Functions Characterizing the Ground State of the XXZ Spin-1/2 Chain in the Thermodynamic Limit*

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Abstract. We establish several properties of the solutions to the linear integral equations describing the infinite volume properties of the XXZ spin-1/2 chain in the disordered regime. In particular, we obtain lower and upper bounds for the dressed energy, dressed charge and density of Bethe roots. Furthermore, we establish that given a fixed external magnetic field (or a fixed magnetization) there exists a unique value of the boundary of the Fermi zone.

Key words: linear integral equations; quantum integrable models; dressed quantities

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1 Introduction

The Bethe Ansatz constitutes a powerful tool allowing one to map the problem of obtaining the spectrum of numerous one-dimensional quantum Hamiltonians onto one of finding solutions to a system of algebraic equations, the so-called Bethe equations. The method was introduced in 1931 by H. Bethe [2] with the example of the XXX spin-1/2 chain. The latter corresponds to the $\Delta = 1$ limit of the so-called XXZ spin-1/2 chain whose Hamiltonian reads

$$H = J \sum_{n=1}^{L} \left\{ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z \right\} - \frac{h}{2} \sum_{n=1}^{L} \sigma_n^z.$$

Here $J > 0$ is a coupling constant measuring the strength of the exchange interaction, Δ is the longitudinal anisotropy in the couplings, h is an external magnetic field and the σ_n^a are Pauli matrices acting non-trivially on the n^{th}-quantum space $V_n \simeq \mathbb{C}^2$ in the tensor product decomposition $\otimes_{n=1}^{L} V_n$ of the Hilbert space on which H acts.

Although, in general, the Bethe equations, whose solutions provide a set of quantum numbers parameterizing the expectation values of observables of the finite system, cannot be solved explicitly by analytic means (except at the free fermion point $\Delta = 0$), they are e.g. the starting point for an exact analysis of the system in the thermodynamic (infinite volume) limit. In this limit, based on still unproven but fairly reasonable assumptions, the Bethe Ansatz method enables an efficient calculation of many of the physical observables of the model. Quantities such as the total energy and momentum per-lattice site, the dressed energy and the momentum of the excitations above the ground state, their dressed charge, etc. – the so-called thermodynamic functions – are directly characterized by solutions to linear integral equations, which, for many

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models, take the form

$$f(\lambda) + \int_{-Q}^{Q} K(\lambda - \mu) f(\mu) \cdot \mathrm{d}\mu = g(\lambda). \tag{1}$$

The integral kernel K and the domain of integration $[-Q; Q]$ are completely fixed by the regime of the integrable model, whereas the driving term g depends on the specific thermodynamic quantity one is interested in. Usually, one calls g the bare thermodynamic function and f its dressed counterpart.

The possibility to use linear integral equations to describe the thermodynamic limit was first observed by Hulthén [14] who built on it so as to propose an integral representation for the ground-state energy per lattice site of the XXX spin-1/2 chain. Then Orbach [25] applied the Bethe Ansatz method to the spin-1/2 XXZ chain. He generalized Hulthén's approach which allowed him to write down an integral representation for the ground-state energy per lattice site of the XXZ spin-1/2 chain at $h = 0$. Orbach's integrand involved the solution to a linear integral equation. Since he was unable to solve it explicitly, he studied its solution numerically. A year later, Walker [27] transformed Orbach's integral equation into one of the type (1) with $Q = \pi/2$ which then, using Fourier series, he was able to solve explicitly. Later, in 1964, Griffiths [13] investigated the linear integral equations associated with the XXZ spin-1/2 chain at non-vanishing magnetic field. The above approaches focused on the evaluation of the ground state energy per-site. In the mid-sixties, des Cloiseaux and Pearson [6] and des Cloiseaux and Gaudin [5] introduced linear integral equations that describe excitations above the ground state, this at zero magnetic field. All these works built on several assumptions on the large-volume behaviour of the Bethe roots describing the ground state and low-lying excited states. For the thermodynamic limit of the XXZ spin-1/2 chain at non-zero magnetic field Yang and Yang [28] were able to prove some of these assumptions. Still, it was only recently that Dorlas and Samsonov [8] proved the condensation property of the ground state Bethe roots for large-L in the regime $-1 < \Delta < 0$ and for $h \neq 0$. Technicalities related with a rigorous justification of its appearance set aside, the machinery of linear integral equations appears utterly useful for understanding the thermodynamic limit of Bethe ansatz solvable models. Over the years, it has been applied in numerous cases (see [21] and references therein).

The usefulness of linear integral equations goes, in fact, beyond the sole setting of a model's thermodynamic limit. Indeed, it is well known since the works [4, 12, 19, 26, 29] that one needs to recourse to non-linear integral equations in order to describe integrable models of finite size L or at finite temperature T. These are also useful to obtain various scaling limits towards massive and conformal quantum field theories [1, 3, 7, 30]. The coefficients arising in the large-L or low-T asymptotic expansion of solutions to these non-linear integral equations satisfy linear integral equations of the type (1). In fact, our interest in the questions considered in this paper arose when we analysed the low-temperature limit of the non-linear integral equations describing the spectrum of the quantum transfer matrix of the XXZ chain [9, 10]. We needed Proposition 1 below in order to prove that the dressed energy provides a lowest order low-temperature approximation to the solution of the corresponding non-linear integral equation.

Although the existence and uniqueness of solutions to equations of the form $(I + K) \cdot f = g$ is usually easily established using the general theory of linear integral equations, their expressions remain implicit. The aim of the present paper is to prove certain overall properties of these solutions in the case of the linear integral equations associated with the thermodynamic limit of the XXZ spin-1/2 chain in the disordered regime, i.e. for $-1 < \Delta < 1$. In particular, we shall obtain explicit lower and upper bounds to the solutions f for various driving terms g of interest. We shall also discuss the construction of the endpoint of integration Q, and we shall show that the latter is well defined. Finally, to make this paper more self-contained, we shall as well review the calculation of the behaviour of solutions at large Q.

It is worth mentioning that, for more complex quantum integrable models, the structure of the linear integral equation changes [11, 23, 24] (the kernel K may also depend on the sum of arguments or can be matrix valued). We do stress, however, that, although we focus on a specific model, the techniques we develop are general and also applicable to other linear integral equations arising in the context of studying the ground-state properties of other quantum integrable models.

The paper is organized as follows. In Section 2 we discuss the thermodynamic functions that we consider (the dressed charge, the density of Bethe roots and the dressed energy) and give a description of our results. Then, in Section 3, we establish some properties of the resolvent kernel. Finally, Sections 4, 6 and 5 are devoted to the proofs of the results presented in Section 2.

2 Functions characterizing the ground state of the XXZ chain

In the case of the XXZ spin-1/2 chain in the disordered regime, as briefly outlined in the introduction, the thermodynamic functions solve the linear integral equation

$$f(\lambda) + \int_{-Q_F}^{Q_F} K(\lambda-\mu|\gamma) f(\mu) \cdot \mathrm{d}\mu = g(\lambda), \tag{2}$$

where $Q_F > 0$ is a parameter called the Fermi rapidity and

$$K(\lambda|\gamma) = \frac{\coth(\lambda - \mathrm{i}\gamma) - \coth(\lambda + \mathrm{i}\gamma)}{2\mathrm{i}\pi} = \frac{\sin(2\gamma)}{2\pi \sinh(\lambda + \mathrm{i}\gamma)\sinh(\lambda - \mathrm{i}\gamma)}$$

is the integral kernel depending parametrically on the variable $\gamma \in]0;\pi[$ which parameterizes the anisotropy parameter $\Delta = \cos(\gamma) \in]-1;1[$. In the following, we shall often omit the auxiliary argument of the integral kernel and denote it simply by $K(\lambda)$. Therefore, throughout the paper, $K(\lambda)$ should *always* be understood as $K(\lambda|\gamma)$. The Fermi rapidity Q_F arising in (2) is fixed by the external parameters of the model, in particular by the magnetic field h. Its precise definition will be given below.

Note also that there is a qualitative change in the structure of the integral equation depending on the sign of Δ, since the integral kernel K changes sign:

$$K(\lambda|\gamma) > 0 \quad \text{for} \quad 0 < \gamma < \frac{\pi}{2} \qquad \text{and} \qquad K(\lambda|\gamma) < 0 \quad \text{for} \quad \frac{\pi}{2} < \gamma < \pi.$$

Although, in the end, this has no big effect on the qualitative behaviour of the solutions, this fact has important consequences for the techniques one needs to develop to establish the theorems presented below. Finally, we recall that the kernel K becomes trivial when $\gamma = \pi/2$. This is the so-called free fermion point of the model, where dressed and bare quantities coincide.

We stress that the linear integral equations discussed above are all of truncated Wiener–Hopf type. They can thus be solved perturbatively for large Q by the Wiener–Hopf method. The latter amounts to solving, asymptotically in Q, a 2×2 Riemann–Hilbert problem. Furthermore, in the $Q = +\infty$ limit, the linear integral equations turn into convolution-type equations and, as such, can be solved explicitly by means of a Fourier transformation.

