A New Class of Solvable Many-Body Problems[⋆]

Francesco CALOGERO and Ge YI

Physics Department, University of Rome "La Sapienza",
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy
E-mail: *francesco.calogero@roma1.infn.it, francesco.calogero@uniroma1.it,*
          ge.yi@roma1.infn.it, yigeking@mail.ustc.edu.cn

Abstract. A *new* class of *solvable* N-body problems is identified. They describe N unit-mass point particles whose time-evolution, generally taking place in the *complex* plane, is characterized by *Newtonian* equations of motion "of goldfish type" (acceleration equal force, with specific velocity-dependent one-body and two-body forces) featuring several arbitrary coupling constants. The corresponding initial-value problems are solved by finding the eigenvalues of a time-dependent $N \times N$ matrix $U(t)$ explicitly defined in terms of the initial positions and velocities of the N particles. Some of these models are *asymptotically isochronous*, i.e. in the remote future they become completely periodic with a period T independent of the initial data (up to exponentially vanishing corrections). Alternative formulations of these models, obtained by changing the dependent variables from the N zeros of a monic polynomial of degree N to its N coefficients, are also exhibited.

Key words: integrable dynamical systems; solvable dynamical systems; solvable Newtonian many-body problems; integrable Newtonian many-body problems; isochronous dynamical systems

2010 Mathematics Subject Classification: 70F10; 70H06; 37J35; 37K10

1 Introduction

In this paper a *new* class of *solvable* N-body problems is identified. They describe an arbitrary number N of unit-mass point particles whose time-evolution, generally taking place in the *complex* plane, is characterized by *Newtonian* equations of motion "of goldfish type" (acceleration equal force, with specific velocity-dependent one-body and two-body forces, see below) featuring several arbitrary coupling constants. The *solvable* character of these models is demonstrated by the possibility to ascertain their time evolution by purely *algebraic* operations. In particular it is shown below that their initial-value problems are solved by finding the eigenvalues of a time-dependent $N \times N$ matrix $U(t)$ explicitly defined in terms of the initial positions and velocities of the N particles. Some of these models are *asymptotically isochronous*, i.e. in the remote future they become completely periodic with a period T independent of the initial data (up to exponentially vanishing corrections). Alternative formulations of these models, obtained by changing the dependent variables from the N zeros of a monic polynomial of degree N to its N coefficients, are also exhibited.

   The main idea to obtain these results is to identify the N coordinates $z_n(t)$ characterizing the positions of the particles of the N-body problem with the N eigenvalues of an $N \times N$ matrix $U(t)$, itself evolving according to a *solvable* (or *integrable*) matrix ODE. This technique was invented long ago by Olshanetsky and Perelomov [1–4] and has been much exploited subsequently, identifying thereby many *solvable* (or *integrable*) many-body problems: see for instance [5] (in

---

particular its Section 2.1.3.2 entitled "The technique of solution of Olshanetsky and Perelomov (OP)") and [6] (in particular its Section 4.2.2 entitled "Goldfishing"), and the more recent papers [7–23]. The present paper provides *new* results of this kind, by taking as point of departure a *solvable* ODE characterizing the time-evolution of the $N \times N$ matrix $U(t)$ which is different or more general than those previously employed to this end. These results are treated in the following section, including its subsections and Appendices A and B where all the equations solved in this paper are listed (hence, the reader wishing to get an immediate glance at them may immediately jump to these appendices). The last section, entitled "Outlook", tersely outlines further developments whose treatment is postponed to subsequent papers.

2 *Solvable N*-body problems

In this section we describe two classes of *solvable N*-body problems. The models of the first class are not new, hence their treatment is not elaborated beyond their identification; several models of the second class are *new,* hence they are fully dealt with.

In Subsection 2.1 we introduce a system of two matrix first-order ordinary differential equations (ODEs) characterizing the time-evolution of the two $N \times N$ matrices $U \equiv U(t)$ and $V \equiv V(t)$, and we indicate how the corresponding initial-value problem can be explicitly solved.

In Subsection 2.2 we show how – via the introduction of two appropriate *ansätze* – the N eigenvalues $z_n(t)$ of the matrix $U(t)$ can be identified with the N coordinates of N unit-mass point particles whose time-evolution, generally taking place in the *complex* plane, is characterized by *Newtonian* equations of motion ("acceleration equal force", with nonlinear one-body and two-body forces). These models are thereby shown to be *solvable.* The first *ansatz* yields various models whose solvability was already known; hence their treatment is limited to the derivation of their equations of motion. The second *ansatz* yields several *new* models – as well as several previously known models – characterized by equations of motion with specific velocity-dependent one-body and two-body forces "of goldfish type" featuring several arbitrary coupling constants. They are all listed in Appendix A. The alternative class of many-body models obtained by changing the dependent variables from the N zeros of a monic polynomial of degree N to its N coefficients is discussed in Subsection 2.3; the corresponding equations of motion are listed in Appendix B.

As it is well known (see for instance [5], in particular Chapter 4, entitled "Solvable and/or integrable many-body problems in the plane, obtained by complexification"), models such as those described below, which describe the motions of N points in the *complex z*-plane, can be reformulated as N-body models describing the motion in a plane of N point-particles, the positions of which are identified by *real* 2-vectors in that plane. While we leave the elaboration of this connection to the interested reader, we feel justified by it to refer to our findings as describing "physical" (if not necessarily "realistic") many-body problems in the plane.

2.1   A *solvable* system of two matrix ODEs

In this subsection we discuss the following system of two matrix ODEs, satisfied by the two $N \times N$ matrices U and V:

$$\dot{U} = \alpha U + \beta U^2 + \gamma V + \eta(UV + VU), \qquad \dot{V} = \rho V. \tag{2.1}$$

Notation. Upper-case Latin letters generally denote $N \times N$ matrices (unless otherwise indicated), with the (scalar!) N being throughout an arbitrary *positive integer*. Here and below the matrices are time-dependent (unless otherwise indicated), in particular $U \equiv U(t)$, $V \equiv V(t)$.

Lower case letters generally denote scalars. The 5 scalar parameters α, β, γ, η, ρ are time-independent but *a priori* arbitrary, until specific restrictions on their values are explicitly mentioned. Superimposed dots always indicate time differentiations.

Remark 2.1. The factor β multiplying the term U^2 in the right-hand side of the first of the two matrix ODEs (2.1) could be eliminated (i.e., replaced by unity) by rescaling U (and by correspondingly replacing γ with an adjusted value, say $\tilde{\gamma}$); then the factor η multiplying the term $(UV + VU)$, or the (adjusted) factor $\tilde{\gamma}$ multiplying the term V, could also be eliminated (i.e., replaced by unity) by rescaling V. It is preferable not to do so in order to keep open the possibility to set one or the other of these parameters, η or γ, to zero (but we will never set *both* of them to zero, to avoid decoupling the time evolution of $U(t)$ from that of $V(t)$). Note moreover that the introduction of a constant scalar term (implicitly multiplied by the $N \times N$ unit matrix I) in the right-hand side of the first equation would amount – up to a redefinition of other parameters – to adding to the matrix U the $N \times N$ unit matrix I multiplied by a new (constant) parameter, implying just a constant shift of all the eigenvalues of the matrix U, a trivial change not worth pursuing; while the introduction of two *different* parameters in front of the two terms UV and VU could be eliminated by adding, in the right-hand side of the first of the two matrix ODEs (2.1), the commutator $[U, V]$ of U and V times a convenient parameter, without any effect on the eigenvalues of U. If some of the parameters in the matrix ODEs (2.1) vanish this model may reduce to one of those that have been previously treated, see [6] (in particular its Section 4.2.2 entitled "Goldfishing") and the more recent papers [7–23]; we do not discard these models in the following, since we consider in any case worthwhile to exploit the unified treatment provided by the present approach based on (2.1). As for the possibility of replacing the right-hand side of the second matrix ODE (2.1) with a more general function of the matrix V, it is a possibility whose exploration is postponed to a future investigation (see Section 3).

Solution of the system of two matrix ODEs (2.1). Clearly the solution of the second of the two matrix ODEs (2.1) reads

$$V(t) = V_0 \exp(\rho t), \qquad V_0 \equiv V(0). \tag{2.2}$$

To solve the first of the two matrix ODEs (2.1) it is convenient to set

$$U = -\frac{\eta}{\beta}V - (\beta F)^{-1}\dot{F}, \tag{2.3}$$

obtaining thereby for the $N \times N$ matrix $F \equiv F(t)$ the following second-order *linear* matrix ODE:

$$\ddot{F} - \alpha\dot{F} + FW = 0, \tag{2.4a}$$
$$W = (-\alpha\eta + \beta\gamma + \eta\rho)V - \eta^2 V^2. \tag{2.4b}$$

It is then plain via (2.2) that the *general solution* of this matrix ODE reads

$$F(t) = F_+ f_+(t; V_0) + F_- f_-(t; V_0), \tag{2.5}$$

where $F_\pm$ are two *arbitrary constant* $N \times N$ matrices and the two *scalar* functions $f_\pm(t; v)$ (which of course become $N \times N$ *matrices* when the *scalar* v is replaced by the $N \times N$ *matrix* V_0, see (2.5)) are two independent solutions of the *scalar* second-order *linear* ODE

$$\ddot{f} - \alpha\dot{f} + \left[(-\alpha\eta + \beta\gamma + \eta\rho)v \exp(\rho t) - \eta^2 v^2 \exp(2\rho t)\right] f = 0. \tag{2.6}$$

And it is easily seen that two independent solutions of this ODE are given by the following formulas:

$$f_+(t;v) = \exp\left[-\frac{\eta v}{\rho}\exp(\rho t)\right]\Phi\left(-\frac{\beta\gamma}{2\eta\rho};1-\frac{\alpha}{\rho};\frac{2\eta v}{\rho}\exp(\rho t)\right), \tag{2.7a}$$

$$f_-(t;v) = \exp\left[-\frac{\eta v}{\rho}\exp(\rho t)\right]\exp(\alpha t)\Phi\left(\frac{2\alpha\eta-\beta\gamma}{2\eta\rho};1+\frac{\alpha}{\rho};\frac{2\eta v}{\rho}\exp(\rho t)\right). \tag{2.7b}$$

Here the (scalar!) function $\Phi(a;c;z)$ is the confluent hypergeometric function (see, for instance, [24]).

Note that the formula (2.3) with (2.5) implies the following *explicit solution* of the *initial-value problem* for the matrix $U(t)$:

$$U(t) = -\frac{1}{\beta}\left\{\eta V_0\exp(\rho t)+\left[f_+(t;V_0)+C\,f_-(t;V_0)\right]^{-1}\left[\dot{f}_+(t;V_0)+C\dot{f}_-(t;V_0)\right]\right\}, \tag{2.8a}$$

with the time-independent matrix C defined in terms of the initial data $U(0)$ and $V_0 \equiv V(0)$ (see (2.2)) as follows:

$$C = -\left\{f_+(0;V_0)[\beta U(0)+\eta V_0]+\dot{f}_+(0;V_0)\right\}\left\{f_-(0;V_0)[\beta U(0)+\eta V_0]+\dot{f}_-(0;V_0)\right\}^{-1}. \tag{2.8b}$$

Remark 2.2. The fact that the expression (2.3) (with (2.2) and (2.4)) of the matrix $U(t)$ satisfies (2.1), and likewise that the functions $f_\pm(t;v)$ (see (2.7)) satisfy the ODE (2.6), can be easily verified. Of course these formulas are valid for *generic* values of the parameters that appear in them, excluding special cases – such as vanishing values of ρ or η (see (2.7)) or of β (see (2.8a)), or values of α and ρ such that α/ρ is an *integer* (in which case the two functions $f_\pm(t;v)$ do *not* provide two *independent* solutions of the ODE (2.6): see Section 6.7 of [24]) – which are clearly problematic (although the formulas may in these cases be reinterpreted via appropriate limiting procedures). We ignore hereafter these issues, even when listing below *solvable* N-body models a few of which belong to these problematic cases. Indeed in this paper we mainly limit our consideration to the *identification* of *solvable* N-body problems; these findings open the way to obtaining a rather detailed understanding of their actual behaviors (see for instance the following Remark 2.3), but such analyses exceed the scope of this paper: they should be done on a case-by-case basis, and shall perhaps be postponed to the moment when some of these N-body models evoke a specific, theoretical or applicative, interest.

Remark 2.3. If the parameter ρ is purely *imaginary* and the parameter α is *real* and *negative*,

$$\rho = \frac{2\pi\mathbf{i}}{T}, \qquad \alpha < 0 \tag{2.9a}$$

(of course with T *real* and *nonvanishing*, and $\mathbf{i}$ the *imaginary unit*, $\mathbf{i}^2 = -1$), then clearly the matrix $U(t)$ (see (2.8) with (2.7)) is *asymptotically isochronous* (i.e., *asymptotically periodic* with the period T independent of the initial data): indeed in this case, as $t \to +\infty$,

$$U(t) = U_+(t) + O[\exp(\alpha t)] \tag{2.9b}$$

with $U_+(t)$ given by the formula (2.8) with $C = 0$,

$$U_+(t) = -\frac{1}{\beta}\left\{\eta V_0\exp(\rho t)+[f_+(t;V_0)]^{-1}[\dot{f}_+(t;V_0)]\right\}, \tag{2.9c}$$

hence (see (2.9a) and (2.7a)) it is *periodic* with period T,

$$U_+(t+T) = U_+(t). \tag{2.9d}$$

This observation entails of course the property of *asymptotic isochrony* for the *solvable* N-body models identified below featuring parameters α and ρ which satisfy the conditions (2.9a).

2.2   Identification of *solvable* many-body models

The starting point is to introduce the N eigenvalues $z_n(t)$ of the matrix $U(t)$, and the corresponding *diagonalizing* matrix $R(t)$, by setting

$$U(t) = R(t)Z(t)[R(t)]^{-1}, \qquad Z(t) = \text{diag}[z_n(t)]. \tag{2.10a}$$

Here and hereafter indices such as n, m, ℓ, k run over the integers from 1 to N (unless otherwise indicated).

