

# $N$ -Bright-Dark Soliton Solution to a Semi-Discrete Vector Nonlinear Schrödinger Equation

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**Abstract.** In this paper, a general bright-dark soliton solution in the form of Pfaffian is constructed for an integrable semi-discrete vector NLS equation via Hirota's bilinear method. One- and two-bright-dark soliton solutions are explicitly presented for two-component semi-discrete NLS equation; two-bright-one-dark, and one-bright-two-dark soliton solutions are also given explicitly for three-component semi-discrete NLS equation. The asymptotic behavior is analysed for two-soliton solutions.

*Key words:* bright-dark soliton; semi-discrete vector NLS equation; Hirota's bilinear method; Pfaffian

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## 1 Introduction

The nonlinear Schrödinger (NLS) equation

$$iu_t = u_{xx} + 2\sigma|u|^2u$$

is a generic model equation describing the evolution of small amplitude and slowly varying wave packets in weakly nonlinear media [2, 6, 8, 18, 23]. It arises in a variety of physical contexts such as nonlinear optics [19, 20], Bose–Einstein condensates [14], water waves [13] and plasma physics [48]. The integrability, as well as the bright-soliton solution in the focusing case ( $\sigma = 1$ ), was found by Zakharov and Shabat [49, 50]. The dark soliton was found in the defocusing NLS equation ( $\sigma = -1$ ) [20, 25, 42], and was observed experimentally in [24, 46].

The integrable discretization of nonlinear Schrödinger equation

$$iq_{n,t} = (1 + \sigma|q_n|^2)(q_{n+1} + q_{n-1})$$

was originally derived by Ablowitz and Ladik [3, 4], so it is also called the Ablowitz–Ladik (AL) lattice equation. Similar to the continuous case, it is known that the AL lattice equation, by Hirota's bilinear method, admits the bright soliton solution for the focusing case ( $\sigma = 1$ ) [28, 41], also the dark soliton solution for the defocusing case ( $\sigma = -1$ ) [27]. The inverse scattering transform (IST) has been developed by several authors in the literature [1, 36, 43, 44]. The geometric construction of the AL lattice equation was given by Doliwa and Santini [15].

The coupled nonlinear Schrödinger equation

$$\begin{aligned} iu_t &= u_{xx} + 2(\sigma_1|u|^2 + \sigma_2|v|^2)u, \\ iv_t &= v_{xx} + 2(\sigma_1|u|^2 + \sigma_2|v|^2)v, \end{aligned} \quad (1.1)$$

where  $\sigma_i = \pm 1$ ,  $i = 1, 2$ , was firstly recognized being integrable by Yajima and Oikawa [47]. For the focusing-focusing case ( $\sigma_1 = \sigma_2 = 1$ ), the system (1.1) solved by Manakov via inverse scattering transform (IST), admits the bright-bright soliton solution [26], so it is also called the Manakov system in the literature. For the defocusing-defocusing case, the Manakov system admits bright-dark and dark-dark soliton solution [35, 37, 38]. However, the focusing-defocusing Manakov system admits all types of soliton solutions such as bright-bright solitons, dark-dark soliton, and bright-dark solitons [22, 33, 45]. The Manakov system can be easily extended to a multi-component case, the so-called vector NLS equation. For the continuous vector NLS equation, the  $N$ -bright soliton solution was obtained in [16, 22, 51]; the general bright-dark and dark-dark soliton solutions were obtained in [16, 17, 25, 34, 45]. The inverse scattering transform with nonvanishing boundary condition was solved by Prinari, Ablowitz and Biondini [35]. We should remark here that the problem of constructing exact soliton solutions to the vector NLS equation and proving their nonsingularity was settled by Dubrovin et al. in their landmark paper [16].

The semi-discrete coupled nonlinear Schrödinger equation

$$\begin{aligned} iq_{n,t}^{(1)} &= (1 + \sigma_1|q_n^{(1)}|^2 + \sigma_2|q_n^{(2)}|^2)(q_{n+1}^{(1)} + q_{n-1}^{(1)}), \\ iq_{n,t}^{(2)} &= (1 + \sigma_1|q_n^{(1)}|^2 + \sigma_2|q_n^{(2)}|^2)(q_{n+1}^{(2)} + q_{n-1}^{(2)}), \end{aligned} \quad (1.2)$$

where  $\sigma_i = \pm 1$ ,  $i = 1, 2$ , is of importance both mathematically and physically. It was solved by the inverse scattering transform (IST) in [39, 40]. The general multi-soliton solution in terms of Pfaffians was found by one of the authors recently [31], which is of bright type for the focusing-focusing case ( $\sigma_1 = \sigma_2 = 1$ ), and is of dark type for the defocusing-defocusing case ( $\sigma_1 = \sigma_2 = -1$ ). However, as far as we know, no general mixed-type (bright-dark) soliton solution is reported in the literature, which motivated the present study.

In the present paper, we consider a  $M$ -coupled semi-discrete NLS equation of all types

$$iq_{n,t}^{(j)} = \left(1 + \sum_{\mu=1}^M \sigma_{\mu} |q_n^{(\mu)}|^2\right) (q_{n+1}^{(j)} + q_{n-1}^{(j)}), \quad j = 1, 2, \dots, M, \quad (1.3)$$

where  $\sigma_{\mu} = \pm 1$  for  $\mu = 1, \dots, M$ . For all-focusing case ( $\sigma_{\mu} = 1$ ,  $\mu = 1, \dots, M$ ), its general  $N$ -bright soliton solution and the interactions among solitons were studied in [5, 29]. However, in contrast with a complete list of the general  $N$ -soliton solution to the vector NLS equation [17], the mixed-type soliton solution of all possible nonlinearities (all possible values of  $\sigma_{\mu}$ ) is missing. The aim of the present paper is to construct a general  $N$ -bright-dark soliton solution to the semi-discrete vector NLS equation. The rest of the paper is organized as follows. In Section 2, we provide a general bright-dark soliton solution in terms of Pfaffians to the semi-discrete vector NLS equation (1.3) and give a rigorous proof by the Pfaffian technique [21, 30, 32]. The one- and two-soliton solutions to two-coupled and three-coupled semi-discrete NLS equation are provided explicitly, respectively, in Section 3. We summarize the paper in Section 4 and present asymptotic analysis for two-soliton solution in Appendix A.