2.1 The thermodynamic functions and the main results

2.1.1 The dressed charge

The dressed charge $Z(\lambda|Q)$ is defined as the solution to the integral equation

$$Z(\lambda|Q) + \int_{-Q}^{Q} K(\lambda-\mu) \cdot Z(\mu|Q) \cdot \mathrm{d}\mu = 1.$$

It can be interpreted as the intrinsic moment of the magnetic excitations in the XXZ spin-1/2 chain. Indeed, the thermodynamic limit of the average magnetization $\langle\sigma^z\rangle/2$ of a state described by Bethe roots that condense on $[-Q;Q]$ is given by

$$\langle\sigma^z\rangle = 1 - 2\int_{-Q}^{Q} Z(\lambda|Q)\cdot K(\lambda|\gamma/2)\cdot \mathrm{d}\lambda.$$

The dressed charge is the thermodynamic function associated with the XXZ spin-1/2 chain which has the simplest non-trivial driving term in (2). It is involved in the characterization of numerous observables in the XXZ chain. For instance, when looking at the form of the low-lying excitations (of the order $1/L$) above the ground state in finite but large volume L, one obtains an expression of the form [20, 21]

$$E_{\mathrm{ex}} - E_{\mathrm{G.S.}} \simeq \frac{2\pi}{L}\cdot v_F \cdot \left\{ \left(\ell\mathcal{Z}\right)^2 + \left(\frac{s}{2\mathcal{Z}}\right)^2 + n \right\}, \tag{3}$$

where ℓ is the Umklapp sector of the excitation, s its spin and n an integer. The formula contains the constant $\mathcal{Z} = Z(Q_F|Q_F)$ which corresponds to the value taken by the dressed charge at the Fermi rapidity Q_F (see Section 2.1.3). The above formula contains v_F which has the interpretation of the velocity of excitations at Q_F. We shall comment on this quantity later on. The dressed charge also appears in the large-distance asymptotic behaviour of the longitudinal spin-spin correlation functions [15]

$$\langle\sigma_1^z\sigma_m^z\rangle = \langle\sigma^z\rangle^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + \sum_{\ell\geq 1}\left|\mathcal{F}\left[\ell Z(*|Q)\right]\right|^2\cdot\frac{\cos(2\pi p_F m)}{m^{2\ell^2\mathcal{Z}^2}}\cdot\left(1 + O\left(\frac{\ln m}{m}\right)\right). \tag{4}$$

In this formula $\mathcal{F}$ is a certain explicit functional which represents the properly normalized form factor of the spin operator taken between the model's ground state and an excited state with an ℓ-Umklapp excitation [16]. The $*$ in (4) indicates the running variable on which $\mathcal{F}$ acts. In the above asymptotic expansion, the constant $\mathcal{Z}$ parameterizes the magnitude of the oscillatory part of the large-distance asymptotics. In particular, depending on whether $\mathcal{Z} > 1$ or $\mathcal{Z} < 1$, the non-oscillatory asymptotics (which rather reflect ferromagnetic order) or the oscillatory ones (which reflect the antiferromagnetic nature of the interactions) will be dominant. The constant p_F will be defined below.

In order to state our results regarding the characterization of the dressed charge we need to introduce the domain

$$\Upsilon_\gamma(Q) = \mathbb{C}\setminus\big\{\{[-Q;Q]+\mathrm{i}\gamma+\mathrm{i}n\pi : n\in\mathbb{Z}\}\cup\{[-Q;Q]-\mathrm{i}\gamma+\mathrm{i}n\pi : n\in\mathbb{Z}\}\big\}.$$

Theorem 1. *The dressed charge is a smooth function of $(\lambda,Q)\in\mathbb{R}\times\mathbb{R}^+$ such that, pointwise in Q, it is an $\mathrm{i}\pi$-periodic, holomorphic function of λ on $\Upsilon_\gamma(Q)$. Furthermore, $\lambda\mapsto Z(\lambda|Q)$ is even and*

- *for $0<\gamma<\pi/2$, is monotonically increasing on $\mathbb{R}^+$ and satisfies the bounds*

$$\frac{1}{2\big(1-\gamma/\pi\big)} < Z(\lambda|Q) < 1; \tag{5}$$

- *for $\pi/2<\gamma<\pi$, is monotonically decreasing on $\mathbb{R}^+$ and satisfies the bounds*

$$1 < Z(\lambda|Q) < \frac{1}{2\big(1-\gamma/\pi\big)}. \tag{6}$$

Finally, one has the large-Q behaviour

$$Z\big(Q|Q\big) = \sqrt{\frac{\pi}{2(\pi-\gamma)}} + O\big(e^{-2Q\epsilon_\gamma}\big) \qquad \textit{with} \quad \epsilon_\gamma = \min\left\{\frac{2\pi}{\pi-\gamma},\frac{\pi}{\gamma}\right\}. \tag{7}$$

2.1.2 The density of Bethe roots and the dressed momentum

Consider the sector of the XXZ spin-1/2 chain corresponding to the magnetization $1 - 2\mathfrak{m}_N$ with $\mathfrak{m}_N = N/L$. The ground state in this sector will be parameterized by N Bethe roots $\lambda_1 < \cdots < \lambda_N$. In the thermodynamic limit, $N, L \to +\infty$ with $\mathfrak{m}_N \to \mathfrak{m} \in [0; 1/2]$, these Bethe roots will condense in an interval $[-Q_{\mathfrak{m}}; Q_{\mathfrak{m}}]$ with a density $\rho(\lambda|Q_{\mathfrak{m}})$. The function $\rho(\lambda|Q)$ solves the linear integral equation

$$\rho(\lambda|Q) + \int_{-Q}^{Q} K(\lambda - \mu) \cdot \rho(\mu|Q) \cdot \mathrm{d}\mu = K(\lambda|\gamma/2). \tag{8}$$

The endpoint $Q_{\mathfrak{m}}$ of the condensation interval, the so-called magnetic Fermi rapidity, is fixed by the equation

$$\int_{-Q_{\mathfrak{m}}}^{Q_{\mathfrak{m}}} \rho(\lambda|Q_{\mathfrak{m}}) \cdot \mathrm{d}\lambda = \mathfrak{m}. \tag{9}$$

In order to affirm that $\rho(\lambda|Q_{\mathfrak{m}})$ does indeed represent a density, one should establish, in particular, that it is a positive function. This property is relatively clear in the regime $\pi/2 < \gamma < \pi$. However, some tricks are necessary to establish it in the other regime $0 < \gamma < \pi/2$. Also, for various practical applications, it is useful to have explicit bounds on $\rho(\lambda|Q)$. Finally, the density of Bethe roots is a highly non-linear function of Q. Therefore (9) is a non-linear equation for the magnetic Fermi rapidity. It is not clear *a priori* whether a solution exists and, if yes, whether it is unique. These properties follow from the theorem below.

Theorem 2. *The density of Bethe roots $\rho(\lambda|Q)$ is an $\mathrm{i}\pi$-periodic meromorphic function on $\Upsilon_\gamma(Q)$ whose sole singularities are simple poles located at the points $\pm\mathrm{i}\gamma/2 + \mathrm{i}n\pi$. It is an even function of λ and, uniformly in Q, is subject to the bounds*

$$\rho_\infty(\lambda) < \rho(\lambda|Q) < K\big(\lambda|\gamma/2\big) \qquad \textit{for} \quad 0 < \gamma < \pi/2, \tag{10}$$

$$K\big(\lambda|\gamma/2\big) < \rho(\lambda|Q) < \rho_\infty(\lambda) \qquad \textit{for} \quad \pi/2 < \gamma < \pi, \tag{11}$$

where

$$\rho_\infty(\lambda) = \lim_{Q \to +\infty} \rho(\lambda|Q) = \left\{ 2\gamma \cosh\left(\frac{\pi\lambda}{\gamma}\right) \right\}^{-1}.$$

Furthermore, for any $\mathfrak{m} \in [0; 1/2]$, there exists a unique magnetic Fermi rapidity $Q_{\mathfrak{m}} \in [0; +\infty[$ solving (9). The map $\mathfrak{m} \mapsto Q_{\mathfrak{m}}$ is smooth and monotonically increasing with derivative given by

$$\partial_{\mathfrak{m}} Q_{\mathfrak{m}} = \big\{ 2\rho\big(Q_{\mathfrak{m}}|Q_{\mathfrak{m}}\big) \cdot Z\big(Q_{\mathfrak{m}}|Q_{\mathfrak{m}}\big) \big\}^{-1}.$$

Finally, for $\pi/5 < \gamma < \pi$ one has the large-Q behaviour

$$\rho(Q|Q) = e^{-\frac{\pi}{\gamma}Q} \cdot \sqrt{\frac{2}{\gamma}} \cdot \big(1 - \gamma/\pi\big)^{\frac{\pi}{2\gamma}} \cdot \frac{\Gamma\big(1 + \pi/2\gamma\big)}{\Gamma((1 + \pi/\gamma)/2)} + O\big(e^{-2Q\epsilon_\gamma}\big),$$

where ϵ_γ is as defined in (7).