Likewise we set

$$V(t) = R(t)Y(t)[R(t)]^{-1}, \qquad Y_{nn}(t) = y_n(t), \tag{2.10b}$$

and we introduce the $N \times N$ matrix $M(t)$ by setting

$$M(t) = [R(t)]^{-1}\dot{R}(t), \qquad M_{nn}(t) = \mu_n(t). \tag{2.10c}$$

Remark 2.4. The diagonalizing matrix $R(t)$ is defined up to right-multiplication by an *arbitrary diagonal* $N \times N$ matrix $D(t)$, since the replacement of $R(t)$ by $\tilde{R}(t) = R(t)D(t)$ does not affect (2.10a). But it changes $M(t)$ (see (2.10c)) into $\tilde{M}(t) = [\tilde{R}(t)]^{-1}\dot{\tilde{R}}(t)$ implying the following change of its diagonal elements: $\mu_n(t) \implies \tilde{\mu}_n(t) = \mu_n(t) + \dot{d}_n(t)/d_n(t)$, where the N quantities $d_n(t)$ are the, *a priori* arbitrary, elements of the diagonal matrix $D(t)$. Hence we retain the privilege to assign at our convenience (see below) the diagonal elements $\mu_n(t)$ of the matrix $M(t)$.

It is now easily seen that via these assignments (2.10) the two matrix ODEs (2.1) get rewritten as follows:

$$\dot{Z} + [M, Z] = \alpha Z + \beta Z^2 + \gamma Y + \eta(ZY + YZ), \qquad \dot{Y} + [M, Y] = \rho Y. \tag{2.11}$$

Here and hereafter the notation $[A, B]$ denotes the commutator of the two matrices A and B: $[A, B] \equiv AB - BA$.

Let us now look, componentwise, at the *diagonal* and *off-diagonal* elements of these two matrix ODEs.

The *diagonal* part of the first of the two ODEs (2.11) reads

$$\dot{z}_n = \alpha z_n + \beta z_n^2 + \gamma y_n + 2\eta z_n y_n, \tag{2.12a}$$

implying

$$y_n = \frac{\dot{z}_n - \alpha z_n - \beta z_n^2}{\gamma + 2\eta z_n}. \tag{2.12b}$$

The *off-diagonal* part of the first of the two ODEs (2.11) reads

$$-(z_n - z_m)M_{nm} = [\gamma + \eta(z_n + z_m)]Y_{nm}, \qquad n \neq m, \tag{2.13a}$$

implying

$$M_{nm} = -\left[\frac{\gamma + \eta(z_n + z_m)}{z_n - z_m}\right]Y_{nm}, \qquad n \neq m. \tag{2.13b}$$

The *diagonal* part of the second of the two ODEs (2.11) reads

$$\dot{y}_n = \rho y_n + \sum_{\ell=1, \ell \neq n}^{N} (Y_{n\ell}M_{\ell n} - M_{n\ell}Y_{\ell n}), \tag{2.14a}$$

implying, via (2.13b),

$$\dot{y}_n = \rho y_n + 2 \sum_{\ell=1,\ell\neq n}^{N} \left\{ Y_{n\ell}Y_{\ell n} \left[\frac{\gamma + \eta(z_n + z_\ell)}{z_n - z_\ell} \right] \right\}. \tag{2.14b}$$

Hence, by time-differentiation of (2.12a) we see via (2.12b) and (2.14b) that the coordinates $z_n(t)$ satisfy the following system of N equations of motion of *Newtonian* type ("acceleration equal force", with one-body and two-body forces):

$$\ddot{z}_n = -\alpha\rho z_n - \beta\rho z_n^2 + (\alpha + \rho)\dot{z}_n + 2\beta\dot{z}_n z_n + \frac{2\eta\dot{z}_n\left(\dot{z}_n - \alpha z_n - \beta z_n^2\right)}{\gamma + 2\eta z_n}$$

$$+ 2(\gamma + 2\eta z_n) \sum_{\ell=1,\ell\neq n}^{N} \left\{ Y_{n\ell}Y_{\ell n} \left[\frac{\gamma + \eta(z_n + z_\ell)}{z_n - z_\ell} \right] \right\}. \tag{2.15}$$

But in these equations of motion the role of "two-body coupling constants" is played by the quantities $Y_{n\ell}Y_{\ell n}$ (with $\ell \neq n$) which are in fact *time-dependent*. Indeed the time evolution of the *off-diagonal* elements Y_{nm} of the matrix Y is determined by the *off-diagonal* part of the second of the two ODEs (2.11) which componentwise read

$$\dot{Y}_{nm} = \rho Y_{nm} + \sum_{k=1}^{N} \left(Y_{nk}M_{km} - M_{nk}Y_{km} \right), \qquad n \neq m, \tag{2.16a}$$

yielding, via (2.10b), (2.10c), (2.12b), (2.13b) and a bit of algebra,

$$\frac{\dot{Y}_{nm}}{Y_{nm}} = \alpha + \rho + \beta(z_n + z_m) - \frac{\dot{z}_n - \dot{z}_m}{z_n - z_m} + \frac{\eta\left(\dot{z}_n - \alpha z_n - \beta z_n^2\right)}{\gamma + 2\eta z_n} + \frac{\eta\left(\dot{z}_m - \alpha z_m - \beta z_m^2\right)}{\gamma + 2\eta z_m}$$

$$- \mu_n + \mu_m + \sum_{\ell=1,\ell\neq n,m}^{N} \left\{ \frac{Y_{n\ell}Y_{\ell m}}{Y_{nm}} \left[2\eta + (\gamma + 2\eta z_\ell) \left(\frac{1}{z_n - z_\ell} + \frac{1}{z_m - z_\ell} \right) \right] \right\},$$

$$n \neq m. \tag{2.16b}$$

So, in order that (2.15) become the N *Newtonian* equations of motion of a genuine N-body problem one must either provide some "physical interpretation" for the quantities $Y_{n\ell}$ (with $\ell \neq n$) – possibly in terms of internal degrees of freedom, an alternative we do not pursue in this paper – or find a way to "get rid" of these quantities, i.e. find a way to express them – if at all possible – via the N coordinates $z_m \equiv z_m(t)$ and possibly also the N velocities $\dot{z}_m \equiv \dot{z}_m(t)$. Previous experience [5–23] suggest that two types of *ansätze* are the appropriate starting points to try and achieve this goal.

2.2.1   First ansatz

The first *ansatz* reads

$$Y_{nm} = \frac{(g_n g_m)^{1/2}}{z_n - z_m}, \qquad n \neq m, \tag{2.17}$$

where we reserve the option to make a convenient assignment for the N functions g_n of the coordinate z_n, $g_n \equiv g_n(z_n) \equiv g_n[z_n(t)]$.

The insertion of this *ansatz* in (2.16b) yields, after a bit of trivial algebra,

$$\frac{\dot{g}_n}{2g_n} - \frac{\alpha + \rho}{2} - \beta z_n - \frac{\eta\left(\dot{z}_n - \alpha z_n - \beta z_n^2\right)}{\gamma + 2\eta z_n} + ((n \to m))$$

$$= \frac{g_n(\gamma + 2\eta z_m) - g_m(\gamma + 2\eta z_n)}{(z_n - z_m)^2} - \mu_n + \mu_m$$

$$+ \sum_{\ell=1,\ell\neq n}^{N} \left\{ g_\ell \left[\frac{\gamma + 2\eta z_n}{(z_n - z_\ell)^2} \right] \right\} - \sum_{\ell=1,\ell\neq m}^{N} \left\{ g_\ell \left[\frac{\gamma + 2\eta z_m}{(z_m - z_\ell)^2} \right] \right\}, \qquad n \neq m. \qquad (2.18)$$

Here and throughout the notation $+((n \to m))$ indicates the addition of whatever comes before it, with the index n replaced by m.

We now take advantage of the freedom (see Remark 2.4) to assign the diagonal elements μ_n of the matrix M by setting

$$\mu_n = \sum_{\ell=1,\ell\neq n}^{N} \left\{ g_\ell \left[\frac{\gamma + 2\eta z_n}{(z_n - z_\ell)^2} \right] \right\}, \qquad (2.19)$$

and we moreover make the assignment

$$g_n = g(\gamma + 2\eta z_n), \qquad (2.20)$$

with g an arbitrary constant (i.e., $\dot{g} = 0$). Thereby the system of $N(N-1)$ equations (2.18) gets reduced to the following, much simpler, system of only N algebraic equations:

$$-\frac{\alpha + \rho}{2} - \beta z_n + \frac{\eta \left(\alpha z_n + \beta z_n^2 \right)}{\gamma + 2\eta z_n} = 0, \qquad (2.21)$$

which amounts to the following 3 equations (recall that we exclude the uninteresting possibility that γ and η both vanish):

$$\beta\eta = 0, \qquad (2.22a)$$
$$\beta\gamma + \eta\rho = 0, \qquad (2.22b)$$
$$\gamma(\alpha + \rho) = 0. \qquad (2.22c)$$

This boils down to the following 3 possibilities:

$$\alpha = \beta = \rho = 0, \qquad (2.23a)$$

or

$$\beta = \gamma = \rho = 0, \qquad (2.23b)$$

or

$$\beta = \eta = 0, \qquad \rho = -\alpha. \qquad (2.23c)$$

The corresponding *solvable* N-body models – obtained by inserting (2.17) with (2.20) and with these assignments of the parameters in the *Newtonian* equations of motion (2.15) – read, in the first 2 of these 3 cases – after conveniently setting

$$\gamma + 2\eta z_n(t) = \exp[2c\zeta_n(t)] \qquad (2.24a)$$

with c an arbitrary nonvanishing constant – as follows:

$$\ddot{\zeta}_n = \left(\frac{g^2\eta^4}{c^2} \right) \frac{d}{d\zeta_n} \sum_{\ell=1,\ell\neq n}^{N} \sinh^{-2}[c(\zeta_n - \zeta_\ell)]; \qquad (2.24b)$$

while in the third of these 3 cases they read

$$\ddot{z}_n = \alpha^2 z_n + g^2 \gamma^4 \frac{d}{dz_n} \sum_{\ell=1,\ell\neq n}^{N} (z_n - z_\ell)^{-2}. \tag{2.25}$$

But these are well-known *solvable* N-body problems, see for instance [5]. Hence we conclude that from the matrix system (2.1) *no new solvable* many-body models are obtained via the *ansatz* (2.17).

2.2.2 Second ansatz

We proceed then to consider a second *ansatz*, reading

$$Y_{nm} = \left[g_n g_m \left(\dot{z}_n + f_n \right) \left(\dot{z}_m + f_m \right) \right]^{1/2}, \qquad n \neq m, \tag{2.26}$$

with g_n and f_n functions of the coordinate z_n that we reserve to assign later: $g_n \equiv g_n(z_n) \equiv g_n[z_n(t)]$, $f_n \equiv f_n(z_n) \equiv f_n[z_n(t)]$. Its insertion in (2.16b) yields, again after a bit of trivial algebra and now the assignment $\mu_n = 0$ (again justified by Remark 2.4), the following system of $N(N-1)$ second-order ODEs:

$$\frac{1}{2} \left(\frac{\ddot{z}_n + \dot{f}_n}{\dot{z}_n + f_n} + \frac{\dot{g}_n}{g_n} - \alpha - \rho \right) - \beta z_n - \eta \frac{\dot{z}_n - \alpha z_n - \beta z_n^2}{\gamma + 2\eta z_n}$$

$$+ \eta(\dot{z}_n + f_n)g_n - \sum_{\ell=1,\ell\neq n}^{N} \left\{ (\dot{z}_\ell + f_\ell)g_\ell \left[\frac{\gamma + \eta(z_n + z_\ell)}{z_n - z_\ell} \right] \right\}$$

$$- \frac{\dot{z}_n[g_n(\gamma + 2\eta z_n) - 1] + f_n g_n(\gamma + 2\eta z_n)}{z_n - z_m} + ((n \to m)) = 0, \qquad n \neq m. \tag{2.27}$$

To reduce this system we clearly must now set

$$g_n = \frac{1}{\gamma + 2\eta z_n}, \tag{2.28a}$$

$$f_n = f^{(0)} + f^{(1)} z_n + f^{(2)} z_n^2, \tag{2.28b}$$

where $f^{(0)}$, $f^{(1)}$, $f^{(2)}$ are 3 *constant* parameters that we reserve to assign later. Thereby the system of $N(N-1)$ ODEs (2.27) gets transformed into the following system of (only) N ODEs:

$$\ddot{z}_n - 2(\dot{z}_n + f_n) \sum_{\ell=1,\ell\neq n}^{N} \left\{ \frac{(\dot{z}_\ell + f_\ell)[\gamma + \eta(z_n + z_\ell)]}{(z_n - z_\ell)(\gamma + 2\eta z_\ell)} \right\}$$

$$= \left[\alpha + \rho + 2\beta z_n + \frac{2\eta \left(\dot{z}_n - \alpha z_n - \beta z_n^2 - f_n \right)}{\gamma + 2\eta z_n} \right] (\dot{z}_n + f_n) + \left(f^{(1)} + 2f^{(2)} z_n \right) f_n. \tag{2.29}$$

Note that to write this system in more compact form we employed a mixed notation, using sometimes the functions f_n or f_ℓ instead of their explicit expressions, see (2.28b).

To complete the task of ascertaining for which values of the (so far arbitrary) 8 *constant* parameters α, β, γ, η, ρ, $f^{(0)}$, $f^{(1)}$, $f^{(2)}$ the system of $N(N-1)$ ODEs (2.29) is satisfied we must utilize the equations of motion (2.15) which, via the *ansatz* (2.26) with (2.28), now read

$$\ddot{z}_n - 2(\dot{z}_n + f_n) \sum_{\ell=1,\ell\neq n}^{N} \left\{ \frac{(\dot{z}_\ell + f_\ell)[\gamma + \eta(z_n + z_\ell)]}{(z_n - z_\ell)(\gamma + 2\eta z_\ell)} \right\}$$

$$= -\alpha\rho z_n - \beta\rho z_n^2 + (\alpha + \rho)\dot{z}_n + 2\beta\dot{z}_n z_n + \frac{2\eta\dot{z}_n\left(\dot{z}_n - \alpha z_n - \beta z_n^2\right)}{\gamma + 2\eta z_n}. \tag{2.30}$$

Comparison of this system of N ODEs to the system (2.29) yields the following system of N algebraic equations (note that – as it were, "miraculously" – the velocities $\dot{z}_n$ have disappeared from these equations):

$$\alpha\rho z_n + \beta\rho z_n^2 + \left[\alpha + \rho + f^{(1)} + 2\left(\beta + f^{(2)}\right)z_n\right]f_n = \frac{2\eta\left(\alpha z_n + \beta z_n^2 + f_n\right)f_n}{\gamma + 2\eta z_n}. \tag{2.31}$$

It is now clear that, in order to satisfy this system – identically, i.e. for any values of the coordinates z_n – one must set either

case (i) : $\ f_n = (a + bz_n)(\gamma + 2\eta z_n)$ \hfill (2.32a)

implying (see (2.28b))

case (i) : $\ f^{(0)} = a\gamma, \qquad f^{(1)} = 2a\eta + b\gamma, \qquad f^{(2)} = 2b\eta,$ \hfill (2.32b)

or

case (ii) : $\ f_n = -\alpha z_n - \beta z_n^2 + (a + bz_n)(\gamma + 2\eta z_n)$ \hfill (2.33a)

implying (see (2.28b))

case (ii) : $\ f^{(0)} = a\gamma, \qquad f^{(1)} = 2a\eta + b\gamma - \alpha, \qquad f^{(2)} = 2b\eta - \beta.$ \hfill (2.33b)

So, in both cases, we hereafter only retain the freedom to assign the 2 constants a, b rather than the 3 constants $f^{(0)}$, $f^{(1)}$, $f^{(2)}$.