## 2 General bright-dark soliton solution to semi-discrete vector NLS equation

Let us consider a general soliton solution consisting of  $m$ -bright solitons and  $(M - m)$ -dark solitons to the semi-discrete vector NLS equation (1.3). To this end, we introduce the following

dependent variable transformations

$$q_n^{(j)} = i^n \frac{g_n^{(j)}}{f_n}, \quad q_n^{(m+l)} = \rho_l (i a_l)^n \frac{h_n^{(l)}}{f_n} e^{\omega_l t}, \quad (2.1)$$

where  $j = 1, \dots, m$ ,  $\omega_l = s(a_l - \bar{a}_l)$ ,  $|a_l| = 1$  with  $\bar{a}_l$  representing the complex conjugate of  $a_l$ ,  $l = 1, \dots, M - m$ . Here,  $f_n$  is a real-valued function, whereas,  $g_n$  and  $h_n$  are complex-valued functions. The transformations convert equation (1.3) into a set of bilinear equations as follows

$$\begin{aligned} D_t g_n^{(j)} \cdot f_n &= s(g_{n+1}^{(j)} f_{n-1} - g_{n-1}^{(j)} f_{n+1}), \quad j = 1, \dots, m, \\ (D_t + \omega_l) h_n^{(l)} \cdot f_n &= s(a_l h_{n+1}^{(l)} f_{n-1} - \bar{a}_l h_{n-1}^{(l)} f_{n+1}), \quad l = 1, \dots, M - m, \\ s f_{n+1} f_{n-1} - f_n^2 &= \sum_{j=1}^m \sigma_j |g_n^{(j)}|^2 + \sum_{l=1}^{M-m} \sigma_{l+m} |\rho_l|^2 |h_n^{(l)}|^2. \end{aligned} \quad (2.2)$$

Here  $s = 1 + \sum_{l=1}^{M-m} \sigma_{l+m} |\rho_l|^2$ .

In what follows, we give a Pfaffian-type solution to the above bilinear equations.

**Theorem 2.1.** *A set of bilinear equations (2.2) admit the following solutions in the form of Pfaffians*

$$\begin{aligned} f_n &= \text{Pf}(a_1, \dots, a_{2N}, b_1, \dots, b_{2N}), \\ g_n^{(j)} &= \text{Pf}(d_0, \beta_j, a_1, \dots, a_{2N}, b_1, \dots, b_{2N}), \quad h_n^{(l)} = \text{Pf}(c_1^{(l)}, \dots, c_{2N}^{(l)}, b_1, \dots, b_{2N}), \end{aligned}$$

with the elements of the Pfaffians defined as follows

$$\begin{aligned} \overline{\text{Pf}(a_j, a_k)_n} &= \frac{p_j - p_k}{p_j p_k - 1} \varphi_j(n) \varphi_k(n), \quad \text{Pf}(d_0, b_j) = \text{Pf}(d_0, \beta_l) = 0, \\ \text{Pf}(c_j^{(l)}, c_k^{(l)})_n &= \frac{p_j - p_k}{p_j p_k - 1} \frac{p_j - a_l}{1 - a_l p_j} \frac{p_k - a_l}{1 - a_l p_k} \varphi_j(n) \varphi_k(n), \\ \text{Pf}(a_j, b_k) &= \delta_{jk}, \quad \text{Pf}(c_j^{(l)}, b_k) = \delta_{jk}, \quad \text{Pf}(d_l, a_k)_n = p_k^l \varphi_k(n), \\ \text{Pf}(b_j, \beta_l) &= \begin{cases} 0, & 1 \leq j \leq N, \\ \alpha_{j-N}^{(l)}, & N + 1 \leq j \leq 2N, \end{cases} \quad \text{Pf}(a_j, \beta_l) = 0, \\ \text{Pf}(b_j, b_k) &= \begin{cases} b_{jk}, & 1 \leq j \leq N, \quad N + 1 \leq k \leq 2N, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

with

$$b_{jk} = \frac{\sum_{l=1}^m \alpha_j^{(l)} \overline{\sigma_l \alpha_{k-N}^{(l)}}}{(p_j - p_k)(p_j p_k - 1) \left( \frac{s}{p_j p_k} - \sum_{l=1}^{M-m} \frac{\sigma_{l+m} |\rho_l|^2 (a_l - \bar{a}_l)^2}{(1 - a_l p_j)(1 - \bar{a}_l p_j)(1 - \bar{a}_l p_k)} \right)},$$

and  $\varphi_j(n) = p_j^n e^{\eta_j}$ ,  $\eta_j = s(p_j - p_j^{-1})t + \eta_{j,0}$  which satisfying  $p_{j+N} = \bar{p}_j$ ,  $\eta_{j+N,0} = \bar{\eta}_{j,0}$ .