The density of Bethe roots is connected with the so-called dressed momentum $p(\lambda|Q)$ in that $\rho(\lambda|Q)$ corresponds to the density of momentum variables

$$p(\lambda|Q) = 2\pi \cdot \int_0^\lambda \rho(\mu|Q) \cdot \mathrm{d}\mu. \tag{12}$$

One can also define the dressed momentum as the solution to the integral equation

$$p(\lambda|Q)+\int_{-Q}^{Q}K(\lambda-\mu)\cdot p(\mu|Q)\cdot \mathrm{d}\mu=p_0(\lambda)+p\big(Q|Q\big)\cdot[\theta(\lambda-Q)+\theta(\lambda+Q)], \tag{13}$$

where p_0 is the bare momentum and θ the bare phase

$$p_0(\lambda)=\mathrm{i}\ln\left(\frac{\sinh(\mathrm{i}\gamma/2+\lambda)}{\sinh(\mathrm{i}\gamma/2-\lambda)}\right) \qquad \text{and} \qquad \theta(\lambda)=\mathrm{i}\ln\left(\frac{\sinh(\mathrm{i}\gamma+\lambda)}{\sinh(\mathrm{i}\gamma-\lambda)}\right).$$

Since $p_0'(\lambda)=2\pi K(\lambda|\gamma/2)$, it is readily seen that the definition (13) coincides with $\partial_\lambda p(\lambda|Q)=2\pi\rho(\lambda|Q)$. It should be also clear that, on the basis of (12), one can deduce the properties of the dressed momentum from those of the density of Bethe roots.

2.1.3 The dressed energy and the Fermi rapidity

The dressed energy $\varepsilon(\lambda|Q)$ satisfies

$$\varepsilon(\lambda|Q)+\int_{-Q}^{Q}K(\lambda-\mu)\cdot\varepsilon(\mu|Q)\cdot\mathrm{d}\mu=\varepsilon_0(\lambda), \tag{14}$$

where $\varepsilon_0(\lambda)=h-4\pi J\sin(\gamma)K(\lambda|\gamma/2)$ is the so-called bare energy. In particular, $\varepsilon(\lambda|Q)$ is an implicit function of the external magnetic field $h>0$ and of the magnitude of the exchange interaction J.

From the point of view of experiments on materials whose magnetic properties are modeled by the XXZ spin-1/2 chain, one imposes an external magnetic field h. The latter forces the magnetization of the material to adjust accordingly. The XXZ-Hamiltonian can be diagonalized in every sector with a fixed magnetization. Each of these sectors admits a fixed magnetization ground state which corresponds to a condensation of Bethe roots in the interval $[-Q_{\mathfrak{m}};Q_{\mathfrak{m}}]$. Then the overall ground state of the model is to be chosen among all these fixed magnetization ground states in such a way that the interaction with the external magnetic field leads to the lowest possible energy. This means that one should choose the Fermi sea of the model $[-Q_F;Q_F]$ in such a way that the energy of excitations exactly at the Fermi boundary Q_F vanishes and is negative inside of $[-Q_F;Q_F]$ and positive outside, viz. on $\mathbb{R}\setminus[-Q_F;Q_F]$. The reason for this is that the ground state built in such a way has the property that creating a hole inside the Fermi zone or adding a particle outside of it necessarily increases the energy. In other words, the Fermi rapidity is defined by the equation

$$\varepsilon(Q_F|Q_F)=0.$$

Note that, just as $\rho(\lambda|Q)$, $\varepsilon(\lambda|Q)$ depends parametrically on Q in a complicated way. Thus, again, it is not clear *a priori* whether the above equation does admit solutions at all and, if, yes whether these are unique or not.

Within this context, the so-called particle-hole excitations above the ground state take the form

$$E_{\text{ex}}-E_{\text{G.S.}}=\sum_{a=1}^{n_p}\varepsilon\big(\lambda_a^{(p)}\big|Q_F\big)-\sum_{a=1}^{n_h}\varepsilon\big(\lambda_a^{(h)}\big|Q_F\big)$$

in which $\lambda_a^{(p)}\in\mathbb{R}\setminus[-Q_F;Q_F]$ are the particle rapidities and $\lambda_a^{(h)}\in[-Q_F;Q_F]$ are the hole rapidities.

At this stage of our discussion we are finally in position to define of the Fermi velocity which appeared in (3). In terms of the dressed energy and the momentum of excitations it is given as

$$v_F = \frac{\partial_\lambda \varepsilon(Q_F|Q_F)}{\partial_\lambda p(Q_F|Q_F)}.$$

Here, ∂_λ refers to the derivative in respect to the first argument of the functions. To close, the constant p_F governing the speed of the oscillations in (4) corresponds to the value of the dressed momentum on the Fermi boundary Q_F, viz. $p(Q_F|Q_F) = p_F$.

Theorem 3. *The dressed energy is a smooth function of $(\lambda, Q) \in \mathbb{R} \times [0; +\infty[$. Furthermore, pointwise in Q, it is an $\mathrm{i}\pi$-periodic meromorphic function of λ on $\Upsilon_\gamma(Q)$ with simple poles at $\lambda = \pm \mathrm{i}\gamma/2 + \mathrm{i}n\pi$.*

$\varepsilon(\lambda|Q)$ satisfies the bounds

$$\forall\, Q \leq Q_0 \qquad \begin{cases} \widetilde{\varepsilon}(\lambda) > \varepsilon(\lambda|Q) > \varepsilon_0(\lambda) & \text{for} \;\; 0 < \gamma < \pi/2, \\ \varepsilon_0(\lambda) > \varepsilon(\lambda|Q) > \widetilde{\varepsilon}(\lambda) & \text{for} \;\; \pi/2 < \gamma < \pi, \end{cases}$$

where Q_0 corresponds to the unique positive zero of $\varepsilon_0(\lambda)$ and the function $\widetilde{\varepsilon}$ is given by

$$\widetilde{\varepsilon}(\lambda) = h - \frac{2\pi J \sin(\gamma)}{\gamma \cosh\left(\frac{\pi}{\gamma}\lambda\right)}.$$

Further, the bound involving $\widetilde{\varepsilon}$ holds, in fact, for any value of Q.

For any $h > 0$, there exists a unique solution $Q_F > 0$ to the equation $\varepsilon(Q_F|Q_F) = 0$. Let $\widetilde{Q} > 0$ be the unique positive zero of $\widetilde{\varepsilon}(\lambda)$. Then the Fermi rapidity Q_F satisfies the bounds

$$\widetilde{Q} < Q_F < Q_0 \quad \text{for} \quad 0 < \gamma < \pi/2 \qquad \text{and} \qquad \widetilde{Q} > Q_F > Q_0 \quad \text{for} \quad \pi/2 < \gamma < \pi.$$

The function $h \mapsto Q_F$ is a smooth, monotonically decreasing, function of h with h-derivative given by

$$\partial_h Q_F = -\frac{Z(Q_F|Q_F)}{\partial_\lambda \varepsilon(Q_F|Q_F)}.$$

For $\pi > \gamma > \pi/5$ it has the small-h asymptotic behaviour given by

$$\exp\left(\frac{\pi}{\gamma} Q_F(h)\right) = \frac{8\pi J \sin(\gamma)}{\sqrt{\gamma} \cdot h} \cdot \left(1 - \frac{\gamma}{\pi}\right)^{\frac{\pi+\gamma}{2\gamma}} \cdot \frac{\Gamma\big(1 + \pi/2\gamma\big)}{\Gamma((1 + \pi/\gamma)/2)} \cdot (1 + o(1)).$$

In fact, the dressed energy evaluated precisely at the Fermi rapidity Q_F has several very natural properties. In particular, its real part is positive on the line $\mathbb{R} - \mathrm{i}\gamma$. This property is crucial for a consistency check of the low-T asymptotic expansion [9] of the solution to the non-linear integral equation [18] driving the finite temperature properties of the XXZ spin-1/2 chain.

Proposition 1. *The dressed energy evaluated at the Fermi rapidity $\lambda \mapsto \varepsilon(\lambda|Q_F)$ is a monotonically increasing function on $\mathbb{R}^+$ with a unique zero at Q_F that satisfies*

$$\Re\big[\varepsilon_+(\lambda - \mathrm{i}\gamma|Q_F)\big] > \frac{h}{4} \qquad \text{for all} \quad \lambda \in \mathbb{R} \qquad \text{and} \qquad 0 < \gamma < \pi/2. \tag{15}$$

Here, ε_+ refers to the boundary value of ε when approaching a point on $\mathbb{R} - \mathrm{i}\gamma$ from above.

