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Euler Equations Related to the Generalized Neveu–Schwarz Algebra

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Abstract. In this paper, we study supersymmetric or bi-superhamiltonian Euler equations related to the generalized Neveu–Schwarz algebra. As an application, we obtain several supersymmetric or bi-superhamiltonian generalizations of some well-known integrable systems including the coupled KdV equation, the 2-component Camassa–Holm equation and the 2-component Hunter–Saxton equation. To our knowledge, most of them are new.

Key words: supersymmetric; bi-superhamiltonian; Euler equations; generalized Neveu–Schwarz algebra

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1 Introduction

For a classical rigid body with a fixed point, the configuration space is the group SO(3) of rotations of three-dimensional Euclidean space. In 1765, L. Euler proposed the equations of motion of the rigid body describing as geodesics in SO(3), where SO(3) is provided with a left-invariant metric. In essence, the Euler theory of a rigid theory is fully described by this invariance.

Let G be an arbitrary (possibly infinite-dimensional) Lie group and $\mathcal{G}$ the corresponding Lie algebra and $\mathcal{G}^*$ the dual of $\mathcal{G}$. V.I. Arnold in [3] suggested a general framework for Euler equations on G, which can be regarded as a configuration space of some physical systems. In this framework Euler equations describe geodesic flows w.r.t. suitable one-side invariant Riemannian metrics on G and can be given to a variety of conservative dynamical systems in mathematical physics, for instance, see [2, 4, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 24, 25, 28, 30, 32, 33, 35, 37, 39] and references therein.

Since V. Ovsienko and B. Khesin in [35] interpreted the Kuper–KdV equation [23] as a geodesic flow equation on the superconformal group w.r.t. an L^2-type metric, it has been attracted a lot of interest in studying super (fermionic or supersymmetric) anologue of Arnold's approach, which has some different characteristic flavors, for instance [2, 11, 16, 23, 24, 25, 36, 38].

In this paper, we are interested in Euler equations related to the $N = 1$ generalized Neveu–Schwarz (GNS in brief) algebra $\mathcal{G}$, which was introduced by P. Marcel, V. Ovsienko and C. Roger in [29] as a generalization of the $N = 1$ Neveu–Schwarz algebra and the extended Virasoro algebra. In [16], P. Guha and P.J. Ovler have studied the Euler equations related to the GNS algebra $\mathcal{G}$ and obtained fermionic versions of the 2-component Camassa–Holm equation and the Ito equation in some special metrics. Our motivations are twofold. One is to study the Euler equation related to $\mathcal{G}$ for a more general metric $M_{c_1,c_2,c_3,c_4,c_5,c_6}$ in (2.1) with six-parameters given by

$$\langle \hat{F}, \hat{G} \rangle = \int_{S^1} \left(c_1 f g + c_2 f_x g_x + c_3 \phi \partial^{-1} \chi + c_4 \phi_x \chi + c_5 a b + c_6 \alpha \partial^{-1} \beta \right) dx + \vec{\sigma} \cdot \vec{\tau},$$

which can be regarded as a super-version of Sobolev-metrics in the super space. The other is to study the condition under which Euler equations are supersymmetric or bi-superhamiltonian. Our main results is to show that

╲ the Euler equation is when the metric is ╲	bi-superhamiltonian	supersymmetric
$M_{c_1, c_2, \frac{1}{4}c_1, c_2, c_5, -c_5}$	Yes	No (if $c_1 \neq 0$)
$M_{c_1, c_2, c_1, c_2, c_5, -c_5}$	only find a superhamiltonian structure (if $c_1 \neq 0$)	Yes
$M_{0, c_2, 0, c_2, c_5, -c_5}$	Yes	Yes

As a byproduct, we obtain some supersymmetric or bi-superhamiltonian generalizations of some well-known integrable systems including the coupled KdV equation, the 2-component Camassa–Holm equation and the 2-component Hunter–Saxton equation.

This paper is organized as follows. In Section 2, we calculate the Euler equation on $\mathcal{G}^*_{\rm reg}$ and discuss their Hamiltonian properties. In Section 3, we study bi-superhamiltonian Euler equations. Section 4 is devoted to describe supersymmetric Euler equations, also including a class of both supersymmetric and bi-superhamiltonian Euler equations. A few concluding remarks are given in the last section.

2 Euler equations related to the GNS algebra

To be self-contained, let us recall the Anorld's approach [4, 20, 21]. Let G be an arbitrary Lie group and $\mathcal{G}$ the corresponding Lie algebra and $\mathcal{G}^*$ the dual of $\mathcal{G}$. Firstly let us fix a energy quadratic form $E(v) = \frac{1}{2}\langle v, \mathcal{A}v \rangle^*$ on $\mathcal{G}$ and consider right translations of this quadratic form on $\mathcal{G}$. Then the energy quadratic form defines a right-invariant Riemannian metric on G. The geodesic flow on G w.r.t. this energy metric represents the extremals of the least action principle, i.e., the actual motions of our physical system. For a rigid body, one has to consider left translations. We next identify $\mathcal{G}$ and its dual $\mathcal{G}^*$ with the help of $E(\cdot)$. This identification $\mathcal{A} : \mathcal{G} \to \mathcal{G}^*$, called an inertia operator, allows us to rewrite the Euler equation on $\mathcal{G}^*$. It turns out that the Euler equation on $\mathcal{G}^*$ is Hamiltonian w.r.t. a canonical Lie–Poisson structure on $\mathcal{G}^*$. Notice that in some cases it turns out to be not only Hamiltonian, but also bihamiltonian. Moreover, the corresponding Hamiltonian function is $-E(m) = -\frac{1}{2}\langle \mathcal{A}^{-1}m, m \rangle^*$ lifted from the Lie algebra $\mathcal{G}$ to its dual space $\mathcal{G}^*$, where $m = \mathcal{A}v \in \mathcal{G}^*$.