**Proof.** It can be shown that

$$\begin{aligned} \frac{d}{dt} f_n &= s \text{Pf}(d_{-1}, d_1, \dots)_n, \quad \text{where} \quad \text{Pf}(d_{-1}, d_1) \equiv 0, \quad \text{Pf}(d_{\pm 1}, b_j) \equiv 0, \\ f_{n+1} &= \text{Pf}(d_0, d_1, \dots)_n, \quad \text{where} \quad \text{Pf}(d_0, d_1) \equiv 1, \end{aligned}$$

$$f_{n-1} = \text{Pf}(d_0, d_{-1}, \dots)_n, \quad \text{where} \quad \text{Pf}(d_0, d_{-1}) \equiv 1,$$

and

$$\begin{aligned} \frac{d}{dt} g_n^{(j)} &= s \text{Pf}(d_0, d_{-1}, d_1, \beta_j, \dots)_n, \\ g_{n+1}^{(j)} &= \text{Pf}(d_1, \beta_j, \dots)_n, \quad g_{n-1}^{(j)} = \text{Pf}(d_{-1}, \beta_j, \dots)_n, \end{aligned}$$

where  $\text{Pf}(d_0, d_1, a_1, \dots, a_{2N}, b_1, \dots, b_{2N})$  is abbreviated by  $\text{Pf}(d_0, d_1, \dots)$ , so as other similar Pfaffians. Thus, an algebraic identity of Pfaffian

$$\begin{aligned} \text{Pf}(d_0, d_{-1}, d_1, \beta_j, \dots)_n \text{Pf}(\dots)_n &= \text{Pf}(d_0, d_{-1}, \dots)_n \text{Pf}(d_1, \beta_j, \dots)_n \\ &\quad - \text{Pf}(d_0, d_1, \dots)_n \text{Pf}(d_{-1}, \beta_j, \dots)_n + \text{Pf}(d_0, \beta_j, \dots)_n \text{Pf}(d_{-1}, d_1, \dots)_n, \end{aligned}$$

together with above Pfaffian relations gives

$$\left( \frac{d}{dt} g_n^{(j)} \right) \times f_n = s g_{n+1}^{(j)} \times f_{n-1} - s g_{n-1}^{(j)} \times f_{n+1} + g_n^{(j)} \times \left( \frac{d}{dt} f_n \right),$$

which is exactly the first bilinear equation. Next we prove the second bilinear equation. It can also be shown that

$$\begin{aligned} h_n^{(l)} &= \text{Pf}(d_0, \bar{d}_0^{(l)}, \dots)_n, \quad h_{n+1}^{(l)} = \bar{a}_l \text{Pf}(d_1, \bar{d}_0^{(l)}, \dots)_n, \quad h_{n-1}^{(l)} = a_l \text{Pf}(d_{-1}, \bar{d}_0^{(l)}, \dots)_n, \\ \left( \frac{d}{dt} + s(a_l - \bar{a}_l) \right) h_n^{(l)} &= s \text{Pf}(d_0, d_{-1}, d_1, \bar{d}_0^{(l)}, \dots)_n, \end{aligned}$$

where

$$\begin{aligned} \text{Pf}(\bar{d}_0^{(l)}, a_j) &= p_j^n \left( \frac{p_j - a_l}{1 - a_l p_j} \right) e^{n_j}, \quad \text{Pf}(\bar{d}_0^{(l)}, b_j) = 0, \quad \text{Pf}(d_0, \bar{d}_0^{(l)}) = 1, \\ \text{Pf}(d_{-1}, \bar{d}_0^{(l)}) &= \bar{a}_l, \quad \text{Pf}(d_1, \bar{d}_0^{(l)}) = a_l. \end{aligned}$$

Therefore, an algebraic identity of Pfaffian

$$\begin{aligned} \text{Pf}(d_0, d_{-1}, d_1, \bar{d}_0^{(l)}, \dots)_n \text{Pf}(\dots)_n &= \text{Pf}(d_0, d_{-1}, \dots)_n \text{Pf}(d_1, \bar{d}_0^{(l)}, \dots)_n \\ &\quad - \text{Pf}(d_0, d_1, \dots)_n \text{Pf}(d_{-1}, \bar{d}_0^{(l)}, \dots)_n + \text{Pf}(d_0, \bar{d}_0^{(l)}, \dots)_n \text{Pf}(d_{-1}, d_1, \dots)_n, \end{aligned}$$

together with above Pfaffian relations gives

$$\left( \frac{d}{dt} + s(a_l - \bar{a}_l) \right) h_n^{(l)} \times f_n = s(a_l h_{n+1}^{(l)} f_{n-1} - \bar{a}_l h_{n-1}^{(l)} f_{n+1}) + h_n^{(l)} \left( \frac{d}{dt} f_n \right),$$

which is nothing but the second bilinear equation. Now let us proceed to the proof of the third bilinear equation. To this end, we need to define

$$\begin{aligned} \text{Pf}(a_j, \bar{\beta}_l) &= 0, \quad \text{Pf}(b_j, \bar{\beta}_l) = \begin{cases} 0, & 1 \leq j \leq N, \\ \alpha_{j-N}^{(l)}, & N+1 \leq j \leq 2N, \end{cases} \\ \text{Pf}(d_0, \bar{d}_0^{(l)}) &= 1, \quad \text{Pf}(\bar{c}_j^{(l)}, \bar{c}_k^{(l)})_n = \frac{p_j - p_k}{p_j p_k - 1} \frac{1 - a_l p_j}{p_j - a_l} \frac{1 - a_l p_k}{p_k - a_l} \varphi_j(n) \varphi_k(n), \\ \text{Pf}(\bar{d}_0^{(l)}, a_j) &= p_j^n \left( \frac{1 - a_l p_j}{p_j - a_l} \right) e^{n_j}, \quad \text{Pf}(\bar{c}_j^{(l)}, b_k) = \delta_{jk}, \quad \text{Pf}(\bar{d}_0^{(l)}, b_j) = 0. \end{aligned}$$

Then from the fact

$$\overline{\text{Pf}(a_j, a_k)} = \text{Pf}(a_{j'}, a_{k'}), \quad \overline{\text{Pf}(b_j, b_k)} = \text{Pf}(b_{j'}, b_{k'}),$$

where  $j' = j + N \bmod(2N)$ ,  $k' = k + N \bmod(2N)$ , we obtain

$$\begin{aligned} \bar{f}_n &= f_n, \quad \bar{g}_n^{(j)} = \text{Pf}(d_0, \bar{\beta}_j, a_1, \dots, a_{2N}, b_1, \dots, b_{2N})_n, \\ \bar{h}_n^{(l)} &= \text{Pf}(\bar{c}_1^{(l)}, \dots, \bar{c}_{2N}^{(l)}, b_1, \dots, b_{2N}) = \text{Pf}(d_0, \bar{d}_0^{(l)}, \dots)_n. \end{aligned}$$