Clearly in case (i) we get the following system of N algebraic equations:

$$\alpha\rho z_n + \beta\rho z_n^2 + \left[\alpha + \rho + 2a\eta + b\gamma + 2(\beta + 2b\eta)z_n\right]\left[a\gamma + (2a\eta + b\gamma)z_n + 2b\eta z_n^2\right]$$
$$= 2\eta\left[\alpha z_n + \beta z_n^2 + (a + bz_n)(\gamma + 2\eta z_n)\right](a + bz_n), \tag{2.34a}$$

and likewise in case (ii)

$$\alpha\rho z_n + \beta\rho z_n^2 + \left[\rho + 2a\eta + b\gamma + 4b\eta z_n\right]\left[a\gamma + (2a\eta + b\gamma - \alpha)z_n + (2b\eta - \beta)z_n^2\right]$$
$$= 2\eta\left[-\alpha z_n - \beta z_n^2 + (a + bz_n)(\gamma + 2\eta z_n)\right](a + bz_n). \tag{2.34b}$$

Hence in case (i) the following set of 4 nonlinear algebraic equations must be satisfied by the 7 parameters a, b, α, β, γ, η, ρ:

$$b\eta(\beta + 2b\eta) = 0, \tag{2.35a}$$
$$(\beta + 2b\eta)(\rho + 2a\eta + 2b\gamma) = 0, \tag{2.35b}$$
$$(\alpha + 2a\eta + b\gamma)(\rho + b\gamma) + 2a(\beta + b\eta)\gamma = 0, \tag{2.35c}$$
$$a(\alpha + \rho + b\gamma)\gamma = 0. \tag{2.35d}$$

Likewise in case (ii) the following set of 4 nonlinear algebraic equations must be satisfied by the 7 parameters a, b, α, β, γ, η, ρ:

$$b\eta(\beta - 2b\eta) = 0, \tag{2.36a}$$
$$b\left[2\eta(2a\eta + 2b\gamma - \alpha + \rho) - \beta\gamma\right] = 0, \tag{2.36b}$$
$$2a\eta\rho + b\gamma(4a\eta + b\gamma - \alpha + \rho) = 0, \tag{2.36c}$$
$$a(\rho + b\gamma)\gamma = 0. \tag{2.36d}$$

Then a trivial if rather tedious computation yields, in case (i) respectively in case (ii), the solutions reported in Table 2.1 respectively in Table 2.2 (note that for $\alpha = \beta = \eta = 0$ these two cases coincide, so the corresponding results are only included in Table 2.2).

Table 2.1

#	a	b	α	β	γ	η	ρ
1	0	0	*	*	*	*	0
2	*	0	*	0	0	*	0
3	*	0	$-2a\eta$	0	0	*	*
4	*	0	$-2a\eta$	0	*	*	$2a\eta$
5	*	0	$-2a\eta$	*	0	*	$-2a\eta$
6	*	0	$2a\eta$	$4a\eta^2/\gamma$	*	*	$-2a\eta$
7	0	*	$-b\gamma$	0	*	0	*
8	*	*	$-b\gamma$	0	*	0	0
9	0	*	*	0	*	0	$-b\gamma$
10	0	*	$-b\gamma$	*	*	0	$-2b\gamma$
11	*	*	$b\gamma$	$b^2\gamma/a$	*	0	$-2b\gamma$
12	0	*	$-b\gamma$	$-2b\eta$	*	*	*
13	0	*	*	$-2b\eta$	*	*	$-b\gamma$
14	*	*	*	$-2b\eta$	0	*	0
15	*	*	$-2a\eta$	$-2b\eta$	0	*	*
16	$-b\gamma$	*	*	$-2b\eta$	*	*	$-\alpha-b\gamma$
17	*	*	$-2a\eta$	$-2b\eta$	*	*	$-\alpha-b\gamma$

This table indicates the 17 sets of values to be assigned to the 7 parameters a, b, α, β, γ, η, ρ in order to satisfy the 4 equations characterizing case (i), see (2.35). Cases with $\gamma = \eta = 0$ are excluded. Asterisks indicate that the corresponding parameters can be assigned freely. Note that each of the 7 lines 1, 12, 13, 14, 15, 16, 17 assigns values to only 3 of the 7 parameters, while each of the other 10 lines assigns values to 4 of the 7 parameters.

Table 2.2

#	a	b	α	β	γ	η	ρ
1	0	0	*	*	*	*	*
2	*	0	*	*	*	*	0
3	0	*	$b\gamma+\rho$	0	*	0	*
4	*	*	0	0	*	0	$-b\gamma$
5	0	*	$b\gamma+\rho$	$2b\eta$	*	*	*
6	0	*	ρ	$2b\eta$	0	*	*
7	*	*	$2a\eta$	$2b\eta$	*	*	$-b\gamma$

This table indicates the 7 sets of values to be assigned to the 7 parameters a, b, α, β, γ, η, ρ in order to satisfy the 4 equations characterizing case (ii), see (2.36). Cases with $\gamma = \eta = 0$ are excluded. Asterisks indicate that the corresponding parameters can be assigned freely. Note that each of the lines 1 and 2 assigns values to only 2 of the 7 parameters, lines 3, 4 and 6 assign values to 4 of the 7 parameters, and lines 5 and 7 assign values to 3 of the 7 parameters.

We therefore conclude that the many-body models of goldfish type characterized by the *Newtonian* equations of motion (2.30) (or, equivalently, (2.29)) are *solvable* provided either the quantities f_n are expressed by (2.32a) and the 7 parameters a, b, α, β, γ, η, ρ are consistent with Table 2.1 or the quantities f_n are expressed by (2.33a) and the 7 parameters a, b, α, β, γ, η, ρ are consistent with Table 2.2. There are therefore altogether 24 *solvable* models. Some of these models are however *not new* (in particular, when some parameters vanish); moreover in some cases the solution (2.8) with (2.7) of the matrix equation (2.1) is only valid in a limiting sense (see Remark 2.2). We leave these issues to whoever will be interested – possibly in specific, theoretical or applicative, contexts – in more detailed investigations of anyone of these models; our focus in this paper is rather in the unified treatment, and the simultaneous display (see the

Table 2.3

#	a	b	α	β	γ	η	ρ
1	0	*	$-b\gamma$	0	*	0	*
2	*	*	$-b\gamma$	0	*	0	0
3	0	*	*	0	*	0	$-b\gamma$
4	0	*	$-b\gamma$	*	*	0	$-2b\gamma$
5	*	*	$b\gamma$	$b^2\gamma/a$	*	0	$-2b\gamma$

This table indicates the 5 sets of values to be assigned to the 7 parameters a, b, α, β, γ, η, ρ in (2.37) with (2.38). Asterisks indicate that the corresponding parameters can be assigned freely. Note that in every case there are 3 free parameters.

Table 2.4

#	a	b	α	β	γ	η	ρ
1	0	*	$b\gamma + \rho$	0	*	0	*
2	*	*	0	0	*	0	$-b\gamma$

This table indicates the set of values to be assigned to the 7 parameters a, b, α, β, γ, η, ρ in (2.37) with (2.39). Asterisks indicate that the corresponding parameters can be assigned freely: in the second line a, b, and γ are 3 free parameters, in the first line b, γ and ρ are 3 free parameters.

appendices below), of all of them (except for the elimination, as already mentioned, of the cases with $\gamma = \eta = 0$).

It is convenient to write the corresponding equations of motion in two different ways, depending whether the parameter η does or does not vanish.

If the parameter η vanishes, $\eta = 0$, the equations of motion read as follows:

$$\ddot{z}_n = -\alpha\rho z_n - \beta\rho z_n^2 + (\alpha + \rho)\dot{z}_n + 2\beta\dot{z}_n z_n + 2(\dot{z}_n + f_n) \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{z}_\ell + f_\ell)}{(z_n - z_\ell)} \right] \tag{2.37}$$

with the following assignments corresponding respectively to case (i) and case (ii).

In case (i)

$$f_n = a\gamma + b\gamma z_n \tag{2.38}$$

and the 7 parameters a, b, α, β, γ, η, ρ are restricted according to Table 2.3 (being the relevant subcase of Table 2.1).

In case (ii)

$$f_n = a\gamma + (b\gamma - \alpha) z_n - \beta z_n^2 \tag{2.39}$$

and the 6 parameters a, b, α, β, γ, ρ are restricted according to Table 2.4 (being the relevant subcase of Table 2.2).

If the parameter η does *not* vanish, $\eta \neq 0$, the equations of motion are more conveniently written in terms of the dependent variables

$$x_n(t) \equiv z_n(t) + \frac{\gamma}{2\eta}, \tag{2.40}$$

reading then as follows:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \lambda\frac{\dot{x}_n}{x_n} + \beta\dot{x}_n x_n + \rho\left(\dot{x}_n + \lambda - \mu x_n - \beta x_n^2\right)$$

$$+ (\dot{x}_n + f_n) \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell + f_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right] \tag{2.41a}$$

Table 2.5

#	a	b	α	β	γ	η	ρ
1	0	0	$*$	$*$	$*$	$*$	0
2	$*$	0	$*$	0	0	$*$	0
3	$*$	0	$-2a\eta$	0	0	$*$	$*$
4	$*$	0	$-2a\eta$	0	$*$	$*$	$2a\eta$
5	$*$	0	$-2a\eta$	$*$	0	$*$	$-2a\eta$
6	$*$	0	$2a\eta$	$4a\eta^2/\gamma$	$*$	$*$	$-2a\eta$
7	0	$*$	$-b\gamma$	$-2b\eta$	$*$	$*$	$*$
8	0	$*$	$*$	$-2b\eta$	$*$	$*$	$-b\gamma$
9	$*$	$*$	$*$	$-2b\eta$	0	$*$	0
10	$*$	$*$	$-2a\eta$	$-2b\eta$	0	$*$	$*$
11	$-b\gamma$	$*$	$*$	$-2b\eta$	$*$	$*$	$-\alpha - b\gamma$
12	$*$	$*$	$-2a\eta$	$-2b\eta$	$*$	$*$	$-\alpha - b\gamma$

This table indicates the 12 sets of values to be assigned to the 7 parameters a, b, α, β, γ, η, ρ in (2.41) with (2.42). Asterisks indicate that the corresponding parameters can be assigned freely (with $\eta \neq 0$). Note that each of the 7 lines 1, 7, 8, 9, 10, 11, 12 assigns values to only 3 of the 7 parameters, while the other 5 assign values to 4 of the 7 parameters.

or equivalently

$$\ddot{x}_n = \left(\lambda - 2f^{(0)}\right)\frac{\dot{x}_n}{x_n} + [(N-2)f^{(1)} + \rho]\dot{x}_n + \left(-2f^{(2)} + \beta\right)\dot{x}_n x_n - \frac{\left(f^{(0)}\right)^2}{x_n} + \rho\lambda$$
$$+ (N-2)f^{(0)}f^{(1)} + \left[-\rho\mu + (N-1)\left(f^{(1)}\right)^2 - 2f^{(0)}f^{(2)}\right]x_n$$
$$+ \left[-\beta\rho + (N-2)f^{(1)}f^{(2)}\right]x_n^2 - \left(f^{(2)}\right)^2 x_n^3 + (\dot{x}_n + f_n)\sum_{k=1}^{N}\left[\frac{\left(\dot{x}_k + f^{(0)}\right)}{x_k} + f^{(2)}x_k\right]$$
$$+ 2\sum_{\ell=1,\ell\neq n}^{N}\left[\frac{(\dot{x}_n + f_n)(\dot{x}_\ell + f_\ell)}{(x_n - x_\ell)}\right] \tag{2.41b}$$

with

$$\lambda = \frac{(2\alpha\eta - \beta\gamma)\gamma}{4\eta^2}, \qquad \mu = \alpha - \frac{\beta\gamma}{\eta} \qquad \text{so that} \qquad \mu^2 = \alpha^2 - \frac{\lambda}{4}, \tag{2.41c}$$

and with the following assignments corresponding respectively to case (i) and case (ii).
    In case (i)

$$f_n = (2a\eta - b\gamma)x_n + 2b\eta x_n^2 \tag{2.42}$$

and the 7 parameters a, b, α, β, γ, η, ρ are restricted according to Table 2.5 (being the relevant subcase of Table 2.1).
    In case (ii)

$$f_n = \frac{(2\alpha\eta - \beta\gamma)\gamma}{4\eta^2} + \left(2a\eta - b\gamma - \alpha + \frac{\beta\gamma}{\eta}\right)x_n + (2b\eta - \beta)x_n^2 \tag{2.43}$$

and the 7 parameters a, b, α, β, γ, η, ρ are restricted according to Table 2.6 (being the relevant subcase of Table 2.2):

These 24 *Newtonian* equations of motion are exhibited in Appendix A. Their *solvable* character is of course implied by the fact that the N coordinates $z_n(t)$ coincide with the N eigenvalues

Table 2.6

#	a	b	α	β	γ	η	ρ
1	0	0	$*$	$*$	$*$	$*$	$*$
2	$*$	0	$*$	$*$	$*$	$*$	0
3	0	$*$	$b\gamma + \rho$	$2b\eta$	$*$	$*$	$*$
4	0	$*$	ρ	$2b\eta$	0	$*$	$*$
5	$*$	$*$	$2a\eta$	$2b\eta$	$*$	$*$	$-b\gamma$

This table indicates the 5 sets of values to be assigned to the 7 parameters a, b, α, β, γ, η, ρ in (2.41) with (2.43). Asterisks indicate that the corresponding parameters can be assigned freely (with $\eta \neq 0$). Note that each of the first 2 lines assigns values to only 2 of the 7 parameters, while the third and fifth lines assign values to 3 of the 7 parameters, and the fourth line assign values to 4 of the 7 parameters.

of the matrix $U(t)$ (see (2.10a)). To ascertain the behavior of these solutions $z_n(t)$ one must in each case take account of the restrictions on the parameters characterizing these models, as detailed above, which are of course also relevant in order to identify the corresponding evolution of the matrix $U(t)$: as implied by inserting in the explicit formula (2.8) with (2.7) – in addition to the parameters α, β, γ, η, ρ associated with the *solvable* N-body model under consideration – the expressions of the *initial* values $U(0)$ and $V(0) \equiv V_0$ of the matrices $U(t)$ and $V(t)$ in terms of the N *initial* values $z_n(0)$ of the N coordinates and the N *initial* values $\dot{z}_n(0)$ of the N velocities. To obtain these expressions it is useful to note that it is possible – and convenient – to assume that the *diagonalizing* matrix $R(t)$ (see (2.10)) is *initially* just the $N \times N$ *unit* matrix I,

$$R(0) = I, \tag{2.44}$$

implying *initially* (see (2.10))

$$U(0) = \text{diag}[z_n(0)], \qquad U_{nm}(0) = \delta_{nm}z_n(0), \tag{2.45a}$$

$$V_0 \equiv V(0) = Y(0), \qquad V_{nm}(0) = \delta_{nm}y_n(0) + (1 - \delta_{nm})Y_{nm}(0). \tag{2.45b}$$

The first of these two formulas provides the explicit expression of $U(0)$ in terms of the initial data $z_n(0)$.