Definition 2.1 ([4, 20])**.** The Euler equation on $\mathcal{G}^*$, corresponding to the right-invariant metric $-E(m) = -\frac{1}{2}\langle \mathcal{A}^{-1}m, m \rangle^*$ on G, is given by the following explicit formula

$$\frac{dm}{dt} = -\mathrm{ad}^*_{\mathcal{A}^{-1}m} m,$$

as an evolution of a point $m \in \mathcal{G}^*$.

In the following, we take $\mathcal{G}$ to be the $N = 1$ generalized Neveu–Schwarz algebra [34]. Let V be a $\mathbb{Z}_2$ graded vector space, i.e., $V = V_B \oplus V_F$. An element v of V_B (resp., V_F) is said to be even (resp., odd). The super commutator of a pair of elements $v, w \in V$ is defined by

$$[v, w] = vw - (-1)^{|v||w|} wv.$$

Let $\mathcal{D}^s(S^1)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the circle and $T_{id}\mathcal{D}^s(S^1)$ the corresponding Lie algebra of vector fields, denoted by $\mathrm{Vect}^s(S^1) = \{f(x)\frac{d}{dx}|f(x)\in H^s(S^1)\}$. We denote

$$V_B = \mathrm{Vect}^s(S^1)\oplus \mathrm{C}^\infty(S^1)\oplus\mathbb{R}^3, \qquad V_F = \mathrm{C}^\infty(S^1)\oplus\mathrm{C}^\infty(S^1).$$

Definition 2.2 ([34])**.** The GNS algebra $\mathcal{G}$ is an algebra $V_B\oplus V_F$ with the commutation relation given by

$$[\hat{F},\hat{G}] = \Bigg(\left(fg_x - f_xg + \frac{1}{2}\phi\chi\right)\frac{d}{dx}, \left(f\chi_x - \frac{1}{2}f_x\chi - g\phi_x + \frac{1}{2}g_x\phi\right)dx^{-\frac{1}{2}},$$
$$fb_x - a_xg + \frac{1}{2}\phi\beta + \frac{1}{2}\alpha\chi, \left(f\beta_x + \frac{1}{2}f_x\beta - \frac{1}{2}a_x\chi - g\alpha_x - \frac{1}{2}g_x\alpha + \frac{1}{2}b_x\phi\right)dx^{\frac{1}{2}}, \vec{\omega}\Bigg),$$

where ϕ, χ, α and β are fermionic functions, and f, g, a and b are bosonic functions, and $\hat{F} = \big(f(x,t)\frac{d}{dx}, \phi(x,t)dx^{-\frac{1}{2}}, a(x,t), \alpha(x,t)dx^{\frac{1}{2}}, \vec{\sigma}\big)\in\mathcal{G}$ and $\hat{G} = \big(g(x,t)\frac{d}{dx}, \chi(x,t)dx^{-\frac{1}{2}}, b(x,t), \beta(x,t)dx^{\frac{1}{2}}, \vec{\tau}\big)\in\mathcal{G}$ and $\vec{\sigma},\vec{\tau}\in\mathbb{R}^3$ and $\vec{\omega} = (\omega_1,\omega_2,\omega_3)\in\mathbb{R}^3$. Here

$$\omega_1(\hat{F},\hat{G}) = \int_{S^1}(f_xg_{xx} + \phi_x\chi_x)dx,$$
$$\omega_2(\hat{F},\hat{G}) = \int_{S^1}(f_{xx}b - g_{xx}a - \phi_x\beta + \chi_x\alpha)dx,$$
$$\omega_3(\hat{F},\hat{G}) = \int_{S^1}(2ab_x + 2\alpha\beta)dx.$$

Let us denote

$$\mathcal{G}^*_{\mathrm{reg}} = \mathrm{C}^\infty(S^1)\oplus\mathrm{C}^\infty(S^1)\oplus\mathrm{C}^\infty(S^1)\oplus\mathrm{C}^\infty(S^1)\oplus\mathbb{R}^3$$

to be the regular part of the dual space $\mathcal{G}^*$ to $\mathcal{G}$, under the following pair

$$\langle\hat{U},\hat{F}\rangle^* = \int_{S^1}(uf + \psi\phi + va + \gamma\alpha)dx + \vec{\varsigma}\cdot\vec{\sigma},$$

where $\hat{U} = \big(u(x,t)dx^2, \psi(x,t)dx^{\frac{3}{2}}, v(x,t)dx, \gamma(x,t)dx^{\frac{1}{2}}, \vec{\varsigma}\big)\in\mathcal{G}^*$ and $\vec{\varsigma} = (\varsigma_1,\varsigma_2,\varsigma_3)\in\mathbb{R}^3$. By the definition, using integration by parts we have

