Constructive Tensor Field Theory[⋆]

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> **Abstract.** We provide an up-to-date review of the recent constructive program for field
> theories of the vector, matrix and tensor type, focusing not on the models themselves but
> on the mathematical tools used.
>
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>
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1   Introduction

This paper is the second one of our twin contributions to the "Special Issue on Tensor Models, Formalism and Applications" of the SIGMA journal. In our first paper [49] the physical motivations behind the tensor models and their connection to quantum gravity are discussed. So we advise strongly the reader not already familiar with tensor models and their relationship to quantum gravity to read [49] first, as we shall not duplicate the corresponding material here, but provide below only a minimal introduction to tensor models. We shall discuss in detail in this paper only the constructive program for (Euclidean) random tensor models of general rank d, which include as special cases vector (rank 1) and matrix (rank 2) models. For other aspects of the burgeoning study of random tensor models and their associated field theories we refer to the other contributions to the special SIGMA issue as well.

Tensor models were introduced as promising candidates for an *ab initio* quantization of gravity [3, 4, 24, 57]. Indeed they are combinatorial objects which do not refer to any background spacetime, nor even to any background topology. However such tensors were initially introduced as symmetric in their indices, a feature which for a long time prevented to investigate them with rigorous analytic methods. In particular in contrast with the famous 't Hooft $1/N$ expansion for random matrix models, there was no way to probe the large N limit of such symmetric random tensors at rank $d \geq 3$. The modern reformulation [12, 27, 30, 34] unlocked the theory by considering *unsymmetrized* random tensors. These objects have a larger, truly tensorial symmetry (typically in the complex case a $U(N)^{\otimes d}$ symmetry at rank d instead of the single $U(N)$ of symmetric tensors). This larger symmetry allows to probe their large N limit through $1/N$ expansions of a new type [10, 11, 26, 28, 31].

Random tensor models can be further divided into fully invariant models, in which both propagator and interaction are invariant, and tensor field theories (hereafter TFT) in which the interaction is invariant but the propagator is not [7]. This propagator can also incorporate a further gauge invariance to make contact with group field theory [14, 15], in which case one calls them tensor group field theories (TGFT).

We restrict ourselves in this paper to the relatively simple case of models with a quartic interaction, for which the typical constructive task is to prove Borel summability of the free energy or of the connected Schwinger functions on the "stable side" of the coupling constant. To be non-trivial and physically interesting, such Borel summability theorems should be either *uniform in N*, the size of the tensor, for tensor models, or *uniform in the ultraviolet cutoff*, hence include renormalization, for TFTs or TGFTs.

Our review starts with a section on general mathematical prerequisites: quartic tensor models, Nevanlinna theorem [58] and the intermediate field representation (IFR) plus key combinatorial constructive tools, namely the forest and jungle formulas [2, 13]. The next section presents three specific constructive methods adapted to tensor models: the loop vertex expansion (LVE), which is a combination of the intermediate field representation and of the forest formula first introduced in [46]; the multiscale loop vertex expansion (MLVE), which is a combination of the intermediate field representation and of a level-2 jungle formula with a bosonic and a fermionic part first introduced in [32]; and iterated Cauchy–Schwarz bounds (ICS bounds), introduced in this context in [43] and [18], for which we provide here a slightly improved version.

Then in the last section we review the recent Borel summability results on tensor models in increasing order of difficulty. The quartic melonic tensor model in any d [29] requires a simple LVE and no ICS bounds. This is also the case for the melonic $U(1) - T_3^4$ TGFT [38]. The general quartic tensor model in any d requires a LVE plus ICS bounds [18]. The melonic $U(1) - T_4^4$ TGFT requires an MLVE but still no ICS bounds [39]. Finally the melonic T_3^4 TFT requires both MLVE and ICS bounds [19]. We discuss also briefly the case of matrix models, for which the LVE has to be completed by an additional topological expansion in order to identify the $1/N$ expansion of the model in a non-perturbative way up to finite genus g [35].

The world of tensor models and field theories is extremely vast, as tensor interactions encode infinitely many triangulations of any piecewise linear manifold in any dimension and of a large class of pseudo-manifolds. It also includes in particular any matrix model, not only invariant ones, but also non-commutative field theories such as the Grosse–Wulkenhaar model which we shall discuss briefly too. The constructive program to explore all these models at the rigorous non-perturbative level has clearly just started. However a rich harvest of non-perturbative Borel-summability results has already been obtained for tensor field theories. Such constructive results essentially have no counterpart yet in the other approaches to quantum gravity. This is very promising for the tensor track towards quantum gravity [47]. In our conclusion we indicate what could be the next steps of this constructive program.

2 Mathematical prerequisites

2.1 Quartic tensor models

For a general introduction to tensor models we refer to [34]. We simply recall in this subsection the notations required for the quartic models discussed thereafter, following roughly [18].

For a complex rectangular random matrix M, called a Wishart matrix, the basic invariant homogeneous of order $2p$ is unique. It is $\mathrm{Tr}(MM^\dagger)^p$. All other polynomial invariants are linear combinations of products of these basic bricks. In particular there is only one[1] matrix model with a connected quartic interaction, namely $\mathrm{Tr}\, MM^\dagger MM^\dagger$. The situation is completely different in the case of tensor models of higher rank d, as there are many more available interactions and in particular many different invariant quartic interactions.

We consider a Hermitian inner product space V of dimension N and e_n, $n = 1, \ldots, N$ an orthonormal basis in V. A covariant tensor of rank d is a multilinear form $\mathbf{T} \colon V^{\otimes d} \to \mathbb{C}$. We

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[1]Of course there is another quartic "disconnected" interaction, namely $[\mathrm{Tr}\, MM^\dagger]^2$. We shall not consider further such "multi-trace" models here. See however [11].

denote its components in the canonical dual tensor product basis by

$$T_{n_1\ldots n_d} \equiv \mathbf{T}(e_{n_1},\ldots,e_{n_d}), \qquad \mathbf{T} = \sum_{n_1,\ldots n_d} T_{n_1\ldots n_d} e^{n_1} \otimes \cdots \otimes e^{n_d}.$$

A priori $T_{n_1\ldots n_d}$ has no symmetry properties, hence its indices have a well defined position. We call the position of an index its *color*, and we denote $\mathcal{D}$ the set of colors $\{1,\ldots,d\}$. Subsets $\mathcal{C} \subset \mathcal{D}$ will be called generalized colors; singleton subsets identify with the ordinary colors.

Using the canonical identification of V with its dual thanks to the Hilbert scalar product, we can also expand the dual tensor as a conjugated multilinear map $\mathbf{T}^\vee$ with matrix elements $\bar{T}_{\bar{n}_1\ldots\bar{n}_d}$ in the canonical dual basis. We write all the indices in subscript, and we denote the contravariant indices with a bar. Indices are always understood to be listed in increasing order of their colors. We denote $\delta_{n_\mathcal{C}\bar{n}_\mathcal{C}} = \prod_{c\in\mathcal{C}} \delta_{n_c\bar{n}_c}$ and $\mathrm{Tr}_\mathcal{C}$ the partial trace over the indices n_c, $c \in \mathcal{C}$.

Under unitary base change, covariant tensors transform under the tensor product of d fundamental representations of $\mathrm{U}(N)$: the group acts independently on each index of the tensor. For $U^{(1)} \cdots U^{(d)} \in \mathrm{U}(N)$,

$$\mathbf{T} \to \big(U^{(1)} \otimes \cdots \otimes U^{(d)}\big)\mathbf{T}, \qquad \mathbf{T}^\vee \to \mathbf{T}^\vee\big(U^{(1)*} \otimes \cdots \otimes U^{(d)*}\big).$$

In components, it reads

$$T_{a_\mathcal{D}} \to \sum_{m_\mathcal{D}} U^{(1)}_{a_1 m_1} \cdots U^{(d)}_{a_d m_d}\, T_{m_\mathcal{D}}, \qquad \bar{T}_{\bar{a}_\mathcal{D}} \to \sum_{\bar{m}_\mathcal{D}} \bar{U}^{(1)}_{\bar{a}_1 \bar{m}_1} \cdots \bar{U}^{(d)}_{\bar{a}_d \bar{m}_d}\, \bar{T}_{\bar{m}_\mathcal{D}}.$$

A *tensor invariant* is a polynomial in the components of the tensor and its conjugate which is invariant under this action of the external tensor product of d independent copies of the unitary group $\mathrm{U}(N)$. Tensor invariants are linear combinations of products of connected invariants (also called bubbles or generalized traces). Such connected invariants are built by contracting indices of the same color of a product of tensor entries into a connected graph. Hence connected invariants are in one to one correspondence with d-regular edge-colored bipartite connected graphs [34] and can be enumerated quite precisely [6].

The unique quadratic trace invariant is the (scalar) Hermitian pairing of $\mathbf{T}^\vee$ and $\mathbf{T}$ which reads

$$\mathbf{T}^\vee \cdot_\mathcal{D} \mathbf{T} = \sum_{n_\mathcal{D}\bar{n}_\mathcal{D}} \bar{T}_{\bar{n}_1\ldots\bar{n}_d} \delta_{\bar{n}_\mathcal{D} n_\mathcal{D}} T_{n_1\ldots n_d},$$

In arbitrary rank d, the most general quartic connected trace invariants are associated to non trivial generalized colors, i.e., to subsets $\mathcal{C} \subset \mathcal{D}$, $\mathcal{C} \neq \varnothing$, $\mathcal{C} \neq \mathcal{D}$. The connected quartic invariant $V_\mathcal{C}$ is

$$V_\mathcal{C}(\mathbf{T}^\vee,\mathbf{T}) = \mathrm{Tr}_\mathcal{C}\left[\left[\mathbf{T}^\vee \cdot_{\mathcal{D}\backslash\mathcal{C}} \mathbf{T}\right] \cdot_\mathcal{C} \left[\mathbf{T}^\vee \cdot_{\mathcal{D}\backslash\mathcal{C}} \mathbf{T}\right]\right],$$

where we denoted $\cdot_\mathcal{C}$ the product of operators from $V^{\otimes\mathcal{C}}$ to $V^{\otimes\mathcal{C}}$. In components this invariant writes:

$$\sum_{n,\bar{n},m,\bar{m}} \big(\bar{T}_{\bar{n}} \delta_{\bar{n}_{\mathcal{D}\backslash\mathcal{C}} n_{\mathcal{D}\backslash\mathcal{C}}} T_n\big) \delta_{n_\mathcal{C}\bar{m}_\mathcal{C}} \delta_{\bar{n}_\mathcal{C} m_\mathcal{C}} \big(\bar{T}_{\bar{m}} \delta_{\bar{m}_{\mathcal{D}\backslash\mathcal{C}} m_{\mathcal{D}\backslash\mathcal{C}}} T_m\big).$$

Remark that the invariant for $\mathcal{C}$ is the same as for its complement $\mathcal{D}\backslash\mathcal{C}$. Hence at rank D there are exactly $2^{D-1}-1$ quartic trace invariants. The 7 invariants at rank $d = 4$ are shown in Fig. 1.

A generic quartic tensor model (with standard scaling and interaction set $\mathcal{Q}$) is then defined by the moments of the (invariant) perturbed Gaussian measure:

$$d\mu = \left(\prod_n N^{D-1} \frac{d\bar{T}_n dT_n}{2\imath\pi}\right) e^{-N^{D-1}\left(\mathbf{T}^\vee \cdot_\mathcal{D} \mathbf{T} + \lambda \sum_{\mathcal{C}\in\mathcal{Q}} V_\mathcal{C}(\mathbf{T}^\vee,\mathbf{T})\right)}, \tag{2.1}$$
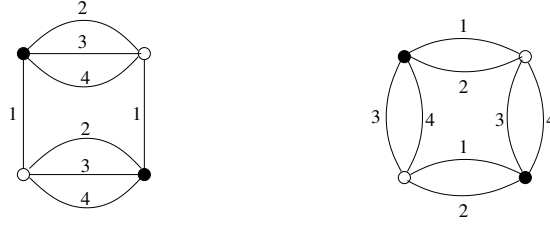
Figure 1. The quartic invariants of rank 4 [18]: there are four types of "melonic" ones (one of them shown left, the others being obtained by permuting the colors) and three types of "necklaces" (one of them shown right, the others being obtained by color permutations).

where $\mathcal{Q}$ is some subset of $\mathcal{C}$s. The global scaling factor N^{D-1} for the action is the unique one such that the model admits a non-trivial $1/N$ expansion [11, 34]. Of course more general models can be considered by giving an independent coupling to each interaction, or by considering dimensions N_c depending on the color, or by adding the disconnected interaction $[\mathbf{T}^{\vee} \cdot_{\mathcal{D}} \mathbf{T}]^2$, but we shall not consider them here.

The "melonic" quartic models [18] are obtained by restricting the interactions to singletons, so that $\mathcal{Q} = \{\mathcal{C}, |\mathcal{C}| = 1\}$ in (2.1). In other words interaction is restricted to the d quartic melonic invariants, one for each color.

Quartic tensor field theories have the same interactions but a non-symmetric propagator. The simplest such models are built around the Hilbert space $V := L^2(\mathbb{Z})$ and as in usual field theory, the propagator is the inverse of a Laplacian operator plus a mass term

$$C(n_{\mathcal{D}}, \bar{n}_{\mathcal{D}}) = \frac{\delta_{n_{\mathcal{D}}, \bar{n}_{\mathcal{D}}}}{\sum\limits_{j=1}^{d} n_j^2 + m^2}. \tag{2.2}$$

The melonic quartic TGFT at rank d with propagator (2.2) and a unique coupling constant λ identical for each coupling is nicknamed T_d^4. We often further simplify the model by putting $m^2 = 1$.