3 The resolvent kernel

The resolvent kernel $R_Q(\lambda,\mu)$ is defined as the integral kernel of the inverse operator $I-R_Q$ to $I+K$ understood as acting on $\mathcal{C}^0([-Q;Q])$, the space of continuous functions on the interval $[-Q;Q]$. It satisfies the integral equation

$$R_Q(\lambda,\mu)+\int_{-Q}^{Q}K(\lambda-\nu)\cdot R_Q(\nu,\mu)\cdot \mathrm{d}\nu = K(\lambda-\mu).$$

Using the resolvent kernel the solution to (2) can be represented as

$$f(\lambda)=g(\lambda)-\int_{-Q}^{Q}R_Q(\lambda,\mu)\cdot g(\mu)\cdot \mathrm{d}\mu.$$

R_Q is thus the most important object associated with this integral equation. In the present section, we establish its existence (i.e. the invertibility of $I+K$) and prove several of its properties.

Also, since the kernel K depends on the difference of its arguments, the integral equations are of truncated Wiener–Hopf type. As such, they are exactly solvable in the $Q\to+\infty$ limit by means of Fourier transformation. In this limit, the Neumann series defines a kernel R solely depending on the difference of variables $\lambda-\mu$ which solves the convolution-type integral equation

$$R(\lambda-\mu)+\int_{\mathbb{R}}K(\lambda-\nu)\cdot R(\nu-\mu)\cdot \mathrm{d}\nu = K(\lambda-\mu).$$

3.1 Overall properties and bounds

Proposition 2. *The operator $I+K:\mathcal{C}^0([-Q;Q])\mapsto\mathcal{C}^0\big([-Q;Q]\big)$ is invertible, and its resolvent kernel is well defined and given in terms of the Neumann series*

$$R_Q(\lambda,\mu)=K(\lambda-\mu)-\sum_{n\geq1}(-1)^{n-1}\int_{-Q}^{Q}K(\lambda-\nu_1)\cdot\prod_{a=1}^{n-1}\big\{K(\nu_a-\nu_{a+1})\big\}\cdot K(\nu_n-\mu)\cdot \mathrm{d}^n\nu.$$

The resolvent kernel $R_Q(\lambda,\mu)$ is a symmetric function of (λ,μ). It is also a smooth function in $(\lambda,\mu,Q)\in\mathbb{R}^2\times\mathbb{R}^+$.

Proof. The Neumann series is convergent in the sup-norm topology $||f||_\infty=\max\big\{|f(\mu)|:\mu\in[-Q;Q]\big\}$ since, for any $f\in\mathcal{C}^0([-Q;Q])$ such that $||f||_\infty\leq1$ one has

$$||K.f||_\infty\leq||f||_\infty\cdot\max_{\lambda\in[-Q;Q]}\int_{\mathbb{R}}\big|K(\lambda-\mu)\big|\cdot\mathrm{d}\mu\leq\Big|1-2\frac{\gamma}{\pi}\Big|<1. \tag{16}$$

The latter implies the uniform convergence of the Neumann series with exponential speed. The bound (16) also ensures that the spectral radius of the integral operator K is bounded by $\big|1-2\frac{\gamma}{\pi}\big|$. Moreover, using standard differentiation under the integral sign theorems, it follows that $R_Q(\lambda,\mu)$ is smooth in $(\lambda,\mu,Q)\in\mathbb{R}^2\times\mathbb{R}^+$. ■

We now focus on the $Q=+\infty$ case. We establish some of the most fundamental properties of the resolvent R at $Q=+\infty$ that will be most useful for the analysis that shall follow:

Lemma 1.

(*i*) *R has the Fourier integral representation*

$$R(\lambda)=\int_{\mathbb{R}}\frac{\sinh\big[\big(\pi/2-\gamma\big)k\big]e^{-\mathrm{i}k\lambda}}{\cosh(\gamma k/2)\sinh[(\pi/2-\gamma/2)k]}\frac{\mathrm{d}k}{4\pi}. \tag{17}$$

(*ii*) *For* $0 < \gamma < \pi/2$, R *is even and positive on* $\mathbb{R}$, *monotonically decreasing on* $\mathbb{R}^+$ *and satisfies* $\lim_{\lambda\to\infty} R(\lambda) = 0$.

Proof. Item (i) follows from the convolution theorem using

$$\mathcal{F}[K](k) = \frac{\sinh\left[\left(\frac{\pi}{2}-\gamma\right)k\right]}{\sinh\left(\frac{\pi k}{2}\right)}, \qquad 1+\mathcal{F}[K](k) = \frac{2\cosh\left(\frac{\gamma k}{2}\right)\sinh\left[\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)k\right]}{\sinh\left(\frac{\pi k}{2}\right)}, \tag{18}$$

where

$$\mathcal{F}[g](k) = \int_{\mathbb{R}} g(\lambda) e^{\mathrm{i}k\lambda}\cdot \mathrm{d}\lambda.$$

In order to prove (ii) observe that

$$\mathcal{F}[g](k) = \frac{\gamma\pi}{\pi-\gamma}\cdot\frac{1}{\cosh\left[\frac{\gamma\pi k}{2(\pi-\gamma)}\right]}, \qquad \text{where} \qquad g(\lambda) = \frac{1}{\cosh\left[(1-\pi/\gamma)\lambda\right]}.$$

This allows one to recast R, for $0 < \gamma < \pi/2$, as

$$R(\lambda) = \frac{\pi}{2\gamma(\pi-\gamma)}\int_{\mathbb{R}} \frac{K\left(\frac{\mu}{1-\gamma/\pi}|\gamma'\right)}{\cosh[\pi(\lambda-\mu)/\gamma]}\cdot \mathrm{d}\mu > 0, \qquad \text{where} \qquad \gamma' = \frac{\gamma/2}{1-\gamma/\pi} \tag{19}$$

(compare [28, Appendix C]). It is clear from the above representation that R is even, asymptotically zero, and that it is monotonically decreasing for $\lambda > 0$. ■

The function R is one of the most prominent functions in the theory of the XXZ model. It appears as the logarithmic derivative of the two-spinon scattering phase and determines the free energy of the six-vertex model in the critical regime. It satisfies nice functional equations and can be expressed in terms of Barnes double-gamma functions.

In order to be able to extract valuable informations out of the resolvent kernel $R_Q(\lambda,\mu)$ one needs to obtain lower (or upper) bounds on it. This is the purpose of the lemma below.

Lemma 2. *The resolvent kernel satisfies the bounds*

$$\begin{aligned} &R_Q(\lambda,\mu) > R(\lambda-\mu) \qquad \textit{for} \quad 0<\gamma<\pi/2, \\ &R(\lambda-\mu) < R_Q(\lambda,\mu) < 0 \qquad \textit{for} \quad \pi/2<\gamma<\pi \end{aligned} \tag{20}$$

uniformly in $(\lambda,\mu)\in\mathbb{R}^2$. *Furthermore, for* $\lambda,\mu>0$, *it also satisfies the inequalities*

$$\begin{aligned} &R_Q(\lambda,\mu) - R_Q(\lambda,-\mu) > 0 \qquad \textit{for} \quad 0<\gamma<\frac{\pi}{2}, \\ &R_Q(\lambda,\mu) - R_Q(\lambda,-\mu) < 0 \qquad \textit{for} \quad \frac{\pi}{2}<\gamma<\pi. \end{aligned} \tag{21}$$

A quite non-trivial consequence of the above lemma is that $R_Q(\lambda,\mu)$ is a strictly positive function when $0 < \gamma < \pi/2$. This statement is highly non-trivial while solely looking at the Neumann series for this operator.

Proof. The bound for $\pi/2 < \gamma < \pi$ is readily deduced by term-wise majorations in the Neumann series, since the series for $-R_Q(\lambda,\mu)$ is a sum of strictly positive terms. In order to establish the second bound, we recast the integral equation satisfied by $R_Q(\lambda,\mu)$ in the form

$$R_Q(\lambda,\mu) + \int_{\mathbb{R}} K(\lambda-\nu)R_Q(\nu,\mu)\cdot\mathrm{d}\nu - \int_{\mathbb{R}\setminus[-Q;Q]} K(\lambda-\nu)R_Q(\nu,\mu)\cdot\mathrm{d}\nu = K(\lambda-\mu).$$

Then, acting with the inverse operator $I - R$ on $I + K$ understood as an integral operator on $\mathcal{C}_b^0(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ that are bounded at infinity, recasts the above integral equation as

$$R_Q(\lambda,\mu) - \int_{\mathbb{R}\setminus[-Q;Q]} R(\lambda-\nu)\cdot R_Q(\nu,\mu)\cdot \mathrm{d}\nu = R(\lambda-\mu).$$

The Neumann series for the resolvent $\mathcal{R}$ of the integral operator $I - R$ understood as acting on $\mathcal{C}_b^0(\mathbb{R}\setminus[-Q;Q])$ converges uniformly, for the very same reasons that were invoked for the operator K acting on $\mathcal{C}_b^0([-Q;Q])$, since, when $0 < \gamma < \pi/2$, cf. (17),

$$\int_{\mathbb{R}} R(\lambda)\cdot \mathrm{d}\lambda = 1 - \frac{\pi}{2(\pi-\gamma)} < 1.$$

