element $\varphi_2 \in \mathrm{Aut}(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})_0$ such that $\varphi_2|\mathfrak{g}_{-1} = \phi$. By Proposition 9.1, there exists $\varphi_3 \in \mathrm{Aut}(\mathfrak{g}(\mathfrak{m}))_0$ such that $\varphi_3|\mathfrak{m} = \varphi_2$. We prove that $\varphi_3(\mathfrak{g}) = \mathfrak{g}$. For $X_0 \in \mathfrak{g}_0$ and $Y \in \mathfrak{e}$, we see that $[\varphi_3(X_0), Y] = \varphi_3([X_0, \varphi_3^{-1}(Y)]) \in \varphi_3(\mathfrak{e}) = \mathfrak{e}$, so $\varphi_3(X_0)(\mathfrak{e}) \subset \mathfrak{e}$. Similarly we get $\varphi_3(X_0)(\mathfrak{f}) \subset \mathfrak{f}$. Thus we obtain that $\varphi_3(\mathfrak{g}_0) = \mathfrak{g}_0$. Now we assume that $\varphi_i(\mathfrak{g}_i) = \mathfrak{g}_i$ for $0 \leqq i \leqq k$. Then for $X_{k+1} \in \mathfrak{g}_{k+1}$ and $Y \in \mathfrak{g}_p$ $(p < 0)$, we see that $[\varphi_3(X_{k+1}), Y] = \varphi_3([X_{k+1}, \varphi_3^{-1}(Y)]) \in \varphi_3(\mathfrak{g}_{p+k+1}) = \mathfrak{g}_{p+k+1}$, so $\varphi_3(\mathfrak{g}_{k+1}) \subset \mathfrak{g}_{k+1}$. Hence $\varphi_3(\mathfrak{g}) = \mathfrak{g}$ and Φ is surjective. Now we assume that $\dim \mathfrak{e} \neq \dim \mathfrak{f}$. Let $\varphi \in \mathrm{Aut}(\mathfrak{g})_0$. Since $\mathfrak{g}_0$-modules $\mathfrak{e}$ and $\mathfrak{f}$ are not isomorphic to each other, we see that (i) $\varphi(\mathfrak{e}) = \mathfrak{e}$, $\varphi(\mathfrak{f}) = \mathfrak{f}$ or (ii) $\varphi(\mathfrak{e}) = \mathfrak{f}$, $\varphi(\mathfrak{f}) = \mathfrak{e}$. According to the assumption $\dim \mathfrak{e} \neq \dim \mathfrak{f}$, we get $\varphi(\mathfrak{e}) = \mathfrak{e}$, $\varphi(\mathfrak{f}) = \mathfrak{f}$, so $\varphi \in \mathrm{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0$. ■

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