Finally tensor group field theories are built around a more general space $V := L^2(G)$ where G is a Lie group [14, 15] or homogeneous space [40], and their propagator incorporates a Boulatov projector

$$C(n_{\mathcal{D}}, \bar{n}_{\mathcal{D}}) = \frac{\delta_{n_{\mathcal{D}}, \bar{n}_{\mathcal{D}}} \delta\left(\sum\limits_{j=1}^{d} n_j\right)}{\sum\limits_{j=1}^{d} n_j^2 + m^2}, \tag{2.3}$$

where the indices n now run over a basis of the dual Fourier space of $L^2(G)$, and n_j^2 is a sloppy notation for the Laplace–Beltrami operator on G (see [14] for the case $G = \mathrm{SU}(2)$). The melonic quartic TGFT at rank d with propagator (2.3) and Lie group G will be nicknamed $G - T_d^4$. For instance in the case $G = \mathrm{U}(1)$ the Hilbert space V is $L^2(\mathbb{Z})$ as in the previous case. We often consider discrete tensor indices as momenta, and the equivalent representation with continuous indices in G^d as the corresponding "direct space" representation.

2.2   Borel summability

The perturbative expansion in quantum field theory is obtained by performing a Taylor expansion of the interaction, and then illegally commuting the sum with the Gaussian integral to

obtain a series indexed by Feynman graphs. Let us return in this section to ordinary quantum field theory, since we want to discuss the relationship between functional integrals and their perturbative expansion in the traditional context before turning to tensor models. The ordinary ϕ_d^4 theory in dimension d has propagator kernel

$$C(x, y) = \int d^d p\, e^{ip(x-y)} \left(p^2 + m^2\right)^{-1}$$

and its partition function is

$$Z(\lambda) = \int d\mu_C\, e^{-\lambda \int d^d x \phi^4(x)} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \sum_G A_G, \tag{2.4}$$

where G runs over labeled Feynman graphs[2] with n vertices. The Feynman amplitude of G is

$$A_G = \int \prod_{v \in V(G)} d^d x_v \prod_{\ell \in E(G)} C(x_{v(\ell)}, x_{v'(\ell)}),$$

where the index v runs over the set of n vertices of G, the index ℓ runs over edges of G and $(v(\ell), v'(\ell))$ is the pair of end vertices of edge ℓ. Of course such amplitudes may diverge if $d \geq 2$ (see, e.g., [33] for a recent review).

Interesting quantum field theories have many degrees of freedom, in which case the important physical quantities are intensive rather than extensive quantities, such as the specific free energy and the connected functions or cumulants G_{2p}^c. Therefore the important quantum field theory quantity is not Z but its logarithm. In particular the connected Green functions are obtained from the generating functional

$$W(J) = \log[Z(\lambda, J)]$$

through

$$G_{2p}^c(x_1, \ldots, x_{2p}) = \frac{\partial^{2p} W(J)}{\partial J_1(x_1) \cdots \partial J_{2p}(x_{2p})}\bigg|_{J=0},$$

where $Z(\lambda, J)$ is defined by adding a source field J to the partition function

$$Z(\lambda, J) = \int d\mu_C\, e^{\int d^d x \phi(x) J(x) - \lambda \int d^d x \phi^4(x)}. \tag{2.5}$$

Similarly the free energy p also is deduced from the logarithm of Z, with interaction restricted to a finite volume Λ. One should divide by the volume cutoff Λ, and finally remove this cutoff, hence $p = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log Z_\Lambda(\lambda)$.

The most important property of the perturbative expansion is that it allows for a quick computation of such physical connected quantities. They are indeed given by the same sums as (2.4), with the same amplitudes, but *restricted to connected graphs*. For instance the free energy is given in perturbation theory by a sum identical to (2.4) but restricted to *connected* rooted Feynman vacuum graphs, in which for each graph a particular root vertex has been fixed to the origin. This rule accounts for the need to break the translation invariance of the theory and divide by the volume. Most of the time in this review we shall consider for simplicity only this free energy.

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[2]Labeled Feynman graphs can be considered as defined by Wick contractions of labeled fields belonging to labeled vertices. Hence we shall avoid in this brief review the subtle issue of automorphim groups of graphs and of their corresponding symmetry factors.

Perturbative series typically diverge. In particular for $d = 0$, in which A_G is 1 for any G, there are $(4n - 1)!!$ labeled Feynman graphs with n vertices, hence the perturbative series for $Z(\lambda)$ is

$$\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}(4n - 1)!! = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \frac{(4n)!}{2^{2n}(2n)!},$$

which has zero radius of convergence. Up to dimension $d \leq 3$ the *renormalized* ϕ^4 series can be shown rigorously to also have zero radius of convergence [17, 36], and it is believed to be also true for $d = 4$, although, to our knowledge, there is no proof of this, and the best result in this direction remains the quite old reference [16].

In the quantum field theory literature we often read that "since the perturbation series diverges it can be asymptotic at best", and that "perturbation theory cannot capture non-perturbative effects, such as instantons". Such statements are both confusing and mathematically wrong.

First, asymptoticity of a series is a very weak notion which has nothing to do with its *convergence*. *Every* power series, no matter how convergent or horribly divergent, is asymptotic to *infinitely many* smooth functions of a real-variable. Hence an asymptotic series never encodes the full information about such a function without additional conditions. The typical example is the function $f(\lambda) = 0$ for $\lambda \leq 0$ and $f(\lambda) = e^{-\frac{1}{\lambda}}$ for $\lambda > 0$ whose asymptotic series at $\lambda = 0$ is 0, the most convergent of all series. It certainly does not encode the information to reconstruct f.

Second, as we shall clarify below, in the case of a Borel summable series, all the information contained in the functional integral (including any non-perturbative effect) is in fact entirely contained in the coefficients of the perturbative series. It is simply difficult to extract in practice.

To correctly relate a function to its power series, the key is to impose an analyticity condition, hence to consider functions of a complex variable. Analyticity is a very strong and rigid condition. For instance the function 0 is the only one asymptotic to the 0 series *in the class of functions analytic in an open neighborhood of the expansion point.*

Fortunately analyticity is a common feature in quantum field theory. The partition function (2.4) is usually analytic in a non-empty domain of the coupling constant. It is certainly analytic for $\Re\lambda > 0$ in dimension $d = 0$, or if we include volume and ultra-violet cutoffs, since in this case the functional integral is uniformly convergent. For $|\lambda|$ sufficiently small this analyticity extends to $\log Z$, since Z is close to 1 and its logarithm well-defined; note however that the domain of analyticity for $\log Z$ typically shrinks as cutoffs are removed.

In the super-renormalizable domain for ϕ^4, namely in dimension 2 or 3, the point $\lambda = 0$ belongs to the boundary of the analyticity domain for the renormalized free energy [17, 36]. Nevertheless the functional integral (2.4) can be fully recovered from the perturbative series through *Borel summability* [21, 44]. The most useful Borel summability theorem for quantum field theory is due to Nevanlinna (1919) and was rediscovered by A. Sokal:

Theorem 2.1 (Nevanlinna–Sokal [58])**.** *A function $f(\lambda)$ with $\lambda \in \mathbb{C}$ is said to be Borel summable in λ if*

- $f(\lambda)$ *is analytic in a disk* $\mathcal{D}_R := \left\{ \lambda \,|\, \Re\big(\lambda^{-1}\big) > R^{-1} \right\}$ *with* $R \in \mathbb{R}_+$.

- *There exists some constants K and σ such that $f(\lambda)$ expands at the origin with uniform Taylor remainder bounds in the disk $\mathcal{D}_R$*

$$f(\lambda) = \sum_{k=0}^{n-1} f_k \lambda^k + R_n(\lambda), \qquad |R_n(\lambda)| \leq K\sigma^n n! |\lambda|^n.$$
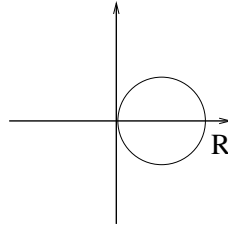
Figure 2. A disk for Nevanlinna's theorem.

If $f(\lambda)$ is Borel summable in λ then

$$B(t) = \sum_{k=0}^{\infty} \frac{1}{k!} f_k t^k$$

is an analytic function for $|t| < \sigma^{-1}$ which admits an analytic continuation in the strip $\{z \,|\, |\Im z| < \sigma^{-1}\}$ such that $|B(t)| \leq B e^{t/R}$ for some constant B and $f(\lambda)$ is given by the absolutely convergent integral

$$f(\lambda) = \frac{1}{\lambda} \int_0^{\infty} dt \, B(t) e^{-\frac{t}{\lambda}}.$$

That is the Taylor expansion of $f(\lambda)$ at the origin is Borel summable, and $f(\lambda)$ is its Borel sum. The set $\left\{\lambda \,|\, \Re(\lambda^{-1}) > R^{-1},\, R \in \mathbb{R}_+\right\}$ is a disk in the complex plane with center at $\frac{R}{2}$ and of radius $\frac{R}{2}$ (see Fig. 2) as, denoting $\lambda = \frac{R}{2} + a e^{\iota\gamma}$,

$$\Re(\lambda^{-1}) > R^{-1} \Leftrightarrow \frac{R^2}{4} > a.$$

The important remark is that Borel summability provides a *uniqueness* criterion: if a power series with coefficients a_n is the Taylor expansion of a Borel summable function $f(\lambda)$ at $\lambda = 0$, then f is *uniquely defined* by the list of a_n. This is the case for the partition function of the ϕ_d^4 model with finite cutoffs, and also for the connected functions without cutoffs after renormalization in the super-renormalizable domain $d \leq 3$ [21, 44]. It is also the case for the "infrared" limit of ϕ_4^4 with fixed ultraviolet cutoff and at the critical point, which is a just renormalizable marginal theory asymptotically free in the infrared direction [22].

When Borel summability holds, the perturbative expansion in fact contains (although in a quite hidden manner) all information about the functional integral, *including instanton effects to all orders*, since a good definition of these instantons is that they occur as discontinuities between different branches of analytic continuation of the functional integral [8]. Of course such effects are very difficult to extract in practice from the perturbative coefficients, because analytic continuation of a function defined by its Taylor germ at a point is typically very difficult.

In summary Borel summability is a key step for the construction of a quantum field theory model, since when it holds, it defines a unique analytic germ synthesizing Feynman's two great ideas: the functional integral and the Feynman graphs. It does not solve all physical issues, such as computing the long range behavior of the model and its bound states, but it renders such questions at least mathematically *well-defined*.

2.3 Intermediate field representation (IFR)

Any quartic (i.e., four-body) interaction can be decomposed as the gluing of two more elementary three body-interactions, by splitting the vertex through a so-called intermediate field.
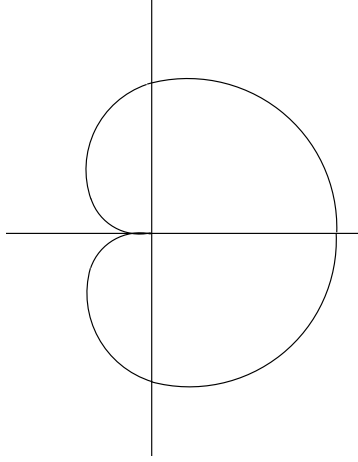
Figure 3. The cardioid domain.

This method (usually called Hubbard–Stratonovic transformation in condensed matter, and Matthews–Salam representation in high energy physics) has been extremely fruitful in physics. In the case of the ordinary ϕ^4 theory defined by (2.5), with a slightly more adapted normalization for the coupling λ, it consists in writing

$$e^{-\frac{\lambda}{2}\int d^dx\phi^4(x)} = \int d\nu(\sigma)e^{i\sqrt{\lambda}\int d^dx\phi^2(x)\sigma(x)},$$

where $d\nu$ is the normalized Gaussian measure with "ultralocal" covariance kernel $\delta(x-y)$. The functional integral over ϕ becomes Gaussian, hence can be explicitly performed, leading to

$$Z(\lambda, J) = \int d\nu(\sigma)\, e^{\langle J, C^{1/2}R(\sigma)C^{1/2}J\rangle}e^{-\frac{1}{2}\,\mathrm{Tr}\log(1-i\sqrt{\lambda}C^{1/2}\sigma C^{1/2})},$$

where we used operator notation. σ is a local multiplication operator, diagonal in direct space, $C^{1/2}$ is the square root in operator sense of the positive operator C, hence is not diagonal in direct space[3]. 1 means the identity operator (with kernel $\delta(x-y)$ in direct space), and $\mathrm{Tr}\,K = \int d^dx K(x,x)$ simply means integration of the diagonal part of a kernel K. Finally the resolvent $R(\sigma)$ is given by

$$R(\sigma) = \left[1 - i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}\right]^{-1},$$

and for any σ and any λ real positive it is a well defined operator norm-bounded by 1. More precisely a moment of investigation reveals that the useful natural domain of definition and analyticity in the λ complex plane for such a resolvent is a cardioid, pictured in Fig. 3.

This cardioid is defined, in polar coordinates $\lambda = \rho e^{i\phi}$, $\rho > 0$, $\phi \in]-\pi, \pi[$ by the inequality $\rho < [\cos(\phi/2)]^2$. Indeed under this condition we have that $R(\sigma)$ is a well defined operator norm-bounded by $[\cos(\phi/2)]^{-1}$. In expansion steps of the resolvents such as

$$R(\sigma) = 1 + i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}R(\sigma)$$

the bad factor $[\cos(\phi/2)]^{-1}$ for $\|R\|$ will be always compensated by a good factor $\sqrt{\rho} < \cos(\phi/2)$ coming from the $\sqrt{\lambda}$ numerator. Remark that any cardioid domain *contains* a Nevanlinna disk of the type shown in Fig. 2.