In the second formula the *initial* values $y_n(0)$ of the *diagonal* elements of the matrix V_0 in terms of the initial coordinates $z_n(0)$ and velocities $\dot{z}_n(0)$ of the N particles read

$$y_n(0) = \frac{\dot{z}_n(0) - \alpha z_n(0) - \beta z_n^2(0)}{\gamma + 2\eta z_n(0)} \tag{2.45c}$$

(see (2.12b)), while the *off-diagonal* elements $Y_{nm}(0)$ (with $n \neq m$) of the matrix V_0 are given by the *ansatz* (2.26) yielding

$$Y_{nm}(0) = \left\{ g_n(0)g_m(0)[\dot{z}_n(0) + f_n(0)][\dot{z}_m(0) + f_m(0)] \right\}^{1/2}, \qquad n \neq m, \tag{2.45d}$$

with the quantities $g_n(0)$ and $f_n(0)$ given by the formulas (see (2.28))

$$g_n(0) = \frac{1}{\gamma + 2\eta z_n(0)}, \qquad f_n = f^{(0)} + f^{(1)}z_n(0) + f^{(2)}z_n^2(0), \tag{2.45e}$$

with the appropriate assignments of the parameters γ, η, $f^{(0)}$, $f^{(1)}$ and $f^{(2)}$ characterizing the various *solvable* many-body models, see above (in particular for $f^{(0)}$, $f^{(1)}$ and $f^{(2)}$ see (2.38) or (2.39) or (2.42) or (2.43), as appropriate).

Note moreover that the *dyadic* character of the *off-diagonal* part of the matrix $V_0 = Y(0)$, see (2.45d), implies a simplification when one must compute functions of this matrix V_0 such as

those appearing in the explicit expression (2.8) with (2.7) of $U(t)$; this simplification becomes particularly significant when the matrix $V_0 = Y(0)$ is altogether *dyadic*, $Y_{nm}(0) = v_n v_m$, since for any *dyadic* matrix, say $X_{nm} = x_n x_m$, there holds the simple formula

$$\varphi(X) = \varphi(0)I + \frac{\varphi(x) - \varphi(0)}{x}X, \qquad x^2 = \sum_{k=1}^{N} x_n^2, \tag{2.46}$$

where $\varphi(x)$ is any (scalar) function for which the (matrix) expression $\varphi(X)$ makes good sense. This simplification clearly happens iff

$$f_n = -\alpha z_n - \beta z_n^2, \tag{2.47a}$$

as implied by (2.12b) and (2.26) with (2.28), hence in case (i) whenever (see (2.32) and Table 2.1)

$$a\gamma = 0, \qquad 2a\eta + b\gamma = -\alpha, \qquad 2b\eta = -\beta, \tag{2.47b}$$

and in case (ii) whenever (see (2.33) and Table 2.2)

$$a = b = 0. \tag{2.47c}$$

Special cases and their (*autonomous*) *isochronous variants*. Certain special models among those identified above as *solvable* can be *isochronized* via the following change of dependent and independent variables,

$$z_n(t) = \exp(\mathbf{i}\sigma\omega t)\zeta_n(\tau), \qquad \tau = \frac{\exp(\mathbf{i}\omega t) - 1}{\mathbf{i}\omega}. \tag{2.48}$$

Here the quantities $\zeta_n(\tau)$ are assumed to satisfy the *Newtonian* equations written above, see (2.37) with (2.38) or (2.39), of course with the new (*complex*) independent variable τ replacing the time t; ω is an *arbitrary real* (for definiteness, *positive*) *constant* to which we associate the period

$$T = \frac{2\pi}{\omega}; \tag{2.49}$$

and the number σ is adjusted so as to produce, for the dependent variables $z_n \equiv z_n(t)$ (with the *real* independent variable t interpreted as "time") *autonomous* equations of motion (the special models providing the starting points for the application of this trick being appropriately selected to allow such an outcome). Since the application of this trick is by now quite standard (see, for instance, Section 2.1 entitled "The trick" of [6]), we dispense here from any detailed discussion of this approach and limit ourselves to reporting the results.

This trick is only applicable to (2.37) with (2.38) in the very special cases with $\alpha = \rho = 0$ and either $\gamma = 0$ or $a = b = 0$ (as long as one is only interested in getting *autonomous* equations of motion). Then the assignment $\sigma = 1$ yields the *isochronous* equations of motion

$$\ddot{z}_n = 3\mathbf{i}\omega\dot{z}_n + 2\omega^2 z_n + 2\beta z_n(\dot{z}_n - \mathbf{i}\omega z_n) + 2(\dot{z}_n - \mathbf{i}\omega z_n) \sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{(\dot{z}_\ell - \mathbf{i}\omega z_\ell)}{z_n - z_\ell}\right]. \tag{2.50a}$$

Likewise the application of this trick to (2.37) with (2.39), again in the very special cases with $\alpha = \rho = 0$ and either $\gamma = 0$ or $a = b = 0$, and again with the assignment $\sigma = 1$, yields the *isochronous* equations of motion

$$\ddot{z}_n = 3\mathbf{i}\omega\dot{z}_n + 2\omega^2 z_n + 2\beta z_n(\dot{z}_n - \mathbf{i}\omega z_n)$$

$$+ 2(\dot{z}_n - \mathbf{i}\omega z_n - \beta z_n^2) \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{z}_\ell - \mathbf{i}\omega z_\ell - \beta z_\ell^2)}{z_n - z_\ell} \right]. \tag{2.50b}$$

Neither one of these two models is however new: see Examples 4.2.2-6 and 4.2.2-7 in [6].

An analogous treatment applied, *mutatis mutandis*, to (2.41) with (2.42) in the special case with $\rho = 0$, $2a\eta = b\gamma$ and $2\alpha\eta = \beta\gamma$ (implying $\lambda = 0$), yields (again via the assignment $\sigma = 1$) the *isochronous* equations of motion

$$\ddot{x}_n = \mathbf{i}\omega\dot{x}_n + \omega^2 x_n + \frac{\dot{x}_n^2}{x_n} + \beta x_n(\dot{x}_n - \mathbf{i}\omega x_n) + \left(\dot{x}_n - \mathbf{i}\omega x_n + 2b\eta x_n^2\right)$$

$$\times \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell - \mathbf{i}\omega x_\ell + 2b\eta x_\ell^2)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right]. \tag{2.50c}$$

Likewise an analogous treatment applied to (2.41) with (2.43) in the special case with $\rho = 0$ and either $2\alpha\eta = \beta\gamma$ (implying $\lambda = 0$) and $\alpha = -2a\eta + b\gamma$ or $\gamma = 0$ (implying $\lambda = 0$) and $\alpha = 2a\eta$, yields (again via the assignment $\sigma = 1$) the same *isochronous* equations of motion (up to the, merely notational, replacement of $2b\eta$ with $2b\eta - \beta$).

While finally this treatment applied, with $\sigma = -1$, to (2.41) with (2.43) in the special case with $b = \beta = \rho = 0$ and $\alpha = 2a\eta$ yields the *isochronous* equations of motion

$$\ddot{x}_n = \mathbf{i}\omega\dot{x}_n - \omega^2 x_n + \frac{\dot{x}_n^2}{x_n} + a\gamma\frac{\dot{x}_n}{x_n} + a\gamma\mathbf{i}\omega$$

$$+ (\dot{x}_n + \mathbf{i}\omega x_n + a\gamma) \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell + \mathbf{i}\omega x_\ell + a\gamma)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right]. \tag{2.50d}$$

2.3   A related class of *solvable* many-body models

In this subsection we consider the *Newtonian* equations of motion that obtain by identifying the N dependent variables of the models discussed above as the N *zeros* of a monic (time-dependent) polynomial of degree N, and by then focussing on the time-evolution of the N *coefficients* of this polynomial. It is again convenient to treat separately the two cases with $\eta = 0$ and with $\eta \neq 0$.

In the $\eta = 0$ case the starting point are the equations of motion (2.37) with (2.38) or (2.39). We then introduce the time-dependent (monic) polynomial $\psi(z,t)$ whose zeros are the N eigenvalues $z_n(t)$ of the $N \times N$ matrix $U(t)$:

$$\psi(z,t) = \det[zI - U(t)], \tag{2.51a}$$

$$\psi(z,t) = \prod_{n=1}^{N}[z - z_n(t)] = z^N + \sum_{m=1}^{N} \left[c_m(t)z^{N-m}\right]. \tag{2.51b}$$

The last of these formulas introduces the N coefficients $c_m \equiv c_m(t)$ of the monic polynomial $\psi(z,t)$; of course it implies that these coefficients are related to the zeros $z_n(t)$ as follows:

$$c_1 = -\sum_{n=1}^{N} z_n, \qquad c_2 = \sum_{n,m=1;n>m}^{N} z_n z_m, \tag{2.51c}$$

and so on.

The fact that the initial-value problem associated with the time evolution of the N coordinates z_n can be *solved* by *algebraic* operations implies that the same *solvable* character can be

attributed to the time evolution of the monic polynomial $\psi(z,t)$ and of the N coefficients $c_m(t)$. The procedure to obtain the equations of motion satisfied by the N coefficients $c_m(t)$ from the N equations of motion satisfied by the N zeros is tedious but standard; a key role in this development are the identities reported, for instance, in Appendix A of [6] (but note that there are two misprints in these formulas: in equation (A.8k) the term $(N+1)$ inside the square brackets should instead read $(N-3)$; in equation (A.8l) the term N^2 inside the square brackets should instead read $N(N-2)$ – these misprints have been corrected in the recent paperback version of this monograph [6]). Here we limit our presentation to reporting the final result.

The equation characterizing the time evolution of the monic polynomial $\psi(z,t)$ implied by the *Newtonian* equations of motion (2.37) with (2.28b) reads as follows:

$$\psi_{tt} - 2\big(f^{(0)} + f^{(1)}z + f^{(2)}z^2\big)\psi_{zt} + \big(p^{(0)} + p^{(1)}z\big)\psi_t$$
$$+ \big(q_2^{(0)} + q_2^{(1)}z + q_2^{(2)}z^2 + q_2^{(3)}z^3 + q_2^{(4)}z^4\big)\psi_{zz}$$
$$+ \big(q_1^{(0)} + q_1^{(1)}z + q_1^{(2)}z^2 + q_1^{(3)}z^3\big)\psi_z + \big(q_0^{(0)} + q_0^{(1)}z + q_0^{(2)}z^2\big)\psi = 0, \tag{2.52}$$

with

$$p^{(0)} = -\alpha - \rho + 2(N-1)f^{(1)} - 2f^{(2)}c_1, \qquad p^{(1)} = 2\big[-\beta + (N-2)f^{(2)}\big]; \tag{2.53a}$$

$$q_2^{(0)} = \big(f^{(0)}\big)^2, \qquad q_2^{(1)} = 2f^{(0)}f^{(1)}, \qquad q_2^{(2)} = 2f^{(0)}f^{(2)} + \big(f^{(1)}\big)^2,$$
$$q_2^{(3)} = 2f^{(1)}f^{(2)}, \qquad q_2^{(4)} = \big(f^{(2)}\big)^2; \tag{2.53b}$$

$$q_1^{(0)} = -2(N-1)f^{(0)}f^{(1)} + 2f^{(0)}f^{(2)}c_1,$$
$$q_1^{(1)} = -\alpha\rho - 2(N-2)f^{(0)}f^{(2)} + 2f^{(1)}f^{(2)}c_1 - 2(N-1)\big(f^{(1)}\big)^2,$$
$$q_1^{(2)} = -\beta\rho - 2(2N-3)f^{(1)}f^{(2)} + 2\big(f^{(2)}\big)^2 c_1,$$
$$q_1^{(3)} = -2(N-2)\big(f^{(2)}\big)^2; \tag{2.53c}$$

$$q_0^{(0)} = N\alpha\rho - 2Nf^{(0)}f^{(2)} + N(N-1)\big(f^{(1)}\big)^2$$
$$\qquad - \big[\beta\rho + 2(N-1)f^{(1)}f^{(2)}\big]c_1 + 2\big(\beta + f^{(2)}\big)\dot{c}_1 + 2\big(f^{(2)}\big)^2 c_2,$$
$$q_0^{(1)} = N\beta\rho + 2N(N-2)f^{(1)}f^{(2)} - 2(N-1)\big(f^{(2)}\big)^2 c_1,$$
$$q_0^{(2)} = N(N-3)\big(f^{(2)}\big)^2, \tag{2.53d}$$

where of course the quantities $f^{(0)}$, $f^{(1)}$, $f^{(2)}$ should be expressed in terms of the other parameters as implied by (2.28b) with (2.38) or (2.39), as the case may be. Note that, while this equation, (2.52), satisfied by the function $\psi(z,t)$ (where of course subscripted variables denote partial differentiations) might seem a *linear PDE*, it is in fact a *nonlinear functional equation*, because some of its coefficients, see (2.53), depend on the quantities c_1 and c_2 which themselves depend on ψ, indeed clearly (see (2.51))

$$c_m \equiv c_m(t) = \frac{\psi^{(N-m)}(0,t)}{(N-m)!}, \tag{2.54a}$$

where we used the shorthand notation $\psi^{(j)}(z,t)$ to denote the j-th partial derivative with respect to the variable z of $\psi(z,t)$,

$$\psi^{(j)}(z,t) \equiv \frac{\partial^j \psi(z,t)}{\partial z^j}, \qquad j = 1,2,\ldots. \tag{2.54b}$$

