Geometry of Spectral Curves
and All Order Dispersive Integrable System

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Abstract. We propose a definition for a Tau function and a spinor kernel (closely related to Baker–Akhiezer functions), where times parametrize slow (of order $1/N$) deformations of an algebraic plane curve. This definition consists of a formal asymptotic series in powers of $1/N$, where the coefficients involve theta functions whose phase is linear in N and therefore features generically fast oscillations when N is large. The large N limit of this construction coincides with the algebro-geometric solutions of the multi-KP equation, but where the underlying algebraic curve evolves according to Whitham equations. We check that our conjectural Tau function satisfies Hirota equations to the first two orders, and we conjecture that they hold to all orders. The Hirota equations are equivalent to a self-replication property for the spinor kernel. We analyze its consequences, namely the possibility of reconstructing order by order in $1/N$ an isomonodromic problem given by a Lax pair, and the relation between "correlators", the tau function and the spinor kernel. This construction is one more step towards a unified framework relating integrable hierarchies, topological recursion and enumerative geometry.

Key words: topological recursion; Tau function; Sato formula; Hirota equations; Whitham equations

2010 Mathematics Subject Classification: 14H70; 14H42; 30Fxx

1 Introduction

Integrable systems are nonlinear dynamical systems, and in many cases, some exact solutions can be produced in terms of algebraic geometry of Riemann surfaces. For instance, Liouville integrable systems can be brought into the form of a linear constant motion with constant velocity in a multidimensional torus which is the Jacobian of some algebraic curve. However, not all solutions are algebro geometric, and an important question is how to find some solutions as perturbations of algebro-geometric ones.

1.1 Goal and motivations

Our goal is to propose a definition of a formal series for a perturbation of an algebro-geometric solution of an integrable system, in a "small" parameter which we call $1/N$. Our definition consists in an all order expansion in powers of $1/N$, whose leading order is the usual algebro-geometric

solution, and whose corrections to each orders contain fast oscillating terms of frequency N, constructed from the invariants of [38].

The motivation for our definition, is to mimick the large N expansion of random $N \times N$ matrix integrals.

Indeed, it is well known [1, 50, 69, 78] that random matrix models are particular examples of Tau-functions of some integrable systems, and also, the formal large N expansion of matrix models can be obtained by formally solving Schwinger–Dyson equations (called loop equations in the context of matrix models), which leads to the invariants of [38], and to an expansion of the Tau-function in terms of them in [35, 37].

Therefore, in this paper, we propose to use the expressions introduced in [35, 37] for matrix models, in a lager context, as candidate Tau-functions associated to an arbitrary algebraic curve. We conjecture that the expression we propose does satisfy (formally) Hirota equation to all orders in $1/N$. We prove it to the two first non-trivial orders.

As a motivation, we recall, that it was proved in [6] that our proposal retrieves the asymptotic expansion for the $(p, 2)$ minimal model reduction of KdV. In the most general case, the matching between our construction and the asymptotic expansion of matrix models has not yet received a proof. However, in some cases like the one-hermitian matrix model with real potentials and some extra assumptions, it has been established to all orders [3, 34] in the one-cut regime (no modulation factors, pure $1/N$ expansion) in [3, 19, 34], and has been addressed up to $o(1)$ (including the modulation by a theta function) in multi-cut regimes in the works [7, 16, 24, 26].

In all examples above appeared a triple $\mathcal{S} = (\mathcal{C}, X, Y)$ consisting of a Riemann surface $\mathcal{C}$ and two functions X and Y defined on it, or some variants, like a curve and 2 meromorphic differentials $\mathrm{d}X$ and $Y\mathrm{d}X$. We call this data a *spectral curve*, and it plays a central role in this article.

There exists several, non tautologically equivalent approaches to integrable systems (see [4, 32] for reviews). In this article, we take the notion of Tau function [53, 54, 56] as a starting point. It is a function of all the times which satisfies Hirota bilinear relations [47]; Tau functions are in correspondence with solutions of the nonlinear integrable PDE's.

Our main goal is to propose that the formal series defined in Definition 5.4 as a functional on the space of spectral curves, is a Tau function. We hope that it will allow the prediction of the full asymptotic expansion (in the small dispersive parameter $1/N$) of some solutions of integrable PDE's in any genus $\mathfrak{g}$ regime, although we postpone precise comparisons to future works (see [64] for perturbation theory of Tau functions).

We just mention that the leading order of our construction retrieves the well-known asymptotic solutions of KdV in the genus 1 regime. More precisely, for such a comparison, we need to consider our construction to 1-form $Y\mathrm{d}X = x\omega_{\infty,1} + 2t\omega_{\infty,3}$ defined on the elliptic curve $\Upsilon^2 = \prod_{i=1}^{3}(X - a_i(x,t))$, where $a_i(x,t)$ satisfy Whitham type equations [81]

$$\partial_t a_i = \frac{\omega_{\infty,3}(a_i)}{\omega_{\infty,1}(a_i)}\partial_x a_i,$$

and where x (resp. t) is the time associated with the 1-form $\omega_{\infty,1}$ (resp. $2\omega_{\infty,3}$). We have used here the parametrization of meromorphic 1-forms introduced in Section 2.2, and ∞ denotes the point at infinity around which $\Upsilon^2 \sim X^3$). The phase $\zeta(x,t)$ coincides with $\zeta(\mathbf{t})$ of Section 3.1 and C a constant depending on the initial data. $\tau(x,t)$ is the time-evolved Riemann period of the elliptic curve, and E_2 the second Eisenstein series. For solutions of the KdV equation

$$u_t + uu_x + \frac{1}{N^2}u_{xxx} = 0 \tag{1.1}$$

with generic initial data, it has been proved [25, 42, 45, 66, 67, 68, 79, 80] that $u_N(x,t)$ for some time after the gradient catastrophe, $u_N(x,t)$ is asymptotic to (in a distributional sense)

$$u_N(x,t) = \frac{2\pi^2 E_2(\tau(x,t))}{3} + \frac{1}{3}\sum_{i=1}^{3} a_i(x,t) + \frac{2}{N^2}\partial_x^2 \ln\theta\big(N\zeta(x,t) + C \,|\, \tau(x,t)\big), \qquad (1.2)$$

where θ is a genus 1 Theta function. The relation between the Tau function and the solution of equation (1.1) is $u(x,t) = 2(\ln\mathcal{T})_{xx}$, and our candidate Tau function defined in Definition 5.4 indeed match in this setting the leading behavior equation (1.2).

We expect our proposal to be of interest in the study of asymptotics of solutions of hierarchies known to govern those matrix models like continuous Toda or nonlinear Schrödinger, in any finite genus regime. We also stress that matrix models are only special cases of our construction, in other words $\mathcal{T}[\mathcal{S}]$ is not in general a matrix integral. We hope that our general construction would describe the all-order asymptotics of solutions of the full dispersive hierarchies associated to Hurwitz spaces, although we do not attempt to make the comparison and do not address the hamiltonian formalism in this article.

1.2 Outline of the article

After a summary of algebraic geometry in Section 2, we review in Section 3 the reconstruction of an isospectral Lax system from its semiclassical spectral curve (which is time-independent). The techniques for this reconstruction are closely related to those developed by Krichever [62, 63] to produce the algebro-geometric solutions of the Zakharov–Shabat hierarchies [77]. We put emphasis on the Baker–Akhiezer spinor kernel $\psi_{\mathrm{cl}}(z_1, z_2)$ [59, 60], and the corresponding Tau function $\mathcal{T}_{\mathrm{cl}}(\mathbf{t})$ in Section 4. Apart from fixing notations, this review is relevant to the present work, as one can illustrate in the case of KdV where the spectral curve does not depend on the times x and t (i.e. a_i and τ are assumed constant) provides an exact solution of KdV [51, 52] which can be obtained by such a reconstruction. Whereas, if one let $a_i(x,t)$ evolve according to Whitham equations as in the second part of the paper, it also describes the leading order of a solution of KdV in the small dispersion limit and for some time after the gradient catastrophe for generic initial data.

Then, for any spectral curve $\mathcal{S}$ whose time evolution is described by Whitham equations [65, 81] (cf. Section 5.1.2), we shall define explicitly in Section 5 a functional $\mathcal{T}[\mathcal{S}]$ (Definition 5.4) as a formal asymptotic series in a small parameter N, as well as a spinor kernel $\psi(z_1, z_2)$ via a Sato-like formula (Definition 5.5), which plays the role of a Baker–Akhiezer spinor kernel. We also introduce in Section 6 the correlators $W_n(z_1, \ldots, z_n)$ (Definition 6.2), which encode the n-th order derivatives of $\mathcal{T}[\mathcal{S}]$ with respect to deformation parameters of $\mathcal{S}$. Here, $z_1, \ldots, z_n$ denotes points on $\mathcal{C}$.

The essential point in this article is the conjecture that $\mathcal{T}[\mathcal{S}]$ satisfies a certain form of Hirota equations to all orders in $1/N$ (Conjecture 7.4), and we check it holds for the two first orders (Appendix A). We present an equivalent conjecture stating that $\psi(z_1, z_2)$ is self-replicating (Conjecture 7.1). This conjecture automatically implies determinantal formulas for the correlators (Theorem 8.1), Christoffel–Darboux formula for the spinor kernel (Theorem 8.3), and a Lax system satisfied by the matrix $\Psi(x_1, x_2) = [\psi(z^i(x_1), z^j(x_2))]_{i,j}$, where $z^i(x) \in \mathcal{C}$ are the points such that $X(z^i(x)) = x$ (Section 8.6).

The coefficient of the so-obtained Lax matrices can be computed in principle order by order in $1/N$. If our conjecture is correct, our approach describes directly a Tau function, but we do not identify the underlying nonlinear hierarchy of equations. The situation is similar to the one evoked in [33], where the dispersive hierarchy is constructed perturbatively in $1/N$, but its resummation for finite N is unknown – except in special cases.
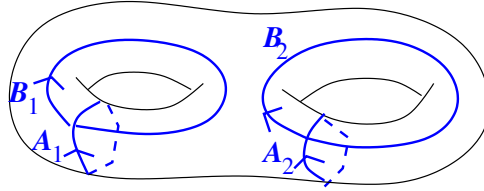
Figure 1. A symplectic basis of $2\mathfrak{g}$ non-contractible cycles on a Riemann surface of genus $\mathfrak{g}$.

Since our approach was strongly motivated by earlier results or heuristics in hermitian matrix models, we recapitulate their relation to the present work in Section 9.

2   Geometry of the spectral curve

We briefly describe some geometric notions attached to a fixed spectral curve $\mathcal{S} = (\mathcal{C}, X, Y)$ [31, 40, 41]. To simplify, we assume in this article that $\mathcal{C}$ is a compact Riemann surface of genus $\mathfrak{g}$, and X and Y are meromorphic functions on $\mathcal{C}$.

2.1   Some notations and properties

2.1.1   Topology and holomorphic 1-forms

The curve $\mathcal{C}$ is either simply connected, and then this is the Riemann sphere $\mathcal{C} = \mathbb{C} \cup \{\infty\}$, or it has genus $\mathfrak{g} > 0$. Then, any maximal open contractible subset of $\mathcal{C}$ is called a fundamental domain. If it is of genus $\mathfrak{g} > 0$, there exist $2\mathfrak{g}$ independent non-contractible cycles (see Fig. 1), and we can choose them in such a way (but not unique) that

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}, \qquad \mathcal{A}_i \cap \mathcal{A}_j = 0, \qquad \mathcal{B}_i \cap \mathcal{B}_j = 0.$$

A basis satisfying these intersection relations is called "symplectic".

From the topological point of view, a genus $\mathfrak{g} > 0$ compact Riemann surface with a basis $(\mathcal{A}_i, \mathcal{B}_i)_{1 \leq i \leq \mathfrak{g}}$ is a $4\mathfrak{g}$ closed polygon Γ, with edges

$$\left[\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_1^{-1}, \mathcal{B}_1^{-1}, \ldots, \mathcal{A}_\mathfrak{g}, \mathcal{B}_\mathfrak{g}, \mathcal{A}_\mathfrak{g}^{-1}, \mathcal{B}_\mathfrak{g}^{-1} \right]$$

glued by pairs. $\mathring{\Gamma}$ is a fundamental domain of $\mathcal{C}$. It is a classical result, that on a curve of genus $\mathfrak{g}$, there exists $\mathfrak{g}$ independent holomorphic 1-forms $\mathrm{d}u_i$ (holomorphic means in particular having no poles), and they can be normalized on the $\mathcal{A}$-cycles

$$\oint_{\mathcal{A}_i} \mathrm{d}u_j = \delta_{i,j}.$$

Then, the $\mathfrak{g} \times \mathfrak{g}$ matrix $\tau_{i,j}$,

$$\tau_{i,j} = \oint_{\mathcal{B}_i} \mathrm{d}u_j,$$

is known to be symmetric $\tau_{i,j} = \tau_{j,i}$ and its imaginary part is definite positive

$$\tau^t = \tau, \qquad \mathrm{Im}\,\tau > 0.$$

τ is called the Riemann matrix of periods of $\mathcal{C}$.

2.1.2 Theta functions

Given any symmetric matrix τ such that $\operatorname{Im}\tau > 0$, one can define the Riemann Theta function

$$\theta(\mathbf{u}|\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^{\mathfrak{g}}} e^{2i\pi\mathbf{n}\cdot\mathbf{u}} e^{i\pi\mathbf{n}^t\cdot\tau\cdot\mathbf{n}}.$$

Since $\operatorname{Im}\tau > 0$, it is a well-defined convergent series for all $\mathbf{u}$ in $\mathbb{C}^{\mathfrak{g}}$. Most often we will not write the τ dependence of the Theta function: $\theta(\mathbf{u}|\tau) \equiv \theta(\mathbf{u})$. This function is quasi-periodic in $\mathbf{u}$: if $\mathbf{n},\mathbf{m} \in \mathbb{Z}^{\mathfrak{g}}$, we have

$$\theta(\mathbf{u} + \mathbf{n} + \tau\mathbf{m}) = e^{-i\pi(2\mathbf{m}^t\cdot\mathbf{u} + \mathbf{m}^t\cdot\tau\cdot\mathbf{m})}\theta(\mathbf{u}). \tag{2.1}$$

It also satisfies the heat equation

$$\partial_{\tau_{i,j}}\theta = \frac{1}{4i\pi}\partial_{u_i}\partial_{u_j}\theta.$$

In this equation, $\tau_{i,j}$ and $\tau_{j,i}$ are considered independent.

2.1.3 Jacobian and Abel map

Let us choose a generic basepoint $o \in \mathcal{C}$ (it will in fact play no role). For any point $z \in \mathcal{C}$, we define

$$\forall i \in \{1,\ldots,\mathfrak{g}\}, \qquad u_i(z) = \int_o^z \mathrm{d}u_i,$$

where the integration path is chosen such that it does not intersect any $\mathcal{A}$-cycle or $\mathcal{B}$-cycle. Then we define the vector

$$\mathbf{u}(z) = [u_i(z)]_{1\le i\le \mathfrak{g}} \in \mathbb{C}^{\mathfrak{g}}.$$

The application $z \mapsto \mathbf{u}(z) \bmod (\mathbb{Z}^{\mathfrak{g}} + \tau\cdot\mathbb{Z}^{\mathfrak{g}})$ is well-defined and analytical, it maps the spectral curve into the Jacobian $\mathbb{J} = \mathbb{C}^{\mathfrak{g}}/(\mathbb{Z}^{\mathfrak{g}} + \tau\cdot\mathbb{Z}^{\mathfrak{g}})$. This defines the Abel map

$$\mathcal{C} \to \mathbb{J},$$
$$z \mapsto \mathbf{u}(z) \bmod (\mathbb{Z}^{\mathfrak{g}} \oplus \tau\cdot\mathbb{Z}^{\mathfrak{g}}).$$

The Jacobi inversion theorem states that every $\mathbf{w} \in \mathbb{J}$ can be represented as $\mathbf{w} = \sum_{j=1}^{\mathfrak{g}} \mathbf{u}(p_j)$ for some points $p_1,\ldots,p_{\mathfrak{g}} \in \mathcal{C}$.

The Theta function can be used with τ the Riemann matrix of periods of a Riemann surface $\mathcal{C}$, and $\mathbf{u}$ the Abel map of a point on $\mathcal{C}$. In this case, it enjoys other important properties. Its zero locus has the following description: there exists $\mathbf{k} \in \mathbb{C}^{\mathfrak{g}}$, so that $\theta(\mathbf{w}|\tau) = 0$ iff there exists $\mathfrak{g} - 1$ points $z_1,\ldots,z_{\mathfrak{g}-1} \in \mathcal{C}$ satisfying $\mathbf{w} = \sum_{j=1}^{\mathfrak{g}-1} \mathbf{u}(z_j) + \mathbf{k}$. $\mathbf{k}$ is called a "Riemann vector of constants", and it depends on the basepoint o used to define the Abel map $\mathbf{u}$.

2.1.4 Prime form

An odd characteristics $\mathbf{c}$ is a vector of the form

$$\mathbf{c} = \frac{\mathbf{n} + \tau\mathbf{m}}{2}, \qquad \mathbf{n},\mathbf{m} \in \mathbb{Z}^{\mathfrak{g}}, \qquad \mathbf{n}^t\cdot\mathbf{m} \in 2\mathbb{Z} + 1.$$

The Theta function vanishes at odd characteristics: $\theta(\mathbf{c}) = 0$, and the following holomorphic form

$$\mathrm{d}h_{\mathbf{c}}(z) = \sum_{i=1}^{\mathfrak{g}} \mathrm{d}u_i(z)\partial_{u_i}\theta(\mathbf{c})$$

has only double zeroes on $\mathcal{C}$, so that we can define its squareroot, and thus one can define the prime form [71, 72, 73]

$$E(z_1, z_2) = \frac{\theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \mathbf{c})}{\sqrt{\mathrm{d}h_{\mathbf{c}}(z_1)\mathrm{d}h_{\mathbf{c}}(z_2)}}.$$

There exists choices of $\mathbf{c}$ such that E is not identically 0 (we say $\mathbf{c}$ is "non singular"), and E is in fact independent of such $\mathbf{c}$. It is a $(-\frac{1}{2}, -\frac{1}{2})$-form on $\mathcal{C} \times \mathcal{C}$, and it vanishes only at $z_1 = z_2$. In any local coordinate $\xi(z)$ we have

$$E(z_1, z_2) \underset{z_1 \to z_2}{=} \frac{\xi(z_1) - \xi(z_2)}{\sqrt{\mathrm{d}\xi(z_1)\mathrm{d}\xi(z_2)}} + O\big((\xi(z_1) - \xi(z_2))^3\big).$$

Because of the Theta function, $E(z_1, z_2)$ is multivalued $\mathcal{C} \times \mathcal{C}$. It transforms according to equation (2.1).

The Theta function associated to a Riemann surface satisfies a non-linear relation called Fay identity [41]: for any $z_1, z_2, z_3, z_4 \in \mathcal{C}$, any $\mathbf{w} \in \mathbb{C}^{\mathfrak{g}}$,

$$\theta(\mathbf{w} + \mathbf{c})\theta(\mathbf{u}_{12} + \mathbf{u}_{34} + \mathbf{w} + \mathbf{c})\frac{E(z_1, z_3)E(z_2, z_4)}{E(z_1, z_4)E(z_2, z_3)}\frac{1}{E(z_1, z_2)E(z_3, z_4)}$$
$$= \frac{\theta(\mathbf{w} + \mathbf{u}_{12} + \mathbf{c})}{E(z_1, z_2)}\frac{\theta(\mathbf{w} + \mathbf{u}_{34} + \mathbf{c})}{E(z_3, z_4)} - \frac{\theta(\mathbf{w} + \mathbf{u}_{14} + \mathbf{c})}{E(z_1, z_4)}\frac{\theta(\mathbf{w} + \mathbf{u}_{32} + \mathbf{c})}{E(z_3, z_2)},$$

where $\mathbf{u}_{jl} = \mathbf{u}(z_j) - \mathbf{u}(z_l)$.

2.1.5 Bergman kernel

We call Bergman kernel the "fundamental (1,1)-form of the second kind" [41], defined as

$$B(z_1, z_2) = \mathrm{d}_{z_1}\mathrm{d}_{z_2}\ln\left(\theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \mathbf{c})\right).$$

It is independent of the choice of a non-singular, odd characteristics $\mathbf{c}$. It is a globally defined, symmetric (1,1)-form, having a double pole at $z_1 = z_2$ with no residue, and no other pole. It is normalized so that

$$\oint_{\mathcal{A}_i} B(\cdot, z) = 0, \qquad \oint_{\mathcal{B}_i} B(\cdot, z) = 2\mathrm{i}\pi \, \mathrm{d}u_i(z).$$

Near $z_1 = z_2$, it behaves, in any local coordinate $\xi(z)$, like

$$B(z_1, z_2) \underset{z_1 \to z_2}{=} \frac{\mathrm{d}\xi(z_1)\mathrm{d}\xi(z_2)}{(\xi(z_1) - \xi(z_2))^2} + O(1).$$

We also define the fundamental 1-form of the third kind

$$\mathrm{d}S_{z_1, z_2}(z) = \int_{z_2}^{z_1} B(\cdot, z),$$

where the integration contour is chosen so that it does not intersect any $\mathcal{A}$-cycle or $\mathcal{B}$-cycle. It is a 1-form in the variable z, and a function of the variable z_1, z_2, and it satisfies

$$\oint_{\mathcal{A}_j} dS_{z_1,z_2} = 0, \qquad \oint_{\mathcal{B}_j} dS_{z_1,z_2} = 2i\pi(u_j(z_1) - u_j(z_2)).$$

It has a simple pole at $z = z_1$ with residue $+1$, a simple pole at $z = z_2$ with residue -1, and no other pole. In other words, in any local coordinate $\xi(z)$

$$dS_{z_1,z_2}(z) \underset{z\to z_1}{\sim} \frac{d\xi(z)}{\xi(z) - \xi(z_1)}, \qquad dS_{z_1,z_2}(z) \underset{z\to z_2}{\sim} \frac{-d\xi(z)}{\xi(z) - \xi(z_2)}.$$

Notice that in the variable z it is globally defined for $z \in \mathcal{C}$ (it has no monodromy if z goes around a non-contractible cycle), whereas in the variable z_1 (resp. z_2) it is defined only on the fundamental domain, it has monodromies when z_1 (resp. z_2) goes around a non-contractible cycle $\mathcal{B}_j$

$$dS_{z_1+\mathcal{B}_j,z_2}(z) = dS_{z_1,z_2}(z) + 2i\pi du_j(z), \qquad dS_{z_1,z_2+\mathcal{B}_j}(z) = dS_{z_1,z_2}(z) - 2i\pi du_j(z).$$

2.1.6 Example in genus $\mathfrak{g} = 1$

When $\mathfrak{g} = 1$, the Abel map is an isomorphism between $\mathcal{C}$ and $\mathbb{J} = \mathbb{C}/\mathbb{L}$ where we set $\mathbb{L} = \mathbb{Z} + \tau\mathbb{Z}$. The $\mathcal{A}$-cycle in $\mathbb{J}$ is the segment $[0,1[$, and the $\mathcal{B}$-cycle is the segment $[0,\tau[$. The Bergman kernel normalized on $\mathcal{A}$-cycles can be expressed as

$$B(u_1, u_2) = du_1 du_2 \left(\wp(u_1 - u_2|\tau) + \frac{\pi^2 E_2(\tau)}{3} \right),$$

where $u_1, u_2 \in \mathbb{J}$, $\wp$ is the Weierstrass function and E_2 the second Eisenstein series

$$\wp(u|\tau) = \frac{1}{u^2} + \sum_{w \in \mathbb{L}\setminus\{0\}} \left(\frac{1}{(u+w)^2} - \frac{1}{w^2} \right),$$

$$E_2(\tau) = \frac{3}{\pi^2} \left(\sum_{n \neq 0} \frac{1}{n^2} + \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(n+m\tau)^2} \right).$$

2.2 Parametrization of meromorphic 1-forms

2.2.1 Sheets, ramification and branchpoints, local coordinate patches

If $\deg X = d$, then for every value x, there are d points $z^1(x), \ldots, z^d(x)$ on the curve $\mathcal{C}$ such that $X(z^i(x)) = x$. $z^i(x)$ is sometimes called the preimage of x in the i-th sheet.

Definition 2.1. We call "ramification points of order k", the zeroes of order $k \geq 1$ of the meromorphic 1-form dX. If $a \in \mathcal{C}$ is a ramification point, the corresponding value $X(a)$ is called a branchpoint. All the other points $z \in \mathcal{C}$ at which $X(z)$ is analytical, are called "regular points".

Definition 2.2. We say that a branchpoint x_a is simple if $X^{-1}(\{x_a\})$ consists in $d-1$ points, one of them being a ramification point of order 1, and all the remaining ones being regular points.

2.2.2 Definition of local coordinates

Near a ramification point a of order k, $\xi_a(z) = (X - X(a))^{1/(k+1)}$ defines a local coordinate on the curve. Simple branchpoints play a special role in Sections 3.6, 5.1 and 6.1. For a simple branchpoint we have

$$\xi_a(z) = \sqrt{X(z) - X(a)}.$$

Since X is a meromorphic function of degree d, it has d poles with multiplicities, i.e. $\infty_1^{d_{\infty_1}}, \ldots,$ $\infty_s^{d_{\infty_s}}$ with $\sum_i d_{\infty_i} = d$. Near ∞_i, a good local variable is

$$\xi_{\infty_i}(z) = X(z)^{-1/d_{\infty_i}}.$$

Besides, we will need to consider also poles of a meromorphic form ω. If p is a pole of ω, but not a pole of X, neither a zero of dX, a good local variable is

$$\xi_p(z) = X(z) - X(p).$$

In this case, the multiplicity of p is $d_p = -1$. We shall now always use the local coordinates $\xi(z)$ defined above. Notice that they depend only on the function $X(z)$.