Since this Neumann series for $\mathcal{R}$ is a sum of strictly positive terms it follows that $R_Q(\lambda,\mu) > R(\lambda-\mu)$. The bounds (21) at $\pi/2 < \gamma < \pi$ follow from the series of multiple integral representations

$$R_Q(\lambda,\mu) - R_Q(\lambda,-\mu) = K(\lambda-\mu) - K(\lambda+\mu) + \sum_{n\geq 1}\int_0^Q \big[K(\lambda-\tau_1) - K(\lambda+\tau_1)\big]$$
$$\times \prod_{\ell=1}^{n-1}\big[K(\tau_\ell-\tau_{\ell+1}) - K(\tau_\ell+\tau_{\ell+1})\big]\cdot\big[K(\tau_n-\mu) - K(\tau_n+\mu)\big]\cdot \mathrm{d}^n\tau$$

and an analogous representation involving R for $0 < \gamma < \pi/2$. ■

3.2 Asymptotic representation at large Q

The resolvent kernel $R_Q(\lambda,\mu)$ of the operator $I + K$ at large Q can be constructed by means of an asymptotic resolution of a 2×2 Riemann–Hilbert problem. This has been carried out in [17]. We recall these results here. The resolvent $R_Q(\lambda,\mu)$ can be decomposed as

$$R_Q(\lambda,\mu) = R_Q^{(0)}(\lambda,\mu) + R_Q^{(\mathrm{pert})}(\lambda,\mu), \tag{22}$$

where

$$R_Q^{(0)}(\lambda,\mu) = \int_{\mathbb{R}} \frac{\mathrm{d}\xi\mathrm{d}\eta}{4\mathrm{i}\pi^2}\mathcal{F}[K](\xi)\cdot\left\{\frac{\alpha_+(\eta)}{\alpha_-(\xi)}e^{\mathrm{i}Q(\xi-\eta)} - \frac{\alpha_+(\xi)}{\alpha_-(\eta)}e^{-\mathrm{i}Q(\xi-\eta)}\right\}\cdot\frac{e^{\mathrm{i}(\mu\eta-\lambda\xi)}}{\xi-\eta} \tag{23}$$

and, uniformly in (λ,μ), one has

$$\left|R_Q^{(\mathrm{pert})}(\lambda,\mu)\right| \leq Ce^{-2\epsilon_\gamma Q} \qquad \text{with} \quad \epsilon_\gamma = \min\left\{\frac{2\pi}{\pi-\gamma},\frac{\pi}{\gamma}\right\}.$$

Note that ϵ_γ corresponds to the imaginary part of the zero of the function $1 + \mathcal{F}[K]$ closest to the real axis. The function α is analytic on $\mathbb{C}\setminus\mathbb{R}$, goes to 1 at ∞ and satisfies the jump condition on $\mathbb{R}$

$$\alpha_+(\xi)\cdot\big(1+\mathcal{F}[K](\xi)\big) = \alpha_-(\xi).$$

Above, $\alpha_\pm(s)$ correspond to the boundary values when z approaches a point $s\in\mathbb{R}$ from the upper $(+)$ or lower $(-)$ half-plane. In fact, α can be expressed in terms of Γ-functions as

$$\alpha(\lambda) = \sqrt{2(\pi-\gamma)}\cdot\frac{(1-\gamma/\pi)^{\frac{\mathrm{i}\lambda}{2}(1-\frac{\gamma}{\pi})}\cdot(\gamma/\pi)^{\frac{\mathrm{i}\lambda\gamma}{2\pi}}\cdot\Gamma(1+\mathrm{i}\lambda/2)}{\Gamma\Big(\frac{1+\mathrm{i}\lambda\gamma/\pi}{2}\Big)\cdot\Gamma(1+\mathrm{i}\frac{\lambda}{2}(1-\frac{\gamma}{\pi}))} \qquad \text{for} \quad \Im(\lambda) < 0,$$

and one has the symmetry $\alpha(\lambda) = 1/\alpha(-\lambda)$, which yields the expression for α in the upper-half plane.

4 The dressed charge

4.1 General bounds

The dressed charge can be explicitly expressed in terms of the resolvent kernel as

$$Z\big(\lambda|Q\big) = 1 - \int_{-Q}^{Q} R_Q(\lambda,\mu)\cdot \mathrm{d}\mu.$$

Thus, it follows from the bounds (20) for $\pi/2 < \gamma < \pi$, that

$$1 < Z\big(\lambda|Q\big) < 1 - \int_{-Q}^{Q} R(\lambda-\mu)\cdot \mathrm{d}\mu < 1 - \int_{\mathbb{R}} R(\mu)\cdot \mathrm{d}\mu = \frac{\pi}{2(\pi-\gamma)}.$$

Since $R(\lambda) > 0$ when $0 < \gamma < \pi/2$ due to (20), one obtains that $R_Q(\lambda,\mu) > 0$ which implies that $Z(\lambda|Q) < 1$ in this regime. Further, repeating the change of integration contour trick, we get that $Z(\lambda|Q)$ solves

$$Z(\lambda|Q) = \frac{\pi}{2(\pi-\gamma)} + \int_{\mathbb{R}\setminus[-Q;Q]} R(\lambda-\mu) Z(\mu|Q)\cdot \mathrm{d}\mu.$$

Using that the resolvent $\mathcal{R}(\lambda,\mu)$ to $I-R$ is positive, we get the lower bound.

It follows from the smoothness properties of the resolvent kernel $R_Q(\lambda,\mu)$ and the compactness of $[-Q;Q]$ that $Z(\lambda|Q)$ is a smooth function of $(\lambda,Q)\in\mathbb{R}\times\mathbb{R}^{+}$. Then, differentiation under the integral sign and integration by parts implies that $\partial_\lambda Z$ solves the integral equation

$$\partial_\lambda Z(\lambda|Q) + \int_{-Q}^{Q} K(\lambda-\mu)\cdot\partial_\mu Z(\mu|Q)\cdot \mathrm{d}\mu = [K(\lambda-Q)-K(\lambda+Q)]\cdot Z(Q|Q).$$

Thus,

$$\partial_\lambda Z(\lambda|Q) = [R_Q(\lambda,Q)-R_Q(\lambda,-Q)]\cdot Z(Q|Q),$$

so that one can conclude about the strict monotonicity of $\lambda\mapsto Z(\lambda|Q)$ in virtue of equation (21).

4.2 Behaviour of $Z(Q|Q)$ at large Q

We now establish the large-Q behaviour by following the steps in [17]. Due to (22), (23)

$$Z(Q|Q) - 1 = -\int_{-Q}^{Q} R_Q(Q,\mu)\cdot \mathrm{d}\mu = \mathcal{V}_Q + O\big(Qe^{-2\epsilon_\gamma Q}\big),$$

where we have used the explicit bounds on $R_Q^{(\mathrm{pert})}$ and have set

$$\mathcal{V}_Q = -\int_{\mathbb{R}} \frac{\mathrm{d}\xi\mathrm{d}\eta}{4\mathrm{i}\pi^2}\cdot\frac{\mathcal{F}[K](\xi)}{\xi-\eta}\left\{\frac{\alpha_+(\eta)}{\alpha_-(\xi)}e^{-\mathrm{i}Q\eta} - \frac{\alpha_+(\xi)}{\alpha_-(\eta)}e^{\mathrm{i}Q\eta-2\mathrm{i}\xi Q}\right\}\cdot\frac{e^{\mathrm{i}\eta Q}-e^{-\mathrm{i}\eta Q}}{\mathrm{i}\eta}.$$

We now carry out the large-Q asymptotic expansion of the above expression. We deform the η-integration contour to $\mathbb{R}+i\kappa$ for some $\kappa>0$ small enough and use the jump condition satisfied by α. This yields

$$\mathcal{V}_Q = -\int_{\mathbb{R}+\mathrm{i}\kappa}\frac{\mathrm{d}\eta}{2\mathrm{i}\pi}\int_{\mathbb{R}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\left\{\alpha_+(\eta)\cdot\frac{\alpha_+^{-1}(\xi)-\alpha_-^{-1}(\xi)}{(\xi-\eta)\eta}\big(1-e^{-2\mathrm{i}Q\eta}\big)\right.$$
$$\left. - e^{-2\mathrm{i}Q\xi}\frac{\alpha_-(\xi)-\alpha_+(\xi)}{\alpha_-(\eta)(\xi-\eta)\eta}\big(e^{2\mathrm{i}Q\eta}-1\big)\right\}.$$