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[3]Cyclicity of the trace means we can also freely replace the operator $C^{1/2}\sigma C^{1/2}$, by $C\sigma$ or σC, leading to slightly shorter formulas, but this would be unwise as Hermiticity would no longer be visible.
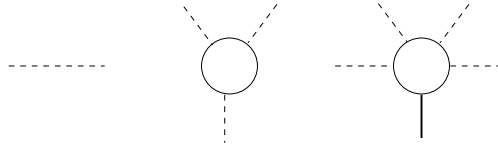
Figure 4. An intermediate field propagator (dashed line), a loop vertex (circle, here with 3 corners or arcs and 3 intermediate field half-propagators) and a ciliated loop vertexwith 4 intermediate field half-propagators and five arcs/corners.
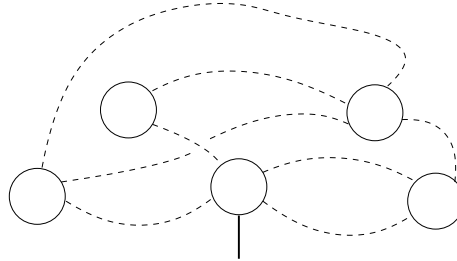

Figure 5. A 2-point graph in the intermediate field theory, here of order 8, has a single *ciliated* loop vertex.

Just like the ordinary representation, the IFR has an associated perturbation theory, obtained by expanding the exponential of the (non-polynomial) interaction $V = \frac{1}{2} \operatorname{Tr} \log \left(1 - i\sqrt{\lambda} C^{1/2} \sigma C^{1/2} \right)$ as

$$e^{-V} = \sum_{n=0}^{\infty} \frac{(-V)^n}{n!},$$

then commuting (again illegally!) integration with respect to $d\nu$ and Gaussian integration over σ, leading to

$$Z(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[e^{\frac{1}{2} \int d^d x \frac{\delta^2}{\delta \sigma(x)^2}} V(\sigma)^n \right]_{\sigma=0}.$$

An intermediated field propagator is represented in Fig. 4 below by a dashed line, which corresponds to the (ultra-local) covariance of a σ intermediate field. The interaction $V = \frac{1}{2} \operatorname{Tr} \log \left(1 - i\sqrt{\lambda} C^{1/2} \sigma C^{1/2} \right)$ is non polynomial, but since $\operatorname{Tr} \log 1 = 0$, it appears in the perturbative expansion only through its functional derivatives which we call *loop vertices*. In contrast with the original ϕ^4 vertices, such loop vertices have arbitrary coordination $q \geq 1$, and bear the initial propagators of the theory on their q *corners* or arcs.

Several interesting remarks are in order. The IFR exchanges the traditional role of propagators and vertices. The IFR loop vertices are non-local, as they contain the propagators of the ordinary theory. The IFR propagators in contrast are local as they correspond to the vertices of the ordinary theory. An even deeper remark is that the cyclic character of the trace allows for an embedding of the graphs in the plane, so that the Feynman graphs of the IFR should really be considered as *combinatorial maps*.

Finally it is easy to add sources to the picture. Any connected function of order $2p$ is obtained by the same expansion in which we add to the ordinary loop vertices p particular *ciliated loop vertices* [29]. The cilium of the j-th ciliated vertex means a cut with two different external variables x_j, x_{j+1} for $j = 1, \ldots, p$ on both sides of the cilium. Apart from the cut, ciliated vertices are similar to loop vertices and bear propagators on each of their corners.

The $2p$-point functions of the theory become sums over Feynman graphs of the intermediate representation with exactly p ciliated vertices, as shown in Fig. 5. The order of perturbation is simply the number of dashed lines.

2.4   The forest formula

A forest formula expands a quantity defined on n points in terms of forests built on these points. Forest formulas, provided they have a positivity property are the key combinatorial component of constructive field theory and were developed systematically in particular by Brydges, Battle and Federbush. The most beautiful such formula is symmetric under action of the permutation group on the n points. It was discovered in [13] and developed with alternative proofs in [2][4].

Consider n points which we identify with the set V_n of vertices of the complete graph K_n; the set of pairs of such points has $n(n-1)/2$ elements $\ell = (i,j)$ for $1 \le i < j \le n$ and can be identified with the set E_n of edges ℓ of K_n. The forest formula is often presented as a Taylor expansion for functions f of $n(n-1)/2$ variables x_ℓ, $\ell \in E_n$ which are smooth, e.g., on an open neighborhood of $[0,1]^{n(n-1)/2}$. Here we expose a variant borrowed from [33] close to constructive applications.

Consider the vector space S_n of symmetric n by n matrices $X = \{X_{ij}\}$, $i,j = 1, \ldots, n$. It has dimension $n(n+1)/2$. The set PS_n of positive symmetric matrices whose diagonal coefficients are all equal to 1 and off-diagonal elements are between 0 and 1 is compact and convex. Symmetric matrices with diagonal elements equal to one and off-diagonal elements in $[0,1]^{n(n-1)/2}$ do not all belong to PS_n, for instance the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is not positive. Any matrix $X \in \mathrm{PS}_n$ can be parametrized by $n(n-1)/2$ elements X_ℓ, where ℓ runs over the edges of the complete graph K_n.

PS_n contains as particularly interesting elements the block matrices X^Π for any partition Π of V_n. The block matrix X^Π has entries $X_{ij}^\Pi = 1$ if i and j belong to the same block of the partition Π, and 0 otherwise. Two extremal cases are the identity matrix Id, corresponding to X^{sing}, that is to the maximal partition made of all singletons, and the matrix $\mathbf{1}$ with all entries equal to one, corresponding to X^{V_n}, that is to the minimal partition made of a single block.

Let us consider a function f defined and smooth in the interior of PS_n with continuous extensions (together with all their derivatives) to PS_n itself. The forest formula can be expressed as a multi-variate Taylor formula with integral remainder which expands such a function between the minimal and maximal block-partition matrices $\mathbf{1}$ and Id. The important point is that the Taylor remainder integrands stay on the PS_n convex set. The precise statement is

Theorem 2.2 (the forest formula).

$$f(\mathbf{1}) = \sum_{\mathcal{F}} \int dw_{\mathcal{F}} \, \partial_{\mathcal{F}} f \left[X^{\mathcal{F}}(w_{\mathcal{F}}) \right],$$

where

- *the sum over $\mathcal{F}$ is over forests over n labeled vertices $i = 1, \ldots, n$, including the empty forest with no edge. Such forests are exactly the acyclic edge-subgraphs of the complete graph K_n,*

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[4]Non-symmetric versions appeared earlier in the constructive literature, but won't be treated here (see [50] for a recent reference).

- $\int dw_{\mathcal{F}}$ means integration from 0 to 1 over one parameter for each forest edge: $\int dw_{\mathcal{F}} \equiv \prod_{\ell \in \mathcal{F}} \int_0^1 dw_\ell$. There is no integration for the empty forest since by convention an empty product is 1. A generic integration point $w_{\mathcal{F}}$ is therefore made of $|\mathcal{F}|$ parameters $w_\ell \in [0,1]$, one for each $\ell \in \mathcal{F}$,

- $\partial_{\mathcal{F}} = \prod_{\ell \in \mathcal{F}} \partial_\ell$ means a product of first-order partial derivatives with respect to the variables X_ℓ corresponding to the edges of $\mathcal{F}$. Again there is no such derivatives for the empty forest since by convention an empty product is 1,

- $X^{\mathcal{F}}(w_{\mathcal{F}})$ is defined by $X_{ii}^{\mathcal{F}}(w_{\mathcal{F}}) = 1 \ \forall i$, and for $i \neq j$, $X_{ij}^{\mathcal{F}}(w_{\mathcal{F}})$ is the infimum of the w_ℓ parameters for ℓ in the unique path $P_{i \to j}^{\mathcal{F}}$ from i to j in $\mathcal{F}$, when such a path exists. If no such path exists, which means that i and j belong to different connected components with respect to the forest $\mathcal{F}$, then by definition $X_{ij}^{\mathcal{F}}(w_{\mathcal{F}}) = 0$,

- the symmetric n by n matrix $X^{\mathcal{F}}(w_{\mathcal{F}})$ defined in this way is positive, hence belongs to PS_n, for any value of $w_{\mathcal{F}}$.

Since $X^{\varnothing} = \mathrm{Id}$, the empty forest term in (2.2) is $f(\mathrm{Id})$, hence (2.2) indeed interpolates f between $\mathbf{1}$ and Id, staying on PS_n as announced.

Proof. We would like to add a new proof to the seven proofs of [1], hence a new item among the 1001 proofs which should exist according to that reference. We thank D. Brydges for suggesting this proof.

We recall first Kruskal's greedy algorithm [37]. Consider a fixed connected graph G, possibly with self-loops and multiple edges. For any *Hepp sector* σ, hence any complete ordering of the edges of G, this algorithm defines a unique particular "leading tree" $T(\sigma)$, which minimizes $\sum_{\ell \in T} \sigma(\ell)$ over all trees of G, where $\sigma(\ell)$ is the order of ℓ in σ. We call $T(\sigma)$, the *leading tree* for σ. The algorithm simply picks the first edge ℓ_1 in σ (i.e., whose order $\sigma(\ell)$ is minimum) which is not a self-loop. Then it picks the next edge ℓ_2 in σ that does not add a cycle to the (usually disconnected) graph with vertex set V and edge set ℓ_1 such that the order $\sigma(\ell_2)$ is minimal among edges with this property, and so on. Another way to explain the algorithm is through a deletion-contraction recursion: following the ordering of the sector σ, every edge is either deleted if it is a self-loop or contracted if it is not. The set of contracted edges is exactly the leading tree $T(\sigma)$.

Returning to the proof of the forest formula, we first bluntly develop the function f at first order with integral remainder over *all* its parameters, hence over all edges of the complete graph K_n. The result is simply in notations compatible with Theorem 2.2

$$f(\mathbf{1}) = \sum_{S \subset K_n} \int dw_S \, \partial_S f\big(Y^S(w_S)\big),$$

where $Y_\ell^S(w_S)$ is simply w_ℓ if $\ell \in S$ and 0 if $\ell \notin S$.

Now for a given S we decompose the w_S integrals according to all Hepp sectors of S

$$f(\mathbf{1}) = \sum_{S \subset K_n} \sum_{\sigma} \int_\sigma dw_S \, \partial_S f\big(Y^S(w_S)\big),$$

and regroup all terms according to the leading forest $\mathcal{F}(S, \sigma)$ which is made of the leading Kruskal tree in each connected component S_j of S for the order on S_j induced by σ. In this way we get

$$f(\mathbf{1}) = \sum_{\mathcal{F}} \sum_{\substack{S \subset K_n, \sigma \\ \mathcal{F}(S,\sigma) = \mathcal{F}}} \int_\sigma dw_S \, \partial_S f\big(Y^S(w_S)\big). \tag{2.6}$$

Now to analyze the result for a fixed $\mathcal{F}$ we remark that any edge $\ell \in K_n - \mathcal{F}$ which does not add a cycle to $\mathcal{F}$ cannot belong to any S in (2.6), hence the value of $\partial_S f$ is always at 0 for such edges. In contrary, any edge $\ell \in K_n - \mathcal{F}$ which adds a cycle to $\mathcal{F}$ can either belong or not belong to S in (2.6). Regrouping the corresponding two terms, we obtain a value at 0 plus an integral of a derivative from 0 to the maximal value allowed by the condition $\mathcal{F}(S,\sigma) = \mathcal{F}$. But this maximal value is nothing but $X^{\mathcal{F}}(w_{\mathcal{F}})$. In this way we have summed all sectors σ inducing a fixed Hepp sector $\tilde{\sigma}$ on $\mathcal{F}$. Finally the remaining sum over $\tilde{\sigma}$ reconstructs exactly the integration range from 0 to 1 for all edges of $\mathcal{F}$.

The fact that $X^{\mathcal{F}}(w_{\mathcal{F}})$ is positive for any ordering σ now stems from the fact that, for any Hepp sector $\tilde{\sigma}$ of $\mathcal{F}$ it is a convex barycentric combination of block matrices

$$X^{\mathcal{F}}(w_{\mathcal{F}}) = (1 - w_{\ell_1})\,\mathrm{Id} + (w_{\ell_1} - w_{\ell_2})X^{\Pi_1} + (w_{\ell_2} - w_{\ell_3})X^{\Pi_2} + \cdots + w_{\ell_k}X^{\Pi_k}.$$

Remark however that this barycentric combination depends on $\tilde{\sigma}$. Hence $X^{\mathcal{F}}$ is in PS_n for any $w_{\mathcal{F}}$, as announced, but for a different reason in each different sector (ordering) of the parameters $w_{\mathcal{F}}$. ∎

We give now a useful corollary of this theorem which expands Gaussian integrals over replicas. Consider indeed a Gaussian measure $d\mu_C$ of covariance C_{pq} on a vector variable $\vec{\tau}$ with N components τ_p. To study approximate factorization properties of the integral of a product of n functions of the variable $\vec{\tau}$ it is useful to first rewrite this integral using a replica trick. It means writing the integral over n identical replicas $\vec{\tau}_i$ for $i = 1, \ldots, n$ with components $\tau_{p,i}$, with the perfectly well-defined measure with covariance $[C \otimes \mathbf{1}]_{p,i;q,j} = C_{pq}\mathbf{1}_{ij} = C_{pq}$:

$$I := \int d\mu_C(\vec{\tau}) \prod_{i=1}^{n} f_i(\vec{\tau}) = \int d\mu_{C\otimes\mathbf{1}}(\vec{\tau}_i) \prod_{i=1}^{n} f_i(\vec{\tau}_i).$$

Applying the forest formula to this Gaussian integral we obtain the following corollary

Corollary 2.3.

$$I = \sum_{\mathcal{F}} \int dw_{\mathcal{F}} \int d\mu_{C\otimes X^{\mathcal{F}}(w_{\mathcal{F}})}(\vec{\tau}_i)\, \partial_{\mathcal{F}}^C \prod_{i=1}^{n} f_i(\vec{\tau}_i),$$

where $\partial_{\mathcal{F}}^C$ means $\displaystyle\prod_{\ell=(i,j)\in\mathcal{F}} \left(\sum_{p,q} \frac{\partial}{\partial\tau_{p,i}} C_{pq} \frac{\partial}{\partial\tau_{q,j}} \right)$.