Definition 2.3. Given a meromorphic 1-form $\omega(z)$ which has no pole at ramification points, let us call

$$\mathcal{P} = \{\text{poles of } \omega\}, \qquad \mathcal{P}_\infty = \{\text{poles of } X\}, \qquad \overline{\mathcal{P}} = \mathcal{P} \cup \mathcal{P}_\infty.$$

To any $p \in \overline{\mathcal{P}}$, we have associated a coordinate patch ξ_p on $\mathcal{C}$ centered on p.

2.2.3 Poles and times, filling fractions

Following Krichever [65], we define

Definition 2.4. For any $p \in \overline{\mathcal{P}}$, we define the "times" near p as the coefficient of the negative part of the Laurent series expansion of ω near p

$$\omega(z) \underset{z \to p}{=} \sum_{j \geq 0} t_{p,j} \left(\xi_p(z)\right)^{-(j+1)} d\xi_p(z) + O(1), \qquad t_{p,j} = \operatorname*{Res}_{z \to p} \omega(z)\xi_p(z)^j.$$

We also write collectively $\vec{t}_p = [t_{p,j}]_{j\in\mathbb{N}}$ and $\vec{t} = (\vec{t}_p)_{p\in\overline{\mathcal{P}}}$. We also define the "filling fractions" (also called "conserved quantities"), associated to non-contractible cycles, by

$$\epsilon_i = \frac{1}{2i\pi} \oint_{\mathcal{A}_i} \omega.$$

Notice that the times $t_{p,0} = \operatorname{Res}_p \omega$ are not independent, because the sum of residues of ω must vanish,

$$\sum_{p\in\overline{\mathcal{P}}} t_{p,0} = 0.$$

There is a form-cycle duality [65]

Definition 2.5. To each time t_k, one can associate a differential meromorphic form $\omega_k(z)$, as well as a dual cycle ω_k^*, and a dual orthogonal cycle $\omega_k^{*\perp}$

$$t_k \longleftrightarrow \omega_k(z) \longleftrightarrow \omega_k^* \longleftrightarrow \omega_k^{*\perp}, \tag{2.2}$$

in such a way that

$$\left.\frac{\partial\omega(z)}{\partial t_k}\right|_{X(z)} = \omega_k(z), \qquad \omega(z) = \sum_k t_k\omega_k(z), \tag{2.3}$$

$$\omega_k(z) = \int_{\omega_k^*} B(\cdot, z), \qquad t_k = \int_{\omega_k^{*\perp}} \omega, \qquad \omega_i^* \cap \omega_j^{*\perp} = \delta_{i,j}.$$

The symbol $\big|_{X(z)}$ means that we differentiate keeping the local coordinates $\xi_p(z)$ fixed (i.e. $X(z)$ fixed).

More explicitly we have

- Filling fractions $\epsilon_i \longrightarrow \omega_j = $ first kind differential

$$\omega_j(z) = 2\mathrm{i}\pi \mathrm{d}u_j(z) = \oint_{\mathcal{B}_j} B(z, \cdot), \qquad \omega_j^* = \mathcal{B}_j, \qquad \omega_j^{*\perp} = \frac{1}{2\mathrm{i}\pi}\mathcal{A}_j.$$

- Residues $t_{p,0} \longrightarrow \omega_{p,0} = $ third kind differential

$$\omega_{p,0}(z) = \mathrm{d}S_{p,o}(z) = \int_o^p B(z, \cdot), \qquad \omega_{p,0}^* = [o, p], \qquad \omega_{p,0}^{*\perp} = \frac{1}{2\mathrm{i}\pi}C_p,$$

where o is an arbitrary basepoint on $\mathcal{C}$, and C_p is a small circle surrounding p with index 1. As we mentioned, the $t_{p,0}$ are not independent variables, and only $(t_{p,0} - t_{p0,0})_{p\neq p_0}$ for a fixed p_0 are independent. As a consequence, we see that only differences $\omega_{p,0} - \omega_{p',0}$ and $\omega_{p,0}^* - \omega_{p',0}^*$ are independent of a choice of basepoint o.

- Higher times $t_{p,j}$ with $j \geq 1 \longrightarrow \omega_{p,j} = $ second kind differential

$$\omega_{p,j}(z) = B_{p,j}(z) = \operatorname*{Res}_{z'\to p} \xi_p(z')^{-j}B(z', z),$$

$$\omega_{p,j}^* = \frac{1}{2\mathrm{i}\pi}\xi_p^{-j}C_p, \qquad \omega_{p,j}^{*\perp} = \frac{1}{2\mathrm{i}\pi}\xi_p^{j+1}C_p.$$

Any meromorphic form ω is a linear combination of those basis meromorphic forms, and almost by definition we have

$$\omega(z) = \sum_k t_k\omega_k(z) = \sum_{i=1}^{\mathfrak{g}} 2\mathrm{i}\pi\epsilon_i\mathrm{d}u_i(z) + \sum_{p\in\overline{\mathcal{P}}} t_{p,0}\mathrm{d}S_{p,o}(z) + \sum_{p\in\overline{\mathcal{P}}, j\geq 1} t_{p,j} B_{p,j}(z). \tag{2.4}$$

2.3 F_0

The fact that $\int_{\omega_i^*}\int_{\omega_j^*} B(z, z')$ is symmetric, implies that there exists a function $F_0(\vec{t})$ such that

$$\frac{\partial F_0}{\partial t_i} \text{ "=" } \int_{\omega_i^*} \omega, \qquad \frac{\partial^2 F_0}{\partial t_i\partial t_j} \text{ "=" } \int_{\omega_i^*}\int_{\omega_j^*} B.$$

The problem (this is why we write quotation marks) is that those integrals are not well-defined for times associated to 3rd kind differentials. Such a statement is correct after an appropriate regularization. When z is in the vicinity of a pole p, we define

$$V_p(z) = -\sum_{j\geq 1}\frac{t_{p,j}}{j}\xi_p(z)^{-j}, \qquad \mathrm{d}V_p(z) = \sum_{j\geq 1}t_{p,j}\frac{\mathrm{d}\xi_p(z)}{\xi_p(z)^{j+1}}.$$

It is such that $\omega - \mathrm{d}V_p$ has at most a simple pole at p. Given an arbitrary base point $o \in \mathcal{C}$, the following integral is well-defined

$$\mu_p = \int_o^{\infty_p}\left(\omega(z) - \mathrm{d}V_p(z) - t_{p,0}\frac{\mathrm{d}\xi_p(z)}{\xi_p(z)}\right) - V_p(o) - t_{p,0}\ln\xi_p(o).$$

μ_p depends on the base point o, but only by an additive constant independent of p. Since $\sum_p t_{p,0} = 0$, the sum $\sum_p t_{p,0}\mu_p$ is thus independent of o. In some sense, μ_p is a regularized version of $\int_{\omega_{p,0}^*}\omega$ (which does not exists). Since for all the other cycles, $\int_{\omega_k^*}\omega$ is well-defined, we can now define F_0

Definition 2.6.

$$F_0(\omega) = \frac{1}{2}\left[\sum_{p\in\overline{\mathcal{P}}}\operatorname*{Res}_p V_p\omega + \sum_{p\in\overline{\mathcal{P}}}t_{p,0}\mu_p + \sum_{i=1}^{\mathfrak{g}}\epsilon_i\oint_{\mathcal{B}_i}\omega\right].$$

This definition is closely related to that of [65] where F_0 appears as a function of the times $t_{p,j}$'s, but here we prefer to define it as a functional of a 1-form ω.

Theorem 2.1 (see e.g. [65]). *The first derivatives of F_0 are given by, for $j \geq 1$,*

$$\frac{\partial F_0}{\partial t_{p,j}} = \oint_{\omega_{p,j}^*}\omega = \operatorname*{Res}_p\xi_p^{-j}\omega, \qquad \frac{\partial F_0}{\partial t_{p,0}} - \frac{\partial F_0}{\partial t_{p',0}} = \mu_p - \mu_{p'}, \qquad \frac{\partial F_0}{\partial\epsilon_i} = \oint_{\mathcal{B}_i}\omega.$$

The proof of this theorem has appeared in many works and contexts, initiated in [30] and generalized in [65]. In the context of Hurwitz spaces, this expression of F_0 specialized to $\omega = $ the primary differential defining the Frobenius structure, coincides with the prepotential [30, Equation (5.64)]. It follows form Theorem 2.1 that

$$F_0 = \frac{1}{2}\sum_k t_k\frac{\partial F_0}{\partial t_k},$$

which means that F_0 is homogeneous of degree 2. Another classical result is

Theorem 2.2 (see e.g. [65]). *The second derivatives of F_0 are given by*

$$\frac{\partial^2 F_0}{\partial t_k\partial t_l} = \int_{\omega_k^*}\int_{\omega_l^*}B,$$

except for the following cases

$$\frac{\partial}{\partial t_k}\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p',0}}\right)F_0 = \int_{\omega_k^*}\int_{\omega_{p,0}^*}B - \int_{\omega_k^*}\int_{\omega_{p',0}^*}B,$$

$$\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p',0}}\right)^2 F_0 = -\ln\left(E(p,p')^2\mathrm{d}\xi_p(p)\mathrm{d}\xi_{p'}(p')\right),$$

$$\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p',0}}\right)\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p'',0}}\right)F_0 = -\ln\left(\frac{E(p,p')E(p,p'')\mathrm{d}\xi_p(p)}{E(p',p'')}\right),$$

$$\left(\frac{\partial}{\partial t_{p,0}} - \frac{\partial}{\partial t_{p',0}}\right)\left(\frac{\partial}{\partial t_{\tilde{p},0}} - \frac{\partial}{\partial t_{\tilde{p}',0}}\right)F_0 = -\ln\left(\frac{E(p,\tilde{p}')E(p',\tilde{p})}{E(p,\tilde{p})E(p',\tilde{p}')}\right).$$

The second derivatives of F_0 do not depend on the 1-form ω, and thus do not depend on the times. Thus we have

$$F_0 = \frac{1}{2} \sum_{k,l} t_k t_l \frac{\partial^2 F_0}{\partial t_k \, \partial t_l}.$$

Theorem 2.3 (see e.g. [65])**.**

$$\frac{\partial^3 F_0}{\partial t_k \partial t_l \partial t_m} = \sum_{a_i = \text{zeroes of } dX} \operatorname*{Res}_{z \to a_i} \frac{\omega_k(z)\omega_l(z)\omega_m(z)}{dX(z)dY(z)}.$$

3 Reconstruction formula

We review in this section the reconstruction [62, 63] of a Lax matrix whose evolution preserves the spectrum, and thus of an integrable system, from the spectral curve (see also the textbook [4] and references therein). The only difference is that, we reformulate it intrinsically in terms of 1-forms ω, instead of using time coordinates $\omega = \sum_k t_k \omega_k$. For this purpose, instead of Baker–Akhiezer functions, we prefer to use a "spinor kernel", which is a convenient special case of Baker–Akhiezer function, which turns out to be a more intrinsic object for our formulation (see also [13, 59, 60]).

3.1 Semiclassical spinor kernel

Given a meromorphic 1-form ω, define the 1-form

$$\chi(z;\omega) = \omega(z) - 2\mathrm{i}\pi \sum_{i=1}^{\mathfrak{g}} \epsilon_i \mathrm{d}u_i(z),$$

which depends linearly on the times (and not on the filling fractions)

$$\chi(z;\omega) = \sum_{p\in\overline{\mathcal{P}}} t_{p,0} \mathrm{d}S_{p,o}(z) + \sum_{p\in\overline{\mathcal{P}}} \sum_{j\geq1} t_{p,j}\omega_{p,j}(z).$$

By construction χ is normalized on $\mathcal{A}$-cycles

$$\oint_{\mathcal{A}_i} \chi = 0.$$

Then we define the vector $\zeta(\omega) = [\zeta_i(\omega)]_{1\leq i\leq\mathfrak{g}}$ with coordinates

$$\zeta_i(\omega) = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}_i} \chi = \frac{1}{2\mathrm{i}\pi} \left(\oint_{\mathcal{B}_i} \omega - \sum_{j=1}^{\mathfrak{g}} \tau_{i,j} \oint_{\mathcal{A}_i} \omega \right), \tag{3.1}$$

which we write for short as

$$\zeta(\omega) = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}-\tau\mathcal{A}} \omega.$$

It can be decomposed as

$$\zeta(\omega) = \sum_{p\in\overline{\mathcal{P}}} \sum_{j\geq0} t_{p,j}\mathbf{v}_{p,j} = \sum_{k=(p,j)} t_k\mathbf{v}_k, \qquad \mathbf{v}_k = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}} \omega_k.$$

The vector $\zeta(\omega)$ is a linear function of the times t_k and is independent of the filling fractions ϵ_i. In other words, it follows a linear motion with constant velocity $\mathbf{v}_k$ in the Jacobian, as a function of any of the times t_k. A well-known property [4, 31, 62, 63] of integrable systems is that, in appropriate variables, the motion (with any of the time t_k) is uniform and linear. The algebraic reconstruction takes the linear evolution in the Jacobian of $\mathcal{C}$ as starting point, and produces more complicated quantities whose evolution is described by a Lax system.

Definition 3.1. We now define the *semiclassical spinor kernel* as the $(1/2, 1/2)$ form

$$\psi_{\mathrm{cl}}(z_1, z_2; \omega) = \frac{\theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta(\omega) + \mathbf{c})}{E(z_1, z_2)} \theta(\zeta(\omega) + \mathbf{c}) e^{\int_{z_2}^{z_1} \chi(z; \omega)}, \tag{3.2}$$

where $\mathbf{c}$ is a non-singular, odd characteristics.

We write a subscript $_{\mathrm{cl}}$ to distinguish the semiclassical spinor kernel from the one proposed in the second part of the article. This kernel was also introduced, in a similar form, in [59, 60] for solving Matrix Riemann–Hilbert problems on branched coverings of $\mathbb{CP}^1$.

Proposition 3.1. $\psi_{\mathrm{cl}}(z_1, z_2; \omega)$ *is a globally defined spinor in* $(z_1, z_2) \in \mathcal{C} \times \mathcal{C}$, *i.e. it is the squareroot of a symmetric* $(1, 1)$-*form.*

- *It has a simple pole at* $z_1 = z_2$: *in any local coordinate* $\xi(z)$

$$\psi_{\mathrm{cl}}(z_1, z_2; \omega) \underset{z_1 \to z_2}{\sim} \frac{1}{E(z_1, z_2)} \sim \frac{\sqrt{\mathrm{d}\xi(z_1)\mathrm{d}\xi(z_2)}}{\xi(z_1) - \xi(z_2)}.$$

- *It has essential singularities when* z_1 *(resp.* z_2) *approaches a pole of* ω.

Proof. The behavior at $z_1 \to z_2$ is obvious, and the essential singularities at the poles of ω come from the exponential term. What we need to prove, is that $\psi_{\mathrm{cl}}(z_1, z_2; \omega)$ is unchanged when z_1 (resp. z_2) goes around a non-trivial cycle. When z_1 (resp. z_2) goes around an $\mathcal{A}$-cycle, the vector $\mathbf{u}(z_1)$ (resp. $\mathbf{u}(z_2)$) is translated by an integer vector, θ is thus unchanged, and thanks to equation (3.1), ψ_{cl} is unchanged when z_1 (resp. z_2) goes around an $\mathcal{A}$-cycle. When z_1 (resp. z_2) goes around a $\mathcal{B}$-cycle, the vector $\mathbf{u}(z_1)$ (resp. $\mathbf{u}(z_2)$) is translated by a lattice vector of the form $\tau \cdot \mathbf{n}$ with $\mathbf{n} \in \mathbb{Z}^{\mathfrak{g}}$, and θ gets multiplied by a phase according to equation (2.1). Remember that the prime form $E(z_1, z_2)$ is also a θ function, and also gets a phase given by equation (2.1). ψ_{cl} is thus changed by

$$\psi_{\mathrm{cl}}(z_1 + \mathbf{n}\mathcal{B}, z_2; \omega) \to \psi_{\mathrm{cl}}(z_1, z_2; \omega) e^{-2\mathrm{i}\pi\mathbf{n}\cdot\zeta(\omega)} e^{\mathbf{n}\cdot\oint_{\mathcal{B}}\chi},$$

and because of equation (3.1), i.e. $\zeta = \frac{1}{2\mathrm{i}\pi}\oint_{\mathcal{B}}\chi$, we see that ψ_{cl} is unchanged when z_1 (resp. z_2) goes around a $\mathcal{B}$-cycle. ∎

3.2 Duality equation

Then we construct the following spinor matrix of size $d \times d$

$$\Psi_{\mathrm{cl}}(x_1, x_2; \omega) = [\psi_{\mathrm{cl}}(z^i(x_1), z^j(x_2); \omega)]_{i,j=1}^d,$$

where we recall that $z^i(x)$ are the d preimages of x on the curve $\mathcal{C}$, i.e. $X(z^i(x)) = x$, and $d = \deg X$. These preimages are distinct and this matrix is well-defined when x_1 (or x_2) is not at a branchpoint.

Proposition 3.2. *We have the "duality" equation*

$$\Psi_{\mathrm{cl}}(x_1, x_2; \omega)\, \Psi_{\mathrm{cl}}(x_2, x_3; \omega) = \frac{(x_1 - x_3)\mathrm{d}x_2}{(x_1 - x_2)(x_2 - x_3)} \Psi_{\mathrm{cl}}(x_1, x_3; \omega).$$

Proof.

$$\frac{1}{\mathrm{d}X(z)} \psi_{\mathrm{cl}}(z^i(x_1), z; \omega)\psi_{\mathrm{cl}}(z, z^j(x_3); \omega)$$

is a meromorphic function of z. Indeed, the product of two $(1/2)$-forms is a 1-form, and when we divide by $\mathrm{d}X$, we get a function. The essential singularities coming from the exponentials cancel in the product, so this function can only have poles, i.e. it is meromorphic. The only possible poles are at $z = z^i(x_1)$ or $z = z^j(x_3)$ or at the zeroes of $\mathrm{d}X(z)$. Then, summing over all sheets, we see that

$$\sum_k \frac{\psi_{\mathrm{cl}}(z^i(x_1), z^k(x_2); \omega)\psi_{\mathrm{cl}}(z^k(x_2), z^j(x_3); \omega)}{\mathrm{d}X(z^k(x_2))}$$

is a symmetric sum of a meromorphic function over all sheets of x_2, therefore it is a meromorphic function of $x_2 \in \widehat{\mathbb{C}}$, i.e. a rational function of the complex variable x_2. It remains to find its poles. $1/\mathrm{d}X(z^k(x_2)))$ behaves like $O(x_2 - X(a_i))^{-1/2}$ at ramification points, and since a rational function of x_2 cannot have a singularity of power $-1/2$, this means that this rational function has no pole at branchpoints. Its only poles can then be at $x_2 = x_1$ or $x_2 = x_3$, and they are simple poles. The residues of the corresponding poles are easily computed and give the theorem. ∎

Proposition 3.3. *We have a refined version of the duality equation*

$$\psi_{\mathrm{cl}}(z_1, z; \omega)\psi_{\mathrm{cl}}(z, z_2; \omega) = -\psi_{\mathrm{cl}}(z_1, z_2; \omega)\left(\mathrm{d}S_{z_1, z_2}(z) - 2\mathrm{i}\pi \sum_{j=1}^{\mathfrak{g}} \alpha_j(z_1, z_2; \omega)\mathrm{d}u_j(z)\right),$$

where

$$\alpha_j(z_1, z_2; \omega) = \frac{\theta_{u_j}(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta(\omega) + \mathbf{c})}{\theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta(\omega) + \mathbf{c})} - \frac{\theta_{u_j}(\zeta(\omega) + \mathbf{c})}{\theta(\zeta(\omega) + \mathbf{c})}.$$

This property, can be viewed as a special case of an "addition formula" for Baker–Akhiezer functions, found in [20]. Notice that Proposition 3.2 is a corollary of Proposition 3.3. Indeed the duality equation (Proposition 3.2) can be obtained by summing the equation above on all sheets $z = z^k(x)$, because $\sum_k \mathrm{d}u_i(z^k(x)) = 0$ and

$$\sum_k \mathrm{d}S_{z_1, z_2}(z^k(x)) = \frac{(X(z_1) - X(z_2))\mathrm{d}X(z)}{(X(z) - X(z_1))(X(z) - X(z_2))}.$$

Proof. Notice that $\psi_{\mathrm{cl}}(z_1, z; \omega)\psi_{\mathrm{cl}}(z, z_2; \omega)$ is a meromorphic 1-form in z, since it has no essential singularity. It has simple poles at $z = z_1$ and $z = z_2$, with residues $\mp\psi_{\mathrm{cl}}(z_1, z_2; \omega)$, and it has no other pole. This means that $\psi_{\mathrm{cl}}(z_1, z; \omega)\psi_{\mathrm{cl}}(z, z_2; \omega) + \psi_{\mathrm{cl}}(z_1, z_2; \omega)\mathrm{d}S_{z_1, z_2}(z)$ is a holomorphic 1-form, with no poles, therefore it must be a linear combination of the $\mathrm{d}u_i(z)$'s, which we choose to write

$$\psi_{\mathrm{cl}}(z_1, z; \omega)\psi_{\mathrm{cl}}(z, z_2; \vec{t}) = -\psi_{\mathrm{cl}}(z_1, z_2; \omega)\left(\mathrm{d}S_{z_1, z_2}(z) - 2\mathrm{i}\pi \sum_{j=1}^{\mathfrak{g}} \alpha_j(z_1, z_2; \omega)\mathrm{d}u_j(z)\right).$$

The left hand side is a well-defined spinor of z_1 and z_2, whereas in the right hand side, $dS_{z_1,z_2}(z) = \int_{z_2}^{z_1} B(z,\cdot)$ gets some shifts when z_1 or z_2 go around non-trivial cycles. This implies the following relations for the coefficients $\alpha_j(z_1, z_2; \omega)$

$$\alpha_j(z_1 + \mathcal{A}_k, z_2; \omega) = \alpha_j(z_1, z_2; \omega), \qquad \alpha_j(z_1, z_2 + \mathcal{A}_k; \omega) = \alpha_i(z_1, z_2; \omega),$$
$$\alpha_j(z_1 + \mathcal{B}_k, z_2; \omega) = \alpha_j(z_1, z_2; \omega) - 2\mathrm{i}\pi\delta_{j,k}, \qquad \alpha_j(z_1, z_2 + \mathcal{B}_k; \omega) = \alpha_j(z_1, z_2; \omega) + 2\mathrm{i}\pi\delta_{j,k}.$$

Moreover, we must have $\alpha_j(z_1, z_1; \omega) = 0$, and $\alpha_j(z_1, z_2; \omega)$ may have poles when $\psi_{\mathrm{cl}}(z_1, z_2; \omega) = 0$. Apart from those poles, $\alpha_j(z_1, z_2; \omega)$ has no other singularities. The following quantity has all the required properties

$$\frac{\theta_{u_j}(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta + \mathbf{c})}{\theta(\mathbf{u}(z_1) - \mathbf{u}(z_2) + \zeta + \mathbf{c})} - \frac{\theta_{u_j}(\zeta + \mathbf{c})}{\theta(\zeta + \mathbf{c})}.$$

So, the difference of α_j and that quantity should be a meromorphic function of z_1 and z_2 without poles, i.e. a constant, and its value is zero by looking at $z_1 = z_2$. ∎

3.3 Link with Baker–Akhiezer functions

3.3.1 Baker–Akhiezer functions

The usual formulation of integrable systems [4, Chapter 5] is obtained by specializing one of the points to $X = \infty$. In some sense, we would like to consider

$$\psi_{\mathrm{cl}|i}(z) \ \text{“=”} \ \psi_{\mathrm{cl}}(z, \infty_i).$$

The problem is, that the expression in the right hand side is divergent, and thus we again need regularizations.

The definitions in this paragraph also apply to the spinor kernel constructed in Section 5.3, so we drop here the $_{\mathrm{cl}}$ index. Recall that the function X has degree d, so the point $X = \infty$ has d preimages ∞_i (counted with multiplicities) on the curve. We define

$$\psi_{i,0}(z) = \lim_{z_2 \to \infty_i} \frac{\psi(z, z_2; \omega)}{\sqrt{\mathrm{d}\xi_{\infty_i}(z_2)}} \mathrm{e}^{V_{\infty_i}(z_2)}(\xi_{\infty_i}(z_2))^{t_{\infty_i,0}}, \tag{3.3}$$

and if $d_{\infty_i} > 1$, we define for $0 \leq j \leq (d_{\infty_i} - 1)$

$$\psi_{i,j}(z) = \frac{\mathrm{d}^j}{\mathrm{d}\xi_{\infty_i}(z_2)^j} \left(\frac{\psi(z, z_2; \omega)}{\sqrt{\mathrm{d}\xi_{\infty_i}(z_2)}} \mathrm{e}^{V_{\infty_i}(z_2)}(\xi_{\infty_i}(z_2))^{t_{\infty_i,0}} \right)_{z_2 = \infty_i}.$$

There are d pairs $I = (i, j)$ such that $0 \leq j \leq d_{\infty_i} - 1$, and therefore the vector $\vec{\psi}(z) = [\psi_I(z)]$ is a d-dimensional vector, and the matrix $\Psi(x; \omega) = [\psi_I(z^k(x))]_{I, 1 \leq k \leq d}$ is a $d \times d$ square matrix.