We now decompose $\alpha_+^{-1}(\xi) - \alpha_-^{-1}(\xi) = \alpha_+^{-1}(\xi) - 1 + 1 - \alpha_-^{-1}(\xi)$ (resp. $\alpha_-(\xi) - \alpha_+(\xi) = \alpha_-(\xi) - 1 + 1 - \alpha_+(\xi)$) in the first (resp. second) term. By deforming the ξ-integration to $-i\infty$ one can then drop all the contribution of $1 - \alpha_-^{-1}(\xi)$ (resp. $\alpha_-(\xi) - 1$) leading to

$$\mathcal{V}_Q = -\int_{\mathbb{R}+\mathrm{i}\kappa} \frac{\mathrm{d}\eta}{2\mathrm{i}\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \left\{ \alpha_+(\eta) \cdot \frac{\alpha_+^{-1}(\xi) - 1}{(\xi - \eta)\eta} \big(1 - e^{-2\mathrm{i}Q\eta}\big) \right\} + \int_{\mathbb{R}+\mathrm{i}\kappa} \frac{\mathrm{d}\eta}{2\mathrm{i}\pi} \int_{\mathbb{R}-\mathrm{i}\kappa} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \left\{ e^{-2\mathrm{i}Q\xi} \frac{1 - \alpha_+(\xi)}{\alpha_-(\eta)(\xi - \eta)\eta} \big(e^{2\mathrm{i}Q\eta} - 1\big) \right\}.$$

The second line is already $O\big(e^{-2\kappa Q}\big)$. The contribution of the first line can be obtained by deforming the ξ-integration to $+\mathrm{i}\infty$ and picking up the pole at $\xi = \eta$. Thus,

$$\begin{aligned} \mathcal{V}_Q &= -\int_{\mathbb{R}+\mathrm{i}\kappa} \frac{\mathrm{d}\eta}{2\mathrm{i}\pi} \frac{1 - \alpha_+(\eta)}{\eta} \big(1 - e^{-2\mathrm{i}Q\eta}\big) + O\big(e^{-2\kappa Q}\big) \\ &= \alpha_+(0) - 1 + \int_{\mathbb{R}-\mathrm{i}\kappa} \frac{1 - \alpha_+(\eta)}{\eta} e^{-2\mathrm{i}Q\eta} \cdot \frac{\mathrm{d}\eta}{2\mathrm{i}\pi} + O\big(e^{-2\kappa Q}\big). \end{aligned}$$

The integral term, again, is $O\big(e^{-2\kappa Q}\big)$. Then, it follows from $\alpha_+(\lambda) = \alpha_-^{-1}(-\lambda)$ that

$$\alpha_+^2(0) = \frac{1}{1 + \mathcal{F}[K](0)} = \frac{\pi}{2(\pi - \gamma)}.$$

In other words, one has

$$Z(Q|Q) = \sqrt{\frac{\pi}{2(\pi - \gamma)}} + O\big(Qe^{-2\epsilon_\gamma Q}\big)$$

whence the optimal value for κ is ϵ_γ.

5 The density of Bethe roots

5.1 General bounds

In this section we establish Theorem 2. The meromorphicity and location of cuts is readily deduced from the linear integral equation defining $\rho(\lambda|Q)$. In order to establish the lower and upper bounds it is enough to repeat the strategy that has already been employed for the dressed charge and use that

$$\frac{p_0'(\lambda)}{2\pi} - \int_{\mathbb{R}} R(\lambda - \mu) \cdot p_0'(\mu) \cdot \frac{\mathrm{d}\mu}{2\pi} = \rho_\infty(\lambda).$$

Thus, it remains to establish the statement relative to the magnetic Fermi rapidity $Q_{\mathfrak{m}}$.

It follows from a differentiation of the linear integral equation satisfied by the density of Bethe roots that

$$\partial_Q \rho(\lambda|Q) = -\rho(Q|Q) \cdot [R_Q(\lambda, Q) + R_Q(\lambda, -Q)].$$

Hence, taking into account the smoothness of $\rho(\lambda|Q)$ in (λ, Q), the function

$$f(Q) = \int_{-Q}^{Q} \rho\big(\lambda|Q\big) \cdot \mathrm{d}\lambda$$

is smooth and

$$f'(Q) = 2\rho(Q|Q) \cdot \left\{ 1 - \int_{-Q}^{Q} R_Q(\lambda, Q) \cdot \mathrm{d}\lambda \right\} = 2\rho(Q|Q) \cdot \mathcal{Z}(Q|Q) > 0.$$

$f(Q)$ is thus a strictly increasing function of Q. Clearly $f(0) = 0$. Moreover, using the explicit solution of Lieb's equation (8) for $Q = +\infty$, we find that $f(+\infty) = 1/2$. f is thus a diffeomorphism from $[0; +\infty[$ onto $[0; 1/2]$. This proves the uniqueness and existence of solutions to (9) along with the smoothness in $\mathfrak{m}$ of the solution $Q_{\mathfrak{m}}$. The monotonicity of $\mathfrak{m} \mapsto Q_{\mathfrak{m}}$ then follows by differentiation of (9).

5.2 Large-Q behaviour of $\rho(Q|Q)$

Proceeding exactly as in the large-Q analysis of $Z(Q|Q)$, namely splitting the integrand of $R^{(0)}(\lambda, \mu)$ in two parts and computing the contributions coming from the functions α_-, we obtain

$$\rho(Q|Q) - K\big(Q|\gamma/2\big) = -\int_{\mathbb{R}+\mathrm{i}\kappa} \frac{\mathrm{d}\eta}{4\mathrm{i}\pi^2} \int \mathrm{d}\xi \alpha_+(\eta) \cdot \frac{\alpha_+^{-1}(\xi) - 1}{\xi - \eta} \int_{-Q}^{Q} \mathrm{d}\mu \cdot K(\mu|\gamma/2) e^{\mathrm{i}(\mu - Q)\eta}$$
$$- \int_{\mathbb{R}+\mathrm{i}\kappa} \frac{\mathrm{d}\eta}{4\mathrm{i}\pi^2} \int \mathrm{d}\xi \alpha_-^{-1}(\eta) \cdot \frac{\alpha_+(\xi) - 1}{\xi - \eta} e^{-2\mathrm{i}\xi Q} \int_{-Q}^{Q} \mathrm{d}\mu \cdot K(\mu|\gamma/2) e^{\mathrm{i}(\mu - Q)\eta} + O\big(e^{-2\epsilon_\gamma Q}\big).$$

The ξ-integral in the first line can be explicitly evaluated whereas the integral term in the second line is already $O\big(e^{-2\kappa Q}\big)$. Thus, all in all,

$$\rho(Q|Q) - K\big(Q|\gamma/2\big) = \int_{\mathbb{R}+\mathrm{i}\kappa} \frac{\mathrm{d}\eta}{2\pi} \left\{ \big(\alpha_+(\eta) - 1\big) \int_{-Q}^{Q} \mathrm{d}\mu \cdot K(\mu|\gamma/2) e^{\mathrm{i}(\mu - Q)\eta} \right\} + O(\delta_Q)$$
$$= -\int_{\mathbb{R}} \frac{\mathrm{d}\eta}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \left\{ \big(\alpha_+(\eta) - 1\big) \cdot \mathcal{F}\big[K(*|\gamma/2)\big](\xi) \cdot \frac{e^{-\mathrm{i}\xi Q} - e^{\mathrm{i}\xi Q - 2\mathrm{i}\eta Q}}{\xi - \eta} \right\} + O(\delta_Q)$$
$$= \int_{\mathbb{R}} \big(\alpha_+(\xi) - 1\big) \cdot \mathcal{F}\big[K(*|\gamma/2)\big](\xi) \cdot e^{-\mathrm{i}\xi Q} \cdot \frac{\mathrm{d}\xi}{2\pi} + O(\delta_Q),$$

where we have set $\delta_Q = e^{-2Q\kappa} + e^{-2Q\epsilon_\gamma}$

Upon identifying $-K\big(Q|\gamma/2\big)$, using the jump condition for $\alpha_+(\xi)$ and remarking that κ is, at most, equal to the imaginary part of the zero of $1 + \mathcal{F}[K]$ closest to $\mathbb{R}$, viz. to ϵ_γ, one is led to the representation

$$\rho(Q|Q) = \int_{\mathbb{R}} \frac{\alpha_-(\xi) \cdot e^{-\mathrm{i}\xi Q}}{2\cosh(\gamma\xi/2)} \cdot \frac{\mathrm{d}\xi}{2\pi} + O\big(e^{-2Q\epsilon_\gamma}\big) = \frac{e^{-\frac{\pi Q}{\gamma}}}{\gamma} \cdot \alpha(-\mathrm{i}\pi/\gamma) + O(\delta_Q). \tag{24}$$

Therefore, for $2\epsilon_\gamma > \frac{\pi}{\gamma}$ viz. $\gamma > \frac{\pi}{5}$, the first term in the r.h.s. of (24) does correspond to the first correction. In the regime $0 < \gamma < \pi/5$, one most probably reaches the same conclusion. However, the analysis is much more technical, so we do not discuss it here.