The proof follows directly from rewriting the Gaussian integral as

$$\int d\mu_C f(x) = e^{\frac{\partial}{\partial\tau_i} C_{ij} \frac{\partial}{\partial\tau_j}} f \Big|_{\tau=0}.$$

This is the corolary used in many quantum field theory applications, such as cluster expansions or loop vertex expansions in which the forest formula is used to decouple terms which are coupled through propagators, i.e., through the covariance of a Gaussian measure.

2.5   Jungle formulas

The forest formula allows to test links for connecting nodes in a kind of minimal and symmetric way. However in many physical situations, and especially in situations with many scales involved, we would like to introduce a *hierarchy* between the links connecting nodes, and we would like to test first connectedness through the most interesting links, those of the highest energy/shortest length.

Jungle formulas [2] provide an abstract solution to this question. They are a simple generalization of forest formulas.

Definition 2.4. Let $m \geq 1$ be an integer. An *m-jungle* over the set V_n of n labeled vertices is a sequence $\mathfrak{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_m)$ of forests on the complete graph K_n built on V_n such that $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m$.

Keeping the same notations than in the previous subsection we now consider a function $f(X^1, \ldots, X^m)$ defined over m copies of the previous space PS_n, hence over $[\mathrm{PS}_n]^m$, which is smooth in the interior with continuous extension of all derivatives to the boundary. We want to Taylor expand with priority to the first matrix X^1, then to X^2 and so on. That is we shall test first how to link the vertices of V_n into larger connected components first through a forest spanned by the links due to Taylor expansion of the off-diagonal part of X_1 then complete this forest (if it has still several connected components) by Taylor expanding in the off diagonal part of the second of the matrix X_2, but so that the new links together with the first ones still form a forest on V_n and so on. The result is the m-level jungle formula of [2].

Theorem 2.5 (the jungle formula)**.**

$$f(\mathbf{1}) = \sum_{\mathfrak{F}} \int dw_{\mathfrak{F}} \, \partial_{\mathfrak{F}} f\left[X^{\mathfrak{F}}(w_{\mathcal{F}})\right], \qquad (2.7)$$

where

- *the sum over $\mathfrak{F}$ is over m-jungles over the set V_n of n labeled vertices $V_n = \{1, \ldots, n\}$, including the empty jungle $\mathfrak{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_m)$ with $\mathcal{F}_i = \varnothing \; \forall i$,*

- $\int dw_{\mathfrak{F}} \equiv \prod_{\ell \in \mathcal{F}_m} \int_0^1 dw_\ell$ *means integration from 0 to 1 over one parameter for each edge in the final forest $\mathcal{F}_m$ of the jungle,*

- $\partial_{\mathfrak{F}} = \prod_{k=1}^{m} \prod_{\ell \in \mathcal{F}_k \backslash \mathcal{F}_{k-1}} \partial_\ell$ *means a product of first-order partial derivatives with respect to the variables X_ℓ^k corresponding to the edges of $\mathcal{F}_k \backslash \mathcal{F}_{k-1}$ (by convention $\mathcal{F}_0 = \varnothing$),*

- $X^{\mathfrak{F}}(w_{\mathfrak{F}})$ *is the sequence of m matrices $X^{\mathfrak{F},k}(w_{\mathfrak{F}})$, $k = 1, \ldots, m$, defined by $X_{ii}^{\mathfrak{F},k}(w_{\mathfrak{F}}) = 1$ $\forall i$, and for $i \neq j$, $X_{ij}^{\mathfrak{F},k}(w_{\mathcal{F}})$ is*

  - 1 *if i and j are connected by $\mathcal{F}_{k-1}$,*
  - 0 *if i and j are not connected by $\mathcal{F}_k$,*
  - *the infimum of the w_ℓ parameters for ℓ in the intersection of $\mathcal{F}_k \backslash \mathcal{F}_{k-1}$ and the unique path of $\mathcal{F}_k$ from i to j, when i and j are connected by $\mathcal{F}_k$ but not by $\mathcal{F}_{k-1}$,*

- *the symmetric n by n matrices $X^{\mathfrak{F},k}(w_{\mathfrak{F}})$ defined in this way are all positive, hence $X^{\mathfrak{F}}(w_{\mathfrak{F}}) \in [\mathrm{PS}_n]^m$, for any value of $w_{\mathfrak{F}}$.*

The proof is an easy generalization of the proof of the forest formula. Remark that the two-level jungle formula is a main ingredient in multiscale loop vertex expansions [32].

3 Constructive tools

3.1 The loop vertex expansion

The loop vertex expansion (LVE) combines an intermediate field functional integral representation for QFT quantities with the forest formula and a replica trick similar to the one of the previous section. It allows the computation of connected functional QFT integrals such as the free energy or connected Schwinger functions as convergent sums indexed by spanning trees of arbitrary size n rather than divergent sums indexed by Feynman graphs.

Initially introduced to analyze *matrix* models with quartic interactions [46], the LVE has been extended to analyze random *tensor* models [18, 19, 29, 43]. In this subsection and the next one we follow [33].

Canonical barycentric weights $w(G,T)$ can then be associated to any pair made of a connected graph G and a spanning tree $T \subset G$ by considering the percentage of Hepp sectors in which the tree is leading for Kruskal's greedy algorithm:

$$w(G,T) = \frac{N(G,T)}{|E|!},$$

where $N(G,T)$ is the number of sectors σ such that $T(\sigma) = T$.

Obviously these weights are barycentric, which simply means that

$$\sum_{T \subset G} w(G,T) = 1,$$

where the sum runs over all spanning trees of G. Moreover we have, in the notations of Theorem 2.2 the integral representation [53].

Lemma 3.1.

$$w(G,T) = \int dw_T \prod_{\ell \in G-T} X^T_{i(\ell)j(\ell)}(w_T).$$

Proof. We introduce first parameters w_ℓ for all the edges in $G - T$, writing

$$X^T_{ij}(\{w\}) = \int_0^1 dw_\ell \left[\prod_{\ell' \in P^T_{i \to j}} \chi(w_\ell < w_{\ell'}) \right],$$

where $\chi(\cdots)$ is the characteristic function of the event $\cdots$. Then we decompose the w integrals according to all possible orderings σ:

$$\int_0^1 \prod_{\ell \in G} dw_\ell \prod_{\ell \notin T} \left[\prod_{\ell' \in P^T_\ell} \chi(w_\ell < w_{\ell'}) \right] = \sum_\sigma \chi(T(\sigma) = T) \int_{0 < w_{\sigma(|E|)} < \cdots < w_{\sigma(1)} < 1} \prod_{\ell \in G} dw_\ell.$$

Indeed in the domain of integration defined by $0 < w_{\sigma(|E|)} < \cdots < w_{\sigma(1)} < 1$ the function $\prod_{\ell \notin T} \left[\prod_{\ell' \in P^T_\ell} \chi(w_\ell < w_{\ell'}) \right]$ is 1 or zero depending whether $T(\sigma) = T$ or not, as this function being 1 is exactly the condition for Kruskal's algorithm to pick exactly T. Strict inequalities are easier to use here: of course equal values of w factors have zero measure anyway. Hence

$$\int dw_T \prod_{\ell \in G-T} X^T_{i(\ell)j(\ell)}(w_T) = \frac{N(G,T)}{|E|!} = w(G,T). \qquad \blacksquare$$

The LVE expressed any Schwinger function S as a convergent sum over trees of the *intermediate field representation*:

$$S = \sum_T A_T, \qquad A_T = \sum_{G \supset T} w(G,T) A_G,$$

with

$$\sum_T |A_T| < +\infty.$$

The usual (divergent) perturbative expansion of S is obtained by the ill defined commutation of the sums over T and G,

$$S = \sum_T \left(\sum_{G \supset T} w(G,T) A_G \right) \text{``=''} \sum_G \sum_{T \subset G} w(G,T) A_G = \sum_G A_G,$$

$$\sum_G |A_G| = \infty.$$

We shall limit ourselves here to introduce the LVE in the particularly simple case of the quartic N-vector models, for which the $1/N$ expansion is governed by rooted plane trees.

More precisely, consider a pair of conjugate vector fields $\{\phi_p\}$, $\{\bar\phi_p\}$, $p = 1, \ldots, N$, with $(\bar\phi \cdot \phi)^2$ interaction. The corresponding functional integral

$$Z(z,N) = \int \frac{d\bar\phi d\phi}{(2i\pi)^N} e^{-(\bar\phi \cdot \phi) + \frac{z}{2N}(\bar\phi \cdot \phi)^2}$$

is convergent for $\Re z < 0$. Note the slightly unusual sign convention for the interaction term. We rewrite it, using a scalar intermediate field σ, as

$$Z(z,N) = \int d\sigma \frac{e^{-\sigma^2/2}}{\sqrt{2\pi}} \int \frac{d\bar\phi d\phi}{(2i\pi)^N} e^{-(\bar\phi \cdot \phi) + \sqrt{z/N}(\bar\phi \cdot \phi)\sigma} = \int \frac{d\sigma}{\sqrt{2\pi}} e^{-\sigma^2/2 - N\log(1 - \sqrt{z/N}\sigma)}.$$

Defining $\tau = \sigma/\sqrt{N}$ one gets

$$Z(z,N) = \int \frac{\sqrt{N}d\tau}{\sqrt{2\pi}} e^{-N[\tau^2/2 + \log(1 - \sqrt{z}\tau)]}. \tag{3.1}$$

The two point function

$$G_2(z,N) = \frac{1}{Z(z,N)} \int \frac{d\bar\phi d\phi}{(2i\pi)^N} \frac{1}{N} \left(\sum_p \bar\phi_p \phi_p \right) e^{-(\bar\phi \cdot \phi) + \frac{z}{2N}(\bar\phi \cdot \phi)^2},$$

can be deduced from the free energy by a Schwinger–Dyson equation

$$0 = \frac{1}{Z(z,N)} \int \frac{d\bar\phi d\phi}{(2i\pi)^N} \frac{1}{N} \sum_p \frac{\partial}{\partial \phi_p} \left[\phi_p e^{-(\bar\phi \cdot \phi) + \frac{z}{2N}(\bar\phi \cdot \phi)^2} \right],$$

which yields

$$G_2(z,N) = 1 + 2z \frac{d}{dz} \left(\frac{1}{N} \log \int \frac{\sqrt{N}d\tau}{\sqrt{2\pi}} e^{-N[\tau^2/2 + \log(1 - \sqrt{z}\tau)]} \right). \tag{3.2}$$

A simple saddle point evaluates the integral (3.1) as $\frac{K e^{-Nf(\tau_c)}}{\sqrt{f''(\tau_c)}}$, where the saddle point of $f(\tau) = \tau^2/2 + \log(1 - \sqrt{z}\tau)]$ is at τ_c with $f'(\tau_c) = 0$ hence $\tau_c = \frac{1}{2\sqrt{z}}[1 - \sqrt{1 - 4z}]$. Also

$$\lim_{N \to \infty} \frac{\log Z(z,N)}{N} = -f(\tau_c),$$

and the two point function in the $N \to \infty$ limit is

$$\lim_{N \to \infty} G_2(z,N) = 1 + 2z \left(-\partial_z f(\tau_c) - \partial_\tau f(\tau_c) \frac{d\tau_c}{dz} \right) = 1 - 2z \frac{-\frac{1}{2\sqrt{z}}\tau_c}{1 - \sqrt{z}\tau_c}$$

$$= \frac{1}{2z}[1 - \sqrt{1 - 4z}], \tag{3.3}$$

which we recognize as the generating function of the Catalan numbers.

Let us now study Borel summability in z of these quantities uniformly as $N \to \infty$, using the loop vertex expansion. We start from the intermediate field representation of the two-point function (3.2) and apply the LVE to get

$$G_2(z, N) = \sum_{\mathcal{T}} \frac{1}{n!} z^n \int dw_{\mathcal{T}} \int d\mu_{\mathcal{T}} \prod_{c \in C(\mathcal{T})} \frac{1}{1 - \sqrt{z}\tau_{i(c)}}, \tag{3.4}$$

where in (3.4)

- the sum over $\mathcal{T}$ is over rooted plane trees, with one ciliated root vertex labeled $i = 0$ plus $n \geq 0$ ordinary vertices labeled $1, \ldots, n$,

- $\int dw_{\mathcal{T}}$ as in Subsection 2.4 means $\left[\prod_{\ell \in \mathcal{T}} \int_0^1 dw_\ell \right]$,

- $d\mu_{\mathcal{T}}$ is the normalized Gaussian measure on the $(n+1)$-dimensional vector field $\vec{\tau} = (\tau_i)$, $i = 0, 1, \ldots, n$, running over the vertices of $\mathcal{T}$, which has covariance $\frac{X_{ij}^{\mathcal{T}}(w_{\mathcal{T}})}{N}$ between vertices i and j. Recall that $X^{\mathcal{T}}(w_{\mathcal{T}})$ is defined in Subsection 2.4,

- the product over c runs over the set $C(\mathcal{T})$ of the $2n+1$ corners of the tree, the cilium creating an additional corner on the plane tree, and $i(c)$ is the index of the vertex to which the corner c belongs.