Since

$$f_{n+1} = \text{Pf}(d_0, d_1, \dots)_n, \quad f_{n-1} = \text{Pf}(d_0, d_{-1}, \dots)_n,$$

we then have

$$\begin{aligned} f_{n+1} &= f_n + \sum_{j=1}^{2N} (-1)^{j-1} \text{Pf}(d_1, a_j) \text{Pf}(d_0, \dots, \hat{a}_j, \dots)_n, \\ f_{n-1} &= f_n + \sum_{j=1}^{2N} (-1)^{j-1} \text{Pf}(d_{-1}, a_j) \text{Pf}(d_0, \dots, \hat{a}_j, \dots)_n, \end{aligned}$$

Then we can show

$$\begin{aligned} f_{n+1}f_{n-1} - f_n f_n &= - \sum_{j < k} (-1)^{j+k} \left( p_j + \frac{1}{p_j} - p_k - \frac{1}{p_k} \right) \text{Pf}(b_j, b_k) \\ &\quad \times \text{Pf}(d_0, \dots, \hat{b}_j, \dots) \text{Pf}(d_0, \dots, \hat{b}_k, \dots). \end{aligned}$$

On the other hand, since

$$h_n^{(l)} = \text{Pf}(d_0, \bar{d}_0^{(l)}, \dots)_n, \quad \bar{h}_n^{(l)} = \text{Pf}(d_0, \bar{d}_0^{(l)}, \dots)_n,$$

we have

$$\begin{aligned} h_n^{(l)} &= f_n + \sum_{j=1}^{2N} (-1)^{j-1} \text{Pf}(d_0, a_j)_n \left( \frac{p_j - a_l}{1 - a_l p_j} \right) \text{Pf}(d_0, \dots, \hat{a}_j, \dots)_n, \\ \bar{h}_n^{(l)} &= f_n + \sum_{j=1}^{2N} (-1)^{j-1} \text{Pf}(d_0, a_j)_n \left( \frac{1 - a_l p_j}{p_j - a_l} \right) \text{Pf}(d_0, \dots, \hat{a}_j, \dots)_n. \end{aligned}$$

Similarly, we can show

$$\begin{aligned} |h_n^{(l)}|^2 - f_n f_n &= - \sum_{j < k} (-1)^{j+k} \left( \frac{p_j - a_l}{1 - a_l p_j} + \frac{1 - a_l p_j}{p_j - a_l} - \frac{p_k - a_l}{1 - a_l p_k} - \frac{1 - a_l p_k}{p_k - a_l} \right) \\ &\quad \times \text{Pf}(b_j, b_k) \text{Pf}(d_0, \dots, \hat{b}_j, \dots) \text{Pf}(d_0, \dots, \hat{b}_k, \dots). \end{aligned}$$

Finally, we have

$$\begin{aligned} s(f_{n+1}f_{n-1} - f_n f_n) &- \sum_{l=1}^{M-m} \sigma_{l+m} |\rho_l|^2 (|h_n^{(l)}|^2 - f_n f_n) \\ &= - \sum_{j < k} (-1)^{j+k} \left\{ s \left( p_j + \frac{1}{p_j} - p_k - \frac{1}{p_k} \right) \right. \end{aligned}$$

$$\begin{aligned}
& - \left. \sum_{l=1}^{M-m} \sigma_{l+m} |\rho_l|^2 \left( \frac{p_j - a_l}{1 - a_l p_j} + \frac{1 - a_l p_j}{p_j - a_l} - \frac{p_k - a_l}{1 - a_l p_k} - \frac{1 - a_l p_k}{p_k - a_l} \right) \right\} \\
& \times \text{Pf}(b_j, b_k) \text{Pf}(d_0, \dots, \hat{b}_j, \dots) \text{Pf}(d_0, \dots, \hat{b}_k, \dots) \\
& = \sum_{j=1}^N \sum_{k=N+1}^{2N} (-1)^{j+k} \sum_{l=1}^m \alpha_j^{(l)} \overline{\sigma_l \alpha_{k-N}^{(l)}} \text{Pf}(d_0, \dots, \hat{b}_j, \dots) \text{Pf}(d_0, \dots, \hat{b}_k, \dots) \\
& = \sum_{l=1}^m \sigma_l \sum_{j=1}^N \sum_{k=N+1}^{2N} (-1)^{j+k} \text{Pf}(b_j, \beta_l) \text{Pf}(b_k, \bar{\beta}_l) \text{Pf}(d_0, \dots, \hat{b}_j, \dots) \text{Pf}(d_0, \dots, \hat{b}_k, \dots) \\
& = \sum_{l=1}^m \sigma_l |g_n^{(l)}|^2.
\end{aligned}$$

The third bilinear equation is proved. ■

The above Pfaffian solutions, with dependent variable transformations (2.1), give general  $N$ -bright-dark soliton solutions to the semi-discrete vector NLS equation (1.3).

### 3 One- and two-soliton solutions for the two- and three-coupled discrete NLS equation

#### 3.1 Two-component semi-discrete NLS equation

In this subsection, we provide and illustrate one- and two-soliton for two-component semi-discrete NLS equation (1.2) explicitly.