$$\langle\mathrm{ad}^*_{\hat{F}}(\hat{U}),\hat{G}\rangle^* = -\langle\hat{U},[\hat{F},\hat{G}]\rangle^* = -\int_{S^1}\Bigg(u\left(fg_x - f_xg + \frac{1}{2}\phi\chi\right)$$
$$+\psi\left(f\chi_x - \frac{1}{2}f_x\chi - g\phi_x + \frac{1}{2}g_x\phi\right) + v\left(fb_x - a_xg + \frac{1}{2}\phi\beta + \frac{1}{2}\alpha\chi\right)$$
$$+\gamma\left(f\beta_x + \frac{1}{2}f_x\beta - \frac{1}{2}a_x\chi - g\alpha_x - \frac{1}{2}g_x\alpha + \frac{1}{2}b_x\phi\right)\Bigg)dx - \vec{\varsigma}\cdot\vec{\omega}$$
$$= \int_{S^1}\left(2uf_x + u_xf - \varsigma_1f_{xxx} + \varsigma_2a_{xx} + \frac{3}{2}\psi\phi_x + \frac{1}{2}\psi_x\phi + \frac{1}{2}\gamma\alpha_x - \frac{1}{2}\gamma_x\alpha + va_x\right)gdx$$
$$+\int_{S^1}\left(\varsigma_1\phi_{xx} - \varsigma_2\alpha_x - \frac{1}{2}u\phi - \frac{1}{2}v\alpha + \frac{3}{2}f_x\psi + f\psi_x + \frac{1}{2}\gamma a_x\right)\chi dx$$
$$+\int_{S^1}\left((vf)_x + \frac{1}{2}(\gamma\phi)_x - \varsigma_2f_{xx} + 2\varsigma_3a_x\right)bdx$$
$$+\int_{S^1}\left(\gamma_xf + \frac{1}{2}\gamma f_x - \frac{1}{2}v\phi + \varsigma_2\phi_x - 2\varsigma_3\alpha\right)\beta dx.$$

So the coadjoint action on $\mathcal{G}^*_{\rm reg}$ is given by

$$\begin{aligned}
\mathrm{ad}^*_{\hat F}(\hat U) = \bigg(&\Big(2uf_x + u_x f - \varsigma_1 f_{xxx} + \varsigma_2 a_{xx} + \frac{3}{2}\psi\phi_x + \frac{1}{2}\psi_x\phi + \frac{1}{2}\gamma\alpha_x - \frac{1}{2}\gamma_x\alpha + va_x\Big)\, dx^2,\\
&\Big(\varsigma_1\phi_{xx} - \varsigma_2\alpha_x - \frac{1}{2}u\phi - \frac{1}{2}v\alpha + \frac{3}{2}f_x\psi + f\psi_x + \frac{1}{2}\gamma a_x\Big)\, dx^{\frac{3}{2}},\\
&\Big((vf)_x + \frac{1}{2}(\gamma\phi)_x - \varsigma_2 f_{xx} + 2\varsigma_3 a_x\Big) dx, \Big(\gamma_x f + \frac{1}{2}\gamma f_x - \frac{1}{2}v\phi + \varsigma_2\phi_x - 2\varsigma_3\alpha\Big) dx^{\frac{1}{2}}, 0\bigg).
\end{aligned}$$

On $\mathcal{G}$, let us introduce an inner product $M_{c_1,c_2,c_3,c_4,c_5,c_6}$ given by

$$\langle \hat F, \hat G\rangle = \int_{S^1} \big(c_1 fg + c_2 f_x g_x + c_3\phi\partial^{-1}\chi + c_4\phi_x\chi + c_5 ab + c_6\alpha\partial^{-1}\beta\big)dx + \vec\sigma\cdot\vec\tau, \tag{2.1}$$

which is a generalization of that in [11, 16]. By the Definition 2.1, the Euler equation on $\mathcal{G}^*_{\rm reg}$ for $M_{c_1,c_2,c_3,c_4,c_5,c_6}$ is

$$\frac{d\hat U}{dt} = -\mathrm{ad}^*_{\mathcal{A}^{-1}\hat U}\hat U \tag{2.2}$$

as an evolution of a point $\hat U = \big(u(x,t)dx^2, \psi(x,t)dx^{\frac{3}{2}}, v(x,t), \gamma(x,t)dx^{\frac{1}{2}}, \vec\varsigma\big) \in \mathcal{G}^*$, where $\mathcal{A} : \mathcal{G} \to \mathcal{G}^*$ is an inertia operator defined by

$$\langle \hat F, \hat G\rangle = \langle \mathcal{A}(\hat F), \hat G\rangle^*.$$

A direct computation shows that the inertia operator $\mathcal{A} : \mathcal{G} \to \mathcal{G}^*$ has the form

$$\mathcal{A}(\hat F) = \big(\Lambda(f)dx^2, \Theta(\phi)dx^{\frac{3}{2}}, c_5 a dx, c_6\partial^{-1}\alpha dx^{\frac{1}{2}}, \vec\sigma\big),$$

where $\Lambda(f) = c_1 f - c_2 f_{xx}$ and $\Theta(\phi) = c_4\phi_x - c_3\partial^{-1}\phi$. Thus we have

Proposition 2.3. *The Euler equation (2.2) on $\mathcal{G}^*_{\rm reg}$ for $M_{c_1,c_2,c_3,c_4,c_5,c_6}$ reads*

$$\begin{aligned}
u_t &= \varsigma_1 f_{xxx} - \varsigma_2 a_{xx} - 2uf_x - u_x f - va_x - \frac{3}{2}\psi\phi_x - \frac{1}{2}\psi_x\phi - \frac{1}{2}\gamma\alpha_x,\\
\psi_t &= \frac{1}{2}u\phi + \frac{1}{2}v\alpha - \varsigma_1\phi_{xx} + \varsigma_2\alpha_x - \frac{3}{2}f_x\psi - f\psi_x - \frac{1}{2}\gamma a_x,\\
v_t &= \varsigma_2 f_{xx} - 2\varsigma_3 a_x - (vf)_x - \frac{1}{2}(\gamma\phi)_x,\\
\gamma_t &= \frac{1}{2}v\phi - \gamma_x f - \frac{1}{2}\gamma f_x - \varsigma_2\phi_x + 2\varsigma_3\alpha,
\end{aligned} \tag{2.3}$$

where $u = \Lambda(f) = c_1 f - c_2 f_{xx}$, $\psi = \Theta(\phi) = c_4\phi_x - c_3\partial^{-1}\phi$, $v = c_5 a$ and $\gamma = c_6\partial^{-1}\alpha$.