Likewise, the equation characterizing the time evolution of the monic polynomial $\phi(x,t)$ implied by the *Newtonian* equations of motion (2.41) via the following assignment (analogous

to (2.51b)),

$$\phi(x,t) = \prod_{n=1}^{N}[x - x_n(t)] = x^N + \sum_{m=1}^{N}\left[c_m(t)x^{N-m}\right], \tag{2.55}$$

reads

$$\phi_{tt} - 2\big(f^{(0)} + f^{(1)}x + f^{(2)}x^2\big)\phi_{xt} + \left(\frac{p^{(-1)}}{x} + p^{(0)} + p^{(1)}x\right)\phi_t$$

$$+\big(q_2^{(0)} + q_2^{(1)}x + q_2^{(2)}x^2 + q_2^{(3)}x^3 + q_2^{(4)}x^4\big)\phi_{xx} \tag{2.56}$$

$$+\left(\frac{q_1^{(-1)}}{x} + q_1^{(0)} + q_1^{(1)}x + q_1^{(2)}x^2 + q_1^{(3)}x^3\right)\phi_x$$

$$+\left(\frac{q_0^{(-1)}}{x} + q_0^{(0)} + q_0^{(1)}x + q_0^{(2)}x^2\right)\phi = 0, \tag{2.57}$$

now with

$$p^{(-1)} = 2f^{(0)} - \lambda, \qquad p^{(0)} = -\rho + Nf^{(1)} - f^{(2)}c_1 + f^{(0)}\frac{c_{N-1}}{c_N} - \frac{\dot{c}_N}{c_N},$$

$$p^{(1)} = 2(N-1)f^{(2)} - \beta; \tag{2.58a}$$

$$q_2^{(0)} = \big(f^{(0)}\big)^2, \qquad q_2^{(1)} = 2f^{(0)}f^{(1)}, \qquad q_2^{(2)} = 2f^{(0)}f^{(2)} + \big(f^{(1)}\big)^2,$$

$$q_2^{(3)} = 2f^{(1)}f^{(2)}, \qquad q_2^{(4)} = \big(f^{(2)}\big)^2; \tag{2.58b}$$

$$q_1^{(-1)} = -\big(f^{(0)}\big)^2, \qquad q_1^{(0)} = \rho\lambda - Nf^{(0)}f^{(1)} + f^{(0)}f^{(2)}c_1 + f^{(0)}\frac{\dot{c}_N}{c_N} - \big(f^{(0)}\big)^2\frac{c_{N-1}}{c_N},$$

$$q_1^{(1)} = -\rho\mu - 2(N-1)f^{(0)}f^{(2)} + f^{(1)}f^{(2)}c_1 - (N-1)\big(f^{(1)}\big)^2 + f^{(1)}\frac{\dot{c}_N}{c_N} - f^{(0)}f^{(1)}\frac{c_{N-1}}{c_N},$$

$$q_1^{(2)} = -\beta\rho - (3N-4)f^{(1)}f^{(2)} + \big(f^{(2)}\big)^2c_1 + f^{(2)}\frac{\dot{c}_N}{c_N} - f^{(0)}f^{(2)}\frac{c_{N-1}}{c_N},$$

$$q_1^{(3)} = -(2N-3)\big(f^{(2)}\big)^2; \tag{2.58c}$$

$$q_0^{(-1)} = \big(\lambda - 2f^{(0)}\big)\frac{\dot{c}_N}{c_N} + \big(f^{(0)}\big)^2\frac{c_{N-1}}{c_N},$$

$$q_0^{(0)} = -\beta\rho c_1 + \beta\dot{c}_1 + \big(f^{(2)}c_1 - Nf^{(1)}\big)\frac{\dot{c}_N}{c_N} + f^{(0)}\big(Nf^{(1)} - f^{(2)}c_1\big)\frac{c_{N-1}}{c_N},$$

$$q_0^{(1)} = N\beta\rho + N(N-2)f^{(1)}f^{(2)} - (N-1)\big(f^{(2)}\big)^2c_1 - Nf^{(2)}\frac{\dot{c}_N}{c_N} + f^{(0)}f^{(2)}\frac{c_{N-1}}{c_N},$$

$$q_0^{(2)} = N(N-2)\big(f^{(2)}\big)^2. \tag{2.58d}$$

The equations of motion of *Newtonian* type satisfied by the N coefficients $c_m(t)$ which obtain from (2.52) hence correspond to the *Newtonian* equations of motion (2.37) read as follows:

$$\ddot{c}_m - 2(N-m+1)f^{(0)}\dot{c}_{m-1} + \big[-2(N-m)f^{(1)} + p^{(0)}\big]\dot{c}_m$$

$$+\big[-2(N-m-1)f^{(2)} + p^{(1)}\big]\dot{c}_{m+1} + (N-m+2)(N-m+1)q_2^{(0)}c_{m-2}$$

$$+(N-m+1)\big[(N-m)q_2^{(1)} + q_1^{(0)}\big]c_{m-1}$$

$$+\big\{(N-m)\big[(N-m-1)q_2^{(2)} + q_1^{(1)}\big] + q_0^{(0)}\big\}c_m$$

$$+\big\{(N-m-1)\big[(N-m-2)q_2^{(3)} + q_1^{(2)}\big] + q_0^{(1)}\big\}c_{m+1}$$

$$+\left\{(N-m-2)\left[(N-m-3)q_2^{(4)}+q_1^{(3)}\right]+q_0^{(2)}\right\}c_{m+2}=0, \qquad (2.59)$$

where c_n vanishes for $n < 0$ and for $n > N$ while $c_0 = 1$ (see (2.51)), and of course the coefficients $p^{(j)}$ and $q_k^{(j)}$ are defined by (2.53). Again, this system of ODEs might seem *linear*, but it is in fact *nonlinear* because some of its coefficients depend on the dependent variables c_1 and c_2, see (2.53). The more explicit version of these equations of motion that obtain by expressing the various coefficients in terms of the free parameters are listed in Appendix B. They are of course just as *solvable* as the *Newtonian* equations of motion satisfied by the N coordinates $x_n(t)$, see Appendix A, to which they correspond via (2.51); and in particular whenever the parameter ρ is *imaginary* and the parameter α is *real* and *negative* they are *asymptotically isochronous* with period T (see Remark 2.3).

Likewise, the equations of motion of *Newtonian* type satisfied by the N coefficients $c_m(t)$ which obtain from (2.57) hence correspond to the *Newtonian* equations of motion (2.41) read as follows:

$$\begin{aligned}
\ddot{c}_m &+ \left[p^{(-1)}-2(N-m+1)f^{(0)}\right]\dot{c}_{m-1}+\left[-2(N-m)f^{(1)}+p^{(0)}\right]\dot{c}_m \\
&+ \left[-2(N-m-1)f^{(2)}+p^{(1)}\right]\dot{c}_{m+1}+(N-m+2)\left[(N-m+1)q_2^{(0)}+q_1^{(-1)}\right]c_{m-2} \\
&+ \left\{(N-m+1)\left[(N-m)q_2^{(1)}+q_1^{(0)}\right]+q_0^{(-1)}\right\}c_{m-1} \\
&+ \left\{(N-m)\left[(N-m-1)q_2^{(2)}+q_1^{(1)}\right]+q_0^{(0)}\right\}c_m \\
&+ \left\{(N-m-1)\left[(N-m-2)q_2^{(3)}+q_1^{(2)}\right]+q_0^{(1)}\right\}c_{m+1} \\
&+ \left\{(N-m-2)\left[(N-m-3)q_2^{(4)}+q_1^{(3)}\right]+q_0^{(2)}\right\}c_{m+2}=0, \qquad (2.60)
\end{aligned}$$

where of course again c_n vanishes for $n < 0$ and for $n > N$ while $c_0 = 1$ (see (2.55)) and of course the coefficients $p^{(j)}$ and $q_k^{(j)}$ are now defined by (2.58). Again, this system of ODEs might seem *linear*, but it is in fact *nonlinear* because some of its coefficients depend on the dependent variables c_1, c_{N-1} and c_N, see (2.58). The more explicit version of these equations of motion that obtains by expressing the various coefficients in terms of the free parameters are listed in Appendix B. They are of course just as *solvable* as the *Newtonian* equations of motion satisfied by the N coordinates $x_n(t)$, see Appendix A, to which they correspond via (2.55); and in particular whenever the parameter ρ is *imaginary* and the parameter α is *real* and *negative* they are *asymptotically isochronous* with period T (see Remark 2.3).

Special cases and their (autonomous) isochronous variants. Certain special models among those identified above (in this subsection) as *solvable* can be *isochronized* by an analogous trick to that employed at the end of the preceding Subsection 2.2. One route to this end takes as starting point the *isochronized* systems of *Newtonian* equations of motion (2.50) and applies to them the same procedure employed above to obtain the equations of motion (2.59) with (2.53) and (2.60) with (2.58). An equivalent procedure is to apply to certain special subcases of these systems of ODEs, (2.59) with (2.53) and (2.60) with (2.58), the following change of dependent and independent variables:

$$c_m(t)=\exp\left(\mathbf{i}\,\sigma m\omega t\right)\chi_m\left(\tau\right),\tau=\frac{\exp\left(\mathbf{i}\omega t\right)-1}{\mathbf{i}\omega}. \qquad (2.61)$$

Here the quantities $\chi_n\left(\tau\right)$ are assumed to satisfy the systems of ODEs written above, see (2.59) with (2.53) and (2.60) with (2.58), of course with the new (*complex*) independent variable τ replacing the time t; ω is an *arbitrary real* (for definiteness, *positive*) *constant* to which we associate the period T, see (2.49); and the number σ is adjusted so as to produce *autonomous* ODEs for the new dependent variables $c_m \equiv c_m(t)$ (with the *real* independent variable t interpreted as "time": the special models providing the starting points for the application of this

trick being appropriately selected in order to allow such an outcome). Since the application of this trick is quite standard, we dispense here from any detailed discussion of this approach and limit ourselves to reporting the results.

The ODEs that follow from (2.59) (with $\alpha = \rho = \eta = 0$, $f^{(0)} = f^{(1)} = f^{(2)} = 0$ and $\sigma = 1$) read as follows:

$$\ddot{c}_m = \mathbf{i}(2m+1)\omega\dot{c}_m - \left[2\beta(\dot{c}_1 - \mathbf{i}\omega c_1) - m(m+1)\omega^2\right]c_m$$
$$+ 2\beta\dot{c}_{m+1} - \mathbf{i}2\beta(m+1)\omega c_{m+1}; \tag{2.62a}$$

those that follow from (2.59) (with $\alpha = \rho = \eta = 0$, $f^{(0)} = f^{(1)} = 0$, $f^{(2)} = -\beta$ and $\sigma = 1$) read instead as follows

$$\ddot{c}_m = [\mathbf{i}(2m+1)\omega - 2\beta c_1]\dot{c}_m + \left[m(m+1)\omega^2 + \mathbf{i}2m\beta\omega c_1 - 2\beta^2 c_2\right]c_m$$
$$+ 2m\dot{c}_{m+1} + 2m\left[\beta^2 c_1 - \mathbf{i}(m+1)\omega\right]c_{m+1} - (m-1)(m+2)\beta^2 c_{m+2}. \tag{2.62b}$$

Neither one of these two *isochronous* many-body problems is new.

The analogous results that follows from (2.60) are instead generally *new*. There are then two sets of cases. The first set of *isochronous* models obtain from the assignment $\sigma = 1$ and read

$$\ddot{c}_m = \left[-\mathbf{i}(N-2m-1)\omega + f^{(2)}c_1 + \frac{\dot{c}_N}{c_N}\right]\dot{c}_m + \left(\beta - 2mf^{(2)}\right)\dot{c}_{m+1}$$
$$+ \left[-m(N-m-1)\omega^2 + (N-m)\omega f^{(2)}c_1 - \frac{\dot{c}_N}{c_N}\left(\mathbf{i}m\omega + f^{(2)}c_1 + \beta(\mathbf{i}\omega c_1 - \dot{c}_1)\right)\right]c_m$$
$$+ \left[-\mathbf{i}(N-2m)(m+1)\omega f^{(2)} - \mathbf{i}(m+1)\omega\beta + m\left(f^{(2)}\right)^2 c_1 + (m+1)f^{(2)}\frac{\dot{c}_N}{c_N}\right]c_{m+1}$$
$$- m(m+2)\left(f^{(2)}\right)^2 c_{m+2}, \tag{2.63a}$$

with the following restriction on the parameters:

$$\rho = \lambda = f^{(0)} = f^{(1)} = 0. \tag{2.63b}$$

The second set of *isochronous* models obtain from the assignment $\sigma = -1$ and read

$$\ddot{c}_m = (2N-2m+1)\lambda\dot{c}_{m-1} + \left[\mathbf{i}(N-2m+1)\omega + \frac{\dot{c}_N}{c_N} - \lambda\frac{c_{N-1}}{c_N}\right]\dot{c}_m$$
$$- (N-m)(N-m+2)\lambda^2 c_{m-2} + \left[\mathbf{i}(m-1)(2N-2m+1)\omega\right.$$
$$+ (N-m+1)(N-m-1)\lambda\left(\lambda\frac{c_{N-1}}{c_N} - \mathbf{i}N\omega - \frac{\dot{c}_N}{c_N}\right)\right]c_{m-1}$$
$$+ \left[-m(N-m+1)\omega^2 + \mathbf{i}m\omega\left(\frac{\dot{c}_N}{c_N} - \lambda\frac{c_{N-1}}{c_N}\right)\right]c_m, \tag{2.64a}$$

with the following restrictions on the parameters:

$$b = \beta = \rho = f^{(1)} = f^{(2)} = 0, \qquad \alpha = 2a\eta, \qquad f^{(0)} = \lambda = a\gamma. \tag{2.64b}$$

3   Outlook

Results analogous, but somewhat more general, than those reported in this paper can be obtained by an analogous treatment based on a somewhat more general – but still *solvable* – system of two $N \times N$ matrix ODEs than (2.1), such as, for instance,

$$\dot{U} = \alpha U + \beta U^2 + \gamma V + \eta(UV + VU), \qquad \dot{V} = \rho_0 + \rho V + \rho_2 V^2, \tag{3.1}$$

which contains the 2 additional scalar constants ρ_0 and ρ_2 (and clearly reduces to (2.1) for $\rho_0 = \rho_2 = 0$). These developments will be reported in subsequent papers.