3.3.2 Dual Baker–Akhiezer functions

Similarly, we would like to define $\phi_{\mathrm{cl}|i}(z) \ \text{“=”} \ \psi_{\mathrm{cl}}(\infty_i, z)$. Thus, we define the dual Baker–Akhiezer functions

$$\phi_{i,0}(z) = \lim_{z_1 \to \infty_i} \frac{\psi(z_1, z; \omega)}{\sqrt{\mathrm{d}\xi_{\infty_i}(z_1)}} \mathrm{e}^{-V_{\infty_i}(z_1)}(\xi_{\infty_i}(z_1))^{-t_{\infty_i,0}},$$

and if $d_{\infty_i} > 1$, we define for each $0 \leq j \leq (d_{\infty_i} - 1)$

$$\phi_{i,j}(z) = \frac{\mathrm{d}^j}{\mathrm{d}\xi_{\infty_i}(z_1)^j} \left(\frac{\psi(z_1, z; \omega)}{\sqrt{\mathrm{d}\xi_{\infty_i}(z_1)}} \mathrm{e}^{-V_{\infty_i}(z_1)}(\xi_{\infty_i}(z_1))^{-t_{\infty_i,0}} \right)_{z_1 = \infty_i}.$$

There are d pairs $I = (i, j)$ such that $0 \leq j \leq d_i - 1$, and therefore the vector $\vec{\phi}(z) = [\phi_I(z)]$ is a d-dimensional vector, and the matrix $\Phi(x; \omega) = [\phi_I(z^k(x))]_{I, 1 \leq k \leq d}$ is a $d \times d$ square matrix. From Corollary 3.3, one retrieves the well-known result that columns of $\Phi(x; \omega)$ are eigenvectors of a Lax matrix.

3.4 Christoffel–Darboux relations

Proposition 3.4. *The matrix*

$$A_{\mathrm{cl}}^{-1} = \frac{1}{\mathrm{d}x} \Phi_{\mathrm{cl}}(x) \Psi_{\mathrm{cl}}^t(x)$$

is invertible, and independent of x. The matrix A_{cl} is called the Christoffel–Darboux matrix. This can also be written $\Psi_{\mathrm{cl}}^t(x) A_{\mathrm{cl}} \Phi_{\mathrm{cl}}(x) = \mathrm{d}x \mathbf{1}_{d \times d}$.

Proof. This is an application of Proposition 3.2, up to a conjugation. Indeed

$$\left(A_{\mathrm{cl}}^{-1}\right)_{(i,k),(i',k')} = \frac{\mathrm{d}^{k'-1}}{\mathrm{d}\xi_{\infty_{i'}}^{k'-1}(z_1)} \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \left[\sum_m \frac{\psi_{\mathrm{cl}}(z_1, z^m) \psi_{\mathrm{cl}}(z^m, z_2)}{\sqrt{\mathrm{d}\xi_{\infty_{i'}}(z_1) \mathrm{d}\xi_{\infty_i}(z_2)}} \right.$$

$$\left. \times \, \mathrm{e}^{V_{\infty_i}(z_2) - V_{\infty_{i'}}(z_1)} \xi_{\infty_i}(z_2)^{t_{\infty_i,0}} \xi_{\infty_{i'}}(z_1)^{-t_{\infty_{i'},0}} \right]_{z_1 = \infty_{i'}}^{z_2 = \infty_i}$$

$$= \frac{\mathrm{d}^{k'-1}}{\mathrm{d}\xi_{\infty_{i'}}^{k'-1}(z_1)} \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \left[\frac{(X(z_1) - X(z_2)) \psi_{\mathrm{cl}}(z_1, z_2) \mathrm{e}^{V_{\infty_i}(z_2) - V_{\infty_{i'}}(z_1)}}{(X(z) - X(z_1))(X(z) - X(z_2))} \right.$$

$$\left. \times \, \frac{\xi_{\infty_i}(z_2)^{t_{\infty_i,0}} \xi_{\infty_{i'}}(z_1)^{-t_{\infty_{i'},0}}}{\sqrt{\mathrm{d}\xi_{\infty_{i'}}(z_1) \mathrm{d}\xi_{\infty_i}(z_2)}} \right]_{z_1 = \infty_{i'}}^{z_2 = \infty_i}.$$

If $i \neq i'$, the quantity

$$\frac{\psi_{\mathrm{cl}}(z_1, z_2) \mathrm{e}^{V_{\infty_i}(z_2) - V_{\infty_{i'}}(z_1)} \xi_{\infty_i}(z_2)^{t_{\infty_i,0}} \xi_{\infty_{i'}}(z_1)^{-t_{\infty_{i'},0}}}{\sqrt{\mathrm{d}\xi_{\infty_{i'}}(z_1) \mathrm{d}\xi_{\infty_i}(z_2)}}$$

has a well-defined limit when $z_1 \to \infty_{i'}$ and $z_2 \to \infty_i$, and the term $\frac{1}{X(z) - X(z_1)} - \frac{1}{X(z) - X(z_2)}$ behaves like

$$\frac{1}{X(z) - X(z_1)} \underset{z_1 \to \infty_{i'}}{\sim} \xi_{\infty_{i'}}(z_1)^{d_{\infty_{i'}}},$$

so we are computing the $(k'-1)$-th derivative of $O(\xi_{\infty_{i'}}(z_1)^{d_{\infty_{i'}}})$, where $k' \leq d_{\infty_{i'}}$, and therefore we get 0, i.e.

$$(A^{-1})_{\mathrm{cl}|(i,k),(i',k')} = 0 \qquad \text{if} \ \ i \neq i'.$$

If $i = i'$, we first take the limit $z_1 \to \infty_i$, and again the term with $\frac{1}{X(z) - X(z_1)}$ vanishes. Then, remember that $\psi_{\mathrm{cl}}(z_1, z_2)$ has a simple pole at $z_1 = z_2$, and thus the derivative with respect to z_1, can generate a pole of degree k' at $z_2 = \infty_i$. Therefore, we are computing the $(k-1)$-th derivative of $O(\xi_{\infty_i}(z_2)^{d_{\infty_i} - k'})$. We get zero if $k + k' \leq d_{\infty_i}$, and therefore

$$(A^{-1})_{\mathrm{cl}|(i,k),(i,k')} = 0 \qquad \text{if} \ \ k + k' \leq d_{\infty_i}.$$

If $i = i'$ and $k + k' = d_{\infty_i} + 1$, the only non-vanishing contribution is

$$
\frac{1}{(k'-1)!} \left(A^{-1} \right)_{\mathrm{cl}|(i,k),(i,k')}
$$

$$
= \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \lim_{z_1 \to \infty_i} \left[\frac{E(z_1, z_2)\psi_{\mathrm{cl}}(z_1, z_2)}{(\xi_{\infty_i}(z_2) - \xi_{\infty_i}(z_1))^{k'}} e^{V_{\infty_i}(z_2) - V_{\infty_i}(z_1)} \right.
$$

$$
\left. \times \xi_{\infty_i}(z_2)^{t_{\infty_i,0} + d_{\infty_i}} \xi_{\infty_i}(z_1)^{-t_{\infty_i,0}} \right]_{z_2 = \infty_i}
$$

$$
= \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \lim_{z_1 \to \infty_i} \left[\psi_{\mathrm{cl}}(z_1, z_2) E(z_1, z_2) e^{V_{\infty_i}(z_2) - V_{\infty_i}(z_1)} \right.
$$

$$
\left. \times \xi_{\infty_i}(z_2)^{t_{\infty_i,0} + d_{\infty_i} - k'} \xi_{\infty_i}(z_1)^{-t_{\infty_i,0}} \right]_{z_2 = \infty_i}
$$

$$
= \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \left[\xi_{\infty_i}(z_2)^{d_{\infty_i} - k'} \right]_{z_2 = \infty_i} = \frac{\mathrm{d}^{k-1}}{\mathrm{d}\xi_{\infty_i}^{k-1}(z_2)} \left[\xi_{\infty_i}(z_2)^{k-1} \right]_{z_2 = \infty_i} = (k-1)! \neq 0.
$$

The matrix A_{cl}^{-1} has thus typically the shape

$$
A_{\mathrm{cl}}^{-1} = \begin{pmatrix} & & * & & \\ & * & * & & \\ * & * & * & & \\ & & & & * \\ & & & * & * \end{pmatrix}
$$

it is made of (inverted) triangular blocks. Since the diagonal of each triangle is non-zero, this proves that the matrix A_{cl}^{-1} is invertible.

Then, if $i = i'$ and $k + k' \geq d_{\infty_i} + 1$, we write that

$$
\frac{1}{X(z) - X(z_1)} = -\frac{1}{X(z_1)} + O\left(1/X(z_1)^2 \right),
$$

and we see that the leading term $\frac{1}{X(z_1)}$ is independent of $X(z)$, and the part which depends on $X(z)$ is $O(1/X(z_1)^2) = O(\xi_{\infty_i}(z_1)^{2d_{\infty_i}})$. A non vanishing contribution to the part which depends on $X(z)$ could occur only if $k + k' > 2d_{\infty_i}$, which can never happen since we assumed $k, k' \leq d_{\infty_i}$. This proves that A_{cl} is independent of $X(z)$. ∎

Corollary 3.1. *The matrices $\Psi_{\mathrm{cl}}(x; \omega)$ and $\Phi_{\mathrm{cl}}(x; \omega)$ are invertible.*

As a consequence, $\psi_{\mathrm{cl}}(z_1, z_2; \omega)/\sqrt{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}$ can be identified with an integrable kernel in the sense of [49], i.e. we have

Proposition 3.5. *Christoffel–Darboux relation:*

$$
\psi_{\mathrm{cl}}(z_1, z_2; \omega) = -\frac{\sum_{I,J} \psi_{\mathrm{cl}|I}(z_1) A_{\mathrm{cl}|I,J} \phi_{\mathrm{cl}|J}(z_2)}{X(z_1) - X(z_2)}.
$$

Proof. This is an application of Property 3.4 and Proposition 3.2. Indeed, the very definition of the $\psi_{\mathrm{cl}|I}$'s, means exactly that there exists a matrix $C_{\mathrm{cl}}(x)$ such that

$$
\Psi_{\mathrm{cl}}^t(X(z_1)) = \lim_{x \to \infty} \Psi_{\mathrm{cl}}(X(z_1), x) C_{\mathrm{cl}}(x),
$$

and similarly, there exists a matrix $\tilde{C}_{\mathrm{cl}}(x)$ such that

$$
\Phi_{\mathrm{cl}}(X(z_2)) = \lim_{x \to \infty} \tilde{C}_{\mathrm{cl}}(x) \Psi_{\mathrm{cl}}(x, X(z_2)).
$$

When $z_1 = z_2$, we have $\Psi_{\mathrm{cl}}(x, X(z_1))\Psi_{\mathrm{cl}}(X(z_1), x) = -\frac{\mathrm{d}x\mathrm{d}X(z_1)}{(x-X(z_1))^2}\mathbf{1}_{d\times d}$, which implies

$$-\lim_{x\to\infty}\frac{\mathrm{d}x\mathrm{d}X(z_1)}{(x-X(z_1))^2}\tilde{C}_{\mathrm{cl}}(x)C_{\mathrm{cl}}(x) = \Phi_{\mathrm{cl}}(X(z_1))\Psi_{\mathrm{cl}}^t(X(z_1)) = A_{\mathrm{cl}}^{-1}\mathrm{d}X(z_1),$$

and therefore

$$\lim_{x\to\infty} A_{\mathrm{cl}}\tilde{C}_{\mathrm{cl}}(x)C_{\mathrm{cl}}(x)\frac{\mathrm{d}x}{x^2} = -\mathbf{1}_{d\times d}. \tag{3.4}$$

Then, we have from the duality equation, for any x

$$(X(z_1) - X(z_2))\,\Psi_{\mathrm{cl}}(X(z_1), X(z_2))$$
$$= \Psi_{\mathrm{cl}}(X(z_1), x)\Psi_{\mathrm{cl}}(x, X(z_2))\frac{(X(z_1) - x)(X(z_2) - x)}{\mathrm{d}x}$$
$$= \Psi_{\mathrm{cl}}(X(z_1), x)C_{\mathrm{cl}}(x)C_{\mathrm{cl}}^{-1}(x)\Psi_{\mathrm{cl}}(x, X(z_2))\frac{(X(z_1) - x)(X(z_2) - x)}{\mathrm{d}x},$$

and in particular, we may take the limit $x \to \infty$ and insert equation (3.4)

$$-(X(z_1) - X(z_2))\Psi_{\mathrm{cl}}(X(z_1), X(z_2))$$
$$= \lim_{x\to\infty}\Psi_{\mathrm{cl}}(X(z_1), x)C_{\mathrm{cl}}(x)A_{\mathrm{cl}}\tilde{C}_{\mathrm{cl}}(x)C_{\mathrm{cl}}(x)C_{\mathrm{cl}}^{-1}(x)\Psi_{\mathrm{cl}}(x, X(z_2))\frac{(X(z_1) - x)(X(z_2) - x)}{x^2}$$
$$= \lim_{x\to\infty}\Psi_{\mathrm{cl}}(X(z_1), x)C_{\mathrm{cl}}(x)A_{\mathrm{cl}}\tilde{C}_{\mathrm{cl}}(x)\Psi_{\mathrm{cl}}(x, X(z_2)) = \Psi_{\mathrm{cl}}^t(X(z_1))A_{\mathrm{cl}}\Phi_{\mathrm{cl}}(X(z_2)). \qquad\blacksquare$$

3.5 Lax matrix

As corollary of Proposition 3.2

Corollary 3.2. *If x_1 and x_2 are not branchpoints, and if $x_2 \neq x_1$, then the matrix $\Psi_{\mathrm{cl}}(x_1, x_2)$ is invertible, and*

$$\Psi_{\mathrm{cl}}(x_1, x_2; \omega)\Psi_{\mathrm{cl}}(x_2, x_1; \omega) = -\frac{\mathrm{d}x_1\mathrm{d}x_2}{(x_1 - x_2)^2}\mathbf{1}_{d\times d}.$$

Proof. Take $x_3 = x_1$ in the duality relation. $\qquad\blacksquare$

Corollary 3.3 (Reconstruction formula)**.** *Let*

$$\tilde{L}(x) = \mathrm{diag}(Y_1(x), \ldots, Y_d(x)) = \mathrm{diag}\big(Y(z^1(x)), Y(z^2(x)), \ldots, Y(z^d(x))\big).$$

For every x_1, the matrix

$$L_{x_1}(x; \vec{t}) = \Psi_{\mathrm{cl}}(x_1, x; \omega)\tilde{L}(x)\Psi_{\mathrm{cl}}^{-1}(x_1, x; \omega) = -\frac{(x_1 - x)^2}{\mathrm{d}x_1\mathrm{d}x}\Psi_{\mathrm{cl}}(x_1, x; \omega)\tilde{L}(x)\Psi_{\mathrm{cl}}(x, x_1; \omega)$$

(it depends on times $\vec{t}$ through $\omega = \sum_k t_k\omega_k$ as in equation (2.4)) is a rational function of x. Its characteristic polynomial is independent of the times $\vec{t}$, and its zero locus defines the semi-classical spectral curve $\det\big(y - L_{x_1}(x; \vec{t})\big) = \det(y - \tilde{L}(x)) = 0$. Changing x_1 just amounts to a conjugation

$$L_{x_1'}(x; \vec{t}) = \Psi_{\mathrm{cl}}(x_1', x_1; \omega)\tilde{L}_{x_1}(x; \vec{t})\Psi_{\mathrm{cl}}^{-1}(x_1', x_1; \omega).$$

Proposition 3.6. *For any of the times $t_{p,j}$ with $j \geq 1$, the matrix $L_{x_1}(x; \vec{t})$ obeys the Lax equation*

$$\frac{\partial}{\partial t_{(p,j)}}L_{x_1}(x; \vec{t}) = [M_{(p,j);x_1}(x; \vec{t}), L_{x_1}(x; \vec{t})], \tag{3.5}$$

where the matrix $M_{(p,j);x_1}$ is $M_{(p,j);x_1}(x; \vec{t}) = \partial_{t_{(p,j)}}\Psi_{\mathrm{cl}}(x_1, x; \omega)\Psi_{\mathrm{cl}}^{-1}(x_1, x; \omega)$.

3.6   Differential systems

Baker–Akhiezer functions satisfy simultaneously several first order differential systems with respect to the spectral parameter and the times [59, 60].

Proposition 3.7. *The matrix* $\Psi_{\mathrm{cl}}(x_1, x; \omega)$ *is solution of a linear ODE with respect to the spectral parameter* x

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathbf{1}_{d\times d}}{x - x_1} - M_{x;x_1}(x; \vec{t}) \right) \frac{\Psi_{\mathrm{cl}}(x_1, x; \omega)}{\sqrt{\mathrm{d}x\mathrm{d}x_1}} = 0,$$

where $M_{x;x_1}$ *is a rational function of* x *having poles only on* $\overline{\mathcal{P}}$, *with the same order as those of* $\omega(z)/\mathrm{d}X(z)$.

Proof. Write $x = X(z)$, we have

$$\mathrm{d}X(z)\big[M_{x;x_1}(X(z); \vec{t})\big]_{i,j} - \delta_{i,j} \frac{\mathrm{d}X(z)}{(X(z) - x_1)}$$
$$= -(X(z) - x_1)^2 \sum_k \mathrm{d}\left(\frac{\psi_{\mathrm{cl}}\big(z^i(x_1), z^k\big)}{\sqrt{\mathrm{d}X(z)\mathrm{d}x_1}} \right) \frac{\psi_{\mathrm{cl}}\big(z^k, z^j(x_1)\big)}{\sqrt{\mathrm{d}X(z)\mathrm{d}x_1}}.$$

In the right hand side, the essential singularities cancel, and only meromorphic singularities remain. Since we perform the sum over all sheets, the result is necessarily a rational function of $X(z)$. Poles could occur at singularities of ψ, or also at zeroes of $\sqrt{\mathrm{d}X(z)}$, or at $X(z) = x_1$.

ψ_{cl} has a simple pole at $z^k = z^i(x_1)$ for some k. Taking the derivative yields a double pole, and multiplying by $(X(z) - x_1)^2$ cancels the double pole. If $i = j$, there is also a simple pole coming from $\psi_{\mathrm{cl}}(z^k, z^j(x_1))$, and if $i \neq j$ there is no pole. It is easy to check that the residue of the pole at $x = x_1$ is $-\delta_{i,j}$. This implies that $M_{x;x_1}(x)$ has no pole at $x = x_1$. From its definition, $M_{x,x_1}(X(z))$ must behave as $o\big((X(z) - X(a))^{-1/2}\big)$ at a ramification point a, and since it is a rational fraction of $X(z)$, this must actually be $O(1)$, meaning that $M_{x;x_1}(x)$ has no poles when x approaches a branchpoint.

Near $z \to p \in \bar{\mathcal{P}}$, the only singularity comes from the exponential term and we have

$$\big[M_{x;x_1}(X(z); \vec{t})\big]_{i,j} = (X(z) - x_1)^2 \sum_k \omega(z^k)\frac{\psi_{\mathrm{cl}}\big(z^i(x_1), z^k\big)\psi_{\mathrm{cl}}\big(z^k, z^j(x_1)\big)}{(\mathrm{d}X(z))^2\mathrm{d}x_1} + O(1).$$

This shows that $\mathrm{d}x\, M_{x;x_1}(x; \vec{t})$ has poles in $\bar{\mathcal{P}}$, of order at most that already present in ω/dX. ∎

Proposition 3.8. *We have with respect to the times* $t_{p,j}$ $(j \geq 1)$

(i) $\big(\partial_{t_{p,j}} - M_{t_{p,j};x_1}(x; \vec{t})\big)\Psi_{\mathrm{cl}}(x_1, x; \omega) = 0.$

(ii) $M_{t_{p,j};x_1}$ *is a rational function of* x, *with possible poles only at* $x = X(p)$.

If $X(p) \neq \infty$, *the pole is of order* j. *If* $X(p) = \infty$, *the pole is of degree* $1 + \lfloor (j-1)/d_p \rfloor$.

The method of the proof is similar to that of Proposition 3.7. There also exists an analog theorem for $\partial_{t_{p,0}} - \partial_{t_{p',0}}$. We retrieve after sending x_1 to ∞ the usual formulation of Lax systems for $\Psi_{\mathrm{cl}}(x; \omega)$ and $\Phi_{\mathrm{cl}}(x; \omega)$ introduced in Section 3.3.1.

4 Semiclassical Tau function

For later convenience, we define $\tilde{F}_0$ as a shifted version of F_0, as follows

$$\tilde{F}_0(\omega) := F_0(\omega) - \sum_{i=1}^{\mathfrak{g}} \epsilon_i \frac{\partial F_0(\omega)}{\partial \epsilon_i} + \mathrm{i}\pi \sum_{j,j'=1}^{\mathfrak{g}} \epsilon_j \epsilon_{j'} \tau_{j,j'}$$

$$= \frac{1}{2} \sum_{k,l=(p,j)} t_k t_l \frac{\partial^2 F_0(\omega)}{\partial t_k t_l} + \frac{1}{2} \sum_{j,j'=1}^{\mathfrak{g}} 2\mathrm{i}\pi \epsilon_j \epsilon_{j'} \tau_{j,j'},$$

where $\epsilon_i = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{A}_i} \omega$, and $\omega = \sum_k \omega_k t_k$.

Definition 4.1. We define

$$\mathcal{T}_{\mathrm{cl}}(\mathcal{C}, \omega) = \mathrm{e}^{\tilde{F}_0(\omega)} \theta(\zeta(\omega) + \mathbf{c}),$$

where we recall that $\zeta(\omega) = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}-\tau\mathcal{A}} \omega = \sum_k v_k t_k$ depends linearly on the times. We shall also sometimes use as notation $\mathcal{T}_{\mathrm{cl}}(\mathcal{C}, \omega) \equiv \mathcal{T}_{\mathrm{cl}}(\vec{t})$ when there is no ambiguity. In this article, we call $\mathcal{T}_{\mathrm{cl}}$ the "semiclassical" Tau function.

$\mathcal{T}_{\mathrm{cl}}(\mathcal{C}, \omega)$ depends as well on the data of a non-singular odd characteristics $\mathbf{c}$ for the Theta function. It is the Tau function [53, 54, 56] associated here to the solution of the problem of isospectral evolution described in Section 3.5. We call Tau function, any function satisfying Hirota bilinear equation, and related to Baker–Akhiezer functions by Sato relation. Sections 4.1 and 4.3 review the fact that $\mathcal{T}_{\mathrm{cl}}$ fits in this definition.