6 The dressed energy

6.1 Main properties and Fermi rapidity

The regularity properties of the solution $\varepsilon(\lambda|Q)$ are readily read off from the integral equation and the smoothness properties of the resolvent $R_Q(\lambda, \mu)$. We shall now establish the bounds on ε. We start with the bound involving $\widetilde{\varepsilon}$ since it follows readily from the previous considerations. Solving the integral equation for ε in terms of the afore-introduced functions leads to the expression

$$\varepsilon(\lambda|Q) = hZ(\lambda|Q) - 4\pi J \sin(\gamma)\rho(\lambda|Q).$$

It then solely remains to apply the (upper or lower, depending on the value of γ) bounds (5) and (10) or (11) and (6) so as to get the bounds involving $\widetilde{\varepsilon}(\lambda)$. We thus focus on the bounds involving ε_0. Starting from the representation

$$\varepsilon(\lambda|Q) = \varepsilon_0(\lambda) - \int_{-Q}^{Q} R_Q(\lambda,\mu)\varepsilon_0(\mu)\cdot \mathrm{d}\mu$$

and using that $R_Q(\lambda,\mu) > 0$ for $0 < \gamma < \pi/2$ and $R_Q(\lambda,\mu) < 0$ for $\pi/2 < \gamma < \pi$, the bounds follow since $-\varepsilon_0(\lambda) > 0$ for any $\lambda \in]-Q;Q[$ as soon as $Q < Q_0$.

We now turn to the proof of existence and uniqueness of the Fermi rapidity. In the regime $0 < \gamma < \pi/2$, we repeat the change of integration domain trick so as to recast the linear integral equation satisfied by ε in the form

$$\varepsilon(\lambda|Q) - \int_{\mathbb{R}\setminus[-Q;Q]} R(\lambda-\mu)\cdot\varepsilon(\mu|Q)\cdot \mathrm{d}\mu = \varepsilon_\infty(\lambda)$$

with

$$\varepsilon_\infty(\lambda) = \lim_{Q\to+\infty}\varepsilon(\lambda|Q) = \frac{h\pi}{2(\pi-\gamma)} - \frac{2\pi J\sin(\gamma)}{\gamma\cosh\left(\frac{\pi\lambda}{\gamma}\right)}.$$

Let, $\mathcal{R}(\lambda,\mu)$ be the resolvent kernel to the operator $I - R$ understood as acting on $\mathcal{C}_b^0(\mathbb{R}\setminus[-Q;Q])$. As already argued, it is strictly positive. Furthermore, manipulations similar to those already discussed lead to

$$\partial_\lambda\varepsilon(\lambda|Q) = \varepsilon(Q|Q)[\mathcal{R}(\lambda,Q) - \mathcal{R}(\lambda,-Q)] + (I+\mathcal{R})[\varepsilon'_\infty](\lambda),$$
$$\partial_Q\varepsilon(\lambda|Q) = -\varepsilon(Q|Q)[\mathcal{R}(\lambda,Q) + \mathcal{R}(\lambda,-Q)]$$

viz.

$$\frac{\mathrm{d}}{\mathrm{d}Q}\varepsilon(Q|Q) = -2\varepsilon(Q|Q)\mathcal{R}(Q,-Q) + \varepsilon'_\infty(\lambda) + \int_{Q}^{+\infty}[\mathcal{R}(\lambda,\nu) - \mathcal{R}(\lambda,-\nu)]\cdot\varepsilon'_\infty(\nu)\cdot \mathrm{d}\nu, \quad (25)$$

where we have used that ε'_∞ is odd. Since $R(\lambda-\mu) - R(\lambda+\mu) > 0$ for $\lambda,\mu \in [Q;+\infty[$, a direct inspection of the difference of Neumann series (see [22, Lemma 2.3] for more details) shows that $\left[\mathcal{R}(\lambda,\nu) - \mathcal{R}(\lambda,-\nu)\right] > 0$. As $\varepsilon'_\infty(\lambda) > 0$ for $\lambda \in \mathbb{R}^+$, one gets that all but the first term in the r.h.s. of (25) are strictly positive. As a consequence, every zero of $Q \mapsto \varepsilon\big(Q|Q\big)$ belongs to an open set on which the function is increasing. Since $Q \mapsto \varepsilon(Q|Q)$ is a continuous function, it has thus at most one zero. Further, the bounds $\widetilde{\varepsilon}(Q) > \varepsilon(Q|Q) > \varepsilon_0(Q)$ ensure that $\varepsilon\big(Q_0|Q_0\big) > 0$ and $\varepsilon\big(\widetilde{Q}|\widetilde{Q}\big) < 0$. Thus, by continuity, there exists $Q_F \in]\widetilde{Q};Q_0[$ such that $\varepsilon(Q_F|Q_F) = 0$.

It remains to treat the case $\pi/2 < \gamma < \pi$. Similar calculations lead to

$$\frac{\mathrm{d}}{\mathrm{d}Q}\varepsilon(Q|Q) = -2\varepsilon(Q|Q)R_Q(Q,-Q)\big]\cdot\varepsilon'_0(\nu) - \int_0^Q [R_Q(\lambda,\nu) - R_Q(\lambda,-\nu)]\cdot\varepsilon'_0(\nu)\cdot \mathrm{d}\nu.$$

Since, as follows from Lemma 2, $-\big[R_Q(\lambda,\nu) - R_Q(\lambda,-\nu)\big] > 0$, one concludes as before regarding to uniqueness of Q_F. Its existence follows from the bounds $\varepsilon_0(Q) > \varepsilon(Q|Q) > \widetilde{\varepsilon}(Q)$.

It remains to justify the smoothness of the map $h \mapsto Q_F$. The latter follows from the implicit function theorem, while the expression for $\partial_h Q_F(h)$ follows from straightforward calculations.

6.2 Positivity of the real part on the other bank

We close the section devoted to the dressed energy by establishing a strictly positive, magnetic field dependent, lower bound on $\Re(\varepsilon_+(\lambda - \mathrm{i}\gamma))$, $\lambda \in \mathbb{R}$. In order to prove this result, we first need to establish a technical lemma.

Lemma 3. *The function $\mathcal{G}$ defined as the Fourier transform*

$$\mathcal{G}(\lambda) = \int_{\mathbb{R}} \frac{\sinh[(\pi/2 - 2\gamma)k]e^{-\mathrm{i}k\lambda}}{\cosh\big(\gamma k/2\big)\sinh[(\pi/2 - \gamma/2)k]} \cdot \frac{\mathrm{d}k}{4\pi} \tag{26}$$

has the following properties:

(*i*) *For $0 < \gamma < \frac{\pi}{4}$ the function $\mathcal{G}$ is real, even and positive on the real axis and monotonically decreasing with vanishing asymptotics for $\lambda > 0$.*

(*ii*) *For $\frac{\pi}{4} < \gamma < \frac{\pi}{2}$ the function $\mathcal{G}$ is real, even and negative on the real axis and has vanishing asymptotics for $\lambda > 0$.*

Proof. In order to prove (i) and (ii) we define

$$\gamma'' = \gamma''(\gamma) = \frac{3\gamma/2}{1 - \gamma/\pi}.$$

This function of γ is monotonically increasing for $0 < \gamma < \pi$, and

$$\gamma''(\pi/4) = \pi/2, \qquad \gamma''(2\pi/5) = \pi.$$

The Fourier transformation formulae (18) only hold as long as $\gamma < \pi$. Thus, we obtain a representation as a convolution, similar to (19), only for $0 < \gamma < 2\pi/5$. For $0 < \gamma < \pi/4$, we have $0 < \gamma'' < \pi/2$ and

$$\mathcal{G}(\lambda) = \frac{\pi}{2\gamma(\pi - \gamma)} \int_{\mathbb{R}} \frac{K\big(\frac{\mu}{1-\gamma/\pi}\big|\gamma''\big)}{\cosh[\pi(\lambda - \mu)/\gamma]} \cdot \mathrm{d}\mu > 0. \tag{27}$$

For $\pi/4 < \gamma < 2\pi/5$ we have the same representation (27) but with $\pi/2 < \gamma'' < \pi$. In this range $K(\lambda|\gamma'')$ is negative, hence $\mathcal{G}(\lambda) < 0$. Using the same reasoning as above we can also say that $\mathcal{G}$ is even, asymptotically vanishing and, for $\lambda > 0$, monotonically decreasing if $0 < \gamma < \pi/4$ and monotonically increasing if $\pi/4 < \gamma < 2\pi/5$.