It is now obvious why (3.3) is true; since the covariance of the τ fields vanishes as $N \to \infty$ the limit of $G_2(z, N)$ is obtained by putting every $\tau_{i(c)}$ factor to 0 in every corner resolvent, in which case we exactly get a weight $z^{n(\mathcal{T})}$ for each rooted plane tree, hence we recover the Catalan generating function (the $1/n!$ is canceled by relabeling of the vertices).

We can now use (3.4) to prove analyticity and Borel summability of the free energy and correlation functions of the model in the variables z and $1/N$ in the cardioid domain of Fig. 3 (see [23] for an early reference to Borel summability of the $1/N$ expansion of vector models).

Let us set $z = |z|e^{i\pi + i\phi}$ for $|\phi| < \pi$. We have $\sqrt{z} = i\sqrt{|z|}e^{i\phi/2}$. Each resolvent $\frac{1}{1 - \sqrt{z}\tau_{i(c)}}$ is bounded in norm by $[\cos(\phi/2)]^{-1}$, hence using the fact that there are $2n+1$ such resolvents, we obtain analyticity of representation (3.4) for $4|z| < [\cos(\phi/2)]^2$, the cardioid domain of Fig. 3.

But in fact we can extend the analyticity domain into the extended cardioid domain of Fig. 6, a domain introduced for quartic vector models in [9]. Indeed using the parametric representation of resolvents

$$\frac{1}{1 - \sqrt{z}\tau} = \int_0^\infty d\alpha_c e^{-\alpha_c(1 - \sqrt{z}\tau)}$$

we can explicitly integrate over the measure $d\mu_{\mathcal{T}}$ and get the integral representation

$$G_2(z, N) = \sum_{\mathcal{T}} \frac{1}{n!} z^n \int dw_{\mathcal{T}} \left[\prod_{c \in \mathcal{T}} \int_0^\infty d\alpha_c e^{-\alpha_c} \right] e^{\frac{z}{2N} \sum_{ij} \left(\sum_{c \in i} \alpha_c \right) X_{ij}^{\mathcal{T}}(w_{\mathcal{T}}) \left(\sum_{c \in j} \alpha_c \right)}.$$

The formula above can be further simplified. Putting $\beta_i = \sum_{c \in i} \alpha_c$ we have

$$G_2(z, N) = \sum_{\mathcal{T}} \frac{1}{n!} z^n \left[\prod_{i=0}^n \int_0^\infty d\beta_i \frac{\beta_i^{d_i - 1}}{(d_i - 1)!} e^{-\beta_i} \right] \int dw_{\mathcal{T}} e^{\frac{z}{2N} \sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}}) \beta_j}, \tag{3.5}$$

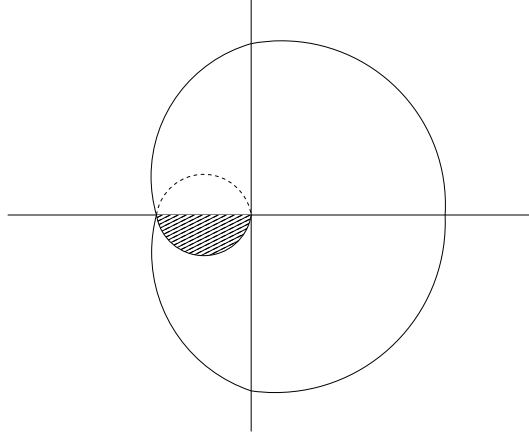where d_i is the degree of i, hence the number of corners of i.

Figure 6. The extended cardioid domain of vector models. The crosshatching represents the analytic continuation above the negative axis cut; there is a symmetric analytic continuation below the cut.

Setting $z = |z|e^{i\pi+i\phi}$ and $\beta = |\beta|e^{i\psi}$ we have, for $-\pi/2 \leq \phi+2\psi \leq \pi/2$ and $-\pi/2 < \psi < \pi/2$, both $\cos(\phi + 2\psi) \geq 0$ and $\cos\psi > 0$, hence

$$
\left| \left[\prod_{i=0}^{n} \int_0^{e^{i\psi}\infty} d\beta_i \frac{\beta_i^{d_i-1}}{(d_i-1)!} e^{-\beta_i} \right] e^{\frac{z}{2N} \sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}})\beta_j} \right|
$$

$$
\leq \prod_{i=0}^{n} \int_0^{\infty} d|\beta_i| \frac{|\beta_i|^{d_i-1}}{(d_i-1)!} e^{-|\beta_i|\cos\psi} \leq (\cos\psi)^{-\sum_{i=0}^{n} d_i} = (\cos\psi)^{-2n-1}.
$$

Therefore $G_2(z, N)$ is analytic in the extended cardioid $C = C_+ \cup C_-$, where C_+ is the union of the quarter-disk $0 \leq \phi < \pi/2$, $4|z| < 1$ and of the domain $\pi/2 \leq \phi < 3\pi/2$, $4|z| < [\cos(\phi/2 - \pi/4)]^2$. C_- is the complex conjugate domain.

To prove that this convergent analytic function is the Borel sum of its perturbative series at any fixed N also requires uniform Taylor estimates of the type $K^p p! |z|^p$ for the Taylor remainder at order p in at least a disk tangent to the imaginary axis (Nevanlinna's criterion). They follow from Taylor expanding the exponential of the β quadratic form with an integral remainder:

$$
e^{\frac{z}{2N} \sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}})\beta_j} = \sum_{q=0}^{p-1} \frac{z^q}{q!(2N)^q} \left[\sum_{ij} \beta_i x_{ij}^{\mathcal{T}}(w_{\mathcal{T}})\beta_j \right]^q
$$

$$
+ \int_0^1 dt \frac{(1-t)^{p-1}}{(p-1)!} \frac{z^p}{(2N)^p} \left[\sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}})\beta_j \right]^p e^{t\frac{z}{2N} \sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}})\beta_j}.
$$

The sum over q, i.e., the p first terms, are exactly the perturbative expansion up to order p hence support a $K^p p! |z|^p$ bound. The Taylor remainder term for any tree $\mathcal{T}$ in the disk $-\pi/2 \leq \phi \leq \pi/2$, where we can take $\psi = 0$ can be bounded as

$$
\left| \left[\prod_{i=0}^{n} \int_0^{\infty} \frac{\beta_i^{d_i-1}}{(d_i-1)!} d\beta_i e^{-\beta_i} \right] \int_0^1 dt \frac{(1-t)^p}{p!} \frac{z^p}{(2N)^p} \right.
$$

$$
\left. \times \int dw_{\mathcal{T}} \left[\sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}})\beta_j \right]^p e^{t\frac{z}{2N} \sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}})\beta_j} \right|
$$

$$
\leq \frac{|z|^p}{p!} \left[\prod_{i=0}^{n} \int_0^{\infty} \frac{\beta_i^{d_i-1}}{(d_i-1)!} d\beta_i e^{-\beta_i} \right] \left[\sum_{i=0}^{n} \beta_i \right]^{2p}
$$

$$= \frac{|z|^p}{p!} \left(\prod_{i=0}^{n} \frac{1}{(d_i - 1)!} \right) \int_0^\infty d\beta e^{-\beta} \beta^{2p} \int_{\beta_1 + \cdots + \beta_n = \beta} \prod_{i=0}^{n} \beta_i^{d_i - 1} d\beta_i$$

$$= \frac{|z|^p}{p!} \left(\prod_{i=0}^{n} \frac{1}{(d_i - 1)!} \right) (2p + 2n + 1)! \int_{u_1 + \cdots + u_n = 1} \prod_{i=0}^{n} u_i^{d_i - 1} du_i \leq 4^n K^p p! |z|^p.$$

These Taylor estimates for a single rooted plane tree can be summed over all rooted plane trees (using the $|z|^n$ factor in (3.5)) in the half-disk defined by $16|z| < 1$ and $-\pi/2 \leq \arg z \leq \pi/2$ (shown in red on Fig. 6). Hence in this half-disk (which is uniform in N) we obtain the desired Taylor estimates, which is more than enough to check that the expansions (3.4) and (3.5) represent indeed for all N the unique Borel sum of the perturbative series.

Interesting functions are the real and imaginary parts along the real axis $0 \leq z < 1/8$ which are

$$G_2^{\mathrm{mean}}(z, N) = \frac{G_2(z, N)_+ + G_2(z, N)_-}{2}, \qquad G_2^{\mathrm{cut}}(z, N) = \frac{G_2(z, N)_+ - G_2(z, N)_-}{2i},$$

where G_+ is analytically continued to $\phi = +\pi$ and G_- is analytically continued to $\phi = -\pi$. Taking $\psi = -\pi/4$ in the first case and $\psi = +\pi/4$ is the second case, one obtains explicitly convergent integral representations for these quantities, namely

$$G_2^{\mathrm{mean}}(z, N) = \sum_{\mathcal{T}} \frac{1}{n!} z^n \left[\prod_{i \in V(\mathcal{T})} \int_0^\infty \frac{\beta_i^{d_i - 1}}{(d_i - 1)!} d\beta_i e^{-\frac{\sqrt{2}}{2} \beta_i} \right]$$

$$\times \int dw_{\mathcal{T}} \cos \left((2n + 1) \frac{\pi}{4} + \frac{\sqrt{2}}{2} \sum_i \beta_i + \frac{z}{2N} \sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}}) \beta_j \right),$$

$$G_2^{\mathrm{cut}}(z, N) = \sum_{\mathcal{T}} \frac{1}{n!} z^n \int dw_{\mathcal{T}} \left[\prod_{i \in V(\mathcal{T})} \int_0^\infty \frac{\beta_i^{d_i - 1}}{(d_i - 1)!} d\beta_i e^{-\frac{\sqrt{2}}{2} \beta_i} \right]$$

$$\times \int dw_{\mathcal{T}} \sin \left((2n + 1) \frac{\pi}{4} + \frac{\sqrt{2}}{2} \sum_i \beta_i + \frac{z}{2N} \sum_{ij} \beta_i X_{ij}^{\mathcal{T}}(w_{\mathcal{T}}) \beta_j \right).$$

where the factors $(2n+1)\frac{\pi}{4}$ come from the rotation of the β integrals, using $\sum d_i = 2n+1$. These convergent integrals extend half-way to the Catalan singularity $z_{\mathrm{Catalan}} = 1/4$. Indeed bounding the cosine or sinus function by 1 we obtain convergence, but loosing a factor $(\sqrt{2})^{\sum d_i} = 2^n \sqrt{2}$.

One can still check easily that the limit for $N \to \infty$ of the mean integral for positive z is the Catalan function. Indeed the cosine function simplifies in that case. Rotating the β integrals back in position we obtain again the factor 1 for each rooted plane tree.

The extended cardioid is an analyticity domain in z which holds for any $N \geq 1$. In other words it is common to all N-vector models, including the particular $N = 1$ scalar case, However as $N \to \infty$ we could hope for larger and larger domains of analyticity which approach the $z = 1/4$ singularity when $N \to \infty$; but we do not know, even in this simple vector model case, how to prove this.

In the case of quartic large N matrix [46] and large N tensor models [18, 19, 29, 43], the LVE also provides analyticity in cardioid-like domains.

The constructive treatment of renormalizable models requires a multiscale analysis, hence a multiscale version of the loop vertex expansion (MLVE). Following [32], we sketch now how this expansion works in the case of a super-renormalizable toy model which is a slight modification of the vector model above.

3.2 Multiscale loop vertex expansion

Consider the same pair of conjugate vector fields $\{\phi_p\}$, $\{\bar{\phi}_p\}$, $p = 1, \ldots, N$, with the same $\frac{\lambda^2}{2}(\bar{\phi} \cdot \phi)^2$ bare interaction as in the previous section, but with a different Gaussian measure $d\mu(\bar{\phi}, \phi)$ which breaks the U(N) invariance of the theory. It has diagonal covariance (or propagator) which decreases as the inverse power of the field index:

$$d\eta(\bar{\phi}, \phi) = \left(\prod_{p=1}^{N} p \frac{d\bar{\phi}_p d\phi_p}{2\pi \imath} \right) e^{-\sum_{p=1}^{N} p \bar{\phi}_p \phi_p}, \qquad \int d\eta(\bar{\phi}, \phi) \bar{\phi}_p \phi_q = \frac{\delta_{pq}}{p}.$$

This propagator renders the perturbative amplitudes of the model finite in the $N \to \infty$ limit, except for a mild divergence of self-loops which yields a logarithmically divergent sum $L_N = \sum_{p=1}^{N} \frac{1}{p} \simeq \log N$. These divergences are easily renormalized by using a vector-Wick-ordered ϕ^4 interaction, namely $\frac{1}{2}[\lambda(\bar{\phi} \cdot \phi - L_N)]^2$. Remark that this interaction (contrary to the ϕ_2^4 case) remains positive for λ real at all values of $(\bar{\phi}, \phi)$. The renormalized partition function of the model is

$$Z(\lambda, N) = \int d\eta(\bar{\phi}, \phi) \, e^{-\frac{\lambda^2}{2}(\bar{\phi} \cdot \phi - L_N)^2}.$$

The intermediate field representation decomposes the quartic interaction using an intermediate scalar field σ:

$$e^{-\frac{\lambda^2}{2}(\bar{\phi} \cdot \phi - L_N)^2} = \int d\nu(\sigma) \, e^{\imath \lambda \sigma (\bar{\phi} \cdot \phi - L_N)},$$

where $d\nu(\sigma) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma^2}{2}}$ is the standard Gaussian measure with covariance 1. Integrating over the initial fields $(\bar{\phi}_p, \phi_p)$ leads to

$$Z(\lambda, N) = \int d\nu(\sigma) \prod_{p=1}^{N} \frac{1}{1 - \imath \frac{\lambda \sigma}{p}} e^{-\imath \frac{\lambda \sigma}{p}} = \int d\nu(\sigma) \, e^{-\sum_{p=1}^{N} \log_2 \left(1 - \imath \frac{\lambda \sigma}{p} \right)},$$

where $\log_2(1 - x) \equiv x + \log(1 - x) = O(x^2)$.