4.1 Sato relation

Sato relation [75] means that the Baker–Akhiezer kernel $\psi_{\mathrm{cl}}(z_1, z_2; \omega)$ is obtained from the Tau function by a Schlesinger transformation. This can be formulated intrinsically in terms of 1-forms, as

Theorem 4.1.

$$\frac{\mathcal{T}_{\mathrm{cl}}(\mathcal{C}, \omega + \mathrm{d}S_{z_1,z_2})}{\mathcal{T}_{\mathrm{cl}}(\mathcal{C}, \omega)} \sqrt{\mathrm{d}X(z_1)\mathrm{d}X(z_2)} = \psi_{\mathrm{cl}}(z_1, z_2; \omega).$$

Proof. We add two simple poles at z_1 and z_2 by considering

$$\omega_\lambda(z) = \omega(z) + \lambda \, \mathrm{d}S_{z_1,z_2}(z).$$

According to Theorem 2.1 we have

$$\left.\frac{\partial F_0}{\partial \lambda}\right|_{\lambda=0} = \mu_{z_1} - \mu_{z_2} = \int_{z_2}^{z_1} \omega, \tag{4.1}$$

and according to Theorem 2.2 we have

$$\left.\frac{\partial^2 F_0}{\partial \lambda^2}\right|_{\lambda=0} = -\ln\left(E(z_1, z_2)^2 \mathrm{d}X(z_1)\mathrm{d}X(z_2)\right). \tag{4.2}$$

Since F_0 is a quadratic functions of the times, and thus a degree 2 polynomial in λ, we have

$$F_0(\omega_\lambda) = F_0(\omega) + \lambda \left.\frac{\partial F_0}{\partial \lambda}\right|_{\lambda=0} + \frac{\lambda^2}{2} \left.\frac{\partial^2 F_0}{\partial \lambda^2}\right|_{\lambda=0}$$

$$= F_0(\omega) + \lambda \int_{z_2}^{z_1} \omega - \lambda^2 \ln\left(E(z_1, z_2)\sqrt{dX(z_1)dX(z_2)}\right).$$

It is also easy to see that

$$\zeta(\omega + dS_{z_1, z_2}) = \zeta(\omega) + \frac{\lambda}{2i\pi} \oint_{\mathcal{B} - \tau \mathcal{A}} dS_{z_1, z_2} = \zeta(\omega) + \lambda(\mathbf{u}(z_1) - \mathbf{u}(z_2)).$$

Taking $\lambda = 1$ implies the theorem.   ■

4.2   Expansion near poles

We claim that Theorem 4.1 (which we wrote intrinsically in terms of 1-forms on $\mathcal{C}$) is equivalent to Sato's formula (which is written in terms of times t_k's). In order to see that, we need to expand for $z_1, z_2 \in U_p$ near a pole $p \in \mathcal{P}$, where U_p denote an open neighborhood of p on which the local coordinate ξ_p is well-defined. The Schlesinger transformation

$$\omega \to \omega + dS_{z_1, z_2}$$

can also be written by Taylor expansion, assuming $||\xi_p(z)| > \max(|\xi_p(z_1)|, |\xi_p(z_2)|)$

$$dS_{z_1, z_2}(z) = \sum_{j \geq 1} \frac{\xi_p(z_1)^j}{j} \omega_{p,j}(z) - \frac{\xi_p(z_2)^j}{j} \omega_{p,j}(z).$$

Thus, the times associated to p after Schlesinger transformation are

$$\forall j \geq 1 \qquad t_{p,j} \longrightarrow t_{p,j} + \left(\frac{\xi_p(z_1)^j}{j} - \frac{\xi_p(z_2)^j}{j}\right).$$

The usual notation for this special infinite collection of times is

$$[z_1]_p = \left(\frac{\xi_p(z_1)^j}{j}\right)_{j \geq 1}. \tag{4.3}$$

Thus Theorem 4.1, together with equation (3.3) gives the Sato formula in its usual presentation involving an infinite set of times

$$\psi_{\mathrm{cl}|(p,0)}(z_1) = \frac{\mathcal{T}_{\mathrm{cl}}(\vec{t} + [z_1]_p)}{\mathcal{T}_{\mathrm{cl}}(\vec{t})}.$$

4.3   Hirota bilinear equation

Definition 4.2. For any point $z \in \mathcal{C}$, we define the insertion operator δ_z, which acts on functions of a meromorphic 1-form ω on $\mathcal{C}$, by considering the one parameter deformation consisting in adding a Bergman kernel $B(z, \cdot)/dX(z)$ (the dot $\cdot$ means the variable on which $\omega = \omega(\cdot)$ depends)

$$\delta_z f(\omega) = dX(z)\frac{\partial}{\partial \lambda} f\left(\omega + \lambda \frac{B(z, \cdot)}{dX(z)}\right)\bigg|_{\lambda = 0}. \tag{4.4}$$

δ_z is a derivation (it satisfies the chain rule $\delta_z(fg) = g\delta_z f + f\delta_z g$). In terms of times coordinates

$$\delta_z f(\vec{t}) = dX(z)\partial_{t_{z,1}} f(\vec{t}).$$

If z is near a pole p, in the local coordinates we have

$$\delta_z \equiv \sum_{j=1}^{\infty} \xi_p(z)^{j-1} \mathrm{d}\xi_p(z) \partial_{t_{p,j}}. \tag{4.5}$$

Again, we prefer equation (4.4), which is intrinsic and does not require the introduction of an infinite number of times. We write equation (4.5) just to make the contact with usual presentation (given for instance in [4]).

Proposition 4.1. *For any two 1-forms ω and $\hat{\omega}$ defined on the same Riemann surface $\mathcal{C}$, we have*

$$\operatorname*{Res}_{z' \to z} \psi_{\mathrm{cl}}(z, z'; \omega) \psi_{\mathrm{cl}}(z', z; \hat{\omega}) = -\delta_z \ln \frac{\mathcal{T}_{\mathrm{cl}}(\omega)}{\mathcal{T}_{\mathrm{cl}}(\hat{\omega})}.$$

Proof. $\psi_{\mathrm{cl}}(z, z'; \omega) \psi_{\mathrm{cl}}(z', z; \hat{\omega})$ has a double pole at $z = z'$, and thus evaluating the residue computes a derivative. ■

As a corollary, we get

Theorem 4.2. *For any data $(\mathcal{C}, \omega)$, the Baker–Akhiezer kernel is self-replicating*

$$\delta_z \psi_{\mathrm{cl}}(z_1, z_2; \omega) = -\psi_{\mathrm{cl}}(z_1, z; \omega) \psi_{\mathrm{cl}}(z, z_2; \omega).$$

Proof. We choose $\hat{\omega} = \omega + \mathrm{d}S_{z_1, z_2}$. Then, $\psi_{\mathrm{cl}}(z, z'; \omega) \psi_{\mathrm{cl}}(z', z; \omega + \mathrm{d}S_{z_1, z_2})$ is a meromorphic form in z' (it has no essential singularity). Its only poles are located at $z' = z$ and $z' = z_2$. Moving the integration contour, we find

$$\delta_z \ln \frac{\mathcal{T}_{\mathrm{cl}}(\omega + dS_{z_1, z_2})}{\mathcal{T}_{\mathrm{cl}}(\omega)} = \operatorname*{Res}_{z' \to z} \psi_{\mathrm{cl}}(z, z'; \omega) \psi_{\mathrm{cl}}(z', z; \omega + \mathrm{d}S_{z_1, z_2})$$

$$= - \operatorname*{Res}_{z' \to z_2} \psi_{\mathrm{cl}}(z, z'; \omega) \psi_{\mathrm{cl}}(z', z; \omega + \mathrm{d}S_{z_1, z_2}).$$

We have

$$\psi_{\mathrm{cl}}(z', z; \omega + \mathrm{d}S_{z_1, z_2}) = \frac{E(z', z_1) E(z, z_2) \mathrm{e}^{\int_z^{z'} \omega + \int_{z_2}^{z_1} \omega}}{E(z', z) E(z', z_2) E(z_1, z) E(z_1, z_2)}$$

$$\times \frac{\theta(\zeta(\omega) + \mathbf{u}(z') - \mathbf{u}(z) + \mathbf{u}(z_1) - \mathbf{u}(z_2) + \mathbf{c})}{\theta(\zeta(\omega) + \mathbf{u}(z_1) - \mathbf{u}(z_2) + \mathbf{c})}$$

and thus near $z' \to z_2$

$$\psi_{\mathrm{cl}}(z', z; \omega + \mathrm{d}S_{z_1, z_2}) \sim -\frac{1}{E(z', z_2)} \frac{\mathrm{e}^{\int_z^{z_1} \omega}}{E(z_1, z)} \frac{\theta(\zeta(\omega) + \mathbf{u}(z_1) - \mathbf{u}(z) + \mathbf{c})}{\theta(\zeta(\omega) + \mathbf{u}(z_1) - \mathbf{u}(z_2) + \mathbf{c})}$$

$$\sim -\frac{1}{E(z', z_2)} \mathrm{e}^{\int_{z_2}^{z_1} \omega} \frac{\psi_{\mathrm{cl}}(z_1, z; \omega)}{\psi_{\mathrm{cl}}(z_1, z_2; \omega)}.$$

Therefore

$$\delta_z \ln \frac{\mathcal{T}_{\mathrm{cl}}(\omega + \mathrm{d}S_{z_1, z_2})}{\mathcal{T}_{\mathrm{cl}}(\omega)} = - \operatorname*{Res}_{z' \to z_2} \psi_{\mathrm{cl}}(z, z'; \omega) \psi_{\mathrm{cl}}(z', z; \omega + \mathrm{d}S_{z_1, z_2})$$

$$= -\psi_{\mathrm{cl}}(z_1, z; \omega) \psi_{\mathrm{cl}}(z, z_2; \omega). \qquad \blacksquare$$

This self-replication property is the analog of the Ricatti equation in [4]. Notice that it can also be obtained by a straightforward computation of $\delta_z \ln \psi_{\mathrm{cl}}(z_1, z_2; \omega)$ from Definition 3.1) and comparison with the refined duality equation (Proposition 3.3).

Hirota equation in terms of times

Theorem 4.2 is written intrinsically in terms of forms ω, but to make contact with usual notations in the literature, let us translate it in terms of times t_k's. Given any derivation ∂_t, one defines a Hirota operator D_t [47] acting on two functions $f(t)$, $g(t)$, such that

$$(D_t f \cdot g)(t) \equiv \partial_u f(t+u) g(t-u)\big|_{u=0} = g^2(t) \partial_t (f/g).$$

In particular, there is a Hirota operator D_z associated to derivation $\delta_z = \partial_{t_{z,1}}$. This allows us to reformulate the self-replication property as

Proposition 4.2. *For $\mathcal{T}_{\mathrm{cl}} = \mathcal{T}_{\mathrm{cl}}(\vec{t})$ written as a function of times, for any pole p, the self-replication property is equivalent to the Hirota bilinear difference equation*

$$D_z \mathcal{T}_{\mathrm{cl}}(\vec{t} + [z_1]_p - [z_2]_p) \cdot \mathcal{T}_{\mathrm{cl}}(\vec{t}) = -\mathcal{T}_{\mathrm{cl}}(\vec{t} + [z_1]_p - [z]_p) \mathcal{T}_{\mathrm{cl}}(\vec{t} + [z]_p - [z_2]_p).$$

Actually, this property for $\mathcal{T}_{\mathrm{cl}}$ is equivalent to the Fay identity (see Section 2.1.2) satisfied by the Theta function $\theta(\cdot|\tau)$. The Hirota equation can also be written in a more symmetric way by setting

$$\vec{t} \longleftarrow \vec{t} - \frac{[z_1]_p}{2} + \frac{[z_2]_p}{2}.$$

Namely

$$D_z \mathcal{T}_{\mathrm{cl}} \left(\vec{t} + \frac{[z_1]_p}{2} - \frac{[z_2]_p}{2} \right) \cdot \mathcal{T}_{\mathrm{cl}} \left(\vec{t} - \frac{[z_1]_p}{2} + \frac{[z_2]_p}{2} \right)$$
$$= -\mathcal{T}_{\mathrm{cl}} \left(\vec{t} - [z]_p + \frac{[z_1]_p}{2} + \frac{[z_2]_p}{2} \right) \mathcal{T}_{\mathrm{cl}} \left(\vec{t} + [z]_p - \frac{[z_1]_p}{2} - \frac{[z_2]_p}{2} \right).$$

The procedure to translate it into a set of differential equations (with respect to the times) is well-known, it is merely obtained by Taylor expansion in $\xi_p(z)$, $\xi_p(z_1)$ and $\xi_p(z_2)$ (see e.g. [55]). This gives an infinite set of partial differential equations involving derivatives of the Tau function with respect to times $t_{p,j}$, which are equations of a KP hierarchy. These equations are equivalent to Hirota bilinear equation. The fact that Theta functions via Definition 4.1 provides solutions to the KP hierarchy (the so-called algebro-geometric solutions) was discovered by [52, 63]. The notion of Tau function was only introduced later in [56], but its expression for the algebro-geometric solution is a straightforward reformulation of those results, and coincides with what we call semiclassical tau function $\mathcal{T}_{\mathrm{cl}}(\vec{t})$.

5 Proposal for a new tau function and spinor kernel

So far, the spectral curve $(\mathcal{C}, X, Y)$ was fixed once for all, and was equipped with a 1-form ω depending linearly on times ($\omega = \sum_k t_k \omega_k$). Now, we shall let the spectral curve itself change around $(\mathcal{C}, X, Y)$, and in particular we may vary the complex structure of the curve $\mathcal{C}$. $Y \mathrm{d} X$ shall play the role of the 1-form.

5.1 More geometry: symplectic invariants

For any spectral curve $\mathcal{S} = (\mathcal{C}, X, Y)$, a sequence of symplectic invariants $F_g[\mathcal{S}]$, and of symplectic covariant forms $\omega_n^{(g)}(\mathcal{S})$ was defined in [38]. Let us recall their definition and main properties.

5.1.1   Topological recursion

Let a_i be the ramification points of $\mathcal{S}$, i.e. the zeroes of $\mathrm{d}X$. We assume that the spectral curve $\mathcal{S}$ is regular, i.e. a_i are simple zeroes of $\mathrm{d}X$ and $\mathrm{d}Y(a_i) \neq 0$. Then, Y behaves like a squareroot near ramification points

$$Y(z) = Y(a_i) + Y'(a_i)\sqrt{X(z) - X(a_i)} + O\big(X(z) - X(a_i)\big),$$

and there is a unique other point $\bar{z} \neq z$ such that $X(z) = X(\bar{z})$, at least for z in a neighborhood U_{a_i} of a_i. Then we define the recursion kernel [38]

$$K(z_0, z) = -\frac{1}{2}\frac{\int_{\bar{z}}^{z} B(z_0, \cdot)}{(Y(z) - Y(\bar{z}))\mathrm{d}X(z)}.$$

In the variable z_0, it is a meromorphic 1-form defined globally for $z_0 \in \mathcal{C}$, and in the variable z, it is the inverse of a 1-form, only defined in $\bigcup_i U_{a_i}$.

Definition 5.1. The "symplectic covariant forms" $\omega_n^{(g)}(z_1, \ldots, z_n)$ are meromorphic $\underbrace{(1, \ldots, 1)}_{n \text{ times}}$-forms defined by the following recursion

$$\omega_1^{(0)}(z) = Y(z)\mathrm{d}X(z), \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2),$$

and for $J = \{z_2, \ldots, z_n\}$

$$\omega_n^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z)\left(\omega_{n+1}^{(g-1)}(z, \bar{z}, J) + \sum_{0 \leq h \leq g, I \subseteq J}' \omega_{1+|I|}^{(h)}(z, I)\omega_{1+n-|I|}^{(g-h)}(\bar{z}, J \setminus I)\right),$$

where $\sum'$ means that terms containing a $\omega_1^{(0)}$ factor are excluded.

One can prove that

Theorem 5.1 ([38]). *For $2 - 2g - n < 0$, $\omega_n^{(g)}(z_1, \ldots, z_n)$ is symmetric in its n variables, and it is a meromorphic form in each variable, having poles only at ramification points, with vanishing residues. The poles are of order at most $6g + 2n - 4$.*

For instance, for $(g, n) = (0, 3)$, this definition yields

$$\omega_3^{(0)}(z_0, z_1, z_2) = \sum_i \operatorname*{Res}_{z \to a_i}\left[-\frac{1}{2}\frac{\int_{\bar{z}}^{z} B(z_0, \cdot)}{(Y(z) - Y(\bar{z}))\mathrm{d}X(z)}\big(B(z, z_1)B(\bar{z}, z_2) + B(\bar{z}, z_1)B(z, z_2)\big)\right]$$

$$= \sum_i \operatorname*{Res}_{z \to a_i} \frac{B(z_0, z)B(z_1, z)B(z_2, z)}{\mathrm{d}X(z)\mathrm{d}Y(z)}. \tag{5.1}$$

We shall comment this expression in Section 5.1.2 below.

Definition 5.2. The "symplectic invariants" F_g are numbers associated to $\mathcal{S}$, as follows:

- For $g = 0$, we define F_0 as in Definition 2.6, using $\omega = Y\,\mathrm{d}X$, i.e., $F_0(\mathcal{S}) := F_0(Y\,\mathrm{d}X)$.

- For $g = 1$, F_1 is defined in terms of the Bergman Tau function $\mathcal{T}_{B,X}$ introduced by Kokotov and Korotkin [58]

$$F_1(\mathcal{S}) := -\frac{1}{2}\ln(\mathcal{T}_{B,X}) - \frac{1}{24}\ln\left(\prod_i Y'(a_i)\right),$$

  where $Y'(a_i) = \lim_{z \to a_i}(Y(z) - Y(a_i))/\sqrt{X(z) - X(a_i)}$. F_1 is related to the logarithm of the determinant of the Laplacian on $\mathcal{C}$ with metrics $|y\mathrm{d}x|^2$.

- For $g \geq 2$

$$F_g(\mathcal{S}) := \frac{1}{2 - 2g} \sum_i \operatorname*{Res}_{z \to a_i} \Phi(z) \omega_1^{(g)}(z),$$

where $\Phi(z)$ is any primitive[1] of $Y \mathrm{d}X$, i.e. $\mathrm{d}\Phi = Y \mathrm{d}X$.

As a notation we write $\omega_0^{(g)} \equiv F_g$.

The name "symplectic invariants" comes from the important property, proved in [38, 39], that F_g and the cohomology class of $\omega_n^{(g)}$ are invariant under the following transformations of spectral curves, each of which let the symbolic symplectic form $\mathrm{d}X \wedge \mathrm{d}Y$ invariant

$$(X, Y) \to (-Y, X), \qquad (X, Y) \to (X, Y + R(X)), \qquad (X, Y) \to (\lambda X, Y/\lambda),$$

where R is any rational function and $\lambda \in \mathbb{C}^*$.

5.1.2   Infinitesimal deformations: special geometry

The times t_k's of the 1-form $Y \mathrm{d}X$ provide locally a set of coordinates of the moduli space of spectral curves $\mathcal{S} = (\mathcal{C}, X, Y)$ [65]. Infinitesimal deformations of the spectral curves are given by flows associated to the 1-forms ω_k dual to the times t_k's. The following theorem holds for the invariants $\omega_n^{(g)}$

Theorem 5.2 ([38]). *For any of the times t_k, we have*

$$\forall n, g \geq 0, \qquad \frac{\partial}{\partial t_k}\bigg|_{X(z_i) \text{ fixed}} \omega_n^{(g)}(z_1, \ldots, z_n) = \int_{\omega_k^*} \omega_{n+1}^{(g)}(\cdot, z_1, \ldots, z_n),$$

where ω_k^* is the dual cycle associated to the time t_k in equation (2.2).

This set of relations is called "special geometry" by physicists in the context of string theory. Let us give some examples:

- First kind deformations/addition of holomorphic forms

$$\frac{\partial \omega_n^{(g)}(z_1, \ldots, z_n)}{\partial \epsilon_i} = \oint_{\mathcal{B}_i} \omega_{n+1}^{(g)}(\cdot, z_1, \ldots, z_n).$$

- Second kind deformations/addition of a double pole

$$\frac{\partial \omega_n^{(g)}(z_1, \ldots, z_n)}{\partial t_{p,1}} = \frac{\omega_{n+1}^{(g)}(p, z_1, \ldots, z_n)}{\mathrm{d}\xi_p(p)}.$$

If p is a pole of $Y \mathrm{d}X$, this does not apply to $(g, n) = (0, 0)$.

- Third kind deformations/addition of simple poles

$$\frac{\partial \omega_n^{(g)}(z_1, \ldots, z_n)}{\partial t_{p,0}} - \frac{\partial \omega_n^{(g)}(z_1, \ldots, z_n)}{\partial t_{p',0}} = \int_{p'}^p \omega_{n+1}^{(g)}(\cdot, z_1, \ldots, z_n).$$

---

[1] Since $\omega_1^{(g)}(z)$ for $g \geq 2$ has no residues, F_g is independent of the choice of Φ.

- $(g, n) = (0, 0)$. We retrieve Theorem 2.1 (Section 2.3) for the derivatives of F_0

$$\frac{\partial F_0}{\partial t_k} = \oint_{\omega_k^*} Y \mathrm{d}X.$$

- $(g, n) = (0, 1)$. We retrieve the definition of the form-cycle duality equation (2.3)

$$\frac{\partial Y \mathrm{d}X}{\partial t_k}(z) = \omega_k(z) = \oint_{\omega_k^*} B(\cdot, z).$$

- $(g, n) = (0, 2)$. Thanks to equation (5.1),

$$\frac{\partial B(z_1, z_2)}{\partial t_k} = \sum_i \operatorname*{Res}_{z \to a_i} \frac{B(z, z_1) B(z, z_2) \omega_k(z)}{\mathrm{d}X(z) \mathrm{d}Y(z)}.$$

If we compare it to Rauch variational formula [74], we retrieve that the variation of the complex structure of $\mathcal{C}$ is such that

$$\frac{\partial X(a_i)}{\partial t_k} = \frac{\omega_k(a_i)}{\mathrm{d}Y(a_i)}. \tag{5.2}$$

In particular, we deduce that the time evolution of the spectral curve that we consider obeys the Whitham equations [65, 81]

$$(\partial_{t_k} X(a_i)) \omega_l(a_i) = (\partial_{t_l} X(a_i)) \omega_k(a_i).$$

Furthermore, if we integrate z_1 and z_2 over the dual cycles ω_l^* and ω_m^*, we retrieve Theorem 2.3 (see also [65]).

Although we make use of the meromorphic deformations (listed above) in this article, the special geometry relations may remain valid for deformations along certain multivalued 1-forms, namely the primaries and their descendents listed in [29] in the context of Hurwitz spaces, or the 1-forms coupled to the Whitham times of [65, Section 7] for a slightly different moduli space.

5.1.3   Finite deformations

Instead of infinitesimal deformations of the spectral curve, one may consider finite deformations: $t_k \to t_k + \frac{1}{N} c_k$ (where N is a formal parameter, which will serve as a formal expansion parameter for Taylor series). In other words, to a spectral curve $\mathcal{S}$ and a meromorphic 1-form $\Omega = \sum_k c_k \omega_k$, we shall associate a new spectral curve, denoted $\mathcal{S} + \frac{1}{N}\Omega$, as push-forward of $\mathcal{S}$ by the flow defined by Ω. This convenient notation should not hide the fact that the conformal structure of $\mathcal{C}$ changes under such global deformations (see equation (5.2)). A more detailed example of computations with global deformation is provided in the proof of Lemma 5.1 below.

Definition 5.3. For any meromorphic 1-form Ω on $\mathcal{C}$, written with its times coordinates c_k (as in equation (2.2))

$$\Omega = \sum_k c_k \omega_k \qquad \text{with dual cycle} \quad \Omega^* = \sum_k c_k \omega_k^*,$$

and for any functional $f[\mathcal{S}] \equiv f(\vec{t})$ depending on a spectral curve $\mathcal{S}$, i.e. on the times t_k's, we take as a definition[2] of $f[\mathcal{S} + \frac{1}{N}\Omega]$ the following formal series in N^{-1}

$$f\left[\mathcal{S} + \frac{1}{N}\Omega\right] := \sum_{k \geq 0} \frac{N^{-k}}{k!} \left(\frac{\partial^k}{\partial \lambda^k} f[\vec{t} + \lambda \vec{c}]\Big|_{\lambda=0}\right).$$

---

[2] With an appropriate regularization when Ω has simple poles.

If the functional f is F_g or ω_n^g, the coefficients in the Taylor series can be computed by the relations of special geometry Theorem 5.2, namely

$$\omega_n^{(g)}\left[\mathcal{S} + \frac{1}{N}\Omega\right](z_1,\ldots,z_n) = \omega_n^{(g)}[\mathcal{S}](z_1,\ldots,z_n) + \sum_{k\geq 1}\frac{N^{-k}}{k!}\underbrace{\int_{\Omega^*}\cdots\int_{\Omega^*}}_{k \text{ times}}\omega_{k+n}^{(0)}[\mathcal{S}](z_1,\ldots,z_n,\bullet),$$

$$F_g\left[\mathcal{S} + \frac{1}{N}\Omega\right] = F_g[\mathcal{S}] + \sum_{k\geq 1}\frac{N^{-k}}{k!}\underbrace{\int_{\Omega^*}\cdots\int_{\Omega^*}}_{k \text{ times}}\omega_k^{(0)}[\mathcal{S}],$$

so that the computation of F_g's or $\omega_n^{(g)}$ for $\mathcal{S} + \frac{1}{N}\Omega$ does not involve derivatives with respect to times, but only integrals at fixed times on $\mathcal{S}$.

5.2 Tau function

5.2.1 Preliminaries

We need Theta functions, and we shall introduce an appropriate notation for our purposes. For any $\mathbf{w} \in \mathbb{C}^{\mathfrak{g}}$, for any $\epsilon \in \mathbb{C}^{\mathfrak{g}}$, and for any $\mathfrak{g} \times \mathfrak{g}$ symmetric matrix τ of positive imaginary part $\mathrm{Im}\,\tau > 0$, and for any $(\mu,\nu) \in \mathbb{C}^{\mathfrak{g}} \times \mathbb{C}^{\mathfrak{g}}$ we define the $\Theta_{[\mu,\nu]}$ function as

$$\Theta_{[\mu,\nu]}(\mathbf{w}|\tau;\epsilon) := \sum_{\mathbf{p}\in\mathbb{Z}^{\mathfrak{g}}} e^{\mathrm{i}\pi(\mathbf{p}+\mu-N\epsilon)\cdot\tau\cdot(\mathbf{p}+\mu-N\epsilon)+(\mathbf{p}+\mu-N\epsilon)\cdot\mathbf{w}+2\mathrm{i}\pi\mathbf{p}\cdot\nu}.$$

It is closely related to the usual Siegel theta function with characteristics $[\mu,\nu]$

$$\Theta_{[\mu,\nu]}(\mathbf{w}|\tau;\epsilon) = e^{\mathrm{i}\pi N^2\epsilon\cdot\tau\cdot\epsilon - N\epsilon\cdot\mathbf{w} - 2\mathrm{i}\pi\mu\cdot\nu}\vartheta\begin{bmatrix}\mu\\\nu\end{bmatrix}(\mathbf{w}/2\mathrm{i}\pi - N\tau\cdot\epsilon|\tau),$$

where

$$\vartheta\begin{bmatrix}\mu\\\nu\end{bmatrix}(\mathbf{w}|\tau) = \sum_{\mathbf{p}\in\mathbb{Z}^{\mathfrak{g}}}\exp\left[\mathrm{i}\pi(\mathbf{p}+\mu)\tau(\mathbf{p}+\mu) + 2\mathrm{i}\pi(\mathbf{p}+\mu)(\mathbf{w}+\nu)\right],$$

$[\mu,\nu]$ plays the role of a characteristics, although we do not require it to be half-integer here. Most often, we shall omit to write the dependance in $[\mu,\nu]$, as well as the dependence in τ, ϵ, and we shall use the notation $\Theta',\Theta'',\ldots,\Theta^{(k)}$ for the tensor of derivatives with respect to $\mathbf{w}$. For instance

$$\Theta' = \left(\frac{\partial\Theta}{\partial w_1},\ldots,\frac{\partial\Theta}{\partial w_{\mathfrak{g}}}\right)^t.$$

This $\Theta_{[\mu,\nu]}$ function satisfies the heat equation

$$\frac{1}{2}\left(\partial_{\tau_{i,j}} + \partial_{\tau_{j,i}}\right)\Theta_{[\mu,\nu]}(\mathbf{w}) = \mathrm{i}\pi\Theta''(\mathbf{w})_{i,j},$$

where, in this equation, $\tau_{i,j}$ and $\tau_{j,i}$ are considered independent.