Let us remark that the system (2.3) has been obtained in [16] with minor typos. But they didn't discuss the condition under which the Euler equation (2.3) is supersymmetric or bi-superhamiltonian.

According to Definition 2.1, the Euler equation (2.3) has a natural Hamiltonian description [4, 20, 21]. Let $F_i : \mathcal{G}^* \to \mathbb{R}$, $i = 1, 2$, be two arbitrary smooth functionals. The dual space $\mathcal{G}^*$ carries a canonical Lie–Poisson bracket

$$\{F_1, F_2\}_2(\hat U) = \left\langle \hat U, \left[\frac{\delta F_1}{\delta \hat U}, \frac{\delta F_2}{\delta \hat U}\right]\right\rangle^*,$$

where $\hat{U} \in \mathcal{G}^*$ and $\frac{\delta F_i}{\delta \hat{U}} = \left(\frac{\delta F_i}{\delta u}, \frac{\delta F_i}{\delta \psi}, \frac{\delta F_i}{\delta v}, \frac{\delta F_i}{\delta \gamma}, \frac{\delta F_i}{\delta \varsigma}\right) \in \mathcal{G}, i = 1, 2$. The induced superhamiltonian operator is given by

$$\mathcal{J}_2 = \begin{pmatrix} \varsigma_1\partial^3 - u\partial - \partial u & -\psi\partial - \frac{1}{2}\partial\psi, & -\varsigma_2\partial^2 - v\partial & -\gamma\partial + \frac{1}{2}\partial\gamma \\ -\partial\psi - \frac{1}{2}\psi\partial & \frac{1}{2}u - \varsigma_1\partial^2 & -\frac{1}{2}\gamma\partial & \frac{1}{2}v + \varsigma_2\partial \\ \varsigma_2\partial^2 - \partial v & -\frac{1}{2}\partial\gamma & -2\varsigma_3\partial & 0 \\ -\partial\gamma + \frac{1}{2}\gamma\partial & \frac{1}{2}v - \varsigma_2\partial & 0 & 2\varsigma_3 \end{pmatrix}. \tag{2.4}$$

Proposition 2.4. *The Euler equation (2.3) could be written as*

$$\frac{d}{dt}(u, \psi, v, \gamma)^{\mathrm{T}} = \mathcal{J}_2 \left(\frac{\delta H_1}{\delta u}, \frac{\delta H_1}{\delta \psi}, \frac{\delta H_1}{\delta v}, \frac{\delta H_1}{\delta \gamma}\right)^{\mathrm{T}} \tag{2.5}$$

with the Hamiltonian $H_1 = \frac{1}{2}\int_{S^1}(uf + \psi\phi + va + \gamma\alpha)dx$*, where* $(\cdot)^{\mathrm{T}}$ *means the transpose of vectors.*

Proof. Indeed, for a functional $F[u, \psi, v, \gamma]$, the variational derivatives $\frac{\delta F}{\delta u}, \frac{\delta F}{\delta \psi}, \frac{\delta F}{\delta v}$ and $\frac{\delta F}{\delta \gamma}$ are defined by

$$\begin{aligned} &\frac{d}{d\epsilon}\,|_{\epsilon=0}\, F[u + \epsilon\delta u, \psi + \epsilon\delta\psi, v + \epsilon\delta v, \gamma + \epsilon\delta\gamma] \\ &\qquad = \int \left(\delta u \frac{\delta F}{\delta u} + \delta\psi \frac{\delta F}{\delta \psi} + \delta v \frac{\delta F}{\delta v} + \delta\gamma \frac{\delta F}{\delta \gamma}\right) dx. \end{aligned} \tag{2.6}$$

By using (2.6), we have

$$\frac{\delta H_1}{\delta f} = \Lambda(f), \qquad \frac{\delta H_1}{\delta \phi} = -\Theta(\phi), \qquad \frac{\delta H_1}{\delta a} = v, \qquad \frac{\delta H_1}{\delta \alpha} = -\gamma.$$

It follows from the definition of u, ψ and γ that

$$\begin{aligned} &\frac{\delta H_1}{\delta u} = \Lambda^{-1}\frac{\delta H_1}{\delta u} = f, \qquad \frac{\delta H_1}{\delta \psi} = -\Theta^{-1}\frac{\delta H_1}{\delta \phi} = \phi, \\ &\frac{\delta H_1}{\delta v} = a, \qquad \frac{\delta H_1}{\delta \gamma} = -\partial\frac{\delta H_1}{\delta \alpha} = \alpha. \end{aligned} \tag{2.7}$$

Hence, (2.5) could be easily verified by using (2.4) and (2.7). ■

3 Bihamiltonian Euler equations on $\mathcal{G}^*_{\mathrm{reg}}$

Unless otherwise stated, in the following we use "(bi)hamiltonian" to denote "(bi)-superhamiltonian". In this section we want to study bihamiltonian Euler equations on $\mathcal{G}^*_{\mathrm{reg}}$ w.r.t. the metric $M_{c_1,c_2,\frac{1}{4}c_1,c_2,c_5,-c_5}$ and propose some new bihamiltonian and fermionic extensions of well-known integrable systems including coupled the KdV equation, the 2-CH equation and the 2-HS equation.