If $\gamma > 2\pi/5$ a representation like (27) does not exist anymore, since then the inverse Fourier transform of $\sinh\big[(\pi/2 - 2\gamma)k\big]/\sinh\big[(\pi/2 - \gamma/2)k\big]$ does not exist. Still, using the elementary identity

$$\frac{\sinh\big[\big(\frac{\pi}{2} - 2\gamma\big)k\big]}{2\cosh\big[\frac{\gamma k}{2}\big]\sinh\big[\big(\frac{\pi}{2} - \frac{\gamma}{2}\big)k\big]} = \frac{\sinh\big[\big(\frac{\pi}{2} - \frac{3\gamma}{2}\big)k\big]}{\sinh\big[\big(\frac{\pi}{2} - \frac{\gamma}{2}\big)k\big]} - \frac{\sinh\big[\big(\frac{\pi}{2} - \gamma\big)k\big]}{2\cosh\big(\frac{\gamma k}{2}\big)\sinh\big[\big(\frac{\pi}{2} - \frac{\gamma}{2}\big)k\big]}$$

we find that

$$\mathcal{G}(\lambda) = \Big(1 - \frac{\gamma}{\pi}\Big)^{-1} \cdot K\left(\frac{\lambda \cdot \pi}{\pi - \gamma}\middle|\gamma'''\right) - R(\lambda), \tag{28}$$

where $\gamma''' = 2\gamma'$ and γ' has been defined in (19). Again γ''' is a monotonically increasing function of γ for $0 < \gamma < \pi$, but now

$$\gamma'''(\pi/3) = \pi/2, \qquad \gamma'''(\pi/2) = \pi.$$

It follows from (28) that $\mathcal{G}(\lambda) < 0$ for $\pi/3 < \gamma < \pi/2$. Since we had already shown that $\mathcal{G}$ is negative for $\pi/4 < \gamma < 2\pi/5$ and since $\pi/3 < 2\pi/5$, we conclude that $\mathcal{G}(\lambda) < 0$ for $\pi/4 < \gamma < \pi/2$. ∎

We are finally in position to prove Proposition 1:

Proof. It follows from (14) that ε is meromorphic in the strip $|\Im(\lambda)| < \gamma$ with simple poles at $\pm \mathrm{i}\gamma/2$ and that

$$\mathrm{Res}\big(\varepsilon(\lambda|Q_F) \cdot \mathrm{d}\lambda, \lambda = \mp \mathrm{i}\gamma/2\big) = \mp 2\mathrm{i}J\sin(\gamma).$$

Then, by deforming the contour in (14) we obtain, for $\lambda \in \mathbb{R}$,

$$\begin{aligned}\varepsilon_+(\lambda - \mathrm{i}\gamma|Q_F) &= \varepsilon_0(\lambda - \mathrm{i}\gamma) + 4\pi J\sin(\gamma)K(\lambda - \mathrm{i}\gamma/2)\\ &\quad + \int_{\mathbb{R}\setminus[-Q_F;Q_F]} K(\lambda - \mu - \mathrm{i}\gamma + \mathrm{i}0) \cdot \varepsilon(\mu|Q_F) \cdot \mathrm{d}\mu - \int_{\mathbb{R}} K(\lambda - \mu)\varepsilon_+(\mu - \mathrm{i}\gamma|Q_F) \cdot \mathrm{d}\mu.\end{aligned}$$

Setting

$$\omega(\lambda) = \Re[\varepsilon_+(\lambda - \mathrm{i}\gamma|Q_F)]$$

and using

$$\begin{aligned}&K(\lambda - \mathrm{i}\gamma + \mathrm{i}0) = \frac{\delta(\lambda)}{2} - \frac{1}{2\pi\mathrm{i}}\mathrm{p.v.}\coth(\lambda) + \frac{1}{2\pi\mathrm{i}}\coth(\lambda - 2\mathrm{i}\gamma),\\ &\varepsilon_0(\lambda - \mathrm{i}\gamma) + 4\pi J\sin(\gamma)K(\lambda - \mathrm{i}\gamma/2) = h + 4\pi J\sin(\gamma)K(\lambda|\gamma/2)\end{aligned}$$

for $\lambda \in \mathbb{R}$, we obtain

$$\begin{aligned}\omega(\lambda) &= h + 4\pi J\sin(\gamma)K(\lambda|\gamma/2) + \frac{\varepsilon_c(\lambda)}{2}\\ &\quad + \frac{1}{2}\int_{\mathbb{R}} K(\lambda - \mu|2\gamma) \cdot \varepsilon_c(\mu) \cdot \mathrm{d}\mu - \int_{\mathbb{R}} K(\lambda - \mu)\omega(\mu) \cdot \mathrm{d}\mu,\end{aligned} \tag{29}$$

which is an equation for real functions on the real line. Above, we agree upon

$$\varepsilon_c(\lambda) = \varepsilon(\lambda|Q_F) \cdot \Theta(|\lambda| - Q_F),$$

where Θ is the Heaviside step function. Equation (29) can be solved by means of Fourier transformation. Using (18) and

$$\frac{\mathcal{F}[K(*|\gamma/2)](k)}{1 + \mathcal{F}[K](k)} = \frac{1}{2\cosh(\gamma k/2)}$$

we obtain

$$\begin{aligned}\omega(\lambda) &= \frac{h\pi}{2(\pi - \gamma)} + \frac{2\pi J\sin(\gamma)/\gamma}{\cosh\left(\frac{\pi\lambda}{\gamma}\right)} + \frac{\varepsilon_c(\lambda)}{2}\\ &\quad + \frac{1}{2}\int_{\mathbb{R}\setminus[-Q_F;Q_F]} \{\mathcal{G}(\lambda - \mu) - R(\lambda - \mu)\} \cdot \varepsilon(\mu|Q_F) \cdot \mathrm{d}\mu.\end{aligned} \tag{30}$$

Here, we recall that R and $\mathcal{G}$ are defined by the Fourier integrals (17) and (26).

By construction, the function ε is positive on $\mathbb{R} \setminus [-Q_F; Q_F]$, where it is also bounded from above by h. We need to find a lower bound for the integral on the right hand side of (30). For this purpose we distinguish two cases.

- $0 < \gamma \le \pi/4$. Then, according to Lemmas 1 and 3,

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}\setminus[-Q_F;Q_F]} \{\mathcal{G}(\lambda - \mu) - R(\lambda - \mu)\}\varepsilon(\mu|Q_F) \cdot \mathrm{d}\mu\\ &\qquad \ge -\frac{1}{2}\int_{\mathbb{R}\setminus[-Q_F;Q_F]} R(\lambda - \mu) \cdot \varepsilon(\mu) \cdot \mathrm{d}\mu > -\frac{h}{2}\int_{\mathbb{R}} R(\lambda) \cdot \mathrm{d}\lambda = -\frac{h}{2}\frac{\frac{1}{2} - \frac{\gamma}{\pi}}{1 - \frac{\gamma}{\pi}}.\end{aligned} \tag{31}$$

- $\pi/4 \le \gamma < \pi/2$. Then, according to Lemmas 1 and 3,

$$\frac{1}{2}\int_{\mathbb{R}\setminus[-Q_F;Q_F]} \Big\{\mathcal{G}(\lambda-\mu) - R(\lambda-\mu)\Big\} \cdot \varepsilon(\mu|Q_F) \cdot \mathrm{d}\mu \\ > \frac{h}{2}\int_{\mathbb{R}} \{\mathcal{G}(\lambda) - R(\lambda)\} \cdot \mathrm{d}\lambda = -\frac{h}{2}\frac{\gamma/\pi}{1-\frac{\gamma}{\pi}}. \tag{32}$$

Using (31), (32) in (30) we obtain

$$\omega(\lambda) > \frac{2\pi J \sin(\gamma)/\gamma}{\cosh(\pi\lambda/\gamma)} + \frac{\varepsilon_c(\lambda)}{2} + \frac{h}{2}\begin{cases} \dfrac{\frac{1}{2}+\frac{\gamma}{\pi}}{1-\frac{\gamma}{\pi}} & \text{for} \quad 0 < \gamma \le \pi/4, \\ 1 & \text{for} \quad \pi/4 \le \gamma < \pi/2, \end{cases}$$

which implies (15). ■

6.3 Dependence of $Q_F(h)$ on h for small h

We have established that $h \mapsto Q_F(h)$ is a strictly decreasing function of h. Thence, since $\varepsilon\big(\lambda| + \infty\big) < 0$ on $\mathbb{R}$, it follows that $\lim_{h\to 0^+} Q_F(h) = +\infty$. Hence, for h small enough,

$$\varepsilon\big(Q_F(h)|Q_F(h)\big) = h\alpha_+(0) - 4\pi J\frac{\sin\gamma}{\gamma}e^{-\pi Q_F(h)/\gamma}\alpha_-(-\mathrm{i}\pi/\gamma) + O\big(e^{-2\varepsilon_\gamma Q_F(h)}\big).$$

Upon explicating the values of α one gets the claimed form of the small-h asymptotics.

7 Conclusion

In the present paper, we have proved several properties of solutions to linear integral equations arising in the description of the ground state of the XXZ spin-1/2 chain in the thermodynamic limit. Although we have focused our analysis on this specific model, we do trust that the method is more general and can be applied to other linear integral equations arising in the context of the thermodynamic limit of more complex quantum integrable models. In particular, with minor modifications, the techniques should work for models based on a $\mathcal{Y}(\mathfrak{g})$ or $U_q(\mathfrak{g})$ symmetry, with $\mathfrak{g}$ a Lie algebra of rank higher than 1.

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