5.2.2 Definition and comments

Definition 5.4 ([35, 37])**.** For any $[\mu,\nu] \in \mathbb{C}^{\mathfrak{g}} \times \mathbb{C}^{\mathfrak{g}}$ and any spectral curve $\mathcal{S} = (\mathcal{C}, X, Y)$ of genus $\mathfrak{g}$, we define

$$\mathcal{T}_{[\mu,\nu]}[\mathcal{S}] = \exp\left(\sum_{g\geq 0}N^{2-2g}F_g\right)\left\{\sum_{k\geq 0}\sum_{l_i>0}\sum_{h_i>1-l_i/2}\frac{N^{\sum_i(2-2h_i-l_i)}}{k!l_1!\cdots l_k!}F_{h_1}^{(l_1)}\cdots F_{h_k}^{(l_k)}\Theta^{(\sum_i l_i)}\right\},$$

where we defined

$$F_h^{(k)} = \overbrace{\oint_{\mathcal{B}} \cdots \oint_{\mathcal{B}}}^{k \text{ times}} \omega_k^{(h)}(\mathcal{S}), \qquad \Theta^{(k)} = \nabla_{\mathbf{w}}^{\otimes k} \Theta_{[\mu,\nu]}(\mathbf{w} = \mathbf{w}_0 \,|\, \tau; \epsilon),$$

$$\epsilon = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{A}} \omega_1^{(0)}, \qquad \mathbf{w}_0 = N F_0' = N \oint_{\mathcal{B}} \omega_1^{(0)}, \qquad \tau = \frac{1}{2\mathrm{i}\pi} F_0'' = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}} \oint_{\mathcal{B}} \omega_2^{(0)}.$$

$\mathcal{T}$ is defined formally order by order in N (in the coefficient of $1/N^k$, each term $\Theta^{(j)}$ is considered formally of order 1). It can be seen as a genuine asymptotic series when $\mathcal{S}$ is such that $\mathbf{w}_0$ is of order $O(1)$.

To the first few orders in $1/N$ we have explicitly

$$\mathcal{T}_{[\mu,\nu]}[\mathcal{S}] = \mathrm{e}^{N^2 F_0} \mathrm{e}^{F_1} \left\{ \Theta + \frac{1}{N} \left(\Theta' F_1' + \frac{1}{6} \Theta''' F_0''' \right) + \frac{1}{N^2} \left(\Theta\, F_2 + \frac{1}{2} \Theta'' F_1'' \right.\right.$$
$$\left.\left. + \frac{1}{2} \Theta'' F_1'^2 + \frac{1}{24} \Theta^{(4)} F_0'''' + \frac{1}{6} \Theta^{(4)} F_0''' F_1' + \frac{1}{72} \Theta^{(6)} F_0'''^2 \right) + o\left(1/N^2 \right) \right\}.$$

Let us emphasize that this definition of $\mathcal{T}$ does not need require the computation of derivatives with respect to times, all the terms consist in cycle integrals on $\mathcal{C}$ at a given time.

Notice that when $\mu + \tau\nu$ is a non-singular half-integer odd characteristics, the leading term when $N \to \infty$ of $\mathcal{T}(\mathcal{S})$ coincides with the semiclassical Tau function of Section 4 computed for the differential form $\omega = NY\mathrm{d}X$, up to an exponential factor

$$\mathcal{T}[\mathcal{S}] \sim e^{N^2 F_0} \Theta(\mathbf{w}_0 | \tau) \sim e^{N^2 \tilde{F}_0} \, \theta(\zeta + \nu + \mu \cdot \tau) e^{2\mathrm{i}\pi\mu \cdot \zeta} \sim \mathcal{T}_{\mathrm{cl}}(NY\mathrm{d}X) e^{2\mathrm{i}\pi\mu \cdot \zeta}.$$

Notice that the Fay identity for θ presented in Section 2.1.2 is also true if we multiply θ by an exponential factor of its argument. Therefore, Theorem 4.2 ensures that the large N limit of $\mathcal{T}[\mathcal{S}]$ satisfies the Hirota bilinear equation (Theorem. 4.2), i.e. is a Tau function to leading order. This is also true for arbitrary characteristics $[\mu,\nu]$, although the times have to be shifted by a (maybe complex) constant in this case. We conjecture (see Section 7) that $\mathcal{T}$ is actually a Tau function, i.e. satisfies Hirota equations to all orders.

We also mention that under a modular transformation (i.e. a change of choice for the $\mathcal{A}$ and $\mathcal{B}$ cycles), $\mathcal{T}$ changes like the Siegel Theta function of characteristics $[\mu,\nu]$ (see [37])

Proposition 5.1 ([37]). *Under a modular* $\mathrm{Sp}(2\mathfrak{g}, \mathbb{Z})$ *transformation* $\tau \to \tilde{\tau} = (\alpha\tau+\beta)(\gamma\tau+\delta)^{-1}$, *the characteristics* $[\mu,\nu]$ *changes as* $\mu \to \tilde{\mu} = \delta\mu - \gamma\nu + \frac{1}{2}(\gamma\delta^t)_{\mathrm{diag}}$, $\nu \to \tilde{\nu} = -\beta\mu + \alpha\nu + \frac{1}{2}(\alpha\beta^t)_{\mathrm{diag}}$, *the Tau function* $\mathcal{T} \equiv \mathcal{T}_{[\mu,\nu]}$ *transforms as*

$$\mathcal{T}_{[\mu,\nu]} \to \zeta_{[\mu,\nu]}(\alpha,\beta,\gamma,\delta) \mathcal{T}_{[\tilde{\mu},\tilde{\nu}]},$$

where $\zeta_{[\mu,\nu]}(\alpha,\beta,\gamma,\delta)$ *is the phase factor, independent of the spectral curve.*

5.2.3 Heuristic motivation for the definition

Let us consider

$$Z_N[\mathcal{S}, \mathbf{n}] = \exp\left(\sum_{g=0}^{\infty} N^{2-2g} F_g(\mathcal{S}, \mathbf{n}) \right),$$

where we emphasize the dependence of $F_g(\mathcal{S}) = F_g(\mathcal{S}, \mathbf{n})$ in the vector $\mathbf{n} = (n_1, \ldots, n_{\mathfrak{g}})$ of filling fractions $n_k = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{A}_k} Y\mathrm{d}X$. Let $F_g(\mathcal{S}, \mathbf{n})$ be the symplectic invariants associated to the curve $\mathcal{S}$

whose vector of filling fractions is $\mathbf{n} = (n_1, \ldots, n_{\mathfrak{g}}) \in \mathbb{C}^{\mathfrak{g}}$. $\ln Z[\mathcal{S}, \mathbf{n}]$ is a formal Laurent series in N (in particular we emphasize that it contains no oscillatory terms, by opposition with the definition to come). The relevance of $Z[\mathcal{S}, \mathbf{n}]$ in topological strings has been pointed out in the work of Dijkgraaf and Vafa [27]. Definition 5.4 gives a precise meaning to the sum over all shifts of filling fractions by integers

$$\mathcal{T}_{\mu,\nu}[\mathcal{S}] \;\text{``=''} \sum_{\mathbf{n}' \in \mathbb{Z}^{\mathfrak{g}}} e^{2\mathrm{i}\pi \mathbf{n}' \cdot \nu} Z[\mathcal{S}', \mathbf{n} + \mathbf{n}' + \mu]. \tag{5.3}$$

Such sums over a lattice have also been considered in the context of the small dispersion limit of KdV in the genus $\mathfrak{g}$ regime [80]. In the context of hermitian matrix integrals, it has been used in [17, 35] to arrive to the formal asymptotic series of Definition 5.4. This series has then been proposed to describe non-perturbative effects in topological strings [37].

5.3 Baker–Akhiezer spinor kernel

We now define the spinor kernel $\psi(z_1, z_2; \mathcal{S})$ through Sato's relation

Definition 5.5.

$$\psi(z_1, z_2; \mathcal{S}) = \frac{\mathcal{T}(\mathcal{S}; \mathcal{S} + \frac{1}{N} \mathrm{d}S_{z_1, z_2})}{\mathcal{T}(\mathcal{S})} \sqrt{\mathrm{d}X(z_1)\mathrm{d}X(z_2)} = \frac{\mathcal{T}(\mathcal{S} + [z_1] - [z_2])}{\mathcal{T}(\mathcal{S})} \sqrt{\mathrm{d}X(z_1)\mathrm{d}X(z_2)},$$

where $[z_1] - [z_2] = \frac{1}{N} dS_{z_1, z_2}$ is Sato's notation, see equation (4.3).

$\psi(z_1, z_2)$ is again defined formally, order by order in $1/N$. The leading order coincides with ψ_{cl} introduced in Definition 3.1. Let us give the first few orders

$$\begin{aligned}
\psi(z_1, z_2; \mathcal{S}) = {} & \frac{e^{N \int_{z_2}^{z_1} Y \mathrm{d}X}}{E(z_1, z_2)} \frac{\Theta\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)}{\Theta\big(\mathbf{w}_0\big)} \Bigg\{ 1 + \frac{1}{N} \Bigg[\frac{1}{6} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \omega_3^{(0)} \\
& + \int_{z_2}^{z_1} \omega_1^{(1)} + \frac{1}{2} \frac{\Theta'\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)}{\Theta\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \oint_{\mathcal{B}} \omega_3^{(0)} \\
& + \frac{1}{2} \frac{\Theta''\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)}{\Theta\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)} \int_{z_2}^{z_1} \oint_{\mathcal{B}} \oint_{\mathcal{B}} \omega_3^{(0)} \\
& + \frac{1}{6} \left(\frac{\Theta'''\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)}{\Theta\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)} - \frac{\Theta'''\big(\mathbf{w}_0\big)}{\Theta\big(\mathbf{w}_0\big)} \right) F_0''' \\
& + \left(\frac{\Theta'\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)}{\Theta\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)} - \frac{\Theta'\big(\mathbf{w}_0\big)}{\Theta\big(\mathbf{w}_0\big)} \right) F_1' \Bigg] + o(1/N) \Bigg\}.
\end{aligned}$$

Lemma 5.1. $\psi(z_1, z_2; \mathcal{S})$ *is a well-defined spinor in z_1 and z_2. Furthermore*

- $\psi(z_1, z_2)$ *has a simple pole at $z_1 = z_2$*

$$\psi(z_1, z_2; \mathcal{S}) \underset{z_1 \to z_2}{\sim} \frac{\sqrt{\mathrm{d}X(z_1)\mathrm{d}X(z_2)}}{X(z_1) - X(z_2)}.$$

- *It has an essential singularity near any pole p of $Y\mathrm{d}X$, of the form $\mathrm{e}^{N \int_{z_2}^{z_1} Y \mathrm{d}X}$.*
- *At all orders in $1/N$ (except at leading order), $\psi(z_1, z_2)$ has poles at the ramification points a_i. Their order increase with the order of $1/N$.*

Proof. When z_1 goes around an $\mathcal{A}$-cycle, $\mathrm{d}S_{z_1,z_2}$ is unchanged. When z_1 goes around a cycle $\mathcal{B}_j$, $\mathrm{d}S_{z_1,z_2}$ is shifted by a holomorphic form $\mathrm{d}S_{z_1,z_2} \to \mathrm{d}S_{z_1,z_2} + 2\mathrm{i}\pi \mathrm{d}u_j$, which is dual to a $\partial/\partial\epsilon_j$, and since the Tau function is background independent (it was proved in [37] that $\partial\mathcal{T}/\partial\epsilon_i = 0$), then it is unchanged. This shows that $\psi(z_1, z_2; \mathcal{S})$ is a well-defined spinor.

Then we compute each term of $\mathcal{T}(\mathcal{S}+[z_1]-[z_2])$ by writing the Taylor expansion (Section 5.1.3)

$$F_g\left[\mathcal{S} + \frac{\lambda}{N}\mathrm{d}S_{z_1,z_2}\right] = \sum_{n\geq 0} \frac{\lambda^n}{n!N^n} \frac{\partial^n F_g}{\partial\lambda^n}\bigg|_{\lambda=0},$$

which we need to evaluate at $\lambda = 1$. The n-th derivatives of F_g at $\lambda = 0$ are computed by the special geometry relations Theorem 5.2, using the dual cycle $(\mathrm{d}S_{z_1,z_2})^* = [z_2, z_1]$

$$\frac{\partial^n F_g}{\partial\lambda^n}\bigg|_{\lambda=0} = \int_{z_2}^{z_1} \cdots \int_{z_2}^{z_1} \omega_n^{(g)}.$$

All the $\omega_n^{(g)}$ with $2-2g-n < 0$ are meromorphic, and have poles only at ramification points, without residues. This implies that their contribution to $\psi(z_1, z_2)$ provides only poles at ramification points. The only terms involving $\omega_n^{(g)}$ with $2 - 2g - n \geq 0$, are $\partial_\lambda F_0$, $\partial_\lambda^2 F_0$ and $\partial_\lambda F_0'$.

- $\frac{\partial F_0}{\partial\lambda}\big|_{\lambda=0} = \int_{z_2}^{z_1} Y\mathrm{d}X$, which contributes to ψ as the essential singularity $\mathrm{e}^{N\int_{z_2}^{z_1} Y\mathrm{d}X}$.

- $\frac{\partial^2 F_0}{\partial\lambda^2}\big|_{\lambda=0} = -\ln\left(E(z_1, z_2)^2\mathrm{d}X(z_1)\mathrm{d}X(z_2)\right)$ which contributes to ψ as $1/E(z_1, z_2)$.

- $\frac{\partial F_0'}{\partial\lambda}\big|_{\lambda=0} = 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))$ which does not yield any singularity.

All the other terms have $2 - 2g - n < 0$, and contribute order by order, only as combinations of meromorphic forms and derivatives of Theta functions, having poles at ramification points conveyed by the $\omega_n^{(g)}$'s with $2 - 2g - n < 0$. ∎

Notice that as a corollary of Proposition 5.1, ψ has nice modular properties:

Corollary 5.1. *Under a modular* $\mathrm{Sp}(2\mathfrak{g}, \mathbb{Z})$ *transformation with the notations of Proposition* 5.1, *the spinor kernel* $\psi_{[\mu,\nu]}(z_1, z_2)$ *transforms as*

$$\psi_{[\mu,\nu]}(z_1, z_2) \to \psi_{[\tilde{\mu},\tilde{\nu}]}(z_1, z_2).$$

6 Correlators

6.1 Second kind deformations of $\mathcal{S}$

Let us recall the definition of the insertion operator δ_z, already encountered in Section 4.3 and adapted now for varying spectral curves.

Definition 6.1. We define the insertion operator δ_z, acting on a functional $f(\mathcal{S})$ of a spectral curve $\mathcal{S}$, as follows

$$\delta_z f = \mathrm{d}X(z)\frac{\partial}{\partial\lambda}f(\mathcal{S}_\lambda)\bigg|_{\lambda=0},$$

where the family of spectral curves $\mathcal{S}_\lambda = \mathcal{S} + \lambda B(z, \cdot)/\mathrm{d}X(z)$ is such that

$$(Y\mathrm{d}X)_\lambda = Y\mathrm{d}X + \lambda\frac{B(z, \cdot)}{\mathrm{d}X(z)}.$$

In other words $\delta_z = \mathrm{d}X(z)\partial/\partial t_{z,1}$ as in Section 4.3.

The dual cycle of $B(z, \cdot)/\mathrm{d}X(z)$ is the contour surrounding z with index 1

$$B(z, \cdot) = \operatorname*{Res}_{z' \to z} B(z', \cdot) \frac{\mathrm{d}X(z)}{(X(z') - X(z))}.$$

Then, the relations of special geometry (Theorem 5.2) for $\omega_n^{(g)}$ imply, for any $n, g \geq 0$

$$\delta_z \omega_n^{(g)}(z_1, \ldots, z_n) = \operatorname*{Res}_{z' \to z} \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n) \frac{\mathrm{d}X(z)}{(X(z') - X(z))} = \omega_{n+1}^{(g)}(z, z_1, \ldots, z_n).$$

For instance

$$\delta_z F_0 = \omega_1^{(0)}(z) = Y(z)\mathrm{d}X(z), \qquad \delta_z F_g = \omega_1^{(g)}(z), \qquad \delta_z \omega_1^{(0)}(z') = B(z, z'),$$

and the conformal structure of $\mathcal{C}$ changes such that

$$\delta_z X(a_i) = \frac{B(z, a_i)}{\mathrm{d}Y(a_i)}.$$

Definition 6.2. For n positive integer, we define the correlators $W_n(z_1, \ldots, z_n)$ and the disconnected correlators $\overline{W}_n(z_1, \ldots, z_n)$ as

$$W_n(z_1, \ldots, z_n) = N^{-n} \delta_{z_1} \cdots \delta_{z_n} \ln \mathcal{T}(\mathcal{S}), \qquad \overline{W}_n(z_1, \ldots, z_n) = \frac{N^{-n} \delta_{z_1} \cdots \delta_{z_n} \mathcal{T}(\mathcal{S})}{\mathcal{T}(\mathcal{S})}.$$

$W_n(z_1, \ldots, z_n)$ and $\overline{W}_n(z_1, \ldots, z_n)$ are $(1, \ldots, 1)$-forms (n times), symmetric in their n variables. Each coefficient, order by order in $1/N$, is a meromorphic form with poles at ramification points.

6.2   Examples

For instance the three first orders of W_1 are

$$W_1(z) = NY(z)\mathrm{d}X(z) + (\ln \Theta)' \cdot 2\mathrm{i}\pi \mathrm{d}\mathbf{u}(z) + \frac{1}{N} \left\{ \omega_1^{(1)}(z) + \frac{\Theta''}{\Theta} \cdot (\mathrm{i}\pi \delta_z \tau) \right.$$

$$\left. + \left[F_1' \left(\frac{\Theta''}{\Theta} - \frac{\Theta'^2}{\Theta^2} \right) + \frac{F_0'''}{6} \left(\frac{\Theta''''}{\Theta} - \frac{\Theta'''\Theta'}{\Theta^2} \right) \right] \cdot 2\mathrm{i}\pi \mathrm{d}\mathbf{u}(z) \right\} + o(1/N),$$

where we recall that

$$\delta_z \tau_{j,k} = \frac{\delta_z F_0''}{2\mathrm{i}\pi} = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{B}_j} \oint_{\mathcal{B}_k} \omega_3^{(0)}(\cdot, \cdot, z)$$

$$= 4\mathrm{i}\pi \sum_l \operatorname*{Res}_{z' \to a_l} K(z, z')\mathrm{d}u_j(z')\mathrm{d}u_k(\overline{z'}) = 2\mathrm{i}\pi \sum_l \operatorname*{Res}_{z' \to a_l} \frac{B(z, z')\mathrm{d}u_j(z')\mathrm{d}u_k(z')}{\mathrm{d}X(z')\mathrm{d}Y(z')},$$

$$\omega_1^{(1)}(z_1) = \sum_l \operatorname*{Res}_{z \to a_l} K(z_1, z)B(z, \overline{z}).$$

For the 2-point correlator, the three first orders are

$$W_2(z_1, z_2) = B(z_1, z_2) + (\ln \Theta)'' \cdot 2\mathrm{i}\pi \mathrm{d}\mathbf{u}(z_1) \otimes 2\mathrm{i}\pi \mathrm{d}\mathbf{u}(z_2) + \frac{1}{N} \left\{ \frac{\Theta'}{\Theta} \cdot \int_{\mathcal{B}} \omega_3^{(0)}(\cdot, z_1, z_2) \right.$$

$$\left. + \left(\frac{\Theta''}{\Theta} \right)' \cdot \left[\mathrm{i}\pi(\delta_{z_1}\tau) \otimes 2\mathrm{i}\pi \mathrm{d}\mathbf{u}(z_2) + 2\mathrm{i}\pi \mathrm{d}\mathbf{u}(z_1) \otimes \mathrm{i}\pi(\delta_{z_2}\tau) \right] \right.$$

$$+ \left[(F_1)' \left(\frac{\Theta''}{\Theta} - \frac{\Theta'^2}{\Theta^2} \right)' + \frac{F_0'''}{6} \left(\frac{\Theta''''}{\Theta} - \frac{\Theta'''\Theta'}{\Theta^2} \right)' \right] \cdot 2i\pi d\mathbf{u}(z_1) \otimes 2i\pi d\mathbf{u}(z_2) \Bigg\}$$

$$+ \, o(1/N).$$

For $n \geq 3$, the leading order of the n-point correlator is a $O(1)$, and is obtained by successive applications of δ_z to the $(\ln \Theta)''$ term

$$W_n(z_1, \ldots, z_n) = (\ln \Theta)^{(n)} \cdot \bigotimes_{j=1}^{n} 2i\pi d\mathbf{u}(z_j) + \frac{1}{N} \Bigg\{ \delta_{n,3} \omega_3^{(0)}(z_1, z_2, z_3)$$

$$+ \sum_{j=1}^{n} \left(\frac{\Theta''}{\Theta} \right)^{(n-1)} \cdot (i\pi \delta_{z_j} \tau) \otimes \bigotimes_{k \neq j} 2i\pi d\mathbf{u}(z_k)$$

$$+ \sum_{1 \leq j < k \leq n} (\ln \Theta)^{(n-1)} \cdot \left(\int_{\mathcal{B}} \omega_3^{(0)}(\cdot, z_j, z_k) \right) \otimes \bigotimes_{l \neq j,k} 2i\pi d\mathbf{u}(z_l)$$

$$+ \left[(F_1)' \left(\frac{\Theta''}{\Theta} - \frac{\Theta'^2}{\Theta^2} \right)^{(n-1)} + \frac{F_0'''}{6} \left(\frac{\Theta''''}{\Theta} - \frac{\Theta'''\Theta'}{\Theta^2} \right)^{(n-1)} \right] \cdot \bigotimes_{j=1}^{n} 2i\pi d\mathbf{u}(z_j) \Bigg\}$$

$$+ \, o(1/N).$$

6.3 Loop equations

Theorem 6.1. *The dispersive Tau function obeys the loop equations. Namely, let us denote by Γ a contour separating the poles of $Y dX$, from the set of preimages of a point $x \in \mathbb{C}$*

$$(i) \quad \oint_{z \in \Gamma} \frac{\delta_z \ln \mathcal{T}[\mathcal{S}] - N(Y dX)(z)}{X(z) - x} = 0,$$

$$(ii) \quad \oint_{z \in \Gamma} \operatorname*{Res}_{z' \to z} \frac{1}{(X(z) - x)(X(z') - X(z))} \left(\frac{1}{\mathcal{T}[\mathcal{S}]} \delta_z \delta_{z'} \mathcal{T}[\mathcal{S}] - \frac{dX(z) dX(z')}{(X(z) - X(z'))^2} \right) = Q(x),$$

where Q is a rational function of x, whose only poles are those of $Y dX$, and with degree one less than that of $Y dX$.

Those loop equations can be written in terms of correlators by applying $\delta_{z_2} \cdots \delta_{z_n}$.

Theorem 6.2. *Let $J = \{z_2, \ldots, z_n\}$. The correlators satisfy*

(i) The linear loop equations. For all $n \geq 1$

$$\oint_{z \in \Gamma} \frac{1}{X(z) - x} \left(W_n(z, J) - \delta_{n,1} N (Y dX)(z) - \delta_{n,2} \frac{dX(z) dX(z_2)}{(X(z) - X(z_2))^2} \right) = 0.$$

(ii) The quadratic loop equations. For all $n \geq 1$

$$\oint_{z \in \Gamma} \operatorname*{Res}_{z' \to z} \frac{1}{(X(z) - x)(X(z') - X(z))} \Bigg\{ \sum_{I \subseteq J} W_{1+|I|}(z, I) \, W_{n-|I|}(z', J \setminus I)$$

$$+ \frac{1}{N^2} W_{n+1}(z, z', J) + dX(z) dX(z') \sum_{z_k \in J} d_{z_k} \left(\frac{W_{n-1}(J)}{(x - X(z_k)) dX(z_k)} \right) \Bigg\}$$

$$= Q_n(x; J)$$

defines a quantity $Q_n(x; J)$ which is a rational function of x, whose only poles are located at those of $Y dX$.

The important information in loop equations, is that those particular combinations of W_n's have no monodromies in the variable x around the branchpoints. Since every W_n has poles at ramification points to all orders in $1/N$, this is a highly non-trivial property.