**One-soliton solution.** the tau-functions for one-soliton solution ( $N = 1$ ) are

$$\begin{aligned}
f_n &= -1 - c_{1\bar{1}} (p_1 \bar{p}_1)^n e^{\eta_1 + \bar{\eta}_1}, \\
g_n^{(1)} &= -\alpha_1^{(1)} p_1^n e^{\eta_1}, \quad h_n^{(1)} = -1 - d_{1\bar{1}}^{(1)} (p_1 \bar{p}_1)^n e^{\eta_1 + \bar{\eta}_1},
\end{aligned}$$

where

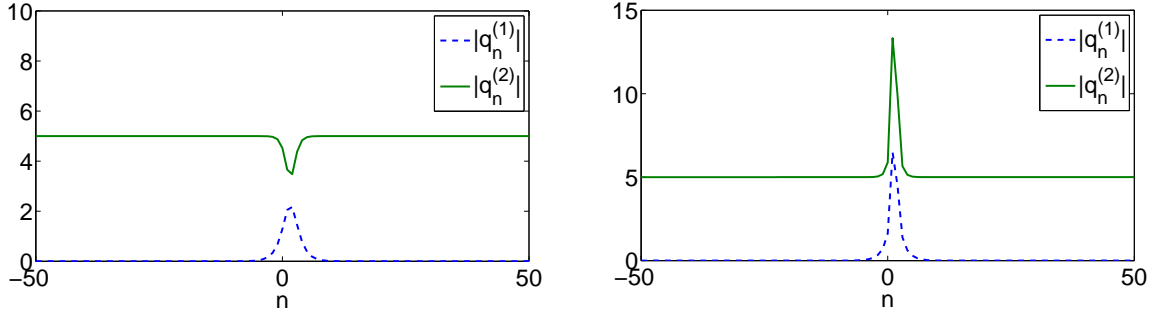
$$c_{1\bar{1}} = \frac{\bar{\alpha}_1^{(1)} \sigma_1 \alpha_1^{(1)}}{(p_1 \bar{p}_1 - 1)^2 \left( \frac{s}{|p_1|^2} + \frac{\sigma_2 |\rho_1|^2 (a_1 - \bar{a}_1)^2}{|1 - a_1 p_1|^2 |1 - \bar{a}_1 \bar{p}_1|^2} \right)}, \quad d_{1\bar{1}}^{(1)} = -\frac{(p_1 - a_1)(\bar{p}_1 - a_1)}{(1 - a_1 p_1)(1 - a_1 \bar{p}_1)} c_{1\bar{1}}.$$

The above tau functions lead to the one-soliton solution as follows

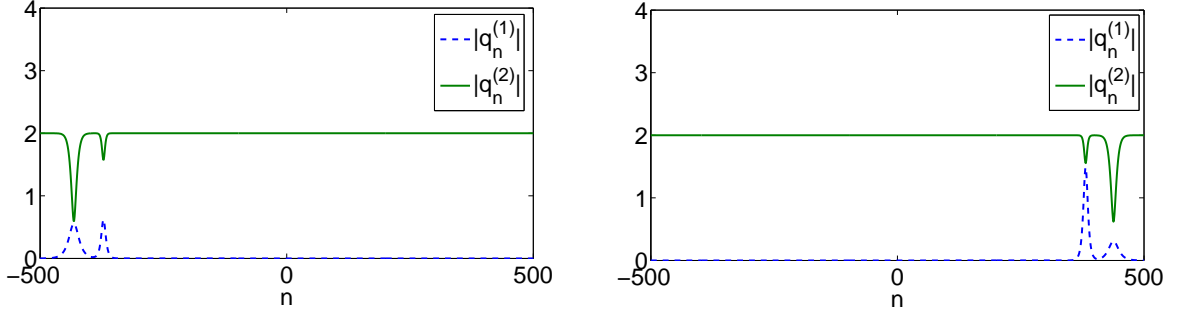
$$\begin{aligned}
q_n^{(1)} &= \frac{\alpha_1^{(1)}}{2\sqrt{c_{1\bar{1}}}} e^{i\xi_{1I}} \text{sech}(\xi_{1R} + \theta_0), \\
q_n^{(2)} &= \frac{1}{2} \rho_1 e^{i\zeta_1} (1 + e^{2i\phi_1} + (e^{2i\phi_1} - 1) \tanh(\xi_{1R} + \theta_0)),
\end{aligned}$$

where  $\xi_1 = \xi_{1R} + i\xi_{1I} = n \ln(ip_1) + s(p_1 - p_1^{-1})t$ ,  $\zeta_1 = n\varphi_1 + n\pi/2 + \omega_1 t$ ,  $e^{i\varphi_1} = a_1$ ,  $e^{2\theta_0} = c_{1\bar{1}}$ ,  $e^{2i\phi_1} = -(p_1 - a_1)(\bar{p}_1 - a_1)/((1 - a_1 p_1)(1 - a_1 \bar{p}_1))$ . Therefore, the amplitude of bright soliton for  $q^{(1)}$  are  $\frac{1}{2} |\alpha_1^{(1)}| / \sqrt{c_{1\bar{1}}}$ . The dark soliton  $q^{(2)}$  approaches  $|\rho_1|$  as  $x \rightarrow \pm\infty$ . In addition, the intensity of the dark soliton is  $|\rho_1| \cos \phi_1$ .

An example of one-bright-dark soliton is illustrated in Fig. 1a  $p_1 = 1.0 + 0.8i$ ,  $\rho_1 = 5.0$ ,  $\alpha_1^{(1)} = 1.0 + i$ ,  $a_1 = 0.8 + 0.6i$ . for focusing-focusing case ( $\sigma_1 = 1.0$ ,  $\sigma_2 = 1.0$ ). Fig. 1b shows the bright-dark soliton for the focusing-defocusing case ( $\sigma_1 = 1.0$ ,  $\sigma_2 = -1.0$ ). It is interesting to note the dark soliton corresponding to defocusing component becomes an anti-dark one.



**Figure 1.** One-bright-dark soliton solution to a two-coupled semi-discrete NLS equation: (a) focusing-focusing case ( $\sigma_1 = 1.0, \sigma_2 = 1.0$ ); (b) focusing-defocusing case ( $\sigma_1 = 1.0, \sigma_2 = -1.0$ ).