Applying the ordinary LVE of the previous section to this functional integral would express $\log Z(\lambda, N)$ as a sum over trees, but there is no simple way to remove the logarithmic divergence of all leaves of the tree without generating many intermediate fields in numerators which, when integrated through the Gaussian measure, would create an apparent divergence of the series. The MLVE is designed to solve this problem.

We fix an integer $M > 1$ and define the j-th *slice*, as made of the indices $p \in I_j \equiv [M^{j-1}, M^j - 1]$. The ultraviolet cutoff N is chosen as $N = M^{j_{\max}} - 1$, with $j_{\max}$ an integer. We can also fix an infrared cutoff $j_{\min}$. Hence there are $j_{\max} - j_{\min}$ slices in the theory, and the ultraviolet limit corresponds to the limit $j_{\max} \to \infty$. The intermediate field representation writes:

$$Z(\lambda, N) = \int d\nu(\sigma) \prod_{j=j_{\min}}^{j_{\max}} e^{-V_j}, \qquad V_j = \sum_{p \in I_j} \log_2 \left(1 - \imath \frac{\lambda \sigma}{p} \right).$$

The factorization of the interaction over the set of slices $\mathcal{S} = [j_{\min}, \ldots, j_{\max}]$ can be encoded into an integral over Grassmann numbers. Indeed,

$$a = \int d\bar{\chi} d\chi \, e^{-\bar{\chi} a \chi} = \int d\mu(\bar{\chi}, \chi) \, e^{-\bar{\chi}(a-1)\chi},$$

where $d\mu(\bar{\chi}, \chi) = d\bar{\chi} d\chi\, e^{-\bar{\chi}\chi}$ is the standard normalized Grassmann Gaussian measure with covariance 1. Hence, denoting $W_j(\sigma) = e^{-V_j} - 1$,

$$Z(\lambda, N) = \int d\nu(\sigma) \left(\prod_{j=j_{\min}}^{j_{\max}} d\mu(\bar{\chi}_j, \chi_j) \right) e^{-\sum\limits_{j=j_{\min}}^{j_{\max}} \bar{\chi}_j W_j(\sigma)\chi_j}.$$

We now rewrite the partition function as

$$Z(\lambda, N) = \int d\nu_{\mathcal{S}}\, e^{-W}, \qquad d\nu_{\mathcal{S}} = d\nu_{\mathbf{1}_{\mathcal{S}}}(\{\sigma_j\})\, d\mu_{\mathbb{I}_{\mathcal{S}}}(\{\bar{\chi}_j, \chi_j\}), \qquad W = \sum_{j=j_{\min}}^{j_{\max}} \bar{\chi}_j W_j(\sigma_j)\chi_j.$$

This is the starting point for the MLVE. The first step is to expand to infinity the exponential of the interaction

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_{\mathcal{S}}\, (-W)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_{\mathcal{S},V} \prod_{a=1}^{n} (-W_a),$$

where the a-th vertex is

$$W_a = \sum_{j=j_{\min}}^{j_{\max}} W_{a,j}, \qquad W_{a,j} = \bar{\chi}_j^a W_j(\sigma_j^a)\chi_j^a,$$

and has now its own (replicated) bosonic and fermionic fields σ_j^a, $\bar{\chi}_j^a$, χ_j^a and the replica measure is completely degenerate:

$$d\nu_{\mathcal{S},V} = d\nu_{\mathbf{1}_{\mathcal{S}} \otimes \mathbf{1}_V}(\{\sigma_j^a\})\, d\mu_{\mathbb{I}_{\mathcal{S}} \otimes \mathbf{1}_V}(\{\bar{\chi}_j^a, \chi_j^a\}).$$

The obstacle to factorize this integral over vertices lies now in the bosonic and fermionic degenerate blocks $\mathbf{1}_V$. In order to deal with these couplings we apply the $m = 2$ jungle formula (2.7) with priority to the bosonic links. It means that in the measure $d\nu$ one introduces first the matrix x_V with coupling parameters $x_{ab} = x_{ba}$, $x_{aa} = 1$ between the vertex bosonic replicas

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_{\mathbf{1}_{\mathcal{S}} \otimes x_V}(\{\sigma_j^a\})\, d\mu_{\mathbb{I}_{\mathcal{S}} \otimes \mathbf{1}_V}(\{\bar{\chi}_j^a, \chi_j^a\}) \prod_{a=1}^{n} \left(-\sum_{j=j_{\min}}^{j_{\max}} W_{a,j} \right) \Big|_{x_{ab}=1}$$

and apply the forest formula. We denote $\mathcal{F}_B$ a bosonic forest with n vertices labelled $\{1, \ldots, n\}$, ℓ_B a generic edge of the forest and $a(\ell_B), b(\ell_B)$ the end vertices of ℓ_B. The result of the first forest formula is

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}_B} \int dw_{\mathcal{F}_B} \int d\nu_{\mathbf{1}_{\mathcal{S}} \otimes X(w_{\ell_B})}(\{\sigma_j^a\})\, d\mu_{\mathbb{I}_{\mathcal{S}} \otimes \mathbf{1}_V}(\{\bar{\chi}_j^a, \chi_j^a\})$$

$$\times \partial_{\mathcal{F}_B} \prod_{a=1}^{n} \left(-\sum_{j=j_{\min}}^{j_{\max}} W_{a,j} \right),$$

where

$$\int dw_{\mathcal{F}_B} = \prod_{\ell_B \in \mathcal{F}_B} \int_0^1 dw_{\ell_B}, \qquad \partial_{\mathcal{F}_B} = \prod_{\ell_B \in \mathcal{F}_B} \left(\sum_{j,k=j_{\min}}^{j_{\max}} \frac{\partial}{\partial \sigma_j^{a(\ell_B)}} \frac{\partial}{\partial \sigma_k^{b(\ell_B)}} \right),$$

and $X_{ab}(w_{\ell_B})$ is the infimum over the parameters w_{ℓ_B} in the unique path in the forest $\mathcal{F}_B$ connecting a and b, and the infimum is set to 1 if $a = b$ and to zero if a and b are not connected by the forest.

The forest $\mathcal{F}_B$ partitions the set of vertices into blocks $\mathcal{B}$ corresponding to its trees. Remark that the blocks can be singletons (corresponding to the trees with no edges in $\mathcal{F}_B$). We denote $a \in \mathcal{B}$ if the vertex a belongs to a bosonic block $\mathcal{B}$. A vertex belongs to a unique bosonic block. Contracting every bosonic block to an "effective vertex" we obtain a graph which we denote $\{n\}/\mathcal{F}_B$. We introduce decoupling parameters $y_{\mathcal{B}\mathcal{B}'} = y_{\mathcal{B}'\mathcal{B}}$ for the fermionic fields $\chi_j^{\mathcal{B}}$ corresponding to the blocks of $\mathcal{F}_B$ (i.e., for the effective vertices of $\{n\}/\mathcal{F}_B$). Applying (a second time) the forest formula, this time for the y's, is exactly the level 2 jungle formula as it leads to a set of fermionic edges $\mathcal{L}_F$ forming a forest in $\{n\}/\mathcal{F}_B$ (hence connecting bosonic blocks). We denote L_F a generic fermionic edge connecting blocks and $\mathcal{B}(L_F), \mathcal{B}'(L_F)$ the end blocks of the fermionic edge L_F. We obtain

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}_B} \sum_{\mathcal{L}_F} \int dw_{\mathcal{F}_B} \int dw_{\mathcal{L}_F} \int d\nu_{\mathbf{1}_{\mathcal{S}} \otimes X(w_{\mathcal{B}})}(\{\sigma_j^a\})$$

$$\times d\mu_{\mathbb{I}_{\mathcal{S}} \otimes Y(w_{L_F})}(\{\bar{\chi}_j^{\mathcal{B}}, \chi_j^{\mathcal{B}}\}) \partial_{\mathcal{F}_B} \partial_{\mathcal{L}_F} \prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \left(-\sum_{j=j_{\min}}^{j_{\max}} \bar{\chi}_j^{\mathcal{B}} W_j(\sigma_j^a) \chi_j^{\mathcal{B}} \right), \qquad (3.6)$$

where

$$\int dw_{\mathcal{L}_F} = \prod_{L_F \in \mathcal{L}_F} \int_0^1 dw_{L_F},$$

$$\partial_{\mathcal{L}_F} = \prod_{L_F \in \mathcal{L}_F} \left(\sum_{j=j_{\min}}^{j_{\max}} \left(\frac{\partial}{\partial \bar{\chi}_j^{\mathcal{B}(L_F)}} \frac{\partial}{\partial \chi_j^{\mathcal{B}'(L_F)}} + \frac{\partial}{\partial \bar{\chi}_j^{\mathcal{B}'(L_F)}} \frac{\partial}{\partial \chi_j^{\mathcal{B}(L_F)}} \right) \right),$$

and $Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F})$ is the infimum over w_{ℓ_F} in the unique path in $\mathcal{L}_F$ connecting $\mathcal{B}$ and $\mathcal{B}'$, this infimum being set to 1 if $\mathcal{B} = \mathcal{B}'$ and to zero if $\mathcal{B}$ and $\mathcal{B}'$ are not connected by $\mathcal{L}_F$. Note that the fermionic edges are oriented. Expanding the sums over j in the last line of equation (3.6) we obtain a sum over slice assignments $J = \{j_a\}$ to the vertices a, where $j_a \in [j_{\min}, j_{\max}]$. Taking into account that $\partial_{\sigma_j^a} W(\sigma_{j_a}^a) = \delta_{jj_a} \partial_{\sigma_{j_a}^a} W(\sigma_{j_a}^a)$ we obtain:

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}_B} \sum_{\mathcal{L}_F} \sum_J \int dw_{\mathcal{F}_B} \int dw_{\mathcal{L}_F}$$

$$\times \int d\nu_{\mathbf{1}_{\mathcal{S}} \otimes X(w_{\mathcal{B}})}(\{\sigma_j^a\}) d\mu_{\mathbb{I}_{\mathcal{S}} \otimes Y(w_{L_F})}(\{\bar{\chi}_j^{\mathcal{B}}, \chi_j^{\mathcal{B}}\})$$

$$\times \partial_{\mathcal{F}_B} \partial_{\mathcal{L}_F} \prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \left(-\bar{\chi}_{j_a}^{\mathcal{B}} W_{j_a}(\sigma_{j_a}^a) \chi_{j_a}^{\mathcal{B}} \right).$$

In order to compute the derivatives in $\partial_{\mathcal{L}_F}$ with respect to the block fermionic fields $\chi_j^{\mathcal{B}}$ and $\bar{\chi}_j^{\mathcal{B}}$ we note that such a derivative acts only on $\prod_{a \in \mathcal{B}} (\chi_{j_a}^{\mathcal{B}} \bar{\chi}_{j_a}^{\mathcal{B}})$ and, furthermore,

$$\frac{\partial}{\partial \bar{\chi}_j^{\mathcal{B}}} \prod_{a \in \mathcal{B}} (\chi_{j_a}^{\mathcal{B}} \bar{\chi}_{j_a}^{\mathcal{B}}) = \left(\sum_{a' \in \mathcal{B}} \delta_{jj_{a'}} \frac{\partial}{\partial \bar{\chi}_{j_{a'}}^{\mathcal{B}}} \right) \prod_{a \in \mathcal{B}} (\chi_{j_a}^{\mathcal{B}} \bar{\chi}_{j_a}^{\mathcal{B}}),$$

$$\frac{\partial}{\partial \chi_j^{\mathcal{B}}} \prod_{a \in \mathcal{B}} (\chi_{j_a}^{\mathcal{B}} \bar{\chi}_{j_a}^{\mathcal{B}}) = \left(\sum_{a' \in \mathcal{B}} \delta_{jj_{a'}} \frac{\partial}{\partial \chi_{j_{a'}}^{\mathcal{B}}} \right) \prod_{a \in \mathcal{B}} (\chi_{j_a}^{\mathcal{B}} \bar{\chi}_{j_a}^{\mathcal{B}}).$$