Finally, let us recall that *Diophantine* findings can be obtained from a *nonlinear autonomous isochronous* dynamical system by investigating its behavior in the *infinitesimal vicinity* of its equilibria. The relevant equations of motion become then generally *linear*, but they of course retain the properties to be *autonomous* and *isochronous*. For a system of *linear autonomous* ODEs, the property of *isochrony* implies that *all* the eigenvalues of the matrix of its coefficients are *integer numbers* (up to a common rescaling factor). When the *linear* system describes the behavior of a *nonlinear autonomous* system in the *infinitesimal vicinity* of its equilibria, these matrices can generally be *explicitly* computed in terms of the values at equilibrium of the dependent variables of the original, *nonlinear* model. In this manner nontrivial *Diophantine* findings and conjectures have been discovered and proposed: see for instance the review of such developments in Appendix C (entitled "Diophantine findings and conjectures") of [6]. Analogous results obtained by applying this approach to the *isochronous* systems of *autonomous nonlinear* ODEs introduced above – and in subsequent papers – will be reported if they turn out to be novel and interesting.

A   First appendix

In this appendix we list the 24 *Newtonian* equations of motion whose *solvable* character has been demonstrated in this paper. In each case the parameters they feature (such as a, b, α, β, γ, η, ρ, as the case may be) are *arbitrary constants*; the (assigned) values of the other ones of these parameters (which also characterize the time-evolution of the solutions of these equations, see (2.8)), are also reported. Let us emphasize that if the parameter ρ is an *imaginary* number and the parameter α is *real* and *negative*, the corresponding many-body problem is *asymptotically isochronous* with period T, see Remark 2.3; and that *isochronous* many-body models are characterized by the 4 *Newtonian* equations of motion (2.50) displayed at the end of Subsection 2.2. Let us also mention again that the equations of motion reported below are *not* all new; in particular *not* new are clearly those whose corresponding equations of motion in the following Appendix B are *linear*.

$\eta = 0$, case (i) (5 models, corresponding to Table 2.3):
(1) $a = \beta = 0$, $\alpha = -b\gamma$, $f_n = -\alpha z_n$:

$$\ddot{z}_n = -\alpha\rho z_n + (\alpha + \rho)\dot{z}_n + 2(\dot{z}_n - \alpha z_n) \sum_{\ell=1,\ell\neq n}^{N} \left(\frac{\dot{z}_\ell - \alpha z_\ell}{z_n - z_\ell} \right); \tag{A.1a}$$

(2) $\alpha = -b\gamma$, $\beta = \rho = 0$, $f_n = -(a/b)\alpha - \alpha z_n$:

$$\ddot{z}_n = -\alpha\dot{z}_n + 2\left(\dot{z}_n - \frac{a\alpha}{b} - \alpha z_n \right) \sum_{\ell=1,\ell\neq n}^{N} \left(\frac{\dot{z}_\ell - a\alpha/b - \alpha z_\ell}{z_n - z_\ell} \right); \tag{A.1b}$$

(3) $a = \beta = 0$, $\rho = -b\gamma$, $f_n = -\rho z_n$:

$$\ddot{z}_n = -\alpha\rho z_n + (\alpha + \rho)\dot{z}_n + 2(\dot{z}_n - \rho z_n) \sum_{\ell=1,\ell\neq n}^{N} \left(\frac{\dot{z}_\ell - \rho z_\ell}{z_n - z_\ell} \right); \tag{A.1c}$$

(4) $a = 0$, $\alpha = -b\gamma$, $\rho = -2b\gamma = 2\alpha$, $f_n = -\alpha z_n$:

$$\ddot{z}_n = -2\alpha^2 z_n - 2\alpha\beta z_n^2 + 3\alpha\dot{z}_n + 2\beta\dot{z}_n z_n + 2(\dot{z}_n - \alpha z_n) \sum_{\ell=1,\ell\neq n}^{N} \left(\frac{\dot{z}_\ell - \alpha z_\ell}{z_n - z_\ell} \right); \tag{A.1d}$$

(5) $\alpha = b\gamma$, $\beta = b^2\gamma/a = b\alpha/a$, $\rho = -2b\gamma = -2\alpha$, $f_n = a\alpha/b + \alpha z_n$:

$$\ddot{z}_n = 2\alpha^2 z_n + \frac{2b}{a}\alpha^2 z_n^2 + \frac{2b}{a}\alpha \dot{z}_n z_n - \alpha \dot{z}_n$$
$$+ 2\left(\dot{z}_n + \alpha\frac{a}{b} + \alpha z_n\right) \sum_{\ell=1,\ell\neq n}^{N} \left(\frac{\dot{z}_\ell + a\alpha/b + \alpha z_\ell}{z_n - z_\ell}\right). \tag{A.1e}$$

$\eta = 0$, case (ii) (2 models, corresponding to Table 2.4):
(1) $a = \beta = 0$, $\alpha = b\gamma + \rho$, $f_n = -\rho z_n$:

$$\ddot{z}_n = -(b\gamma + \rho)\rho z_n + (b\gamma + 2\rho)\dot{z}_n + 2(\dot{z}_n - \rho z_n) \sum_{\ell=1,\ell\neq n}^{N} \left(\frac{\dot{z}_\ell - \rho z_\ell}{z_n - z_\ell}\right); \tag{A.2a}$$

(2) $\alpha = \beta = 0$, $\rho = -b\gamma$, $f_n = a\gamma + b\gamma z_n$:

$$\ddot{z}_n = -b\gamma\dot{z}_n + 2(\dot{z}_n + a\gamma + b\gamma z_n) \sum_{\ell=1,\ell\neq n}^{N} \left(\frac{\dot{z}_\ell + a\gamma + b\gamma z_\ell}{z_n - z_\ell}\right). \tag{A.2b}$$

$\eta \neq 0$, case (i) (12 models, corresponding to Table 2.5):
(1) $a = b = \rho = 0$, $f_n = 0$, $\lambda = (2\alpha\eta - \beta\gamma)\gamma/(4\eta^2)$, $\mu = \alpha - \beta\gamma/\eta$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \lambda\frac{\dot{x}_n}{x_n} + \beta\dot{x}_n x_n + \dot{x}_n \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{\dot{x}_\ell(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3a}$$

(2) $b = \beta = \gamma = \rho = 0$, $f_n = 2a\eta x_n$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + (\dot{x}_n + 2a\eta x_n) \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell + 2a\eta x_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3b}$$

(3) $b = \beta = \gamma = 0$, $\alpha = -2a\eta$, $f_n = -\alpha x_n$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \rho(\dot{x}_n - \alpha x_n) + (\dot{x}_n - \alpha x_n) \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell - \alpha x_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3c}$$

(4) $b = \beta = 0$, $\alpha = -2a\eta$, $\rho = -\alpha$, $f_n = -\alpha x_n$, $\lambda = \alpha\gamma/(2\eta)$, $\mu = \alpha$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \frac{\alpha\gamma}{2\eta}\frac{\dot{x}_n}{x_n} - \alpha\left(\dot{x}_n + \frac{\alpha\gamma}{2\eta} - \alpha x_n\right)$$
$$+ (\dot{x}_n - \alpha x_n) \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell - \alpha x_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3d}$$

(5) $b = \gamma = 0$, $\alpha = \rho = -2a\eta$, $f_n = -\alpha x_n$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \beta\dot{x}_n x_n + \alpha\left(\dot{x}_n - \alpha x_n - \beta x_n^2\right)$$
$$+ (\dot{x}_n - \alpha x_n) \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell - \alpha x_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3e}$$

(6) $b = 0$, $\alpha = 2a\eta$, $\beta = 4a\eta^2/\gamma = 2\eta\alpha/\gamma$, $\rho = -\alpha$, $f_n = \alpha z_n$, $\lambda = 0$, $\mu = -\alpha$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \frac{2\alpha\eta}{\gamma}\dot{x}_n x_n - \alpha\left(\dot{x}_n + \alpha x_n - \frac{2\alpha\eta}{\gamma}x_n^2\right)$$
$$+ (\dot{x}_n + \alpha x_n)\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{(\dot{x}_\ell + \alpha x_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3f}$$

(7) $a = 0$, $\alpha = -b\gamma$, $\beta = -2b\eta$, $f_n = \alpha x_n - \beta x_n^2$, $\lambda = 0$, $\mu = -\alpha$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \beta\dot{x}_n x_n + \rho\left(\dot{x}_n + \alpha x_n + \beta x_n^2\right)$$
$$+ \left(\dot{x}_n + \alpha x_n - \beta x_n^2\right)\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{\left(\dot{x}_\ell + \alpha x_\ell - \beta x_\ell^2\right)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3g}$$

(8) $a = 0$, $\beta = -2b\eta$, $\rho = -b\gamma$, $f_n = -b\gamma x_n + 2b\eta x_n^2$, $\lambda = (\alpha + b\gamma)\gamma/(2\eta)$, $\mu = \alpha + 2b\gamma$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \frac{(\alpha + b\gamma)\gamma}{2\eta}\frac{\dot{x}_n}{x_n} - 2b\eta\dot{x}_n x_n - b\gamma\left[\dot{x}_n + \frac{(\alpha + b\gamma)\gamma}{2\eta} - (\alpha + 2b\gamma)x_n + 2b\eta x_n^2\right]$$
$$+ \left(\dot{x}_n - b\gamma x_n + 2b\eta x_n^2\right)\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{\left(\dot{x}_\ell - b\gamma x_\ell + 2b\eta x_\ell^2\right)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3h}$$

(9) $\gamma = \rho = 0$, $\beta = 2b\eta$, $f_n = 2a\eta x_n + 2b\eta x_n^2$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} - 2b\eta\dot{x}_n x_n + \left(\dot{x}_n + 2a\eta x_n + 2b\eta x_n^2\right)$$
$$\times\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{\left(\dot{x}_\ell + 2a\eta x_\ell + 2b\eta x_\ell^2\right)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3i}$$

(10) $\gamma = 0$, $\alpha = -2a\eta$, $\beta = -2b\eta$, $f_n = -\alpha x_n - \beta x_n^2$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \beta\dot{x}_n x_n + \rho\left(\dot{x}_n - \alpha x_n - \beta x_n^2\right)$$
$$+ \left(\dot{x}_n - \alpha x_n - \beta x_n^2\right)\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{\left(\dot{x}_\ell - \alpha x_\ell - \beta x_\ell^2\right)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right]; \tag{A.3j}$$

(11) $a = -b\gamma$, $\beta = -2b\eta$, $\rho = -\alpha - b\gamma$, $f_n = -(2\eta + 1)b\gamma x_n + 2b\eta x_n^2$, $\lambda = (\alpha + b\gamma)\gamma/(2\eta)$, $\mu = \alpha + 2b\gamma$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \frac{(\alpha + b\gamma)\gamma}{2\eta}\frac{\dot{x}_n}{x_n} - 2b\eta\dot{x}_n x_n - (\alpha + b\gamma)\left[\dot{x}_n + \frac{(\alpha + b\gamma)\gamma}{2\eta} - (\alpha + 2b\gamma)x_n + 2b\eta x_n^2\right]$$
$$+ \left[\dot{x}_n - (2\eta + 1)b\gamma x_n + 2b\eta x_n^2\right]$$
$$\times\sum_{\ell=1,\,\ell\neq n}^{N}\left\{\frac{\left[\dot{x}_\ell - (2\eta + 1)b\gamma x_\ell + 2b\eta x_\ell^2\right](x_n + x_\ell)}{(x_n - x_\ell)x_\ell}\right\}; \tag{A.3k}$$

(12) $\alpha = -2a\eta$, $\beta = -2b\eta$, $\rho = -\alpha - b\gamma$, $f_n = (2a\eta - b\gamma)x_n + 2b\eta x_n^2$, $\lambda = (-2a\eta + b\gamma)\gamma/(2\eta)$, $\mu = -2a\eta + 2b\gamma$:

$$\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \frac{(b\gamma - 2a\eta)\gamma}{2\eta}\frac{\dot{x}_n}{x_n} - 2b\eta\dot{x}_n x_n$$

$$
- (\alpha + b\gamma) \left[\dot{x}_n + \frac{(b\gamma - 2a\eta)\gamma}{2\eta} + (2a\eta - 2b\gamma)x_n + 2b\eta x_n^2 \right]
$$

$$
+ \left[\dot{x}_n + (2a\eta - b\gamma)x_n + 2b\eta x_n^2 \right]
$$

$$
\times \sum_{\ell=1,\,\ell\neq n}^{N} \left\{ \frac{\left[\dot{x}_\ell + (2a\eta - b\gamma)x_\ell + 2b\eta x_\ell^2 \right](x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right\}. \tag{A.3l}
$$

$\eta \neq 0$, case (ii) (5 models, corresponding to Table 2.6):
(1) $a = b = 0$, $f_n = \lambda - \mu x_n - \beta x_n^2$, $\lambda = (2\alpha\eta - \beta\gamma)\gamma/(4\eta^2)$, $\mu = \alpha - \beta\gamma/\eta$:

$$
\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \lambda\frac{\dot{x}_n}{x_n} + \beta\dot{x}_n x_n + \rho\left(\dot{x}_n + \lambda - \mu x_n - \beta x_n^2 \right)
$$

$$
+ \left(\dot{x}_n + \lambda - \mu x_n - \beta x_n^2 \right) \sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{\left(\dot{x}_\ell + \lambda - \mu x_\ell - \beta x_\ell^2 \right)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right]; \tag{A.4a}
$$