**Figure 2.** A two-soliton solution of mixed type for a two-coupled semi-discrete NLS equation: (a) before the collision  $t = -80$ ; (b) after the collision  $t = 80$ .

**Two-soliton solution.** The tau functions for two-soliton are of the following form

$$\begin{aligned} f_n &= 1 + c_{1\bar{1}}E_1\bar{E}_1 + c_{2\bar{1}}E_2\bar{E}_1 + c_{1\bar{2}}E_1\bar{E}_2 + c_{2\bar{2}}E_2\bar{E}_2 + c_{12\bar{1}\bar{2}}E_1E_2\bar{E}_1\bar{E}_2, \\ g_n &= \alpha_1^{(1)}E_1 + \alpha_2^{(1)}E_2 + c_{12\bar{1}}^{(j)}E_1E_2\bar{E}_1 + c_{12\bar{2}}^{(j)}E_1E_2\bar{E}_2, \\ h_n &= 1 + d_{1\bar{1}}^{(1)}E_1\bar{E}_1 + d_{2\bar{1}}^{(1)}E_2\bar{E}_1 + d_{1\bar{2}}^{(1)}E_1\bar{E}_2 + d_{2\bar{2}}^{(1)}E_2\bar{E}_2 + d_{12\bar{1}\bar{2}}^{(1)}E_1E_2\bar{E}_1\bar{E}_2, \end{aligned}$$

where  $E_j = p_j^n e^{\eta_j}$ ,

$$\begin{aligned} c_{i\bar{j}} &= \frac{\bar{\alpha}_i^{(1)}\sigma_1\alpha_j^{(1)}}{(p_i\bar{p}_j - 1)^2 \left( \frac{s}{p_i\bar{p}_j} + \frac{\sigma_2|\rho_1|^2(a_1 - \bar{a}_1)^2}{|1 - a_1p_i|^2|1 - \bar{a}_1\bar{p}_j|^2} \right)}, & d_{i\bar{j}}^{(1)} &= -\frac{(p_i - a_1)(\bar{p}_j - a_1)}{(1 - a_1p_i)(1 - a_1\bar{p}_j)}c_{i\bar{j}}, \\ c_{12\bar{1}\bar{2}} &= |p_2 - p_1|^2 \left( \frac{c_{1\bar{1}}c_{2\bar{2}}}{(p_1 + \bar{p}_2)(p_2 + \bar{p}_1)} - \frac{c_{1\bar{2}}c_{2\bar{1}}}{(p_1 + \bar{p}_1)(p_2 + \bar{p}_2)} \right), \\ c_{12\bar{j}} &= (p_2 - p_1) \left( \frac{\alpha_2^{(1)}c_{1\bar{j}}}{p_2 + \bar{p}_j} - \frac{\alpha_1^{(1)}c_{2\bar{j}}}{p_1 + \bar{p}_j} \right), \\ d_{12\bar{1}\bar{2}} &= \frac{(p_1 - a_1)(\bar{p}_1 - a_1)(p_2 - a_2)(\bar{p}_1 - a_2)}{(1 - a_1p_1)(1 - a_1\bar{p}_1)(1 - a_1p_2)(1 - a_1\bar{p}_2)}c_{12\bar{1}\bar{2}}. \end{aligned}$$

A two bright-dark soliton solution is shown in Fig. 2 before and after the collision for parameters  $\sigma_1 = 1.0, \sigma_2 = -1, p_1 = 1.0 + 0.5i, \rho = 2.0, \alpha_1^{(1)} = 1.0 + 0.8i, \alpha_2^{(1)} = 1.0 + 0.5i, a_1 = 0.6 + 0.8i$ . It can be seen that the collision is elastic which is the same as for the continuous two-component NLS equation.

### 3.2 Bright-dark soliton solution for three-component semi-discrete NLS equation

In this subsection, we will give the mixed-type soliton solution to the following three-component semi-discrete NLS equation

$$\begin{aligned} iq_{n,t}^{(1)} &= \left(1 + \sum_{k=1}^3 \sigma_k |q_n^{(k)}|^2\right) (q_{n+1}^{(1)} + q_{n-1}^{(1)}), \\ iq_{n,t}^{(2)} &= \left(1 + \sum_{k=1}^3 \sigma_k |q_n^{(k)}|^2\right) (q_{n+1}^{(2)} + q_{n-1}^{(2)}), \\ iq_{n,t}^{(3)} &= \left(1 + \sum_{k=1}^3 \sigma_k |q_n^{(k)}|^2\right) (q_{n+1}^{(3)} + q_{n-1}^{(3)}). \end{aligned}$$

**Two-bright-one-dark soliton solution.** The tau-functions for one-soliton solution ( $N = 1$ ) are

$$f_n = -1 - c_{1\bar{1}}(p_1\bar{p}_1)^n e^{\eta_1 + \bar{\eta}_1}, \quad g_n^{(j)} = -\alpha_1^{(j)} p_1^n e^{\eta_1}, \quad h_n^{(1)} = -1 - d_{1\bar{1}}^{(1)}(p_1\bar{p}_1)^n e^{\eta_1 + \bar{\eta}_1},$$

where

$$c_{1\bar{1}} = \frac{\sum_{k=1}^2 \bar{\alpha}_1^{(k)} \sigma_k \alpha_1^{(k)}}{(p_1\bar{p}_1 - 1)^2 \left( \frac{s}{|p_1|^2} + \frac{\sigma_3 |\rho_1|^2 (a_1 - \bar{a}_1)^2}{|1 - a_1 p_1|^2 |1 - \bar{a}_1 \bar{p}_1|^2} \right)}, \quad d_{1\bar{1}}^{(1)} = -\frac{(p_1 - a_1)(\bar{p}_1 - a_1)}{(1 - a_1 p_1)(1 - a_1 \bar{p}_1)} c_{1\bar{1}},$$