Proof. The F_g were precisely introduced such that for any **n**

$$Z[\mathcal{S}, \mathbf{n}] = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S}, \mathbf{n} + \mu)\right)$$

is a solution of the loop equations. Since $\tau[\mathcal{S}]$ is constructed formally as a linear combination (see equation (5.3)) of such objects, $\mathcal{T}[\mathcal{S}]$ satisfy the same loop equations. As a matter of fact, $\mathcal{T}[\mathcal{S}]$ was introduced in [35] so as to solve those loop equations while preserving modularity. ∎

7 Hirota equations

We mentioned in Theorem 4.2 that the self-replication property of the Baker–Akhiezer spinor kernel ψ_{cl} is equivalent to an infinitesimal Fay identity for the semiclassical spectral curve, which is known in turn to be equivalent to Hirota equations. In a similar way, we conjecture here

Conjecture 7.1. ψ is self-replicating

$$\frac{1}{N}(\delta_z \psi)(z_1, z_2) = -\psi(z_1, z)\psi(z, z_2).$$

We have not been able to prove this conjecture. We prove in Appendix A that it holds up to $o(1/N)$. We argue in Section 9 that it is compatible with what is known for spectral curves coming from the one matrix model (hyperelliptic curves), or the two matrix model. Besides, these matrix models do not allow to reach all plane curves $\mathcal{S}$. The difficulty in finding a proof of Conjecture 7.1 comes from the singularities at ramification points. For instance, one can always write

$$\frac{1}{N}(\delta_z \psi)(z_1, z_2; \mathcal{S}) = \operatorname*{Res}_{z' \to z} \psi(z, z'; \mathcal{S})\psi(z', z; \mathcal{S} + [z_1] - [z_2]),$$

where $\mathcal{S} + [z_1] - [z_2] = \mathcal{S} + \frac{1}{N}\mathrm{d}S_{z_1, z_2}$. Since the integrand is a differential form on the Riemann surface $\mathcal{C}$ underlying $\mathcal{S}$, we can move the contour to the poles at the ramification points, and the pole at $z' = z_2$

$$\frac{1}{N}(\delta_z \psi)(z_1, z_2; \mathcal{S}) = -\psi(z_1, z; \mathcal{S})\psi(z, z_2; \mathcal{S}) - \sum_i \operatorname*{Res}_{z' \to a_i} \psi(z, z'; \mathcal{S})\psi(z', z; \mathcal{S} + [z_1] - [z_2]).$$

Then, it remains to show that the sum of residues at ramification points vanishes. So, Conjecture 7.1 is equivalent to

Conjecture 7.2.

$$\sum_i \operatorname*{Res}_{z' \to a_i} \psi(z, z'; \mathcal{S})\psi(z', z; \mathcal{S} + [z_1] - [z_2]) = 0.$$

In Appendix A, we check that this residue at each a_i is $o(1/N)$. This involves already non-trivial identities between Theta functions associated to a complex curve, like Fay identity, its degenerations and differentiations with respect to the moduli of the curve, and involves the precise expression of $\omega_3^{(0)}$ (equation (5.1)). We have not been able yet to find a general way to show that this residue is 0 to all orders in $1/N$.

In terms of the $\mathcal{T}$ function, Conjecture 7.1 can be rephrased

Conjecture 7.3. $\mathcal{T}$ *satisfy an infinitesimal version of Hirota equations*

$$\mathcal{T}[\mathcal{S}](\delta_z \mathcal{T})[\mathcal{S} + [z_1] - [z_2]] - (\delta_z \mathcal{T})[\mathcal{S}]\mathcal{T}[\mathcal{S} + [z_1] - [z_2]]$$
$$= -N\mathcal{T}[\mathcal{S} + [z_1] - [z]]\mathcal{T}[\mathcal{S} + [z] - [z_2]].$$

There is also a global version of the former conjecture. First, notice from our definition in Section 5 that

$$\mathcal{T}[(\mathcal{S} + [z_1] - [z_2]) + [z_3] - [z_4]] = \mathcal{T}[(\mathcal{S} + [z_3] - [z_4]) + [z_1] - [z_2]],$$

so omitting the parentheses makes sense, but

$$\mathcal{T}[\mathcal{S} + [z_1] - [z_4] + [z_3] - [z_2]] = -\mathcal{T}[\mathcal{S} + [z_1] - [z_2] + [z_3] - [z_4]].$$

This sign comes from the fact that the definition of $\mathcal{T}$ contains a regularization procedure (for $\int_{z_i}^{z_j} \int_{z_i}^{z_j} B$), whose result depends on the way we form the pairs of simple poles to add to $\mathcal{S}$.

Conjecture 7.4. $\mathcal{T}$ *satisfies Hirota equations*

$$\mathcal{T}[(\mathcal{S} + [z_1] - [z_2]) + [z_3] - [z_4]]\mathcal{T}[\mathcal{S}]$$
$$= \mathcal{T}[\mathcal{S} + [z_1] - [z_2]]\mathcal{T}[\mathcal{S} + [z_3] - [z_4]] - \mathcal{T}[\mathcal{S} + [z_3] - [z_2]]\mathcal{T}[\mathcal{S} + [z_1] - [z_4]].$$

Provided our conjectures hold, $\mathcal{T}$ and ψ are actually the Tau function and the "wave function" of a dispersive integrable system.

Proof of equivalence of Conjectures 7.3 and 7.4. We can obtain Conjecture 7.3 from Conjecture 7.4 by letting z_1 and z_2 merge to a point z. In the other direction, we use shorter notations

$$\mathcal{T}_{ijkl} = \mathcal{T}[\mathcal{S} + [z_i] - [z_j] + [z_k] - [z_l]], \qquad \mathcal{T}_{ij} = \mathcal{T}[\mathcal{S} + [z_i] - [z_j]], \qquad \mathcal{T} = \mathcal{T}[\mathcal{S}],$$

and we apply Conjecture 7.3 to the spectral curve $\mathcal{S} + [z_3] - [z_4]$

$$\delta_z \mathcal{T}_{1234} = \frac{\delta_z \mathcal{T}_{34}}{\mathcal{T}_{34}}\mathcal{T}_{1234} - N\frac{\mathcal{T}_{1z34}\mathcal{T}_{z234}}{\mathcal{T}_{34}} = \frac{\delta_z \mathcal{T}}{\mathcal{T}}\mathcal{T}_{1234} - N\frac{\mathcal{T}_{3z}\mathcal{T}_{z4}\mathcal{T}_{1234}}{\mathcal{T}\mathcal{T}_{34}} - N\frac{\mathcal{T}_{1z34}\mathcal{T}_{z234}}{\mathcal{T}_{34}}.$$

Exchanging the roles of $z_1 \leftrightarrow z_3$ and $z_2 \leftrightarrow z_4$ also gives

$$\delta_z \mathcal{T}_{1234} = \frac{\delta_z \mathcal{T}}{\mathcal{T}}\mathcal{T}_{1234} - N\frac{\mathcal{T}_{1z}\mathcal{T}_{z2}\mathcal{T}_{1234}}{\mathcal{T}\mathcal{T}_{12}} - N\frac{\mathcal{T}_{3z12}\mathcal{T}_{z412}}{\mathcal{T}_{12}},$$

and comparing the two, we may get rid of the terms involving δ_z

$$\mathcal{T}_{12}\mathcal{T}_{3z}\mathcal{T}_{z4}\mathcal{T}_{1234} + \mathcal{T}\mathcal{T}_{12}\mathcal{T}_{1z34}\mathcal{T}_{z234} = \mathcal{T}_{34}\mathcal{T}_{1z}\mathcal{T}_{z2}\mathcal{T}_{1234} + \mathcal{T}\mathcal{T}_{34}\mathcal{T}_{3z12}\mathcal{T}_{z412}. \tag{7.1}$$

Let us define

$$U_{1234} = \mathcal{T}\mathcal{T}_{1234} - \mathcal{T}_{12}\mathcal{T}_{34} + \mathcal{T}_{14}\mathcal{T}_{32},$$

which is the quantity which should vanish at the end of our computation. Notice that

$$\lim_{z_1 \to z_2} \mathcal{U}_{1234} = 0.$$

Indeed

$$\mathcal{U}_{1234} \underset{z_1 \to z_2}{\sim} \left(\frac{\mathcal{T}\mathcal{T}_{34}}{E_{12}} + \mathcal{T}\delta_1 \mathcal{T}_{34} - \frac{\mathcal{T}\mathcal{T}_{34}}{E_{12}} - \delta_1 \mathcal{T}\mathcal{T}_{34} + \mathcal{T}_{14}\mathcal{T}_{31}\right)$$

$$\underset{z_1 \to z_2}{\sim} \left(\mathcal{T} \delta_1 \mathcal{T}_{34} - \delta_1 \mathcal{T} \mathcal{T}_{34} + \mathcal{T}_{14} \mathcal{T}_{31} \right),$$

where $E_{ij} = E(z_i, z_j)$, and this expression vanish by application of Conjecture 7.3 to the spectral curve $\mathcal{S}$. Notice also that the remark about $\mathcal{T}$ made above Conjecture 7.4 implies $\mathcal{U}_{ijkl} = \mathcal{U}_{klij} = -\mathcal{U}_{ilkj}$. Let us rewrite equation (7.1) in terms of $\mathcal{U}_{ijkl}$ only

$$\begin{aligned}
\mathcal{T}_{12} \mathcal{U}_{1z34} \mathcal{U}_{z234} &- \mathcal{T}_{34} \mathcal{U}_{3z12} \mathcal{U}_{z412} + \left(\mathcal{T}_{12} \mathcal{T}_{3z} \mathcal{T}_{z4} - \mathcal{T}_{1z} \mathcal{T}_{z2} \mathcal{T}_{34} \right) \mathcal{U}_{1234} \\
&+ \left(\mathcal{T}_{1z} \mathcal{T}_{34} - \mathcal{T}_{14} \mathcal{T}_{3z} \right) \mathcal{T}_{12} \mathcal{U}_{z234} + \left(\mathcal{T}_{z2} \mathcal{T}_{34} - \mathcal{T}_{32} \mathcal{T}_{z4} \right) \mathcal{T}_{12} \mathcal{U}_{1z34} \\
&+ \left(\mathcal{T}_{32} \mathcal{T}_{1z} - \mathcal{T}_{12} \mathcal{T}_{3z} \right) \mathcal{T}_{34} \mathcal{U}_{z412} + \left(\mathcal{T}_{z2} \mathcal{T}_{14} - \mathcal{T}_{12} \mathcal{T}_{z4} \right) \mathcal{T}_{34} \mathcal{U}_{3z12} = 0. \quad (7.2)
\end{aligned}$$

The left hand side may have simple poles when $z_i \to z_j$ due to $\mathcal{T}_{ij}$, but not higher degree poles since we know that $\mathcal{U}_{ijkl}$ is actually regular when $z_i \to z_j$ or z_l. For equation (7.2) to hold, in particular, the coefficient of the simple pole when $z_2 \to z_3$ must vanish

$$-\mathcal{T}_{13} \mathcal{T}_{z4} \mathcal{U}_{1z34} + \mathcal{T}_{1z} \mathcal{T}_{34} \mathcal{U}_{z413} = 0,$$

which we can also write after reindexing the points and using the symmetries of $\mathcal{U}$

$$\mathcal{T}_{31} \mathcal{T}_{z4} \mathcal{U}_{1z34} = \mathcal{T}_{3z} \mathcal{T}_{14} \mathcal{U}_{z134}. \quad (7.3)$$

Similarly, the coefficient of the simple pole when $z_1 \to z_2$ must vanish

$$\mathcal{U}_{1z34} \mathcal{U}_{z134} + \left(\mathcal{T}_{z1} \mathcal{T}_{34} - \mathcal{T}_{z4} \mathcal{T}_{31} \right) \mathcal{U}_{1z34} + \left(\mathcal{T}_{1z} \mathcal{T}_{34} - \mathcal{T}_{14} \mathcal{T}_{3z} \right) \mathcal{U}_{z134} = 0.$$

Now, we may combine the latter with equation (7.3), set $z = z_2$ for convenience, and isolate $\mathcal{U}_{1234}$

$$\frac{\mathcal{T}_{31} \mathcal{T}_{24}}{\mathcal{T}_{32} \mathcal{T}_{14}} \mathcal{U}_{1234}^2 + \left(\left(\mathcal{T}_{21} \mathcal{T}_{34} - \mathcal{T}_{24} \mathcal{T}_{31} \right) + \frac{\mathcal{T}_{31} \mathcal{T}_{24}}{\mathcal{T}_{32} \mathcal{T}_{14}} \left(\mathcal{T}_{12} \mathcal{T}_{34} - \mathcal{T}_{14} \mathcal{T}_{32} \right) \right) \mathcal{U}_{1234} = 0.$$

If $\mathcal{U}_{1234}$ were not identically zero, we would have (by continuity of all the coefficients of the series) for any points z_1, z_2, z_3, z_4 on the curve

$$\mathcal{U}_{1234} = \frac{\mathcal{T}_{32} \mathcal{T}_{14}}{\mathcal{T}_{31} \mathcal{T}_{24}} \left(\mathcal{T}_{24} \mathcal{T}_{31} - \mathcal{T}_{21} \mathcal{T}_{34} \right) + \left(\mathcal{T}_{14} \mathcal{T}_{32} - \mathcal{T}_{12} \mathcal{T}_{34} \right).$$

But matching the coefficient of the simple pole when $z_1 \to z_4$ in this equation yields

$$0 = \frac{\mathcal{T}_{32}}{\mathcal{T}_{31} \mathcal{T}_{21}} \left(\mathcal{T}_{21} \mathcal{T}_{31} - \mathcal{T}_{21} \mathcal{T}_{31} \right) + \mathcal{T}_{32} = \mathcal{T}_{32},$$

which is not true. Therefore, $\mathcal{U}_{1234} \equiv 0$. $\blacksquare$

8 Consequences

8.1 Exponential formula

We start with a remark which does not rely on the conjectures of Section 7, but which is natural to present now. Recall that the kernel is defined by Sato's formula

$$\psi(z_1, z_2; \mathcal{S}) = \frac{\mathcal{T}(\mathcal{S} + \frac{1}{N} \mathrm{d} S_{z_1, z_2})}{\mathcal{T}(\mathcal{S})} \sqrt{\mathrm{d} X(z_1) \mathrm{d} X(z_2)}.$$

Adding a double pole can be realized by adding two simple poles and taking the limit where the two simple poles collapse. In other words, we can write

$$\mathrm{d} S_{z_1, z_2} = \int_{z_2}^{z_1} B = (X(z_1) - X(z_2)) B(\cdot, z_1) / \mathrm{d} X(z_1) + O((X(z_1) - X(z_2)))^2,$$

and then express ψ in terms of second kind deformations of $\mathcal{T}$, i.e. in terms of the correlators W_n. However, when we substitute second kind deformations instead of third kind deformations, we must pay attention to regularization for the term $\int_{z_2}^{z_1} \int_{z_2}^{z_1} B$, like in Sections 4.1 and 5.3. Besides, for any n, W_n has a finite number of terms which are not $O(1/N)$. For this reason we define the following quantities, which are $O(1/N)$

$$\widehat{W}_1(z) = W_1(z) - N\,Y\,\mathrm{d}X(z) - (\ln\Theta)' \cdot 2\mathrm{i}\pi\mathrm{d}\mathbf{u}(z),$$
$$\widehat{W}_2(z_1, z_2) = W_2(z_1, z_2) - B(z_1, z_2) - (\ln\Theta)'' \cdot 2\mathrm{i}\pi\mathrm{d}\mathbf{u}(z_1) \otimes 2\mathrm{i}\pi\mathrm{d}\mathbf{u}(z_2),$$

and for $n \geq 3$

$$\widehat{W}_n(z_1, \ldots, z_n) = W_n(z_1, \ldots, z_n) - (\ln\Theta)^{(n)} \cdot \bigotimes_{j=1}^n 2\mathrm{i}\pi\mathrm{d}\mathbf{u}(z_j).$$

Then, we have the exponential formula

Proposition 8.1.

$$\psi(z_1, z_2) = \frac{e^{N\int_{z_2}^{z_1} Y\mathrm{d}X}}{E(z_1, z_2)} \frac{\Theta_{12}}{\Theta} \exp\left(\sum_{n\geq 1} \frac{1}{n!} \int_{z_2}^{z_1} \cdots \int_{z_2}^{z_1} \widehat{W}_n \right),$$

where $\Theta_{12} = \Theta\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big)$ and $\Theta = \Theta(\mathbf{w}_0)$. This formula is an equality if we collect on both sides all the terms of the same order.

Proof. The Taylor formula allows to express $\ln\mathcal{T}\big[\mathcal{S}+[z_1]-[z_2]\big] - \ln\mathcal{T}\big[\mathcal{S}\big]$ to all orders in $1/N$. For the first and second order, we use equations (4.1) and (4.2)

$$\ln\mathcal{T}\big[\mathcal{S}+[z_1]-[z_2]\big] - \ln\mathcal{T}\big[\mathcal{S}\big] = \int_{z_2}^{z_1} N^{-1}\delta_\zeta(\ln\mathcal{T}) + \frac{1}{2} \text{``} \int_{z_2}^{z_1}\int_{z_2}^{z_1} N^{-2}\delta_\zeta\delta_{\zeta'}(\ln\mathcal{T})\big[\mathcal{S}\big] \text{''}$$
$$+ \sum_{n\geq 3} \frac{N^{-n}}{n!} \underbrace{\int_{z_2}^{z_1}\delta_{\zeta_1}\cdots\int_{z_2}^{z_1}\delta_{\zeta_n}}_{n\text{ times}}(\ln\mathcal{T})\big[\mathcal{S}\big]$$
$$= \int_{z_2}^{z_1} W_1 + \frac{1}{2}\text{``}\int_{z_2}^{z_1}\int_{z_2}^{z_1} W_2\text{''} + \sum_{n\geq 3}\frac{1}{n!}\underbrace{\int_{z_2}^{z_1}\cdots\int_{z_2}^{z_1}}_{n\text{ times}} W_n$$
$$= N\int_{z_2}^{z_1} Y\mathrm{d}X - \frac{1}{2}\ln\big((E(z_1,z_2))^2\mathrm{d}X(z_1)\mathrm{d}X(z_2)\big)$$
$$+ \sum_{n\geq 1}\frac{1}{n!}(\ln\Theta)^{(n)} \cdot \big(2\mathrm{i}\pi(\mathbf{u}(z_1)-\mathbf{u}(z_2))\big)^{\otimes n} + \sum_{n\geq 1}\frac{1}{n!}\underbrace{\int_{z_2}^{z_1}\cdots\int_{z_2}^{z_1}}_{n\text{ times}}\widehat{W}_n.$$

In the last step, we have used the expression for the leading order of W_n found in Section 6.2. Then, the second line can be resummated into $\ln\Theta_{12}$, and the whole result exponentiated leads to the announced formula. ∎

Notice that, when expanding the exponential, to any given order $O(N^{-k})$, all the W_n give a contribution involving derivatives of theta functions contracted with tensor products $(\mathbf{u}(z_1) - \mathbf{u}(z_2))^{\otimes n}$. These contributions have to be resummated into a single theta function (or derivatives thereof) with argument shifted by $\mathbf{u}(z_1) - \mathbf{u}(z_2)$, then producing an expression at the order sought involving only a finite number of terms.

8.2   Determinantal formulas

Conversely, the correlators $W_n(z_1, \ldots, z_n)$ can be retrieved from the spinor kernel $\psi(z_1, z_2)$, they are the determinantal correlation functions built with $\psi(z_i, z_j)$. Let us consider first the case of W_1, which does not rely on the conjectures of Section 7.

Lemma 8.1.

$$W_1(z) = N Y \mathrm{d}X(z) + \lim_{z' \to z} \left(\psi(z', z) e^{-N \int_{z_2}^{z_1} Y \mathrm{d}X} - \frac{\sqrt{\mathrm{d}X(z') \mathrm{d}X(z)}}{X(z') - X(z)} \right).$$

Proof. First, we notice that adding a double pole can be realized by adding two simple poles and take the limit where the two simple poles collapse. More precisely, when $z' \to z$, we have

$$\mathrm{d}S_{z',z}(z_0) \underset{z' \to z}{\sim} (X(z') - X(z)) \frac{B(z, z_0)}{\mathrm{d}X(z)}.$$

We can thus use the definition of δ_z with $\lambda = X(z') - X(z)$. For any regular functional $f[\mathcal{S}]$ of the spectral curve, we thus have

$$f \left[\mathcal{S} + \frac{1}{N} \mathrm{d}S_{z',z} \right] = f[\mathcal{S}] + \frac{X(z') - X(z)}{N \mathrm{d}X(z)} \delta_z f[\mathcal{S}] + O\big((X(z') - X(z))^2\big).$$

In some sense, we trade a variation of $\mathcal{S}$ by a second kind differential with a variation with a third kind differential.

In particular, F_g with $g \geq 1$, or every Θ-term in Definition 5.4 are regular functionals of the spectral curve. We just have to pay attention to the F_0 term, because the derivative of F_0 with respect to third kind differentials involves a regularization procedure, whereas the derivative with respect to second kind differentials does not. We have, by Taylor expansion, and computing all derivatives from special geometry (Theorem 5.2)

$$
\begin{aligned}
F_0\big[\mathcal{S} + \lambda \, \mathrm{d}S_{z',z}\big] &= F_0[\mathcal{S}] + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{\partial^n}{\partial \lambda^n} F_0\big[\mathcal{S} + \lambda \mathrm{d}S_{z',z}\big]\big|_{\lambda=0} \\
&= F_0[\mathcal{S}] + \lambda \int_z^{z'} Y \mathrm{d}X - \frac{\lambda^2}{2} \ln \big(E(z, z')^2 \mathrm{d}X(z) \mathrm{d}X(z')\big) \\
&\quad + \sum_{n=3}^{\infty} \frac{\lambda^n}{n!} \int_z^{z'} \cdots \int_z^{z'} W_n^{(0)} \\
&= F_0(\mathcal{S}) + \lambda \int_z^{z'} Y \mathrm{d}X - \frac{\lambda^2}{2} \ln \big(E(z, z')^2 \mathrm{d}X(z) \mathrm{d}X(z')\big) \\
&\quad + O\big((X(z') - X(z))^2\big).
\end{aligned}
$$

Taking $\lambda = 1/N$ gives

$$
\begin{aligned}
&F_0 \left[\mathcal{S} + \frac{1}{N} \mathrm{d}S_{z',z} \right] + \frac{1}{N^2} \ln \big(E(z', z) \sqrt{\mathrm{d}X(z) \mathrm{d}X(z')}\big) \\
&\qquad = F_0[\mathcal{S}] + \frac{X(z') - X(z)}{N \, \mathrm{d}X(z)} \delta_z F_0[\mathcal{S}] + O\big((X(z') - X(z))^2\big).
\end{aligned}
$$

Finally we have

$$\psi(z', z) \mathrm{e}^{-N \int_z^{z'} Y \mathrm{d}X} E(z', z) = 1 + \frac{X(z') - X(z)}{N \mathrm{d}X(z)} \delta_z \big(\ln \mathcal{T} - N^2 F_0\big) + O\big((X(z') - X(z))^2\big),$$

and thus

$$\psi(z',z)\mathrm{e}^{-N\int_z^{z'} Y\mathrm{d}X} - \frac{1}{E(z',z)} = \frac{X(z')-X(z)}{N\,\mathrm{d}X(z)E(z',z)}\delta_z\big(\ln\mathcal{T}-N^2F_0\big) + O(X(z')-X(z)).$$

Taking the limit $z' \to z$, and noticing that

$$\frac{1}{E(z',z)} = \sqrt{\mathrm{d}X(z)\mathrm{d}X(z')}/(X(z')-X(z)) + O(X(z')-X(z)),$$

we find

$$\frac{1}{N}\delta_z\big(\ln\mathcal{T}-N^2F_0\big) = W_1(z) - NY(z)\mathrm{d}X(z)$$

$$= \lim_{z'\to z}\left(\psi(z',z)\mathrm{e}^{-N\int_z^{z'}Y\mathrm{d}X} - \frac{\sqrt{\mathrm{d}X(z')\mathrm{d}X(z)}}{X(z')-X(z)}\right),$$

hence the lemma. ∎

Theorem 8.1. *If Conjecture 7.1 holds, then*

$$\forall\,n \geq 2, \qquad W_n(z_1,\ldots,z_n) = (-1)^{n+1}\sum_{\sigma\text{ cyclic perm.}}\prod_{i=1}^{n}\psi(z_i,z_{\sigma(i)}).$$

Equivalently

$$\overline{W}_n(z_1,\ldots,z_n) = \text{``det''}\,\psi(z_i,z_j),$$

where "det" means that when we decompose the determinant as a sum of permutations, each factor $\psi(z_i,z_i)$ should be replaced by $W_1(z_i)$.

Proof. The formula for W_1 is proved in Lemma 8.1. Then, we get the formula for W_n by recursively applying δ_{z_i} and using the self-replication of ψ. ∎

It is thus clear that this determinantal structure relies on the (conjectured) existence of Hirota equations for $\mathcal{T}$, i.e. on integrability.