(2) $b = \rho = 0$, $f_n = \lambda + (2a\eta - \mu)x_n - \beta x_n^2$, $\lambda = (2\alpha\eta - \beta\gamma)\gamma/(4\eta^2)$, $\mu = \alpha - \beta\gamma/\eta$:

$$
\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \lambda\frac{\dot{x}_n}{x_n} + \beta\dot{x}_n x_n + \left[\dot{x}_n + \lambda + (2a\eta - \mu)x_n - \beta x_n^2 \right]
$$

$$
\sum_{\ell=1,\,\ell\neq n}^{N} \left\{ \frac{\left[\dot{x}_\ell + \lambda + (2a\eta - \mu)x_\ell - \beta x_\ell^2 \right](x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right\}; \tag{A.4b}
$$

(3) $a = 0$, $\alpha = b\gamma + \rho$, $\beta = 2b\eta$, $f_n = \lambda - \rho x_n$, $\lambda = (\rho - b\gamma)\gamma/(2\eta)$, $\mu = \rho - b\gamma$:

$$
\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \lambda\frac{\dot{x}_n}{x_n} + 2b\eta\dot{x}_n x_n + \rho\left[\dot{x}_n + \lambda - (\rho - b\gamma)x_n - 2b\eta x_n^2 \right]
$$

$$
+ \left(\dot{x}_n + \lambda - \rho x_n \right) \sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell + \lambda - \rho x_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right]; \tag{A.4c}
$$

(4) $a = \gamma = 0$, $\alpha = \rho$, $\beta = 2b\eta$, $f_n = -\alpha x_n$, $\lambda = 0$, $\mu = \alpha$:

$$
\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \beta\dot{x}_n x_n + \alpha\left(\dot{x}_n - \alpha x_n - \beta x_n^2 \right)
$$

$$
+ \left(\dot{x}_n - \alpha x_n \right) \sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell - \alpha x_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right]; \tag{A.4d}
$$

(5) $\alpha = 2a\eta$, $\beta = 2b\eta$, $\rho = -b\gamma$, $f_n = \lambda + b\gamma x_n$, $\lambda = (2\alpha\eta - \beta\gamma)\gamma/(2\eta)$, $\mu = 2a\eta - 2b\gamma$:

$$
\ddot{x}_n = \frac{\dot{x}_n^2}{x_n} + \lambda\frac{\dot{x}_n}{x_n} + 2b\eta\dot{x}_n x_n - b\gamma\left[\dot{x}_n + \lambda - (2\alpha\eta - \beta\gamma)x_n - 2b\eta x_n^2 \right]
$$

$$
+ \left(\dot{x}_n + \lambda + b\gamma x_n \right) \sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{(\dot{x}_\ell + \lambda + b\gamma x_\ell)(x_n + x_\ell)}{(x_n - x_\ell)x_\ell} \right]. \tag{A.4e}
$$

B   Second appendix

In this appendix we list the second series of 24 *Newtonian* equations of motion whose *solvable* character has been demonstrated in this paper; they correspond to those reported in Appendix A via the transformation among the N zeros z_n and the N coefficients c_m of a monic polynomial,

see (2.51) and (2.55). In each case the parameters they feature (such as a, b, α, β, γ, η, ρ, as the case may be) are *arbitrary constants*; the (assigned) values of the other ones of these parameters (which also characterize the time-evolution of the solutions of these equations, see (2.8)), are also reported. Let us emphasize that if the parameter ρ is an *imaginary* number and the parameter α is *real* and *negative*, the corresponding many-body problem is *asymptotically isochronous* with period T, see Remark 2.3; and that *isochronous* many-body models are characterized by the 4 *Newtonian* equations of motion (2.62), (2.63) and (2.64) displayed at the end of Subsection 2.3. Let us also mention again that the equations of motion reported below are *not* all new; in particular all those that are *linear* are of course *not* new.

Let us recall that it is always assumed that $c_n = 0$ for $n < 0$ and for $n > N$, and $c_0 = 1$.

$\eta = 0$, case (i) (5 models, corresponding to Table 2.3):

(1) $a = \beta = 0$, $\alpha = -b\gamma$:

$$\ddot{c}_m + [(1 - 2m)\alpha - \rho]\dot{c}_m + m\alpha[(m - 1)\alpha + \rho]c_m = 0; \tag{B.1a}$$

(2) $\beta = \rho = 0$, $\alpha = -b\gamma$:

$$\ddot{c}_m - 2(N - m - 1)a\gamma\dot{c}_{m-1} + (2m - 1)b\gamma\dot{c}_m + (N - m + 2)(N - m + 1)a^2\gamma^2 c_{m-2}$$
$$- 2(N - m + 1)(m - 1)ab\gamma^2 c_{m-1} + m(m - 1)b^2\gamma^2 = 0; \tag{B.1b}$$

(3) $\alpha = \beta = 0$, $\rho = -b\gamma$:

$$\ddot{c}_m + [(1 - 2m)\rho - \alpha]\dot{c}_m + m\rho[(m - 1)\rho + \alpha]c_m = 0; \tag{B.1c}$$

(4) $a = 0$, $\alpha = -b\gamma$, $\rho = 2\alpha$:

$$\ddot{c}_m - (2m + 1)\alpha\dot{c}_m - 2\beta\dot{c}_{m+1} + [m(m + 1)\alpha^2 - 2\alpha\beta c_1 + 2\beta\dot{c}_1]c_m$$
$$+ 2(m + 1)\alpha\beta c_{m+1} = 0; \tag{B.1d}$$

(5) $\alpha = b\gamma$, $\beta = b^2\gamma/a$, $\rho = -2b\gamma$:

$$\ddot{c}_m - 2(N - m + 1)a\gamma\dot{c}_{m-1} + (2m - 1)b\gamma\dot{c}_m - \frac{2b^2\gamma}{a}\dot{c}_{m+1}$$
$$+ (N - m + 2)(N - m + 1)a^2\gamma^2 c_{m-2} - 2(m - 1)(N - m + 1)ab\gamma^2 c_{m-1}$$
$$+ \left[m(m - 3)b^2\gamma^2 + \frac{2b^3\gamma^2}{a}c_1 + \frac{2b^2\gamma}{a}\dot{c}_1\right]c_m - (m + 1)\frac{2b^3\gamma^2}{a}c_{m+1} = 0. \tag{B.1e}$$

$\eta = 0$, case (ii) (2 models, corresponding to Table 2.4):

(1) $\alpha = \beta = 0$, $\rho = -b\gamma$:

$$\ddot{c}_m - 2(N - m + 1)a\gamma\dot{c}_{m-1} + (2m - 1)b\gamma\dot{c}_m + (N - m + 2)(N - m + 1)a^2\gamma^2 c_{m-2}$$
$$- 2(m - 1)(N - m + 1)ab\gamma^2 c_{m-1} + m(m - 1)b^2\gamma^2 c_m = 0; \tag{B.2a}$$

(2) $a = \beta = 0$, $\alpha = b\gamma + \rho$:

$$\ddot{c}_m - (2m\rho + b\gamma)\dot{c}_m + \left[mb\gamma\rho - \left(N^2 - m^2\right)\rho^2\right]c_m = 0. \tag{B.2b}$$

$\eta \neq 0$, case (i) (12 models, corresponding to Table 2.5):

(1) $a = b = \rho = 0$, $\lambda = [(2\alpha\eta - \beta\gamma)\gamma]/(2\eta)^2$, $\mu = \alpha - \beta\gamma/\eta$:

$$\ddot{c}_m - \lambda\dot{c}_{m-1} - \frac{\dot{c}_N}{c_N}\dot{c}_m - \beta\dot{c}_{m+1} + \lambda\frac{\dot{c}_N}{c_N}c_{m-1} + \beta\dot{c}_1 c_m = 0; \tag{B.3a}$$

(2) $b = \beta = \gamma = \rho = 0$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{c}_m - \left[(2N - 4m)a\eta - \frac{\dot{c}_N}{c_N}\right]\dot{c}_m - 2ma\eta\left(2a\eta + \frac{\dot{c}_N}{c_N}\right)c_m = 0; \tag{B.3b}$$

(3) $b = \beta = \gamma = 0$, $\alpha = -2a\eta$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{c}_m + \left[(N - 2m)\alpha - \rho - \frac{\dot{c}_N}{c_N}\right]\dot{c}_m - \left[m(N - m)\alpha^2 + (N - m)\alpha\rho - ma\frac{\dot{c}_N}{c_N}\right]c_m = 0; \tag{B.3c}$$

(4) $b = \beta = 0$, $\alpha = -2a\eta$, $\rho = 2a\eta = -\alpha$, $\lambda = -a\gamma$, $\mu = \alpha$:

$$\ddot{c}_m + a\gamma\dot{c}_{m-1} - \left[(N - 2m + 1)\alpha + \frac{\dot{c}_N}{c_N}\right]\dot{c}_m + a\gamma\left[(N - m - 1)\alpha - \frac{\dot{c}_N}{c_N}\right]c_{m-1}$$
$$+ \left[-(N - m)\alpha^2 + 2(N - m)\alpha^2 + ma\frac{\dot{c}_N}{c_N}\right]c_m = 0; \tag{B.3d}$$

(5) $b = \gamma = 0$, $\alpha = \rho = -2a\eta$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{c}_m + \left[(N - 2m - 1)\alpha - \frac{\dot{c}_N}{c_N}\right]\dot{c}_m - \left[(m + 1)(N - m)\alpha^2 + ma\frac{\dot{c}_N}{c_N} + \alpha\beta c_1 - \beta\dot{c}_1\right]c_m$$
$$+ (m + 1)\alpha\beta c_{m+1} = 0; \tag{B.3e}$$

(6) $b = 0$, $\alpha = 2a\eta$, $\beta = 4a\eta^2/\gamma$, $\rho = -2a\eta$, $\lambda = 0$, $\mu = -2a\eta$:

$$\ddot{c}_m - \left[2(N - 2m + 1)a\eta + \frac{\dot{c}_N}{c_N}\right]\dot{c}_m - \frac{4a\eta^2}{\gamma}\dot{c}_{m+1} \tag{B.3f}$$
$$- 2a\eta\left[2(m - 1)(N - m)a\eta + m\frac{\dot{c}_N}{c_N} - \frac{4a\eta^2}{\gamma}c_1 - \frac{2\eta}{\gamma}\dot{c}_1\right]c_m - (m + 1)\frac{8a^2\eta^3}{\gamma}c_{m+1} = 0;$$

(7) $a = 0$, $\alpha = -b\gamma$, $\beta = -2b\eta$, $\lambda = 0$, $\mu = -\alpha$:

$$\ddot{c}_m - \left[(N - 2m)\alpha + \rho + \beta c_1 + \frac{\dot{c}_N}{c_N}\right]\dot{c}_m - (2m + 1)\beta\dot{c}_{m+1}$$
$$- \left[m(N - m)\alpha^2 - (N - m)\alpha\rho + (N - m)\alpha\beta c_1 + (\beta c_1 + m\alpha)\frac{\dot{c}_N}{c_N} + \beta\rho c_1 - \beta\dot{c}_1\right]c_m$$
$$- \left\{(m + 1)\beta\rho - [2N^2 - (3m + 5)N + 4m + 4]\alpha\beta + m\beta^2 c_1 + (m + 1)\beta\frac{\dot{c}_N}{c_N}\right\}c_{m+1}$$
$$- m(m + 2)\beta^2 c_{m+2} = 0; \tag{B.3g}$$

(8) $a = 0$, $\beta = -2b\eta$, $\rho = -b\gamma$, $\lambda = (\alpha + b\gamma)\gamma/(2\eta)$, $\mu = \alpha + 2b\gamma$:

$$\ddot{c}_m - \lambda\dot{c}_{m-1} + \left[(N - 2m + 1)b\gamma - 2b\eta c_1 - \frac{\dot{c}_N}{c_N}\right]\dot{c}_m + 4(m - 1)\dot{c}_{m+1}$$
$$+ \lambda\left[-(N - m + 1)b\gamma + \frac{\dot{c}_N}{c_N}\right]c_{m-1} - \left[(m + 4)(N - m)b^2\gamma^2 + 2(2N - 2m + 1)b^2\gamma\eta c_1\right.$$
$$\left. - (N - m)b\alpha\gamma + 2b\eta\dot{c}_1 + (mb\gamma - 2b\eta c_1)\frac{\dot{c}_N}{c_N}\right]c_m$$
$$+ 2\left[(m + 1)(N - 2m + 1)b^2\gamma\eta - 2mb^2\eta^2 c_1 - (m + 1)b\eta\frac{\dot{c}_N}{c_N}\right]c_{m+1}$$
$$+ 4m(m + 2)c_{m+2} = 0; \tag{B.3h}$$