It follows that the Grassmann Gaussian integral is

$$
\Bigg[e^{\sum_{\mathcal{B},\mathcal{B}'} Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F}) \sum_{a\in\mathcal{B},b\in\mathcal{B}'} \delta_{j_a j_b} \frac{\partial}{\partial \bar{\chi}^{\mathcal{B}}_{j_a}} \frac{\partial}{\partial \chi^{\mathcal{B}'}_{j_b}}}
$$

$$
\times \prod_{L_F\in\mathcal{L}_F} \left(\sum_{a\in\mathcal{B}(L_F),b\in\mathcal{B}'(L_F)} \delta_{j_a j_b} \left(\frac{\partial}{\partial \bar{\chi}^{\mathcal{B}(L_F)}_{j_a}} \frac{\partial}{\partial \chi^{\mathcal{B}'(L_F)}_{j_b}} + \frac{\partial}{\partial \bar{\chi}^{\mathcal{B}'(L_F)}_{j_b}} \frac{\partial}{\partial \chi^{\mathcal{B}(L_F)}_{j_a}} \right) \right)
$$

$$
\times \prod_{\mathcal{B}} \prod_{a\in\mathcal{B}} (\chi^{\mathcal{B}}_{j_a} \bar{\chi}^{\mathcal{B}}_{j_a}) \Bigg]_{\chi^{\mathcal{B}}_j, \bar{\chi}^{\mathcal{B}}_j = 0}.
$$

The sums over $a \in \mathcal{B}(\ell_F)$ and $b \in \mathcal{B}'(\ell_F)$ yield a sum over all the possible ways to hook the edge $L_F \in \mathcal{L}_F$ to vertices in its end blocks. Each term represents a detailed fermionic edge ℓ_F in the original graph (having the same $w_{\ell_F} = w_{L_F}$ parameter). The sum over $\mathcal{L}_F$ becomes therefore a sum over detailed fermionic forests $\mathcal{F}_F$ in the original graph (in which the bosonic blocks are not contracted)and we obtain

$$
Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}} \sum_{J} \int dw_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \prod_{\mathcal{B}} \prod_{a\in\mathcal{B}} (W_{j_a}(\sigma^a_{j_a}) \chi^{\mathcal{B}}_{j_a} \bar{\chi}^{\mathcal{B}}_{j_a}),
$$

where

- the sum over J means $\displaystyle\sum_{j_1=j_{\min}}^{j_{\max}} \cdots \sum_{j_n=j_{\min}}^{j_{\max}}$,

- the sum over $\mathcal{J}$ runs over all two level jungles, hence over all ordered pairs $\mathcal{J} = (\mathcal{F}_B, \mathcal{F}_F)$ of two (each possibly empty) disjoint forests on V, such that $\bar{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ is still a forest on V. The forests $\mathcal{F}_B$ and $\mathcal{F}_F$ are the bosonic and fermionic components of $\mathcal{J}$. The edges of $\mathcal{J}$ are partitioned into bosonic edges ℓ_B and fermionic edges ℓ_F,

- $\int dw_{\mathcal{J}}$ means integration from 0 to 1 over parameters w_ℓ, one for each edge $\ell \in \bar{\mathcal{J}}$. $\int dw_{\mathcal{J}} = \prod_{\ell\in\bar{\mathcal{J}}} \int_0^1 dw_\ell$. A generic integration point $w_{\mathcal{J}}$ is therefore made of $|\bar{\mathcal{J}}|$ parameters $w_\ell \in [0,1]$, one for each $\ell \in \bar{\mathcal{J}}$,

-
$$
\partial_{\mathcal{J}} = \prod_{\substack{\ell_B\in\mathcal{F}_B \\ \ell_B=(c,d)}} \left(\frac{\partial}{\partial \sigma^c_{j_c}} \frac{\partial}{\partial \sigma^d_{j_d}} \right) \prod_{\substack{\ell_F\in\mathcal{F}_F \\ \ell_F=(a,b)}} \delta_{j_a j_b} \left(\frac{\partial}{\partial \bar{\chi}^{\mathcal{B}(a)}_{j_a}} \frac{\partial}{\partial \chi^{\mathcal{B}(b)}_{j_b}} + \frac{\partial}{\partial \bar{\chi}^{\mathcal{B}(b)}_{j_b}} \frac{\partial}{\partial \chi^{\mathcal{B}(a)}_{j_a}} \right),
$$

  where $\mathcal{B}(a)$ denotes the bosonic blocks to which a belongs,

- the measure $d\nu_{\mathcal{J}}$ has covariance $X(w_{\ell_B}) \otimes \mathbf{1}_{\mathcal{S}}$ on bosonic variables and $Y(w_{\ell_F}) \otimes \mathbb{I}_{\mathcal{S}}$ on fermionic variables, hence $\int d\nu_{\mathcal{J}} f$ is the value at $\sigma = \bar{\chi} = \chi = 0$ of

$$
e^{\frac{1}{2}\sum_{a,b=1}^{n} X_{ab}(w_{\ell_B}) \frac{\partial}{\partial \sigma^a_{j_a}} \frac{\partial}{\partial \sigma^b_{j_b}} + \sum_{\mathcal{B},\mathcal{B}'} Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F}) \sum_{a\in\mathcal{B},b\in\mathcal{B}'} \delta_{j_a j_b} \frac{\partial}{\partial \bar{\chi}^{\mathcal{B}}_{j_a}} \frac{\partial}{\partial \chi^{\mathcal{B}'}_{j_b}}} f,
$$

- $X_{ab}(w_{\ell_B})$ is the infimum of the w_{ℓ_B} parameters for all the bosonic edges ℓ_B in the unique path $P^{\mathcal{F}_B}_{a\to b}$ from a to b in $\mathcal{F}_B$. The infimum is set to zero if such a path does not exists and to 1 if $a = b$,

- $Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F})$ is the infimum of the w_{ℓ_F} parameters for all the fermionic edges ℓ_F in any of the paths $P^{\mathcal{F}_B\cup\mathcal{F}_F}_{a\to b}$ from some vertex $a \in \mathcal{B}$ to some vertex $b \in \mathcal{B}'$. The infimum is set to 0 if there are no such paths, and to 1 if such paths exist but do not contain any fermionic edges.

Remember that the symmetric n by n matrix $X_{ab}(w_{\ell_B})$ is positive for any value of $w_{\mathcal{J}}$, hence the Gaussian measure $d\nu_{\mathcal{J}}$ is well-defined. The matrix $Y_{\mathcal{BB'}}(w_{\ell_F})$ is also positive, with all elements between 0 and 1. Since the slice assignments, the fields, the measure and the integrand are now factorized over the connected components of $\bar{\mathcal{J}}$, the logarithm of Z is exactly the same sum but restricted to the two-levels spanning trees:

$$\log Z(\lambda, N) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{J} \int dw_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \left[W_{j_a}(\sigma_{j_a}^a) \chi_{j_a}^{\mathcal{B}} \bar{\chi}_{j_a}^{\mathcal{B}} \right], \tag{3.7}$$

where the sum is the same but conditioned on $\bar{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ being a *spanning tree* on $V = [1, \ldots, n]$. In [32], it is proven in detail that

Theorem 3.2. *The series* (3.7) *is absolutely convergent for* $\lambda \in [-1, 1]$ *uniformly in* $j_{\max}$.

Theorem 3.3. *The series* (3.7) *is absolutely convergent for* $\lambda \in \mathbb{C}$, $\lambda = |\lambda| e^{\iota\gamma}$ *in the domain* $|\lambda|^2 < (\cos 2\gamma)$ *uniformly in* $j_{\max}$.

We sketch below the proof of Theorem 3.2, referring the reader to [32] for details. By Cayley's theorem the number of two level trees over $n \geq 1$ vertices is exactly $2^{n-1} n^{n-2}$.

The Grassmann Gaussian integral evaluates to

$$\left(\prod_{\mathcal{B}} \prod_{\substack{a,b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right) \left(\prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F = (a,b)}} \delta_{j_a j_b} \right) \left(\mathbf{Y}_{\hat{a}_1 \ldots \hat{a}_k}^{\hat{b}_1 \ldots \hat{b}_k} + \mathbf{Y}_{\hat{b}_1 \ldots \hat{a}_k}^{\hat{a}_1 \ldots \hat{b}_k} + \cdots + \mathbf{Y}_{\hat{b}_1 \ldots \hat{b}_k}^{\hat{a}_1 \ldots \hat{a}_k} \right),$$

where the sum runs over the 2^k ways to exchange an a_i and a b_i. Each $\mathbf{Y}_{\hat{b}_1 \ldots \hat{a}_k}^{\hat{a}_1 \ldots \hat{b}_k}$ factor is a determinant of a matrix made of $Y_{\mathcal{BB'}}(w_{\ell_F})$ interpolating factors (see [32] for the precise definition). Its absolute value is therefore bounded by 1 thanks to Hadamard's inequality, because the corresponding matrix is positive with diagonal entries equal to 1.

The bosonic integral is a bit more cumbersome, as one should first evaluate the effect of the bosonic derivatives on the exponential vertex kernels W_j through the Faà di Bruno formula, whose combinatoric is easy to control. It leads to a sum over similar exponential kernels but multiplied by some polynomials.

To bound the remaining bosonic functional integral one first separates the exponential kernels from the polynomials by some Cauchy–Schwarz estimate with respect to the bosonic Gaussian measure. The exponential terms being positive, the corresponding piece is bounded by 1. The polynomial piece is then explicitly evaluated. This generates a dangerous product of local factorials of the number of fields in the bosonic blocks, but allows also a good factor M^{-j} from the propagator of scale j for each occupied bosonic scale j.

But here comes the key point. The Grassmann Gaussian integrals ensure that the occupied scales in any bosonic block of the first forest formula are all *distinct*. Therefore the good factor collected from the propagator easily beats the local factorials. The worst case is indeed when the p occupied scales in the block are lowest, in which case $\prod_{j=1}^{p} M^{-j} = M^{-p(p+1)/2}$ which easily beats $p!$.

For just renormalizable theories it is not so easy to beat the dangerous factors by the decay of the propagators, and the constructive expansion must proceed even more carefully, essentially expanding the functional integral in each scale in a much more detailed way.

3.3   Iterated Cauchy–Schwarz bounds

A Cauchy Schwarz bound requires to decompose an expression as a scalar product between two halves, $I = \langle S_1.S_2 \rangle \implies |I| \leq \langle S_1.S_1 \rangle^{1/2} \langle S_2.S_2 \rangle^{1/2}$. A key problem in the LVE is to bound

the resolvents still present in the tree amplitudes by using their uniform bound in norm (in a cardioid domain of the coupling constant). The idea of ICS bounds is to get rid of resolvents inductively, by applying many times this inequality. Typically at any inductive step a resolvent is sandwiched in the scalar product between S_1 and S_2 and disappears in the right hand side of the bound (thanks to the bound on its norm), resulting in tree amplitudes with at least one resolvent less.

ICS bounds for a general quartic tensor model were introduced in [18], but we describe here a new slightly different version. More precisely in [18] a tree amplitude of the LVE is decomposed in two halves by identifying a pair of opposite resolvents. However in this process the two halves may not have the same number of vertices, and the bound obtained can involve new trees of different orders. Here we shall describe how to work at fixed order by pairing a resolvent with its opposite corner in the tree, so that the two halves A and B remain of the same order.

Consider a tensor model with quartic interactions for generalized colors $\mathcal{C} \in \mathcal{D}$. Let N be the range for each color index. The perturbative amplitudes at order n, are simply $\lambda^n N^{F-(D-1)n}$, where F is the number of *faces* of the graph.

Among the graphs of the IFR are in particular the trees. A planar colored tree of order n, is a tree joining $n + 1$ loop vertices, in which each edge bears a generalized color $\mathcal{C}$, corresponding to the subset $\mathcal{C}$ of indices of the quartic invariant for $\mathcal{C}$. Such a tree has therefore exactly $2n$ *corners*, and an amplitude $N^{F-(d-1)n}$.

However in an LVE or MLVE a typical term is indexed by such planar colored trees but in which some corners of the tree can bear resolvents operators R instead of ordinary propagators 1. The goal of the ICBS bounds is to prove that in a cardioid domain of the coupling constant, the amplitude for any such tree is bounded (uniformly in σ) by the supremum over all trees at the same order n of the same amplitude but of the ordinary perturbative type, that is without any resolvent.

A resolvent-dressed tree is a pair $(T, A, A^\dagger)$ made of a tree T and a subset S of the corners of T. Its amplitude is

$$A_T^{A,A^\dagger} = \operatorname{Tr} \prod_{a \in A} C^{1/2} R C^{1/2} \prod_{a \in A^\dagger} C^{1/2} R^\dagger C^{1/2} \prod_{a \notin A \cup A^\dagger} C \prod_e \delta_{\mathcal{C}_e},$$

where the product over e runs over all edges of the tree, the edge factors are δ functions

$$\delta_{\mathcal{C}_e} = \prod_{c \in \mathcal{C}_e} \delta_{i_c, j_c} \delta_{i'_c j'_c},$$

where the operator $\delta_{i_c, j_c} \delta_{i'_c j'_c}$ joins the color indices i_c and j_c of the four corners touched by edge e, and finally the trace means that indices have to be summed over all faces of the tree.

We assume that in a cardioid domain of the complex plane for the coupling constant a uniform N and σ-independent bound for the resolvent holds:

$$\|R\| \leq K.$$

The same independent bound automatically follows for the matrix coefficients of the resolvent

$$|R_{n,\bar{n}}| \leq K \qquad \forall n, \bar{n},$$

since any coefficient of a matrix is bounded in absolute value by its norm.

Iterated Cauchy–Schwarz bounds then allow to bound the absolute value of any resolvent-dressed tree amplitude by the supremum over similar tree amplitudes but without any resolvent for trees T' of the same order than T:

Theorem 3.4. *Under the hypotheses above, any resolvent-dressed tree A_T^S of order n obeys the bound*

$$\left| A_T^{A,A^\dagger} \right| \leq K^{2n} \sup_{T'} A_{T'}^\varnothing.$$

Proof. Consider a dressed tree $(T, A, A^\dagger)$ of order n with $p = |A| + |A^\dagger|$ resolvents. If $p > 0$ we select an arbitrary corner γ containing a resolvent R of T (if the resolvent is of the conjugate type $R^\dagger$ the reasoning is identical). Turning around the tree, there is a unique corner $\tilde{\gamma}$ opposite to γ, that is such that the path from γ to $\tilde{\gamma}$ has $n - 1$ corners whether we turn around the tree clockwise or counterclockwise. If $p = 0$, we select any corner and its opposite pair.