3.1 The frozen Lie–Poisson bracket on $\mathcal{G}^*_{\mathrm{reg}}$

For the purpose of discussing possible bihamiltonian Euler equations, we introduce a frozen Lie–Poisson bracket on $\mathcal{G}^*_{\mathrm{reg}}$ defined by

$$\{F_1, F_2\}_1(\hat{U}) = \left\langle \hat{U}_0, \left[\frac{\delta F_1}{\delta \hat{U}}, \frac{\delta F_2}{\delta \hat{U}}\right]\right\rangle^*,$$

for a fixed point $\hat{U}_0 \in \mathcal{G}^*$. The corresponding Hamiltonian equation is given by

$$\frac{d\hat{U}}{dt} = -\mathrm{ad}^*_{\frac{\delta H_2}{\delta \hat{U}}} \hat{U}_0 \tag{3.1}$$

for a functional $H_2 : \mathcal{G}^*_{\mathrm{reg}} \to \mathbb{R}$. If we could find a functional H_2 and a suitable point $\hat{U}_0 \in \mathcal{G}^*_{\mathrm{reg}}$ such that the system (3.1) coincides with (2.3). This means that the Euler equation (2.3) is bihamiltonian and could be written as

$$\frac{d}{dt}(u, \psi, v, \gamma)^{\mathrm{T}} = \mathcal{J}_1 \left(\frac{\delta H_2}{\delta u}, \frac{\delta H_2}{\delta \psi}, \frac{\delta H_2}{\delta v}, \frac{\delta H_2}{\delta \gamma} \right)^{\mathrm{T}} = \mathcal{J}_2 \left(\frac{\delta H_1}{\delta u}, \frac{\delta H_1}{\delta \psi}, \frac{\delta H_1}{\delta v}, \frac{\delta H_1}{\delta \gamma} \right)^{\mathrm{T}}$$

with Hamiltonian operators $\mathcal{J}_2$ in (2.4) and $\mathcal{J}_1 = \mathcal{J}_2|_{\hat{U}=\hat{U}_0}$. Moreover, according to Proposition 5.3 in [20], $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are compatible for every freezing point $\hat{U}_0$.

3.2 Bihamiltonian Euler equations on $\mathcal{G}^*_{\mathrm{reg}}$ w.r.t. $M_{c_1,c_2,\frac{1}{4}c_1,c_2,c_5,-c_5}$

In this case, we have

$$c_1 = 4\,c_3, \qquad c_2 = c_4, \qquad c_6 = -c_5.$$

By setting $\phi = \eta_x$ and $\alpha = \mu_x$, then

$$u = \Lambda(f) = c_1 f - c_2 f_{xx}, \qquad \psi = \Pi(\eta) = c_2 \eta_{xx} - \frac{1}{4} c_1 \eta, \qquad v = c_5 a, \qquad \gamma = -c_5 \mu \tag{3.2}$$

and the Euler equation becomes

$$\begin{aligned}
u_t &= \varsigma_1 f_{xxx} - \varsigma_2 a_{xx} - 2u f_x - u_x f - \frac{3}{2}\psi \eta_{xx} - \frac{1}{2}\psi_x \eta_x - v a_x - \frac{1}{2}\gamma \mu_{xx}, \\
\psi_t &= \frac{1}{2} u \eta_x - \varsigma_1 \eta_{xxx} + \varsigma_2 \mu_{xx} - \frac{3}{2} f_x \psi - f \psi_x + \frac{1}{2}(v\mu)_x, \\
v_t &= \varsigma_2 f_{xx} - 2\varsigma_3 a_x - (vf)_x - \frac{1}{2}(\gamma \eta_x)_x, \\
\gamma_t &= \frac{1}{2} v \eta_x - \gamma_x f - \frac{1}{2}\gamma f_x - \varsigma_2 \eta_{xx} + 2\varsigma_3 \mu_x.
\end{aligned} \tag{3.3}$$

We are now in a position to state our main theorem.

Theorem 3.1. *The system (3.3) is bihamiltonian on $\mathcal{G}^*_{\mathrm{reg}}$ with a freezing point $\hat{U}_0 = \left(\frac{c_1}{2} dx^2, 0, 0, 0, \left(c_2, 0, \frac{c_5}{2}\right)\right) \in \mathcal{G}^*_{\mathrm{reg}}$ and a Hamiltonian functional*

$$\begin{aligned}
H_2 = \int_{S^1} \Big(&-\frac{\varsigma_1}{2} f f_{xx} + \frac{c_1}{2} f^3 - \frac{c_2}{4} f^2 f_{xx} - \frac{c_2}{2} f \eta_x \eta_{xx} - \frac{3c_1}{8} f \eta \eta_x + \frac{\varsigma_1}{2} \eta \eta_{xx} \\
&- \varsigma_2 a f_x + \frac{1}{2} a v f + \varsigma_3 a^2 + \frac{1}{2} a \gamma \eta_x + \varsigma_3 \mu \mu_x + \varsigma_2 \mu \eta_{xx} + \frac{1}{2} \gamma \mu_x f \Big) dx.
\end{aligned}$$