8.3 Baker–Akhiezer functions

We recall the notations of Section 3. With the kernel $\psi(z_1,z_2)$ that we have just constructed, we introduce a $d\times d$ matrix $\Psi(x_1,x_2) = \psi(z^i(x_1),z^j(x_2))_{i,j=1,\ldots,d}$ for $x_1,x_2\in\mathbb{C}$, where $z^j(x)$ are the d preimages of x on the curve. This definition has to be regularized when x_1 or x_2 is equal to $X(p)$ where p is a pole of X or Y. In particular, it was explained in Section 3.3 how to take x_1 or $x_2 \to \infty$

$$\Psi(x) \text{ ``=''} \left[\Psi\big(z^j(x),\infty_I\big)\right]_{I,1\leq j\leq d}, \qquad \Phi(x) \text{ ``=''} \left[\Psi\big(\infty_I,z^j(x)\big)\right]_{I,1\leq j\leq d}.$$

8.4 Duality equation

Theorem 8.2. *If Conjecture 7.1 holds, we have for any spectral curve $\mathcal{S}$*

$$\Psi(x_1,x_2)\Psi(x_2,x_3) = \frac{(x_1-x_3)\mathrm{d}x_2}{(x_2-x_1)(x_3-x_1)}\Psi(x_1,x_3).$$

Proof. For $\mathcal{S} = (\mathcal{C}, x, y)$, we introduce an auxiliary spectral curve

$$\hat{\mathcal{S}}_{ij} = \mathcal{S} + \frac{1}{N} \, \mathrm{d}S_{z^i(x_1), z^j(x_3)}.$$

To compute the matrix element, we use the self-replication of ψ

$$[\Psi(x_1, x_2)\Psi(x_2, x_3)]_{ij} = \sum_m \psi(z^i(x_1), z^m(x_2))\psi(z^m(x_2), z^j(x_3))$$

$$= -\sum_m \delta_{z^m(x_2)} \ln\left(\frac{\mathcal{T}[\hat{\mathcal{S}}_{ij}]}{\mathcal{T}[\mathcal{S}]}\right).$$

Now, we use the fact that δ_z is a derivation, and by definition, $\delta_z \ln \mathcal{T}[\hat{\mathcal{S}}_{ij}] = W_1[z; \hat{\mathcal{S}}_{ij}]$

$$[\Psi(x_1, x_2)\Psi(x_2, x_3)]_{ij}$$
$$= -\sum_m W_1[z^m(x_2); \hat{\mathcal{S}}_{ij}]\psi(z^i(x_1), z^j(x_3)) + \sum_m W_1[z^m(x_2); \mathcal{S}]\psi(z^i(x_1), z^j(x_3)).$$

The linear loop equation (Theorem 6.1) tells us the sum over sheets of $W_1[z^m(x_2); \hat{\mathcal{S}}_{ij}]$

$$[\Psi(x_1, x_2)\Psi(x_2, x_3)]_{ij} = [\Psi(x_1, x_3)]_{ij}$$

$$\times \left(-\sum_m NY \mathrm{d}X(z^m(x_2)) - \mathrm{d}S_{z^i(x_1), z^j(x_3)}(z^m(x_2)) + NY \mathrm{d}X(z^m(x_2))\right)$$

$$= -[\Psi(x_1, x_3)]_{ij} \sum_m \mathrm{d}S_{z^i(x_1), z^j(x_3)}(z^m(x_2)) = \frac{(x_1 - x_3)\mathrm{d}x_2}{(x_1 - x_2)(x_2 - x_3)}[\Psi(x_1, x_3)]_{ij}. \qquad \blacksquare$$

The Baker–Akhiezer spinor kernel $\psi_{\mathrm{cl}}(z_1, z_2)$ of Section 3 was regular at ramification points. So, we could find a formula for $\psi_{\mathrm{cl}}(z_1, z)\psi_{\mathrm{cl}}(z, z_2)$ even before summing over the sheets where z is located (refined duality equation, Proposition 3.3). Here, the spinor kernel $\psi(z_1, z_2)$ does have, order by order in $1/N$, poles at ramification points. So, we do not have a simple expression for $\psi(z_1, z)\psi(z, z_2)$. However, Theorem 8.2 shows all contributions from the ramification points cancel in the sum over sheets.

8.5 Christoffel–Darboux relations

The matrix $\Psi(x_1, x_2)$ is invertible, since it is defined as a series for which the leading term coincides with $\Psi_{\mathrm{cl}}(x_1, x_2)$ which is invertible (see Lemma 3.2). We also have a duality relation to express the inverse

Corollary 8.1. *If Conjecture 7.1 holds,*

$$\Psi^{-1}(x_1, x_2) = -\frac{(x_1 - x_2)^2}{\mathrm{d}x_1 \mathrm{d}x_2}\Psi(x_2, x_1).$$

Theorem 8.3. *If Conjecture 7.1 holds, the matrices $\Psi(x)$ and $\Phi(x)$ are invertible, and we have the Christoffel–Darboux relation*

$$\psi(z_1, z_2) = \frac{\sum_{I,J} \psi_I(z_1) A_{I,J} \phi_J(z_2)}{X(z_1) - X(z_2)},$$

where the matrix A is invertible, independent of x and given by

$$A^{-1} = \frac{1}{\mathrm{d}x}\Phi(x)\Psi^t(x).$$

The proofs can be copied from the semiclassical case (Section 3.4), because they are based only on the duality equation.

8.6 Differential systems

$\Psi(x_1, x_2)$ is the solution of a system of differential equations with respect to the positions of the poles $X(p)$ ($x = \infty$ is a fixed pole) and the times $t_{p,j}$.

Theorem 8.4. *If Conjecture 7.1 holds, then for any deformation parameter $\lambda = t_{p,j}$ or $\lambda = X(p) \neq \infty$, there exists a $d \times d$ matrix $M_\lambda(x_1, x_2)$, such that*

(i) $(\partial_\lambda - M_\lambda)\Psi(x_1, x_2) = 0$;

(ii) M_λ *is a rational function of x_2;*

(iii) M_λ *has no pole at branchpoints.*

These deformations are compatible (since Ψ is invertible) and isomonodromic.

Implicitly, for $j = 0$, only deformations $\partial_\lambda = \partial_{t_{p,0}} - \partial_{t_{p',0}}$ are considered.

Proof. By definition

$$M_\lambda = \partial_\lambda \Psi(x_1, x_2)\, \Psi^{-1}(x_1, x_2) = -\frac{(x_1 - x_2)^2}{\mathrm{d}x_1 \mathrm{d}x_2} \partial_\lambda \Psi(x_1, x_2)\, \Psi(x_2, x_1).$$

When λ is a time $t_{p,k}$ ($k \geq 1$), we have by self-replication

$$\partial_{t_{p,k}} \psi(z_1, z_2) = -\int_{\omega^*_{p,k}} \psi(z_1, \cdot)\psi(\cdot, z_2).$$

We compute

$$[M_{t_{p,k}}]_{ij} = \frac{(x_1 - x_2)^2}{\mathrm{d}x_1 \mathrm{d}x_2} \sum_m \int_{\omega^*_{p,k}} \psi(z_1^i, z)\psi(z, z_2^m)\psi(z_2^m, z_1^j)$$

$$= \frac{1}{\mathrm{d}x_1} \int_{\omega^*_{p,k}} \frac{(X(\cdot) - x_1)(x_2 - x_1)}{(X(\cdot) - x_2)} \psi(z_1^i, \cdot)\psi(\cdot, z_1^j).$$

It is clear that $M_{t_{p,k}}(x_1, x_2)$ is a rational function of x_2 without poles at branchpoints. If we assume $X(p) \neq \infty$, it has a pole only at $x_2 = X(p)$, and this pole is of order $k + 1$. If we assume $X(p) = \infty$, it has a pole only at $x_2 = \infty$, and this pole is of order $1 + \lfloor k/d_p \rfloor$ (we recall that d_p is the multiplicity of order of the pole p of X). It is straightforward to adapt this proof for $k = 0$, namely $\partial_\lambda = \partial_{t_{p,0}} - \partial_{t_{p',0}}$.

Now, we turn to $\lambda = X(p) \neq \infty$. We have assumed that $\mathrm{d}X(p) \neq 0$, so the preimages $p^1, \ldots, p^d$ of $X(p)$ are distinct, and $(X(z) - X(p))$ is a local coordinate near each p^m. The Laurent expansion of $Y\mathrm{d}X(z)$ when $z \to p^m$ is

$$Y\mathrm{d}X(z) = \sum_{k \geq 0} t_{p,k} \frac{\mathrm{d}X(z)}{(X(z) - X(p))^{k+1}} + O(1),$$

where only a finite number of $t_{p,k'} = 0$ are non zero. If we perform $X(p) \to X(p) + \lambda$ while the times are fixed, we change $Y\mathrm{d}X$ to $(Y\mathrm{d}X)_\lambda$, with Laurent expansion at p^m

$$(Y\mathrm{d}X)_\lambda = Y\mathrm{d}X + \lambda \left(\sum_{k \geq 1} \frac{k t_{p,k-1}}{(X(z) - X(p))^{k+1}} \right) + O(1).$$

So, we can identify

$$\frac{\partial}{\partial X(p)} \longrightarrow \sum_{k \geq 1} k t_{p,k-1} \frac{\partial}{\partial t_{p,k}},$$

which is a finite sum. Hence

$$[M_{X(p)}]_{ij} = -\frac{1}{dx_1} \sum_m \operatorname*{Res}_{z \to z^m(X(p))} \frac{dY(z)}{dX(z)} \frac{(X(z) - x_1)(x_2 - x_1)}{(X(z) - x_2)} \psi(z_1^i, z) \psi(z, z_1^j).$$

This is a rational fraction of x_2, with a pole at $x_2 = X(p)$ of order

$$1 + \lfloor (\max_m \operatorname{ord}_{z^m(p)} Y dX)/d_p \rfloor. \qquad\qquad \blacksquare$$

One could also define a matrix L with the reconstruction formula presented in Section 3.5. But now, the function Y depends on the spectral curve, the evolution of L under the flows generated by the times is not isospectral. However, since the position and the order of the poles is fixed, the deformation parameters $t_{p,j}$ preserve the monodromies of Ψ. Such a reconstruction has also been performed at leading order by Bertola and Gekhtman [13], and our expressions up to $o(1)$ match their results.

9 Dictionary for matrix models

We recall here the correspondence between the definitions in integrability (and in this article), and the observables defined in matrix models. Let M be a random square matrix of size $N \times N$, diagonalizable by a unitary conjugation, with eigenvalues restricted to some contour Γ in the complex plane. The probability measure $d\mu(M)$ is a data of the matrix model. Its normalization defines the partition function $Z_N = \int d\mu(M)$. For a function f, we note $\langle f(M) \rangle$ the expectation value of the random variable $f(M)$

$$\langle f(M) \rangle = \frac{\int d\mu(M) f(M)}{\int d\mu(M)}.$$

9.1 Examples of integrable matrix models

Important examples of integrable matrix models, described in details in [69], include:

The one hermitian matrix model. $M \in H_N$ is a $N \times N$ hermitian matrix, with eigenvalues on the real axis

$$d\mu(M) = e^{-N \operatorname{Tr} V(M)} dM, \qquad dM = \prod_{i=1}^{N} dM_{i,i} \prod_{1 \le i < j \le N} d\operatorname{Re} M_{i,j} d\operatorname{Im} M_{i,j},$$

and $V(M)$ is a "semiclassical potential" (see [7]), i.e. $V'(x)$ is a rational function of its variable x, chosen such that $\int_{\mathbb{R}} e^{-V(x)} dx$ is absolutely convergent.

The one normal matrix model, with eigenvalues on a contour Γ. We define

$$H_N(\Gamma) = \left\{ U \operatorname{diag}(x_1, \ldots, x_N) U^\dagger, \ U \in U(N) \text{ and } (x_1, \ldots, x_N) \in \Gamma^N \right\}$$

equipped with a "measure" (not necessarily real or normalized)[3]

$$dM = dU \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^{N} dx_i,$$

---

[3] For a given $M \in H_N(\Gamma)$, there are many choices of U and x_i's (in fact U can be multiplied by any element of $U(1)^N$ and the x_i's can be permuted, and dU is in fact the measure on the quotient $U(N)/\mathfrak{S}_N \times U(1)^N$ by the Haar measure on $U(N)$.

where dU is the Haar measure on the unitary group $U(N)$ and dx_i is the curvilinear[4] measure along Γ. This measure is always invariant under unitary transformations. We then define the measure $d\mu(M)$ on $H_N(\Gamma)$

$$d\mu(M) = e^{-N\operatorname{Tr}V(M)}dM,$$

where $V'(x) \in \mathbb{C}(x)$, i.e. $V'(x)$ is a rational function of its variable x, chosen such that $\int_\Gamma e^{-V(x)}dx$ is absolutely convergent. This means that for a given potential $V(x)$, Γ must go to ∞ or to the poles of V', only in sectors where $\operatorname{Re}V(x) \to +\infty$. When $\Gamma = \mathbb{R}$, this definition is the same as the hermitian matrix model $H_N(\mathbb{R}) = H_N$, with the usual Lebesgue measure on H_N. When Γ is the unit circle $\mathbb{S}_1$, this definition is that of the unitary matrix model $H_N(\mathbb{S}_1) = U(N)$.

The two normal matrices model. Given two paths Γ_1, Γ_2, we define a measure on $H_N(\Gamma_1) \times H_N(\Gamma_2)$

$$d\mu_2(M_1, M_2) = e^{-N\operatorname{Tr}[V_1(M_1)+V_2(M_2)-M_1M_2]}dM_1dM_2,$$

where V_1 and V_2 are semiclassical potentials (V_1' and V_2' are rational functions), chosen such that integrals on $\Gamma_1 \times \Gamma_2$ are absolutely convergent. Upon integration on M_2, this measure induces a measure $d\mu(M)$ on M_1 that we rename M

$$d\mu(M_1) = \int_{M_2 \in H_N(\Gamma_2)} d\mu_2(M, M_2).$$

The chain of matrices. This is the natural generalization of the case of two matrices. Consider k paths $\Gamma_1, \ldots, \Gamma_k$, and k semiclassical potentials $V_1, \ldots, V_k$, and define the measure on $H_N(\Gamma_1) \times \cdots \times H_N(\Gamma_k)$ as

$$d\mu_k(M_1, \ldots, M_k) = e^{-N\operatorname{Tr}\left[\sum\limits_{i=1}^{k} V_i(M_i) - \sum\limits_{i=1}^{k-1} M_iM_{i+1}\right]}dM_1 \cdots dM_k.$$

For any $i \in \{1, \ldots, k\}$, we integrate on M_j's with $j \neq i$, and renaming $M = M_i$, this measure induces a measure $d\mu(M)$:

$$d\mu(M_i) = \int_{M_j \in H_N(\Gamma_j), j \neq i} d\mu_k(M_1, \ldots, M_k).$$

9.2   Correspondences

9.2.1   Partition function

All the listed matrix models above have the property to be integrable, in a sense explained below. There is a huge literature on the subject, let us mention among others the early works in physics [57, 61], and in mathematics [1, 46, 78]. For the one and two matrices models, it is established that the partition function $Z_N = \int d\mu(M)$ is an isomonodromic Tau function [10, 11, 14], but such a result is not known at present for the chain of matrices. The so-called double scaling limit of matrix models has been also intensively from the point of view of isomonodromic deformations [28, 43, 50, 70].

---

[4]If $\Gamma \subset \mathbb{C}$ is a path in the complex plane parametrized by a C^1 function $\gamma : \mathbb{R} \to \Gamma \subset \mathbb{C}$, i.e. $\Gamma = \{\gamma(s), s \in \mathbb{R}\}$, at $x = \gamma(s)$ we define the curvilinear measure $dx = \gamma'(s)ds$ where ds is the Lebesgue measure on $\mathbb{R}$.

9.2.2   Correlators, spectral curves, loop equations

For all the matrix models above, the correlators are

$$\mathcal{W}_n(x_1, \ldots, x_n) = N^{-n} \Big\langle \prod_{i=1}^{n} \mathrm{Tr}\, \frac{1}{x_i - M} \Big\rangle_C, \qquad \overline{\mathcal{W}}_n(x_1, \ldots, x_n) = N^{-n} \Big\langle \prod_{i=1}^{n} \mathrm{Tr}\, \frac{1}{x_i - M} \Big\rangle,$$

where C means 'cumulant'. $\mathcal{W}_n$ is often called n-point function, or *connected* n-point function, and $\overline{\mathcal{W}}_n$ is called disconnected n-point function.

The one point function $\mathcal{W}_1(x)$ (also called *resolvent*) plays an important role. In all the matrix models above, under reasonable assumptions, it has a large N limit denoted

$$\frac{1}{N} \mathcal{W}_1(x) \underset{N \to \infty}{\sim} \mathcal{W}_1^{(0)}(x) = \mathcal{Y}(x),$$

which is furthermore an algebraic function of x, i.e. there exists a polynomial $\mathcal{E}(x, y)$ such that $\mathcal{E}(x, \mathcal{Y}(x)) = 0$. Hence, there exists a spectral curve $\mathcal{S}_{\mathrm{MM}} = (\mathcal{C}, X, Y)$, such that

$$\forall\, z \in \mathcal{C}, \qquad \mathcal{Y}(X(z)) = Y(z).$$

If we define

$$W_n(z_1, \ldots, z_n) = N^n\, \mathcal{W}_n(X(z_1), \ldots, X(z_n)) \bigotimes_{i=1}^{n} \mathrm{d}X(z_i) + \delta_{n,2} \frac{\mathrm{d}X(z_1) \otimes \mathrm{d}X(z_2)}{(X(z_1) - X(z_2))^2},$$

it is a classical result of random matrix theory (for instance it can be proved by integration by parts in the matrix integral) that, for all matrix models listed above, W_n satisfy the loop equations of Theorem 6.1.

9.2.3   Large N asymptotic expansion

It is conjectured that the large N asymptotic expansion of the partition function Z_N matches with that of $\mathcal{T}[\mathcal{S}_{\mathrm{MM}}]$ function for the spectral curve $\mathcal{S}$ introduced in Definition 5.4, i.e. has an asymptotic expansion

$$\ln Z_N = N^2 F_0 + F_1 + \ln \Theta + \frac{1}{N} \left(F_1' \frac{\Theta'}{\Theta} + F_0''' \frac{\Theta'''}{\Theta} \right) + o(1/N).$$

The characteristics $[\mu, \nu]$ of the Theta function Θ is determined by the choice of the integration contour Γ. This conjecture was derived heuristically in [17, 35].

When the semiclassical spectral curve is of genus 0, there is no Theta function and the expansion involves only powers of $1/N$ (in fact, powers of $1/N^2$). This happens for the so-called "one-cut regime", and for the one matrix model, the existence of such an expansion has been proved for the one matrix model for real-valued, analytic potential [3], and then the coefficients are necessarily given by the symplectic invariants of the semiclassical spectral curve [36]. Beyond the one-cut regime, the Riemann–Hilbert steepest descent analysis [26, 50] has been applied to find explicitly the asymptotics up to $o(1)$ in the one hermitian matrix model with real-valued, polynomial potential [16, 24]. It features in general a pseudo-periodic behavior with N, encoded in the Theta function, but the one-cut regime can also be retrieved with this Riemann–Hilbert method [34]. These results have later been extended to the one normal matrix model with complex-valued polynomial potential [15]. This method can be used in principle to find recursively the subleading orders, although it does not allow to write the answer a priori explicitly to all orders. Definition 5.4 is expected to give the correct answer to all orders.

9.2.4 Baker–Akhiezer spinor kernel

The spinor kernel is related to the expectation value of ratios of characteristic polynomials

$$\psi(z_1, z_2) = \left\langle \frac{\det(X(z_1) - M)}{\det(X(z_2) - M)} \right\rangle \frac{\sqrt{\mathrm{d}X(z_1)\mathrm{d}X(z_2)}}{X(z_1) - X(z_2)}.$$

The Baker–Akhiezer spinor kernel $\psi_{\mathrm{cl}}(z_1, z_2)$ is the large N limit of $\psi(z_1, z_2)$. When sending $X(z_2) \to \infty$, one gets, after proper renormalization

$$\psi(z_1) = \text{``}\psi(z_1, \infty)\text{''} = \left\langle \det(X(z_1) - M) \right\rangle,$$

which is clearly a polynomial in $X(z_1)$ of degree N. It is a classical result of random matrix theory [69] that this expectation value of the characteristic polynomial of M is the orthogonal polynomial of degree N, for the orthogonality given by the measure $\mathrm{d}\mu$ for matrices of size 1. The dual function for matrix models is

$$\phi(z_1) = \text{``}\psi(\infty, z_1)\text{''} = \left\langle \frac{1}{\det(X(z_1) - M)} \right\rangle.$$

It can be derived from orthogonality relations of the orthogonal polynomials that those spinor kernels satisfy Hirota equations, also called "determinantal formula", for instance,

$$\frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_3 - x_4)} \left\langle \frac{\det(x_1 - M)\det(x_2 - M)}{\det(x_3 - M)\det(x_4 - M)} \right\rangle$$
$$= \frac{1}{(x_1 - x_3)(x_2 - x_4)} \left\langle \frac{\det(x_1 - M)}{\det(x_3 - M)} \right\rangle \left\langle \frac{\det(x_2 - M)}{\det(x_4 - M)} \right\rangle$$
$$- \frac{1}{(x_1 - x_4)(x_2 - x_3)} \left\langle \frac{\det(x_1 - M)}{\det(x_4 - M)} \right\rangle \left\langle \frac{\det(x_2 - M)}{\det(x_3 - M)} \right\rangle.$$

In that case, N is an integer (the size of the matrices), and Hirota equation is an equality between sequences indexed by N, not only of formal asymptotic series. This relations have been proved in [44] for the one matrix model, and in [5, 2] for the two matrices model, from which the case of chain of matrices can be deduced. It shows that, modulo the heuristic derivation of asymptotics of matrix integrals, Hirota equations (as stated as in Conjecture 7.4) hold for the semiclassical spectral curves of matrix models (however, not all algebraic curves can be reached in this way).

9.2.5 Differential systems

The orthogonal polynomials, as well as their duals, in all cases above, do satisfy ODE's of order d (d depends on the degrees of potentials). As we have seen in Section 8, the proof is a consequence of Hirota equations. For instance for the one matrix model, this is a second order ODE ($d = 2$). For the two matrices model with potentials V_1 and V_2, we always have $d = 1 + \deg V_2'$ (in case V_2' is a rational function, $\deg V_2'$ is the sum of degrees of all poles). In all cases we have a d-dimensional vector

$$\vec{\Psi}(z) = \begin{pmatrix} \psi(z_1) \\ \vdots \end{pmatrix}, \tag{9.1}$$

where the first entry is $\psi(z_1)$. The other entries are obtained from $\psi(z_1)$ with a procedure described in [8] and very similar to Section 3.3. This vector satisfies an ODE

$$\frac{\mathrm{d}}{\mathrm{d}X(z)} \vec{\Psi}(z) = \mathcal{D}(X(z)) \vec{\Psi}(z),$$

where $\mathcal{D}(x)$ is a $d \times d$ matrix, whose entries are rational functions of x, that depend implicitly on N, on the coefficients of the potentials, and on the choice of integration contour Γ. The locus of eigenvalues of $\mathcal{D}(x)$, i.e. the polynomial equation

$$\mathcal{E}_N(x, y) = \det(y - \mathcal{D}(x)) = 0 \tag{9.2}$$

defines a spectral curve for any finite N. The semiclassical spectral curve (i.e. its large N limit) coincides with the spectral curve $\mathcal{S}_{\mathrm{MM}}$

$$\mathcal{E}(x, y) = \lim_{N \to \infty} \mathcal{E}_N(x, y).$$

There are also $d \times d$ differential systems for derivatives with respect to all coefficients of the potentials, and there is also a linear recursion relation on $N \to N + 1$ (see [23] for the one matrix model). All these systems are compatible as shown in [9] in full generality for the chain of matrices.

9.2.6 Symplectic invariance

Notice that, in the two matrices model

$$Z = \int_{H_N(\Gamma) \times H_N(\tilde{\Gamma})} \mathrm{d}\mu(M_1, M_2),$$

we have defined our semiclassical spectral curve $\mathcal{S} = (\mathcal{C}, X, Y)$ from the large N limit of the resolvent $\mathcal{W}_1(x) = \langle \mathrm{tr} \frac{1}{x - M_1} \rangle$ associated to the matrix M_1. Since M_1 and M_2 play a symmetric role, it is clear that we would have obtained the same partition function, starting from the semiclassical spectral curve $\tilde{\mathcal{S}} = (\tilde{\mathcal{C}}, \tilde{X}, \tilde{Y})$ associated to the resolvent of matrix M_2, and thus we must have

$$\mathcal{T}[\mathcal{S}] = \mathcal{T}[\tilde{\mathcal{S}}].$$

One can easily find that the two spectral curves $\mathcal{S}_{\mathrm{MM}} = (\mathcal{C}, X, Y)$ and $\tilde{\mathcal{S}}_{\mathrm{MM}} = (\tilde{\mathcal{C}}, \tilde{X}, \tilde{Y})$ are related by $\mathcal{C} = \tilde{\mathcal{C}}$ and $\tilde{X} = Y$, $\tilde{Y} = X$, in other words they arse symplectically equivalent. Hence, the fact that $\mathcal{T}[\mathcal{S}_{\mathrm{MM}}] = \mathcal{T}[\tilde{\mathcal{S}}_{\mathrm{MM}}]$ can be seen at all order in the large N expansion as a consequence of the symplectic invariance of the F_g's. In fact, this is a manifestation at large N of an exact result for finite N. The orthogonal polynomials $\vec{\Psi}(x)$ associated to matrix M_1 satisfy an ODE of some order d (see equation (9.1))

$$\frac{\mathrm{d}}{\mathrm{d}X(z)} \vec{\Psi}(z) = \mathcal{D}(X(z)) \vec{\Psi}(z),$$

whereas the orthogonal polynomials $\vec{\tilde{\Psi}}(y)$ associated to matrix M_2 satisfy another ODE of some order $\tilde{d}$ (in general $\tilde{d} \neq d$)

$$\frac{\mathrm{d}}{\mathrm{d}Y(z)} \vec{\tilde{\Psi}}(z) = \tilde{\mathcal{D}}(Y(z)) \vec{\tilde{\Psi}}(z).$$

It was discovered in [12] that

$$\det\left(y \, \mathbf{1}_{d \times d} - \mathcal{D}(x)\right) = \det\left(x \, \mathbf{1}_{\tilde{d} \times \tilde{d}} - \tilde{\mathcal{D}}(y)\right).$$

This also implies that the semiclassical spectral curve defined from M_1 or from M_2 are related as explained above.