where  $\omega = s(a_1 - \bar{a}_1)$ ,  $|a_1| = 1$ ,  $s = 1 + \sigma_3 |\rho_1|^2$  with  $\bar{a}_1$  representing the complex conjugate of  $a_1$ . The above tau functions lead to the one-soliton as follows

$$\begin{aligned} q_n^{(j)} &= \frac{\alpha_1^{(j)}}{2\sqrt{c_{1\bar{1}}}} e^{i\xi_{1I}} \operatorname{sech}(\xi_{1R} + \theta_0), \quad j = 1, 2 \\ q_n^{(3)} &= \frac{1}{2} \rho_1 e^{i\zeta_1} (1 + e^{2i\phi_1} + (e^{2i\phi_1} - 1) \tanh(\xi_{1R} + \theta_0)), \end{aligned}$$

where  $\xi_1 = \xi_{1R} + i\xi_{1I} = n \ln(ip_1) + s(p_1 - p_1^{-1})t$ ,  $\zeta_1 = n\varphi_1 + n\pi/2 + \omega_1 t$ ,  $e^{i\varphi_1} = a_1$ ,  $e^{2\theta_0} = c_{1\bar{1}}$ ,  $e^{2i\phi_1} = -(p_1 - a_1)(\bar{p}_1 - a_1)/((1 - a_1 p_1)(1 - a_1 \bar{p}_1))$ . Therefore, the amplitude of bright soliton for  $q^{(j)}$  are  $\frac{1}{2} |\alpha_1^{(j)}| / \sqrt{a_{1\bar{1}}}$ . The dark soliton  $q^{(3)}$  approaches  $|\rho_1|$  as  $x \rightarrow \pm\infty$ . In addition, the intensity of the dark soliton is  $|\rho_1| \cos \phi_1$ .

The tau functions for two-soliton are of the following form

$$\begin{aligned} f_n &= 1 + c_{1\bar{1}} E_1 \bar{E}_1 + c_{2\bar{1}} E_2 \bar{E}_1 + c_{1\bar{2}} E_1 \bar{E}_2 + c_{2\bar{2}} E_2 \bar{E}_2 + c_{12\bar{1}\bar{2}} E_1 E_2 \bar{E}_1 \bar{E}_2, \\ g_n^{(j)} &= \alpha_1^{(j)} E_1 + \alpha_2^{(j)} E_2 + c_{12\bar{1}}^{(j)} E_1 E_2 \bar{E}_1 + c_{12\bar{2}}^{(j)} E_1 E_2 \bar{E}_2, \quad j = 1, 2, \\ h_n^{(1)} &= 1 + d_{1\bar{1}}^{(1)} E_1 \bar{E}_1 + d_{2\bar{1}}^{(1)} E_2 \bar{E}_1 + d_{1\bar{2}}^{(1)} E_1 \bar{E}_2 + d_{2\bar{2}}^{(1)} E_2 \bar{E}_2 + d_{12\bar{1}\bar{2}}^{(1)} E_1 E_2 \bar{E}_1 \bar{E}_2, \end{aligned}$$

where

$$\begin{aligned} c_{i\bar{j}} &= \frac{\sum_{k=1}^2 \bar{\alpha}_i^{(k)} \sigma_k \alpha_j^{(k)}}{(p_i \bar{p}_j - 1)^2 \left( \frac{s}{|p_i \bar{p}_j|^2} + \frac{\sigma_3 |\rho_1|^2 (a_1 - \bar{a}_1)^2}{|1 - a_1 p_i|^2 |1 - \bar{a}_1 \bar{p}_j|^2} \right)}, \quad d_{i\bar{j}}^{(1)} = -\frac{(p_i - a_1)(\bar{p}_j - a_1)}{(1 - a_1 p_i)(1 - a_1 \bar{p}_j)} c_{i\bar{j}}, \\ c_{12\bar{1}\bar{2}} &= |p_2 - p_1|^2 \left( \frac{c_{1\bar{1}} c_{2\bar{2}}}{(p_1 + \bar{p}_2)(p_2 + \bar{p}_1)} - \frac{c_{1\bar{2}} c_{2\bar{1}}}{(p_1 + \bar{p}_1)(p_2 + \bar{p}_2)} \right), \end{aligned}$$



$$c_{12\bar{j}} = (p_2 - p_1) \left( \frac{\alpha_2^{(1)} c_{1\bar{j}}}{p_2 + \bar{p}_j} - \frac{\alpha_1^{(1)} c_{2\bar{j}}}{p_1 + \bar{p}_j} \right),$$

$$d_{12\bar{1}\bar{2}}^{(1)} = \frac{(p_1 - a_1)(\bar{p}_1 - a_1)(p_2 - a_2)(\bar{p}_1 - a_2)}{(1 - a_1 p_1)(1 - a_1 \bar{p}_1)(1 - a_1 p_2)(1 - a_1 \bar{p}_2)} c_{12\bar{1}\bar{2}}.$$

It is found that two-bright-one-dark soliton solution given above is nonsingular for any other combinations of nonlinearities if the following quantity

$$\left( \sum_{k=1}^2 \bar{\alpha}_i^{(k)} \sigma_k \alpha_j^{(k)} \right) \left( \frac{s}{|p_i|^2} + \frac{\sigma_3 |\rho_1|^2 (a_1 - \bar{a}_1)^2}{|1 - a_1 p_i|^2 |1 - \bar{a}_1 p_i|^2} \right)$$

is positive.