Having selected opposite corners we can divide the tree T into two halves S_1 and S_2, made of the clockwise and counterclockwise paths form γ to $\tilde{\gamma}$. Every edge from S_1 to S_2 is a scalar product because of the form of the δ_e factors. Hence in a certain tensor product Hilbert space of many elements we can write $A_T^{A,A^\dagger}$ as

$$\left| A_T^{A,A^\dagger} \right| = |\langle A_{S_1} R A_{S_2} \rangle| \leq K \langle A_{S_1} A_{S_1}^\dagger \rangle^{1/2} \langle A_{S_2} A_{S_2}^\dagger \rangle^{1/2}.$$

In this way we have bounded the initial tree with p resolvents by the arithmetic mean of two trees with total number of resolvents at most $2(p - 1)$. Iterating this bound for all trees still containing resolvents, we obtain, after r iterations

$$\left| A_T^{A,A^\dagger} \right| \leq \prod K^{k_r} \prod_j |A_{S_j}|^{2^{-r}}.$$

It is not true that in such a process after a finite number of steps all resolvents disappear (counterexamples are easily built). Nevertheless they *rarefy* at each step, and this is enough to complete the proof of Theorem 3.4. Let us define p_r the total number of resolvents contained in all the trees S_j and m_r the number of trees containing at least one resolvent. Since any tree is of order n it can contain at most $2n$ resolvents, hence $m_r \geq p_r / (2n)$. Furthermore $p_{r+1} \leq 2p_r - 2m_r$. Therefore the sequence $q_r = 2^{-r} p_r$ satisfies $q_{r+1} \leq q_r (1 - \frac{1}{2n})$ hence tends to 0 as $r \to \infty$.

Now a rough relatively trivial bound to evaluate a tree with p resolvents consists in paying a full additional factor $\prod_f N^{2q_f}$ for each face f meeting q_f resolvents along the face. Since a resolvent is along at most d different faces we certainly find that for any r

$$\left| A_T^{A,A^\dagger} \right| \leq [f(N)]^p \sup_{T'} A_{T'}^\varnothing$$

for a certain function $f(N)$. Hence by the induction

$$\left| A_T^{A,A^\dagger} \right| \leq [f(N)]^{q_r} \sup_{T'} A_{T'}^\varnothing,$$

and letting $r \to \infty$ completes the proof of Theorem 3.4, since $q_r \to 0$ as $r \to \infty$. ∎

4 Results

We now review the various tensor models for which (uniform) Borel summability has been proved. Rather than summarizing or paraphrasing the original papers, we focus on explaining the reader which of the previous techniques are required for which model and why.

4.1 Melonic quartic models

This is the simplest tensor model, restricted to a quartic melonic interactions, hence the set $\mathcal{Q}$ of generalized colors in (2.1) is restricted to *singletons*. This model is fully treated in [29] via a single loop vertex expansion and does not require any ICS bounds. The LVE graphs in which all resolvents R are replaced by one are exactly the tree graphs of the IFR of the model, and they coincide exactly with the melonic graphs of the initial representation of the tensor model. Hence the LVE automatically computes the leading order of quartic melonic tensor models as the $\sigma = 0$, $R(\sigma) = 1$ approximation of the LVE. Sub leading orders are of course obtained by further expanding the resolvent factors of the LVE.

Why is it that the melonic quartic model does not require the technique of ICS bounds to prove Borel summability? This is essentially because in this case the propagators of the IFR correspond to insertions of the type $1 \otimes \cdots \otimes \sigma_c \otimes \cdots \otimes 1$ which all *commute*.

As a result one can use the parametric representation of the resolvent

$$\left(1 + i\sqrt{\lambda}\sigma\right) = \int_0^\infty e^{-\alpha(1+i\sqrt{\lambda}\sigma)} d\alpha \tag{4.1}$$

and regroup all identical colors. The bound now factorizes over the *subforests of the same color* in the tree and this allows for a constructive bound of the same order than the leading melonic graphs of the model.

4.2 General quartic models

For these models the interactions are no longer restricted to the melonic case. Intermediate fields are matrices corresponding to generalized colors $\mathcal{C}$ which are not necessarily singletons. In particular they no longer commute when the subsets $\mathcal{C}$ and $\mathcal{C}'$ have a non-trivial intersection $\mathcal{C} \cap \mathcal{C}' \neq \varnothing$. This is why the parametric representation (4.1) is not enough to get rid of resolvents in the loop vertex expansion, and ICS bounds have to be used [18].

4.3 $\mathbf{U(1) - T_3^4}$ TGFT

This field theory is built on the U(1) group, for tensors of rank 3, with quartic melonic interactions and a Boulatov-type projector, hence the propagator, in momentum space is

$$C(n) = \frac{\delta(n_1 + n_2 + n_3)}{n_1^2 + n_2^2 + n_3^2 + 1}$$

for $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$.

The delta projector reduces by 1 the effective dimension of the model, hence this model has the same power counting than a rank 2 non-commutative field theory without any Boulatov projector, hence is fully convergent (no ultraviolet divergencies). Therefore it can be built by a single loop vertex expansion [38].

4.4 $\mathbf{U(1) - T_4^4}$ TGFT

This field theory [39] is built on the U(1) group, for tensors of rank 4, with quartic melonic interactions and a Boulatov-type projector, hence the propagator, in momentum space is

$$C(n) = \frac{\delta(n_1 + n_2 + n_3 + n_4)}{n_1^2 + n_2^2 + n_3^2 + n_4^2 + 1}$$

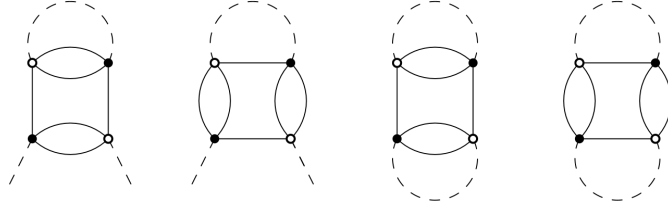for $n = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$.

Figure 7. From left to right, the divergent tadpole, the convergent tadpole and the two vacuum divergent graphs.

The delta projector reduces by 1 the effective dimension of the model, hence this model has the same power counting than a rank 3 quartic tensor model without any Boulatov projector, hence requires a rather mild renormalization.

Therefore it requires a multi-loop vertex expansion. However it does not require any iterated Cauchy–Schwarz bounds, since insertions of different colors commute! This is surprising, given that a priori they do not commute with the C propagator, whose denominators mixes colors. However it is a peculiarity and simplifying feature that the Boulatov projector, when combined with quartic melonic interactions leads to a global momentum conservation rule along all the propagators of a given loop vertex.

Indeed consider a σ insertion of color c in a loop vertex. A priori the n_c index may change into m_c. However the propagators δ functions immediately before and after the insertion enforce that $n_c = - \sum_{c' \neq c} n_{c'}$. Therefore since the $n_{c'}$ are conserved through the σ_c insertions, we obtain that $n_c = m_c$. In fact the random intermediate field matrices σ_c reduce to their diagonal part in momentum space, and the intermediate fields can therefore be considered as vectors rather than matrices [41].

Now this conservation rule along any loop vertex ensures the commutation of all intermediate fields along the loop vertex, hence the argument with the parametric representation of [29] can be adapted to this situation and ICS bounds are not necessary.

4.5 T_3^4 TFT

This super-renormalizable field theory is built on the U(1) group, for tensors of rank 3, with quartic melonic interactions and an ordinary Laplacian-based projector which in momentum space is

$$C(n) = \frac{1}{n_1^2 + n_2^2 + n_3^2 + 1}$$

for $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$.

The model has a power counting almost similar to the one of the ordinary ϕ_2^4 theory. It has for each color c two vacuum divergent graphs, one linearly divergent and one logarithmically divergent. It has also a single logarithmically divergent two-point graph (the "melonic tadpole"), again with a single vertex, which requires a mass renormalization (see Fig. 7).

Therefore the proof of Borel summability for this model requires both a multi-loop vertex expansion and ICS bounds [19]. Indeed insertions of intermediate fields of different colors along a loop vertex no longer need to commute, since they do not commute with the propagators along a given loop vertex. Indeed these propagators may have now different momenta running through them, since there is no longer the conservation rule of the quartic melonic models with Boulatov projectors.

An interesting aspect of the renormalization of this model is that it is performed directly in the IFR representation. Hence the initial $\mathrm{Tr}\log(1+i\sqrt{\lambda}\Sigma)$ interaction is replaced by a subtracted

interaction $\text{Tr} \log_2(1 + i\sqrt{\lambda}\Sigma)$, where $\log_2(1 + x) := -x + \log(1 + x)$. In addition to the ICS bounds to get rid of resolvents in the trees created by the bosonic forest formula, there is also a non-trivial argument to bound the non perturbative part left in the exponential by the MLVE. This non trivial "non-perturbative" bound can be traced to the divergence of the vacuum energy graphs, and can be considered a kind of tensor analog in the IFR representation of the famous Nelson bound for ϕ_2^4, which was the birth act of constructive field theory [45].

4.6   Quartic matrix model

The paper [35] deals with a Gaussian complex random matrix model with a quartic perturbation. Its partition function is defined by the integral over complex $N \times N$ matrices M:

$$Z(\lambda, N) = \int dM e^{-\text{Tr}(MM^\dagger - \frac{\lambda}{2N}MM^\dagger MM^\dagger)}.$$

The model can be shown Borel summable with a single loop vertex expansion but the trees of that expansion form only a part of the planar sector of the theory which is the leading order at large N. To extract this leading sector is possible through an additional expansion called the topological expansion in [35]. Starting form the LVE trees, one can perform a first expansion step on the resolvents, testing the presence of an additional field σ. This field is then contracted with the Gaussian measure, and the process is continued until a first non-planar Wick contraction appear. The part of the expansion for planar graphs does not diverge since the number of planar graphs at order n is bounded by $(\text{const})^n$.

Continuing this additional topological expansion until at most $p + 1$ independent crossing, it is possible to extract the complete terms of the $1/N$ expansion up to order p and to prove that the remainder is still Borel summable, with a uniform bound in $N^{-(p+1)}$.

A similar expansion could be used for tensor models such as those of [11], which mix a melonic sector and a planar sector behavior.

5   Conclusion: open questions

Super-renormalizable and just renormalizable tensor field theories form a rich world [5]. A key physical issue is to move towards realistic quantum gravity models, and to better connect the "tensor field theory" approach to other approaches to quantum gravity. In particular understanding which tensor models admit a realistic emergent four dimensional (Euclidean) space time with a real time analytic continuation seems to us the key problem. However we shall not discuss this issue here, addressing the reader to our twin review [49] and focus here only on a brief discussion of the next technical steps of the constructive tensor program.

The next obvious step in this program is to build super-renormalizable theories with power counting of the same level than ordinary ϕ_3^4, namely the T_4^4 model with propagator $(p^2 + 1)^{-1}$. This is well under way and it seems that it can be treated like T_3^4 by just the same 2-jungle formula of the MLVE and ICS bounds; but the list of divergent graphs is much longer (there are 12 primitively divergent graphs) and the $\text{Tr} \log$ interaction should therefore be pushed further, resulting in particular in much more complicated non perturbative estimates compared to T_3^4 [51].

The model $\text{U}(1) - T_5^4$ with Boulatov projectors has a comparable finite amount of divergent graphs. It should be easier than T_4^4, since intermediate fields can be represented by diagonal matrices in the momentum basis.

The next, more difficult step is the construction of a just renormalizable model such as the T_5^4 model or the $\text{U}(1) - T_6^4$ model with Boulatov projector [41, 55, 56]. Both models are asymptotically free [5, 48, 54], hence the construction should be possible but it is not clear

whether it can be done with the same tools or if a more advanced expansion (such as a multiscale loop vertex expansion with higher-order jungle formulas) will be required.

Another important direction is to treat models with higher-order interactions, for instance the just renormalizable models with six-order melonic interactions introduced in [7] and [14]. Here the starting point might be the intermediate field representation introduced in [52] and improved in [42]. Beware however that even in zero dimension we do not know yet how to perform a loop vertex expansion in this representation [42].

We would like to complete this review by indicating two other challenging research directions for matrix models and non-commutative field theory

- The first direction is to extend the analyticity domain in the coupling constant of quartic models beyond the cardioid characteristically obtained through the LVE. In the case of vector models we already know that this is possible and that the optimal angular domain of analyticity extends to 3π (see Fig. 6 and the last section of [33]). In matrix or tensor models nothing of this type is known.

  This is a key issue for many reasons, in particular to understand the instanton cuts which are responsible for the usual non-perturbative effects of ϕ^4 theory, and how they should disappear in the limit $N \to \infty$, unveiling a new singularity at finite distance on the negative coupling constant which is the one responsible for the single and double scaling limits of quantum gravity in two dimensions, hence for the emergence of continuous surfaces [20].

- Construct non-perturbatively the full non-planar sector of the Grosse–Wulkenhaar model [25]. Although in a certain limit $\theta \to \infty$ of infinite non-commutativity, this sector disappears, again it would be very interesting to understand rigorously how to construct the full theory at finite θ. It would be extremely interesting, as the theory is asymptotically safe, to check that this property persists at the non-perturbative level.

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