10 Conclusion

For integrable systems with a small dispersive parameter $1/N$, using the theory of symplectic invariants [38], we have introduced a formal object $\mathcal{T}$, which is conjectured to be a Tau function in the sense that it satisfies Hirota equations. It is challenging to find a full proof of Conjecture 7.4, and that would certainly prove that a certain quantity constructed out of $\mathcal{T}$ (like $u(x,t) = 2(\ln \mathcal{T})_{xx}$ for KdV) provides the all-order asymptotics of solutions of nonlinear integrable PDE's in the small dispersion limit.

One can wonder how to generalize the construction. First, to algebraic curves in $\mathbb{C}^* \times \mathbb{C}^*$, having in view the mirror curves appearing in Gromov–Witten theory, which all are of the form $\mathrm{Pol}(e^x, e^y) = 0$. In a recent work, we have described an application of the expressions for $\mathcal{T}$ and ψ for such curves to the computation of perturbative knot invariants [18]. This explicit example suggests that the construction remains meaningful in this context. Second, it should be possible to extend our construction to the bundles appearing in generalized matrix models like the $O(n)$ model, which might be related to isomonodromic deformations on Riemann surfaces of positive genus and to Hitchin systems [48]. And, even further to $\mathcal{D}$-modules, for which an adapted topological recursion (the so-called β-deformation of the topological recursion) is being developed with similar properties [21, 22].

Eventually, it remains to compare this construction with other approaches (Frobenius manifolds [29] and construction of integrable hierarchies of topological type [33], Poisson bracket structures, Segal–Wilson formalism in the Grassmannian [76], etc.), study its consequences and better understand the underlying geometry.

A Proof of Conjecture 7.1 up to $o(1/N)$

Proposition A.1. $\psi(z_1, z_2)$ is self-replicating at least up to $o(1/N)$,

$$\frac{1}{N}\delta_z \psi(z_1, z_2) + \psi(z_1, z)\psi(z, z_2) = o(1/N).$$

Proof. Let us start from $\psi_{12} = \psi(z_1, z_2)$ written as

$$\psi_{12} = \frac{\mathrm{e}^{N \int_2^1 Y \mathrm{d}X}}{E_{12}} \frac{\Theta_{12}}{\Theta} \left\{ 1 + \frac{1}{N}\hat{\psi}_{12} + o(1/N) \right\},$$

with

$$\hat{\psi}_{12} = \int_2^1 \omega_1^{(1)} + \frac{1}{6}\int_2^1 \int_2^1 \int_2^1 \omega_3^{(0)} + \frac{1}{2}\frac{\Theta_{12}''}{\Theta_{12}} \oint \oint \int_2^1 \omega_3^{(0)} + \frac{1}{2}\frac{\Theta_{12}'}{\Theta_{12}} \oint \int_2^1 \int_2^1 \omega_3^{(0)}$$
$$+ \left(\frac{\Theta_{12}'}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right) F_1' + \frac{1}{6}\left(\frac{\Theta_{12}'''}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right) F_0''',$$

where, to shorten notations, 1 means z_1, 2 means z_2, $\oint$ means the contour integral around $\mathcal{B}$-cycles (indices are understood in tensor notations, i.e. contracted with the indices of derivatives of Θ), and Θ_{12} means

$$\Theta_{12} = \Theta\big(\mathbf{w}_0 + 2\mathrm{i}\pi(\mathbf{u}(z_1) - \mathbf{u}(z_2))\big), \qquad \Theta = \Theta(\mathbf{w}_0).$$

We remind that (equation (5.1))

$$\omega_3^{(0)}(z_0, z_1, z_2) = \sum_i \operatorname*{Res}_{z \to a_i} \frac{B(z_0, z)B(z_1, z)B(z_2, z)}{\mathrm{d}X(z)\,\mathrm{d}Y(z)},$$

and by special geometry

$$F_0''' = \oint_{\mathcal{B}} \oint_{\mathcal{B}} \oint_{\mathcal{B}} \omega_3^{(0)} = \sum_i \operatorname*{Res}_{z \to a_i} \frac{(\mathrm{d}\mathbf{v}(z))^3}{\mathrm{d}X(z)\mathrm{d}Y(z)},$$

where we have set

$$\mathrm{d}\mathbf{v}(z) = 2\mathrm{i}\pi\mathrm{d}\mathbf{u}(z) = \oint_{\mathcal{B}} B(z, \cdot).$$

The expression for $\hat{\psi}_{12}$ is thus

$$
\begin{aligned}
\hat{\psi}_{12} &= \int_2^1 \omega_1^{(1)} + \frac{1}{6}\int_2^1\int_2^1\int_2^1 \omega_3^{(0)} + \frac{1}{2}\frac{\Theta_{12}''}{\Theta_{12}}\oint\oint\int_2^1 \omega_3^{(0)} + \frac{1}{2}\left(\frac{\Theta_{12}'}{\Theta_{12}}\right)\oint\int_2^1\int_2^1 \omega_3^{(0)} \\
&\quad + \left(\frac{\Theta_{12}'}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)F_1' + \frac{1}{6}\left(\frac{\Theta_{12}'''}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)F_0''' \\
&= \int_2^1 \omega_1^{(1)} + \left(\frac{\Theta_{12}'}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)F_1' + \sum_i \operatorname*{Res}_{z \to a_i} \frac{1}{\mathrm{d}X(z)\mathrm{d}Y(z)}\left\{\frac{1}{6}(\mathrm{d}S_{12}(z))^3\right. \\
&\quad + \frac{1}{2}\left(\frac{\Theta_{12}'}{\Theta_{12}}\right)\mathrm{d}\mathbf{v}(z)(\mathrm{d}S_{12}(z))^2 \\
&\quad \left. + \frac{1}{2}\left(\frac{\Theta_{12}''}{\Theta_{12}}\right)(\mathrm{d}\mathbf{v}(z))^2\mathrm{d}S_{12}(z) + \frac{1}{6}\left(\frac{\Theta_{12}'''}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)(\mathrm{d}\mathbf{v}(z))^3\right\}.
\end{aligned}
$$

We need to apply the insertion operator $\frac{1}{N}\delta_z$ to $\hat{\psi}_{12}$, and obtain the result up to $o(1)$. Only the variation of NF_0' appearing in the Theta functions contributes to this order

$$
\begin{aligned}
\frac{1}{N}\delta_z\hat{\psi}_{12} &= \left[\left(\frac{\Theta_{12}'}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)'F_1' + \sum_i \operatorname*{Res}_{z' \to a_i} \frac{1}{\mathrm{d}X(z')\mathrm{d}Y(z')}\left\{\frac{1}{2}\left(\frac{\Theta_{12}'}{\Theta_{12}}\right)'\mathrm{d}\mathbf{v}(z')(\mathrm{d}S_{12}(z'))^2\right.\right. \\
&\quad \left.\left. + \frac{1}{2}\left(\frac{\Theta_{12}''}{\Theta_{12}}\right)'(\mathrm{d}\mathbf{v}(z'))^2\mathrm{d}S_{12}(z') + \frac{1}{6}\left(\frac{\Theta_{12}'''}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)'(\mathrm{d}\mathbf{v}(z'))^3\right\}\right]\mathrm{d}\mathbf{v}(z) + o(1).
\end{aligned}
$$

This allows us to compute

$$
\begin{aligned}
\frac{1}{N}\delta_z\ln\psi_{12} &= \mathrm{d}S_{12}(z) + \left(\frac{\Theta_{12}'}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)\mathrm{d}\mathbf{v}(z) + \frac{1}{2N}\int_2^1\int_2^1 \omega_3^{(0)}(z, \cdot, \cdot) \\
&\quad + \frac{1}{N}\left(\frac{\Theta_{12}'}{\Theta_{12}}\right)\oint\int_2^1 \omega_3^{(0)}(z, \cdot, \cdot) + \frac{1}{2N}\left(\frac{\Theta_{12}''}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)\oint\oint \omega_3^{(0)}(z, \cdot, \cdot) \\
&\quad + \frac{1}{N^2}\delta_z\hat{\psi}_{12} + o(1/N) \\
&= \mathrm{d}S_{12}(z) + \mathrm{d}\mathbf{v}(z)\left(\frac{\Theta_{12}'}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right) + \frac{1}{N}\sum_i \operatorname*{Res}_{z' \to a_i} \frac{B(z', z)}{\mathrm{d}X(z')\mathrm{d}Y(z')}\left\{\frac{1}{2}(\mathrm{d}S_{12}(z))^2\right. \\
&\quad \left. + \left(\frac{\Theta_{12}'}{\Theta_{12}}\right)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z') + \frac{1}{2}\left(\frac{\Theta_{12}''}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)(\mathrm{d}\mathbf{v}(z'))^2\right\} + \frac{1}{N^2}\delta_z\hat{\psi}_{12} + o(1/N) \\
&= \mathrm{d}S_{12}(z) + \left(\frac{\Theta_{12}'}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)\mathrm{d}\mathbf{v}(z) + \left(\frac{\Theta_{12}'}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)'F_1'\mathrm{d}\mathbf{v}(z) \\
&\quad + \frac{1}{N}\sum_i \operatorname*{Res}_{z' \to a_i} \frac{1}{\mathrm{d}X(z')\mathrm{d}Y(z')}\left\{\frac{1}{2}(\mathrm{d}S_{12}(z'))^2 B(z', z)\right.
\end{aligned}
$$

$$+ \left(\frac{\Theta'_{12}}{\Theta_{12}}\right) d\mathbf{v}(z') dS_{12}(z') B(z',z) + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)(d\mathbf{v}(z'))^2 B(z',z)$$

$$+ \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}}\right)' (d\mathbf{v}(z'))^2 d\mathbf{v}(z') dS_{12}(z) + \frac{1}{2}\left(\frac{\Theta'_{12}}{\Theta_{12}}\right)' d\mathbf{v}(z')(dS_{12}(z'))^2 d\mathbf{v}(z)$$

$$+ \frac{1}{6}\left(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)' (d\mathbf{v}(z'))^3 d\mathbf{v}(z)\Bigg\} + o(1/N).$$

We may transform the first line using the refined duality equation established in Proposition 3.3, which is a consequence of the Fay identity satisfied by the Theta function of the spectral curve. It can be rephrased as

$$dS_{12}(z) + \left(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right) d\mathbf{v}(z) = -\frac{E_{12}}{E_{1z}E_{z2}}\frac{\Theta_{1z}\Theta_{z2}}{\Theta\Theta_{12}}. \tag{A.1}$$

On the other hand we have

$$\frac{\psi_{1z}\psi_{z2}}{\psi_{12}} = \frac{E_{12}}{E_{1z}E_{z2}}\frac{\Theta_{1z}\Theta_{z2}}{\Theta\Theta_{12}}\left(1 + \frac{1}{N}\left(\hat{\psi}_{1z} + \hat{\psi}_{z2} - \hat{\psi}_{12}\right) + o(1/N)\right)$$

$$= \frac{E_{12}}{E_{1z}E_{z2}}\frac{\Theta_{1z}\Theta_{z2}}{\Theta\Theta_{12}}\left(1 + \frac{1}{N}\left[\frac{\Theta'_{1z}}{\Theta_{1z}} + \frac{\Theta'_{z2}}{\Theta_{z2}} - \frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right]F'_1\right.$$

$$+ \frac{1}{N}\sum_i \operatorname*{Res}_{z'\to a_i}\frac{1}{dX(z')dY(z')}\left\{\frac{1}{2}\left(\frac{\Theta'_{1z}}{\Theta_{1z}}(dS_{1z}(z'))^2 + \frac{\Theta'_{z2}}{\Theta_{z2}}(dS_{z2}(z'))^2\right.\right.$$

$$\left.- \frac{\Theta'_{12}}{\Theta_{12}}(dS_{12}(z'))^2\right)d\mathbf{v}(z') + \frac{1}{2}\left(\frac{\Theta''_{1z}}{\Theta_{1z}}dS_{1z}(z') + \frac{\Theta''_{z2}}{\Theta_{z2}}dS_{z2}(z')\right.$$

$$\left.- \frac{\Theta''_{12}}{\Theta_{12}}dS_{12}(z')\right)(d\mathbf{v}(z'))^2 + \frac{1}{6}\left(\frac{\Theta'''_{1z}}{\Theta_{1z}} + \frac{\Theta'''_{z2}}{\Theta_{z2}} - \frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)(d\mathbf{v}(z'))^3\Bigg\}$$

$$\left.- \frac{1}{2}dS_{1z}(z')dS_{z2}(z')dS_{12}(z') + o(1/N)\right).$$

Let us now compute

$$\frac{1}{N}\delta_z \ln\psi_{12} + \frac{\psi_{1z}\psi_{z2}}{\psi_{12}} = \left\{\left(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)' d\mathbf{v}(z)\right.$$

$$+ \frac{E_{12}}{E_{1z}E_{z2}}\frac{\Theta_{1z}\Theta_{z2}}{\Theta\Theta_{12}}\left(\frac{\Theta'_{1z}}{\Theta_{1z}} + \frac{\Theta'_{z2}}{\Theta_{z2}} - \frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)\Bigg\}F'_1$$

$$+ \sum_i \operatorname*{Res}_{z'\to a_i}\frac{1}{dX(z')dY(z')}\left\{\frac{1}{2}(dS_{12}(z'))^2 B(z',z) + \left(\frac{\Theta'_{12}}{\Theta_{12}}\right)d\mathbf{v}(z')dS_{12}(z')B(z',z)\right.$$

$$+ \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)(d\mathbf{v}(z'))^2 B(z',z) + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}}\right)'(d\mathbf{v}(z'))^2 dS_{12}(z')d\mathbf{v}(z)$$

$$+ \frac{1}{2}\left(\frac{\Theta'_{12}}{\Theta_{12}}\right)' d\mathbf{v}(z')(dS_{12}(z'))^2 d\mathbf{v}(z) + \frac{1}{6}\left(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)'(d\mathbf{v}(z'))^3 d\mathbf{v}(z)$$

$$+ \frac{E_{12}}{E_{1z}E_{z2}}\frac{\Theta_{1z}\Theta_{z2}}{\Theta\Theta_{12}}\left[\frac{1}{2}\left(\frac{\Theta'_{1z}}{\Theta_{1z}}(dS_{1z}(z'))^2 + \frac{\Theta'_{z2}}{\Theta_{z2}}(dS_{z2}(z'))^2 - \frac{\Theta'_{12}}{\Theta_{12}}(dS_{12}(z'))^2\right)d\mathbf{v}(z')\right.$$

$$+ \frac{1}{2}\left(\frac{\Theta''_{1z}}{\Theta_{1z}}dS_{1z}(z') + \frac{\Theta''_{z2}}{\Theta_{z2}}dS_{z2}(z') - \frac{\Theta''_{12}}{\Theta_{12}}dS_{12}(z')\right)(d\mathbf{v}(z'))^2$$

$$+ \frac{1}{6}\left(\frac{\Theta'''_{1z}}{\Theta_{1z}} + \frac{\Theta'''_{z2}}{\Theta_{z2}} - \frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)(\mathrm{d}\mathbf{v}(z'))^3 - \frac{1}{2}\mathrm{d}S_{1z}(z')\mathrm{d}S_{z2}(z')\mathrm{d}S_{12}(z')\bigg]\bigg\}$$

$$+ o(1/N). \tag{A.2}$$

The coefficient of F'_1 vanishes, as we can see by computing the gradient of equation (A.1) to leading order in N (recall that N enters in the definition of our Θ through the point $\mathbf{w}_0 = NF'_0$)

$$\left(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)' \mathrm{d}\mathbf{v}(z) - \frac{E_{12}}{E_{1z}E_{z2}} \frac{\Theta_{1z}\Theta_{2z}}{\Theta\Theta_{12}}\left(\frac{\Theta'_{1z}}{\Theta_{1z}} + \frac{\Theta'_{z2}}{\Theta_{z2}} - \frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right). \tag{A.3}$$

This identity can also be seen as a consequence of Fay identity.

Let us now study the residue term of equation (A.2), which we write

$$\sum_i \operatorname*{Res}_{z'\to a_i} \frac{H_{12}(z',z)}{\mathrm{d}X(z')\mathrm{d}Y(z')}.$$

First, notice that by construction, $H_{12}(z',z)$ is a meromorphic 1-form in the variable z, which means that it has trivial monodromy when z goes around a non-trivial cycle. It may have simple poles at $z = z_1$ or $z = z_2$ coming from the ratio of prime forms, but the expression in $[\cdots]$ vanish when $z = z_1$ or z_2, so $H_{12}(z,z')$ is actually regular at $z = z_1$ or $z = z_2$. It may also have a singularity at $z = z'$ coming from the term $[\cdots]$, which is at most a double pole. To leading order when $z \to z'$, we find

$$H_{12}(z',z) = \frac{1}{2}\mathrm{d}S_{12}(z')^2 + \left(\frac{\Theta'_{12}}{\Theta_{12}}\right)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z') + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)(\mathrm{d}\mathbf{v}(z'))^2$$

$$\frac{E_{12}}{E_{1z'}E_{z'2}}\frac{\Theta_{1z'}\Theta_{z'2}}{\Theta\Theta_{12}}\left\{\frac{1}{2}\left(\frac{\Theta'_{1z'}}{\Theta_{1z}} + \frac{\Theta'_{z'2}}{\Theta_{z'2}}\right)\mathrm{d}\mathbf{v}(z') + \frac{1}{2}\mathrm{d}S_{12}(z')\right\} + O\big(\xi_{z'}^{-1}(z)\big),$$

where $\xi_{z'}$ is our notation for a local coordinate centered at z'. We use again equation (A.1) and find

$$H_{12}(z',z) = \frac{1}{2}\mathrm{d}S_{12}(z')^2 + \left(\frac{\Theta'_{12}}{\Theta_{12}}\right)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z') + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)(\mathrm{d}\mathbf{v}(z'))^2$$

$$- \left(\mathrm{d}S_{12}(z') + \left(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)\mathrm{d}\mathbf{v}(z)\right)\left\{\frac{1}{2}\left(\frac{\Theta'_{1z'}}{\Theta_{1z}} + \frac{\Theta'_{z'2}}{\Theta_{z'2}}\right)\mathrm{d}\mathbf{v}(z') + \frac{1}{2}\mathrm{d}S_{12}(z')\right\}$$

$$+ O\big(\xi_{z'}^{-1}(z)\big)$$

$$= \bigg\{ - \mathrm{d}S_{12}(z')\left(-\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta} + \frac{\Theta'_{1z'}}{\Theta_{1z'}} + \frac{\Theta'_{z'2}}{\Theta_{z'2}}\right)$$

$$- \left(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)\left(\frac{\Theta'_{1z'}}{\Theta_{1z'}} + \frac{\Theta'_{z'2}}{\Theta_{z'2}}\right)\mathrm{d}\mathbf{v}(z') + \left(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)\mathrm{d}\mathbf{v}(z')\bigg\}\frac{\mathrm{d}\mathbf{v}(z')}{2}$$

$$+ O\big(\xi_{z'}^{-1}(z)\big).$$

We can rearrange the terms

$$H_{12}(z',z) = \bigg\{ -\left[\mathrm{d}S_{12}(z') + \left(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)\mathrm{d}\mathbf{v}(z')\right]\left(\frac{\Theta'_{1z'}}{\Theta_{1z'}} + \frac{\Theta'_{z'2}}{\Theta_{z'2}} - \frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)$$

$$+ \left(\frac{\Theta'_{12}}{\Theta_{12}} - \frac{\Theta'}{\Theta}\right)'\mathrm{d}\mathbf{v}(z')\bigg\}\frac{\mathrm{d}\mathbf{v}(z')}{2} + O\big(\xi_{z'}^{-1}(z)\big),$$

and according to equation (A.3), we see that $H_{12}(z',z) \in O\big(\xi_{z'}^{-1}(z)\big)$. Therefore, $H_{12}(z',z)$ is a meromorphic function whose only singularity is a pole at most simple at $z = z'$. But

a meromorphic function cannot have a single simple pole, so $H_{12}(z, z')$ must be holomorphic, and we can write it

$$H_{12}(z, z') = h_{12}(z')\mathrm{d}\mathbf{v}(z).$$

Since the prefactor $h_{12}(z')$ is independent of z, we may compute it by specializing to $z = z_1$ in $H_{12}(z, z_1)$ defined from equation (A.2). Doing so, we obtain

$$\begin{aligned}
h_{12}(z')\mathrm{d}\mathbf{v}(z_1) &= \frac{1}{2}\big(\mathrm{d}S_{12}(z')\big)^2 B(z', z_1) + \left(\frac{\Theta'_{12}}{\Theta_{12}}\right)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z')B(z', z_1) \\
&\quad + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)\big(\mathrm{d}\mathbf{v}(z')\big)^2 B(z', z_1) + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}}\right)'\big(\mathrm{d}\mathbf{v}(z')\big)^3\mathrm{d}\mathbf{v}(z_1) \\
&\quad + \frac{1}{2}\left(\frac{\Theta'_{12}}{\Theta_{12}}\right)'\mathrm{d}\mathbf{v}(z')\big(\mathrm{d}S_{12}(z')\big)^2\mathrm{d}\mathbf{v}(z_1) + \frac{1}{6}\left(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)'\big(\mathrm{d}\mathbf{v}(z')\big)^3\mathrm{d}\mathbf{v}(z_1) \\
&\quad - \mathrm{d}_{z=z_1}\left[\frac{1}{2}\left(\frac{\Theta'_{1z}}{\Theta_{1z}}\big(\mathrm{d}S_{1z}(z')\big)^2 + \frac{\Theta'_{z2}}{\Theta_{z2}}\big(\mathrm{d}S_{z2}(z')\big)^2 - \frac{\Theta'_{12}}{\Theta_{12}}\big(\mathrm{d}S_{12}(z')\big)^2\right)\mathrm{d}\mathbf{v}(z')\right. \\
&\quad + \frac{1}{2}\left(\frac{\Theta''_{1z}}{\Theta_{1z}}\mathrm{d}S_{1z}(z') + \frac{\Theta''_{z2}}{\Theta_{z2}}\mathrm{d}S_{z2}(z') - \frac{\Theta''_{12}}{\Theta_{12}}\mathrm{d}S_{12}(z')\right)\big(\mathrm{d}\mathbf{v}(z')\big)^2 \\
&\quad \left. + \frac{1}{6}\left(\frac{\Theta'''_{1z}}{\Theta_{1z}} + \frac{\Theta'''_{z2}}{\Theta_{z2}} - \frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)\big(\mathrm{d}\mathbf{v}(z')\big)^3 - \frac{1}{2}\mathrm{d}S_{1z}(z')\mathrm{d}S_{z2}(z')\mathrm{d}S_{12}(z')\right] \\
&= \frac{1}{2}\big(\mathrm{d}S_{12}(z')\big)^2 B(z', z_1) + \left(\frac{\Theta'_{12}}{\Theta_{12}}\right)\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z')B(z', z_1) \\
&\quad + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)\big(\mathrm{d}\mathbf{v}(z')\big)^2 B(z', z_1) + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}}\right)'\big(\mathrm{d}\mathbf{v}(z')\big)^3\mathrm{d}\mathbf{v}(z_1) \\
&\quad + \frac{1}{2}\left(\frac{\Theta'_{12}}{\Theta_{12}}\right)'\mathrm{d}\mathbf{v}(z')\big(\mathrm{d}S_{12}(z')\big)^2\mathrm{d}\mathbf{v}(z_1) + \frac{1}{6}\left(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)'\big(\mathrm{d}\mathbf{v}(z')\big)^3\mathrm{d}\mathbf{v}(z_1) \\
&\quad - \left[\left(\frac{\Theta'_{12}}{\Theta_{12}}\right)B(z_1, z')\mathrm{d}\mathbf{v}(z')\mathrm{d}S_{12}(z') + \frac{1}{2}\left(\frac{\Theta'_{12}}{\Theta_{12}}\right)'\mathrm{d}\mathbf{v}(z')\mathrm{d}\mathbf{v}(z_1)\big(\mathrm{d}S_{12}(z')\big)^2\right. \\
&\quad + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}} - \frac{\Theta''}{\Theta}\right)\big(\mathrm{d}\mathbf{v}(z')\big)^2 B(z_1, z') + \frac{1}{2}\left(\frac{\Theta''_{12}}{\Theta_{12}}\right)'\big(\mathrm{d}\mathbf{v}(z')\big)^2\mathrm{d}\mathbf{v}(z_1)\mathrm{d}S_{12}(z') \\
&\quad \left. + \frac{1}{6}\left(\frac{\Theta'''_{12}}{\Theta_{12}} - \frac{\Theta'''}{\Theta}\right)'\big(\mathrm{d}v(z')\big)^3\mathrm{d}\mathbf{v}(z_1) + \frac{1}{2}B(z_1, z')\big(\mathrm{d}S_{12}(z')\big)^2\right] = 0.
\end{aligned}$$

All the terms eventually cancel each other. Thus $H_{12}(z, z') = 0$, and coming back to equation (A.2), this proves

$$\frac{1}{N}\delta_z \ln \psi(z_1, z_2) + \frac{\psi(z_1, z)\psi(z, z_2)}{\psi(z_1, z_2)} = o(1/N).\qquad\blacksquare$$

Acknowledgments

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