Proof. Direct computation gives

$$\begin{aligned}
\frac{\delta H_2}{\delta f} &= \varsigma_2 a_x - \varsigma_1 f_{xx} + \frac{3c_1}{2} f^2 - c_2 f f_{xx} - \frac{c_2}{2} f_x^2 - \frac{c_2}{2} \eta_x \eta_{xx} - \frac{3c_1}{8} \eta \eta_{xx} + \frac{1}{2} a v + \frac{1}{2} \gamma \mu_x, \\
\frac{\delta H_2}{\delta \eta} &= \frac{3c_2}{2} f_x \eta_{xx} + c_2 f \eta_{xxx} + \frac{c_2}{2} f_{xx} \eta_x - \frac{3c_1}{4} f \eta_x - \frac{3c_1}{8} f_x \eta + \varsigma_1 \eta_{xx} - \varsigma_2 \mu_{xx} + \frac{1}{2}(a\gamma)_x, \\
\frac{\delta H_2}{\delta a} &= 2\varsigma_3 a + v f + \frac{1}{2} \gamma \eta_x - \varsigma_2 f_x,
\end{aligned} \tag{3.4}$$

$$\frac{\delta H_2}{\delta \mu} = \gamma_x f + \frac{1}{2}\gamma f_x - \frac{1}{2}\upsilon \eta_x + \varsigma_2 \eta_{xx} - 2\varsigma_3 \mu_x.$$

Under the special freezing point

$$\hat{U}_0 = \left(\frac{c_1}{2}dx^2, 0, 0, 0, \left(c_2, 0, \frac{c_5}{2}\right)\right) \in \mathcal{G}^*_{\text{reg}},$$

the system (3.1) reads

$$\begin{aligned} &u_t = c_2 \left(\frac{\delta H_2}{\delta u}\right)_{xxx} - c_1 \left(\frac{\delta H_2}{\delta u}\right)_x, \qquad \upsilon_t = -c_5 \left(\frac{\delta H_2}{\delta \upsilon}\right)_x, \\ &\psi_t = \frac{c_1}{4}\frac{\delta H_2}{\delta \psi} - c_2 \left(\frac{\delta H_2}{\delta \psi}\right)_{xx}, \qquad \gamma_t = c_5 \frac{\delta H_2}{\delta \gamma}. \end{aligned} \tag{3.5}$$

Using (3.2), we have

$$\frac{\delta H_2}{\delta u} = \Lambda^{-1}\left(\frac{\delta H_2}{\delta f}\right), \qquad \frac{\delta H_2}{\delta \psi} = \Pi^{-1}\left(\frac{\delta H_2}{\delta \eta}\right), \qquad c_5 \frac{\delta H_2}{\delta \upsilon} = \frac{\delta H_2}{\delta a}, \qquad c_5 \frac{\delta H_2}{\delta \gamma} = -\frac{\delta H_2}{\delta \mu}.$$

The system (3.5) becomes

$$u_t = -\left(\frac{\delta H_2}{\delta f}\right)_x, \qquad \psi_t = -\frac{\delta H_2}{\delta \eta}, \qquad \upsilon_t = -\left(\frac{\delta H_2}{\delta a}\right)_x, \qquad \gamma_t = -\frac{\delta H_2}{\delta \mu},$$

which is the desired system (3.3) due to (3.2) and (3.4). We thus complete the proof of the theorem. ■

3.3 Examples

Example 3.2 (an L^2-type metric $M_{1,0,\frac{1}{4},0,1,-1}$)**.** The systems (3.3) reduces to

$$\begin{aligned} &f_t = \varsigma_1 f_{xxx} - \varsigma_2 a_{xx} - 3 f f_x + \frac{3}{8}\eta\eta_{xx} - a a_x + \frac{1}{2}\mu\mu_{xx}, \\ &\eta_t = 4\varsigma_1 \eta_{xxx} - 3 f \eta_x - \frac{3}{2} f_x \eta - 4\varsigma_2 \mu_{xx} - 2(a\mu)_x, \\ &a_t = \varsigma_2 f_{xx} - 2\varsigma_3 a_x - (af)_x + \frac{1}{2}(\mu\eta_x)_x, \\ &\mu_t = \varsigma_2 \eta_{xx} - 2\varsigma_3 \mu_x - \frac{1}{2} a\eta_x - \mu_x f - \frac{1}{2}\mu f_x. \end{aligned} \tag{3.6}$$

We call this system (3.6) to be a Kuper-2KdV equation. Especially, (1) if we set $\eta = \mu = 0$, we have

$$f_t = \varsigma_1 f_{xxx} - \varsigma_2 a_{xx} - 3 f f_x - a a_x, \qquad a_t = \varsigma_2 f_{xx} - 2\varsigma_3 a_x - (af)_x,$$

which is a two-component generalization of the KdV equation with three parameters including the Ito equation in [17] for $\varsigma_1 \neq 0$, $\varsigma_2 = \varsigma_3 = 0$; (2) if we set $\varsigma_1 = \frac{1}{2}$, $\varsigma_2 = 0$, $a = 0$ and $\mu = 0$, we have

$$f_t = \frac{1}{2} f_{xxx} - 3 f f_x + \frac{3}{8}\eta\eta_{xx}, \qquad \eta_t = 2\eta_{xxx} - 3 f \eta_x - \frac{3}{2} f_x \eta,$$

which is the Kuper–KdV equation in [23].

Let us remark that when we choose $\varsigma_1 = \frac{1}{4}$ and $\varsigma_2 = \varsigma_3 = 0$, up to a rescaling, the Kuper–2KdV equation (3.6) is the super-Ito equation (equation (4.14b) in [1]) proposed by M. Antonowicz and A.P. Fordy, which has three Hamiltonian structures. According to our terminologies, we would like to call it the Kuper–Ito equation.