(9) $\gamma = \rho = 0$, $\beta = -2b\eta$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{c}_m - \left[2(N-2m) + 2b\eta c_1 + \frac{\dot{c}_N}{c_N}\right]\dot{c}_m + 2(2m+1)b\eta\dot{c}_{m+1}$$

$$+ \left[-4m(N-m)a^2\eta^2 + 4(N-m)ab\eta^2 c_1 + \beta\dot{c}_1 + 2(b\eta c_1 - ma\eta)\frac{\dot{c}_N}{c_N}\right]c_m$$

$$- 2\left[2(m+1)(N-2m)ab\eta^2 + 2mb^2\eta^2 c_1 + (m+1)b\eta\frac{\dot{c}_N}{c_N}\right]c_{m+1}$$

$$+ 4m(m+2)c_{m+2} = 0; \tag{B.3i}$$

(10) $\gamma = 0$, $\alpha = -2a\eta$, $\beta = -2b\eta$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{c}_m + \left[(N-2m)\alpha - \rho + \beta c_1 - \frac{\dot{c}_N}{c_N}\right]\dot{c}_m - (2m+1)\beta\dot{c}_{m+1}$$

$$- \left[m(N-m)\alpha^2 + (N-m)\alpha\rho - (N-m)\alpha\beta c_1 + \beta\rho c_1 - \beta\dot{c}_1 + (\beta c_1 - m\alpha)\frac{\dot{c}_N}{c_N}\right]c_m$$

$$- \left[(m+1)(N-2m)\alpha\beta - (m+1)\beta\rho + m\beta^2 c_1 - (m+1)\beta\frac{\dot{c}_N}{c_N}\right]c_{m+1}$$

$$+ m(m+2)c_{m+2} = 0; \tag{B.3j}$$

(11) $a = -b\gamma$, $\beta = -2b\eta$, $\rho = -\alpha - b\gamma$, $\lambda = (\alpha + b\gamma)\gamma/(2\eta)$, $\mu = \alpha + 2b\gamma$:

$$\ddot{c}_m - \lambda\dot{c}_{m-1} + \left\{[2(N-2m)\eta + N - 2m + 1]b\gamma + \alpha - 2b\eta c_1 - \frac{\dot{c}_N}{c_N}\right\}\dot{c}_m$$

$$+ 2(2m+1)\dot{c}_{m+1} + \lambda\left[\frac{\dot{c}_N}{c_N} - (N-m+1)(\alpha + b\gamma)\right]c_{m-1}$$

$$- \left\{m(N-m)b^2\gamma^2(2\eta+1)^2 - (N-m)(\alpha + b\gamma)(\alpha + 2b\gamma) + 2(N-m)b^2\gamma\eta(2\eta+1)c_1\right.$$

$$\left. + 2b\eta(\alpha + b\gamma)c_1 + 2b\eta\dot{c}_1 - [2b\eta c_1 + mb\gamma(2\eta+1)]\frac{\dot{c}_N}{c_N}\right\}c_m$$

$$+ 2\left[(m+1)(N-2m)b^2\gamma\eta(2\eta+1) + (m+1)b\eta(\alpha+b\gamma) - 2mb^2\eta^2 c_1\right.$$

$$\left. - (m+1)b\eta\frac{\dot{c}_N}{c_N}\right]c_{m+1} + m(m+2)c_{m+2} = 0; \tag{B.3k}$$

(12) $\alpha = -2a\eta$, $\beta = -2b\eta$, $\rho = -\alpha - b\gamma = 2a\eta - b\gamma$, $\lambda = (b\gamma - 2a\eta)\gamma/(2\eta)$, $\mu = 2b\gamma - 2a\eta$:

$$\ddot{c} - \lambda\dot{c}_{m-1} - \left[(N-2m+1)(2a\eta - b\gamma) + 2b\eta c_1 + \frac{\dot{c}_N}{c_N}\right]\dot{c}_m$$

$$+ 2(2m+1)b\eta\dot{c}_{m+1} + \lambda\left[(N-m-1)(2a\eta - b\gamma) + \frac{\dot{c}_N}{c_N}\right]c_{m-1}$$

$$+ \left\{-m(N-m)(2a\eta - b\gamma)^2 + 2(N-m+1)(2a\eta - b\gamma)b\eta c_1\right.$$

$$\left. + (N-m)(2a\eta - b\gamma)(2a\eta - 2b\gamma) - 2b\eta\dot{c}_1 + [2b\eta c_1 - m(2a\eta - b\gamma)]\frac{\dot{c}_N}{c_N}\right\}c_m$$

$$- 2\left[(m+1)(N-2m+1)(2a\eta - b\gamma)b\eta + 2mb^2\eta^2 c_1 + (m+1)b\eta\frac{\dot{c}_N}{c_N}\right]c_{m+1}$$

$$+ 4m(m+2)b^2\eta^2 c_{m+2} = 0. \tag{B.3l}$$

$\eta \neq 0$, case (ii) (5 models, corresponding to Table 2.6):

(1) $a = b = 0$, $\lambda = [(2\alpha\eta - \beta\gamma)\gamma]/(2\eta)^2$, $\mu = \alpha - \beta\gamma/\eta$:

$$\ddot{c}_m - (2N - 2m + 1)\lambda\dot{c}_{m-1} + \left[(N - 2m)\mu - \rho + \beta c_1 + \lambda\frac{c_{N-1}}{c_N} - \frac{\dot{c}_N}{c_N}\right]\dot{c}_m$$

$$- (2m+1)\beta\dot{c}_{m+1} + (N - m + 2)(N - m)\lambda^2 c_{m-2} - \left[(N - 2m)(N - m + 1)\lambda\mu\right.$$

$$- (N - m + 1)\rho\lambda + (N - m + 1)\beta\lambda c_1 + \lambda\frac{\dot{c}_N}{c_N} - \lambda^2\frac{c_{N-1}}{c_N}\bigg]c_{m-1}$$

$$+ \left[m(N - m)(2\beta\lambda - \mu^2) + (\beta c_1 - m\mu)\left(\lambda\frac{c_{N-1}}{c_N} - \frac{\dot{c}_N}{c_N}\right)\right.$$

$$-(N - m)\rho\mu + (N - m)\beta\mu c_1 - \beta\rho c_1 + \beta\dot{c}_1\bigg]c_m$$

$$- \left[(m + 1)(N - 2m)\beta\mu + (m + 1)\beta\rho + m\beta^2 c_1 - (m + 1)\beta\frac{\dot{c}_N}{c_N}\right.$$

$$- (N - m - 2)\beta\lambda\frac{c_{N-1}}{c_N}\bigg]c_{m+1} + m(m + 2)\beta^2 c_{m+2} = 0; \tag{B.4a}$$

(2) $b = \rho = 0$, $\lambda = [(2\alpha\eta - \beta\gamma)\gamma]/(2\eta)^2$, $\mu = \alpha - \beta\gamma/\eta$:

$$\ddot{c}_m - (2N - 2m + 1)\lambda\dot{c}_{m-1} + \left[(N - 2m)(\mu - 2a\eta) + \beta c_1 + \lambda\frac{c_{N-1}}{c_N} - \frac{\dot{c}_N}{c_N}\right]\dot{c}_m$$

$$- (2m + 1)\beta\dot{c}_{m+1} + (N - m)(N - m + 2)\lambda^2 c_{m-2}$$

$$+ \left[(N - 2m)(N - m + 1)\lambda(2a\eta - \mu) - (N - m + 1)\beta\lambda c_1\right.$$

$$+ (N - m)\lambda\frac{\dot{c}_N}{c_N} - (N - m)\lambda^2\frac{c_{N-1}}{c_N}\bigg]c_{m-1} + \left\{m(N - m)\left[2\beta\lambda - (2a\eta - \mu)^2\right]\right.$$

$$- (N - m)(2a\eta - \mu)\beta c_1 + \beta\dot{c}_1 + [\beta c_1 + m(2a\eta - \mu)]\left(\lambda\frac{c_{N-1}}{c_N} - \frac{\dot{c}_N}{c_N}\right)\bigg\}c_m$$

$$- \left[(m + 1)(N - 2m)\beta\mu + (m + 1)\beta\rho + m\beta^2 c_1\right.$$

$$- (m + 1)\beta\frac{\dot{c}_N}{c_N} - (N - m - 2)\beta\lambda\frac{c_{N-1}}{c_N}\bigg]c_{m+1} + m(m + 2)\beta^2 c_{m+2} = 0; \tag{B.4b}$$

(3) $a = 0$, $\alpha = b\gamma + \rho$, $\beta = 2b\eta$, $\lambda = \gamma\rho/(2\eta)$, $\mu = \rho - b\gamma$:

$$\ddot{c}_m - (2N - 2m + 1)\lambda\dot{c}_{m-1} - \left[(N - 2m + 1)\rho - \lambda\frac{c_{N-1}}{c_N} + \frac{\dot{c}_N}{c_N}\right]\dot{c}_m$$

$$- 2b\eta\dot{c}_{m+1} + (N - m)(N - m + 2)\lambda^2 c_{m-2}$$

$$+ \left[(N - m + 1)(N - 2m + 1)\lambda\rho + (N - m)\lambda\left(\frac{\dot{c}_N}{c_N} - \lambda\frac{c_{N-1}}{c_N}\right)\right]c_{m-1}$$

$$- \left[(m + 1)(N - m)\rho^2 - (N - m)b\gamma\rho + m\rho\frac{\dot{c}_N}{c_N} - m\lambda\rho\frac{c_{N-1}}{c_N}\right.$$

$$- 2b\eta\rho c_1 - 2b\eta\dot{c}_1\bigg]c_m + 2(m + 1)b\eta\rho c_{m+1} = 0; \tag{B.4c}$$

(4) $a = \gamma = 0$, $\alpha = \rho$, $\beta = 2b\eta$, $\lambda = 0$, $\mu = \alpha$:

$$\ddot{c}_m + \left[(N - 2m - 1)\alpha - \frac{\dot{c}_N}{c_N}\right]\dot{c}_m - \beta\dot{c}_{m+1}$$

$$-\left[(m+1)(N-m)\alpha^2 - m\alpha\frac{\dot{c}_N}{c_N} + \alpha\beta c_1 - \beta\dot{c}_1\right]c_m + (m+1)\alpha\beta c_{m+1} = 0; \qquad \text{(B.4d)}$$

(5) $\alpha = 2a\eta$, $\beta = 2b\eta$, $\rho = -b\gamma$, $\lambda = [(2a\eta - b\gamma)\gamma]/(2\eta)$, $\mu = 2a\eta - 2b\gamma$:

$$\ddot{c}_m - (2N - 2m + 1)\lambda\dot{c}_{m-1} - \left[(N - 2m - 1)b\gamma - \lambda\frac{c_{N-1}}{c_N} + \frac{\dot{c}_N}{c_N}\right]\dot{c}_m$$

$$- \left[(N + 2m + 1)(N - m + 1)b\gamma\lambda + (N - m)\lambda\left(\lambda\frac{c_{N-1}}{c_N} - \frac{\dot{c}_N}{c_N}\right)\right]c_{m-1}$$

$$+ \left[-(m+2)(N-m)b^2\gamma^2 + mb\gamma\left(\lambda\frac{c_{N-1}}{c_N} - \frac{\dot{c}_N}{c_N}\right)\right.$$

$$\left. + 2(N-m)ab\gamma\eta + 2b^2\gamma\eta c_1 + 2b\eta\dot{c}_1\right]c_m - 2(m+1)b^2\gamma\eta c_{m+1} = 0. \qquad \text{(B.4e)}$$

Acknowledgements

References

[1] Olshanetsky M.A., Perelomov A.M., Explicit solution of the Calogero model in the classical case and geodesic flows on symmetric spaces of zero curvature, *Lett. Nuovo Cimento* **16** (1976), 333–339.

[2] Olshanetsky M.A., Perelomov A.M., Completely integrable Hamiltonian systems connected with semisimple Lie algebras, *Invent. Math.* **37** (1976), 93–108.

[3] Olshanetsky M.A., Perelomov A.M., Explicit solutions of some completely integrable systems, *Lett. Nuovo Cimento* **17** (1976), 97–101.

[4] Olshanetsky M.A., Perelomov A.M., Classical integrable finite-dimensional systems related to Lie algebras, *Phys. Rep.* **71** (1981), 313–400.

[5] Calogero F., Classical many-body problems amenable to exact treatments, *Lecture Notes in Physics. New Series m: Monographs*, Vol. 66, Springer-Verlag, Berlin, 2001.

[6] Calogero F., Isochronous systems, Oxford University Press, Oxford, 2008.[1]

[7] Bruschi M., Calogero F., Droghei R., Integrability, analyticity, isochrony, equilibria, small oscillations, and Diophantine relations: results from the stationary Burgers hierarchy, *J. Phys. A: Math. Theor.* **42** (2009), 475202, 9 pages.

[8] Bruschi M., Calogero F., Droghei R., Integrability, analyticity, isochrony, equilibria, small oscillations, and Diophantine relations: results from the stationary Korteweg–de Vries hierarchy, *J. Math. Phys.* **50** (2009), 122701, 19 pages.

[9] Calogero F., A new class of solvable dynamical systems, *J. Math. Phys.* **49** (2008), 052701, 9 pages.

[10] Calogero F., Two new solvable dynamical systems of goldfish type, *J. Nonlinear Math. Phys.* **17** (2010), 397–414.

[11] Calogero F., Isochronous dynamical systems, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **369** (2011), 1118–1136.

[12] Calogero F., A new goldfish model, *Theoret. and Math. Phys.* **167** (2011), 714–724.

[13] Calogero F., Another new goldfish model, *Theoret. and Math. Phys.* **171** (2012), 629–640.

[14] Calogero F., An integrable many-body problem, *J. Math. Phys.* **52** (2011), 102702, 5 pages.

[15] Calogero F., New solvable many-body model of goldfish type, *J. Nonlinear Math. Phys.* **19** (2012), 1250006, 19 pages.

---

[1] A paperback edition of this monograph has been published by Oxford University Press in September 2012: it coincides with the hardback version, except for the corrections of several misprints, and the addition of a two-page "Preface to the paperback edition" reporting a number of relevant new references.

[16] Calogero F., Two quite similar matrix ODEs and the many-body problems related to them, *Int. J. Geom. Methods Mod. Phys.* **9** (2012), 1260002, 6 pages.

[17] Calogero F., Another new solvable many-body model of goldfish type, *SIGMA* **8** (2012), 046, 17 pages, arXiv:1207.4850.

[18] Calogero F., Iona S., Isochronous dynamical system and Diophantine relations. I, *J. Nonlinear Math. Phys.* **16** (2009), 105–116.

[19] Calogero F., Leyvraz F., Examples of isochronous Hamiltonians: classical and quantal treatments, *J. Phys. A: Math. Theor.* **41** (2008), 175202, 11 pages.

[20] Calogero F., Leyvraz F., A new class of isochronous dynamical systems, *J. Phys. A: Math. Theor.* **41** (2008), 295101, 14 pages.

[21] Calogero F., Leyvraz F., Isochronous oscillators, *J. Nonlinear Math. Phys.* **17** (2010), 103–110.

[22] Calogero F., Leyvraz F., Solvable systems of isochronous, multi-periodic or asymptotically isochronous nonlinear oscillators, *J. Nonlinear Math. Phys.* **17** (2010), 111–120.

[23] Droghei R., Calogero F., Ragnisco O., An isochronous variant of the Ruijsenaars–Toda model: equilibrium configurations, behavior in their neighborhood, Diophantine relations, *J. Phys. A: Math. Theor.* **42** (2009), 445207, 9 pages.

[24] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G. (Editors), Higher transcendental functions, Vol. I, *Bateman Manuscript Project*, McGraw-Hill Book Co., New York, 1953.