Example 3.3 (an H^1-type metric $M_{1,1,\frac{1}{4},1,1,-1}$)**.** The systems (3.3) reduces to

$$\begin{aligned}
&f_t - f_{xxt} = \varsigma_1 f_{xxx} - \varsigma_2 a_{xx} - 3ff_x + 2f_x f_{xx} + ff_{xxx} + \frac{3}{8}\eta\eta_{xx} + \frac{1}{2}\eta_x\eta_{xxx} - aa_x + \frac{1}{2}\mu\mu_{xx},\\
&\eta_{xxt} - \frac{1}{4}\eta_t = \frac{3}{4}f\eta_x + \frac{3}{8}f_x\eta - f\eta_{xxx} - \frac{1}{2}f_{xx}\eta_x - \frac{3}{2}f_x\eta_{xx} - \varsigma_1\eta_{xxx} + \varsigma_2\mu_{xx} - \frac{1}{2}(a\mu)_x,\\
&a_t = \varsigma_2 f_{xx} - 2\varsigma_3 a_x - (af)_x + \frac{1}{2}(\mu\eta_x)_x,\\
&\mu_t = \varsigma_2\eta_{xx} - 2\varsigma_3\mu_x - \frac{1}{2}a\eta_x - \mu_x f - \frac{1}{2}\mu f_x.
\end{aligned} \tag{3.7}$$

We call this system (3.7) to be a Kuper–2CH equation. Especially, (1) if we set $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$ and $\eta = \mu = 0$, we have

$$f_t - f_{xxt} = 2f_x f_{xx} + ff_{xxx} - 3ff_x - aa_x, \qquad a_t = -(af)_x,$$

which is the 2-CH equation in [6, 13]; (2) if by setting $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$, $a = 0$ and $\mu = 0$, the system (3.7) becomes

$$\begin{aligned}
&f_t - f_{xxt} = ff_{xxx} + 2f_x f_{xx} - 3ff_x + \frac{3}{8}\eta\eta_{xx} + \frac{1}{2}\eta_x\eta_{xxx},\\
&\eta_{xxt} - \frac{1}{4}\eta_t = \frac{3}{4}f\eta_x + \frac{3}{8}f_x\eta - f\eta_{xxx} - \frac{1}{2}f_{xx}\eta_x - \frac{3}{2}f_x\eta_{xx},
\end{aligned}$$

which is the Kuper–CH equation in [10, 38].

4 Supersymmetric Euler equations on $\mathcal{G}^*_{\mathrm{reg}}$

In this section, we want to discuss a class of supersymmetric Euler equations on $\mathcal{G}^*$ associated to a special metric $M_{c_1,c_2,c_1,c_2,c_5,-c_5}$. Moreover, we present a class of supersymmetric and bihamiltonian Euler equations.

4.1 Supersymmetric Euler equations on $\mathcal{G}^*_{\mathrm{reg}}$ w.r.t. $M_{c_1,c_2,c_1,c_2,c_5,-c_5}$

In this case, we have

$$c_1 = c_3, \qquad c_2 = c_4, \qquad c_6 = -c_5.$$

By setting $\phi = \eta_x$ and $\alpha = \mu_x$, we obtain

$$u = c_1 f - c_2 f_{xx}, \qquad \psi = c_2\eta_{xx} - c_1\eta, \qquad v = c_5 a, \qquad \gamma = -c_5\mu.$$

Let us define a superderivative D by $D = \partial_\theta + \theta\partial_x$ and introduce two superfields

$$\Phi = \eta + \theta f, \qquad \Omega = \mu + \theta a,$$

where θ is an odd coordinate. A direct computation gives

Theorem 4.1. *The Euler equation (2.3) on $\mathcal{G}^*_{\rm reg}$ w.r.t. $M_{c_1,c_2,c_1,c_2,c_5,-c_5}$ is invariant under the supersymmetric transformation*

$$\delta f = \theta\eta_x, \qquad \delta\eta = \theta f, \qquad \delta a = \theta\mu_x, \qquad \delta\mu = \theta a$$

and could be rewritten as

$$\begin{aligned} c_1\Phi_t - c_2D^4\Phi_t &= \varsigma_1 D^6\Phi - \frac{3}{2}c_1\big(\Phi D^3\Phi + D\Phi D^2\Phi\big) \\ &\quad + c_2\left(D\Phi D^6\Phi + \frac{1}{2}D^2\Phi D^5\Phi + \frac{3}{2}D^3\Phi D^4\Phi\right) \\ &\quad - \varsigma_2 D^4\Omega + \frac{1}{2}c_5\big(D\Omega D^2\Omega + \Omega D^3\Omega\big), \\ c_5\Omega_t &= \varsigma_2 D^4\Phi - 2\varsigma_3 D^2\Omega - \frac{1}{2}c_5\big(D\Omega D^2\Phi + 2D^2\Omega D\Phi + \Omega D^3\Phi\big). \end{aligned} \tag{4.1}$$

4.2 Examples

Example 4.2 (another L^2-type metric $M_{1,0,1,0,1,-1}$)**.** The system (4.1) reduces to

$$\begin{aligned} \Phi_t &= \varsigma_1 D^6\Phi - \frac{3}{2}\big(\Phi D^3\Phi + D\Phi D^2\Phi\big) - \varsigma_2 D^4\Omega + \frac{1}{2}\big(D\Omega D^2\Omega + \Omega D^3\Omega\big), \\ \Omega_t &= \varsigma_2 D^4\Phi - 2\varsigma_3 D^2\Omega - \frac{1}{2}\big(D\Omega D^2\Phi + 2D^2\Omega D\Phi + \Omega D^3\Phi\big). \end{aligned} \tag{4.2}$$

We call this system (4.2) to be a super-2KdV equation. Especially, (1) if we set $\eta = \mu = 0$, we recover the two-component KdV equation again; (2) but if we choose $\varsigma_1 = \frac{1}{2}$, $\varsigma_2 = \varsigma_3 = 0$ and $\Omega = 0$, the system (4.2) becomes

$$\Phi_t = \frac{1}{2}D^6\Phi - \frac{3}{2}\big(\Phi D^3\Phi + D\Phi D^2\Phi\big),$$

equivalently in componentwise forms,

$$f_t = \frac{1}{2}f_{xxx} - 3ff_x + \frac{3}{2}\eta\eta_{xx}, \qquad \eta_t = \frac{1}{2}\eta_{xxx} - \frac{3}{2}(f\eta)_x,$$

which is the super-KdV equation in [31].

Example 4.3 (another H^1-type metric $M_{1,1,1,1,1,-1}$)**.** The system (4.1) reduces to

$$\begin{aligned} \Phi_t - D^4\Phi_t &= \varsigma_1 D^6\Phi - \frac{3}{2}(\Phi D^3\Phi + D\Phi D^2\Phi) + \left(D\Phi D^6\Phi + \frac{1}{2}D^2\Phi D^5\Phi + \frac{3}{2}D^3\Phi D^4\Phi\right) \\ &\quad - \varsigma_2 D^4\Omega + \frac{1}{2}\big(D\Omega D^2\Omega + \Omega D^3\Omega\big), \\ \Omega_t &= \varsigma_2 D^4\Phi - 2\varsigma_3 D^2\Omega - \frac{1}{2}\big(D\Omega D^2\Phi + 2D^2\Omega D\Phi + \Omega D^3\Phi\big). \end{aligned} \tag{4.3}$$

We call this system (4.3) to be a super-2CH equation. Especially, (1) if we set $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$ and $\eta = \mu = 0$, we obtain the 2-CH equation in [6, 13] again; (2) but if by setting $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$ and $\Omega = 0$, the system (4.3) becomes

$$\Phi_t - D^4\Phi_t = \left(D\Phi D^6\Phi + \frac{1}{2}D^2\Phi D^5\Phi + \frac{3}{2}D^3\Phi D^4\Phi\right) - \frac{3}{2}\big(\Phi D^3\Phi + D\Phi D^2\Phi\big),$$

which is the super-CH equation in [11].

4.3 Supersymmetric and bihamiltonian Euler equations w.r.t. $M_{0,c_2,0,c_2,c_5,-c_5}$ on $\mathcal{G}^*_{\rm reg}$

Let us combine with Theorem 3.1 and Theorem 4.1, we have

Theorem 4.4. *The Euler equation on $\mathcal{G}^*_{\rm reg}$ w.r.t. the metric $M_{0,c_2,0,c_2,c_5,-c_5}$ is supersymmetric and bihamiltonian.*

Example 4.5 (an $\dot{H}^1$-type metric $M_{0,1,0,1,1,-1}$)**.** The systems (4.1) reduces to

$$\begin{aligned} -D^4\Phi_t &= \varsigma_1 D^6\Phi + \left(D\Phi D^6\Phi + \frac{1}{2}D^2\Phi D^5\Phi + \frac{3}{2}D^3\Phi D^4\Phi\right) \\ &\quad - \varsigma_2 D^4\Omega + \frac{1}{2}\big(D\Omega D^2\Omega + \Omega D^3\Omega\big), \\ \Omega_t &= \varsigma_2 D^4\Phi - 2\varsigma_3 D^2\Omega - \frac{1}{2}\big(D\Omega D^2\Phi + 2D^2\Omega D\Phi + \Omega D^3\Phi\big). \end{aligned} \tag{4.4}$$

We call this system (4.4) to be a super-2HS equation. Especially, (i) if we set $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$ and $\eta = \mu = 0$, we have

$$-f_{xxt} = 2f_x f_{xx} + f f_{xxx} - a a_x, \qquad a_t = -(af)_x,$$

which is a 2-HS equation in [39]; (ii) if by setting $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$ and $\Omega = 0$, the system (4.4) becomes

$$-D^4\Phi_t = D\Phi D^6\Phi + \frac{1}{2}D^2\Phi D^5\Phi + \frac{3}{2}D^3\Phi D^4\Phi,$$

which is the super-HS equation in [5, 24].

5 Concluding remarks

We have described Euler equations associated to the GNS algebra and shown that under which conditions there are superymmetric or bihamiltonian. Here we only obtain some sufficient conditions but not necessary conditions. As an application, we have naturally presented several generalizations of some well-known integrable systems including the Ito equation, the 2-CH equation and the 2-HS equation. It is well-known that the Virasoro algebra, the extended Virasoro algebra and the Neveu–Schwarz algebras are subalgebras of the GNS algebra. Thus our result could be regarded as a generalization of that related to those subalgebras, see for instances [2, 4, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 24, 25, 28, 30, 32, 33, 35, 37, 39] and references therein. In the past twenty years, in this subject it has grown in many different directions, please see [21] and references therein. Finally let us point out that in this paper all super-Hamiltonian operators are even. Recently, in [5, 26, 27], the classical Harry–Dym equation is supersymmetrized in two ways, either by even supersymmetric Hamiltonian operators or by odd supersymmetric Hamiltonian operators. Notice that the HS equation is one of a member of negative Harry–Dym hierarchy. It would be interesting to investigate whether the above point of view has an extension to the odd supersymmetric integrable system, for instance, the odd HS equation.

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