**One-bright-two-dark soliton solution.** The tau-functions for one-soliton solution ( $N = 1$ ) are

$$f_n = -1 - c_{1\bar{1}} (p_1 \bar{p}_1)^n e^{\eta_1 + \bar{\eta}_1}, \quad g_n^{(1)} = -\alpha_1^{(1)} p_1^n e^{\eta_1}, \quad h_n^{(l)} = -1 - d_{1\bar{1}}^{(l)} (p_1 \bar{p}_1)^n e^{\eta_1 + \bar{\eta}_1},$$

where

$$c_{1\bar{1}} = \frac{\bar{\alpha}_1^{(1)} \sigma_1 \alpha_1^{(1)}}{(p_1 \bar{p}_1 - 1)^2 \left( \frac{s}{|p_1|^2} + \frac{\sum_{l=1}^2 \sigma_{l+1} |\rho_l|^2 (a_l - \bar{a}_l)^2}{|1 - a_1 p_1|^2 |1 - \bar{a}_1 p_1|^2} \right)}.$$

Here  $\omega_l = s(a_l - \bar{a}_l)$ ,  $|a_l| = 1$ ,  $s = 1 + \sigma_2 |\rho_1|^2 + \sigma_3 |\rho_2|^2$ ,  $\bar{a}_l$  represents the complex conjugate of  $a_l$ . The above tau functions lead to the one-soliton as follows

$$q_n^{(1)} = \frac{\alpha_1^{(1)}}{2} \sqrt{c_{1\bar{1}}} e^{i\xi_{1l}} \operatorname{sech}(\xi_{1R} + \theta_0),$$

$$q_n^{(l+1)} = \frac{1}{2} \rho_l e^{i\zeta_l} (1 + e^{2i\phi_l} + (e^{2i\phi_l} - 1) \tanh(\xi_{1R} + \theta_0)), \quad l = 1, 2,$$

where  $\xi_1 = \xi_{1R} + i\xi_{1l} = n \ln(ip_1) + s(p_1 - p_1^{-1})t$ ,  $\zeta_l = n\varphi_l + n\pi/2 + \omega_l t$ ,  $e^{i\varphi_l} = a_l$ ,  $e^{2\theta_0} = c_{1\bar{1}}$ ,  $e^{2i\phi_l} = -(p_1 - a_l)(\bar{p}_1 - a_l)/((1 - a_1 p_1)(1 - a_1 \bar{p}_1))$ . Therefore, the amplitude of bright soliton for  $q^{(1)}$  are  $\frac{1}{2} |\alpha_1^{(1)}| / \sqrt{a_{1\bar{1}}}$ . The dark soliton  $q^{(l+1)}$  approaches  $|\rho_l|$  as  $x \rightarrow \pm\infty$ . In addition, the intensity of the dark soliton is  $|\rho_l| \cos \phi_l$ .

The tau functions for two-soliton are of the following form

$$f_n = 1 + c_{1\bar{1}} E_1 \bar{E}_1 + c_{2\bar{1}} E_2 \bar{E}_1 + c_{1\bar{2}} E_1 \bar{E}_2 + c_{2\bar{2}} E_2 \bar{E}_2 + c_{12\bar{1}\bar{2}} E_1 E_2 \bar{E}_1 \bar{E}_2,$$

$$g_n^{(1)} = \alpha_1^{(1)} E_1 + \alpha_2^{(1)} E_2 + c_{12\bar{1}}^{(1)} E_1 E_2 \bar{E}_1 + c_{12\bar{2}}^{(1)} E_1 E_2 \bar{E}_2,$$

$$h_n^{(l)} = 1 + d_{1\bar{1}}^{(l)} E_1 \bar{E}_1 + d_{2\bar{1}}^{(l)} E_2 \bar{E}_1 + d_{1\bar{2}}^{(l)} E_1 \bar{E}_2 + d_{2\bar{2}}^{(l)} E_2 \bar{E}_2 + d_{12\bar{1}\bar{2}}^{(l)} E_1 E_2 \bar{E}_1 \bar{E}_2, \quad l = 1, 2,$$

where

$$c_{i\bar{j}} = \frac{\bar{\alpha}_i^{(1)} \sigma_1 \alpha_j^{(1)}}{(p_i \bar{p}_j - 1)^2 \left( \frac{s}{p_i \bar{p}_j} + \frac{\sum_{l=1}^2 \sigma_{l+1} |\rho_l|^2 (a_l - \bar{a}_l)^2}{|1 - a_l p_i|^2 |1 - \bar{a}_l p_j|^2} \right)}, \quad d_{i\bar{j}}^{(l)} = -\frac{(p_i - a_l)(\bar{p}_j - a_l)}{(1 - a_l p_i)(1 - a_l \bar{p}_j)} c_{i\bar{j}},$$

$$c_{12\bar{1}\bar{2}} = |p_2 - p_1|^2 \left( \frac{c_{1\bar{1}} c_{2\bar{2}}}{(p_1 + \bar{p}_2)(p_2 + \bar{p}_1)} - \frac{c_{1\bar{2}} c_{2\bar{1}}}{(p_1 + \bar{p}_1)(p_2 + \bar{p}_2)} \right),$$

$$c_{12\bar{j}} = (p_2 - p_1) \left( \frac{\alpha_2^{(1)} c_{1\bar{j}}}{p_2 + \bar{p}_j} - \frac{\alpha_1^{(1)} c_{2\bar{j}}}{p_1 + \bar{p}_j} \right),$$

$$d_{12\bar{1}\bar{2}}^{(l)} = \frac{(p_1 - a_l)(\bar{p}_1 - a_l)(p_2 - a_l)(\bar{p}_1 - a_l)}{(1 - a_l p_1)(1 - a_l \bar{p}_1)(1 - a_l p_2)(1 - a_l \bar{p}_2)} c_{12\bar{1}\bar{2}}.$$

For all possible combinations of mixed type in three-coupled NLS equation, one-bright-two-dark soliton solution exists if the following quantity

$$(\bar{\alpha}_i^{(1)} \sigma_1 \alpha_i^{(1)}) \left( \frac{s}{|p_i|^2} + \frac{\sum_{l=1}^2 \sigma_{l+1} |\rho_l|^2 (a_l - \bar{a}_l)^2}{|1 - a_l p_i|^2 |1 - \bar{a}_l p_i|^2} \right)$$

is positive.

The asymptotic analysis for two-soliton solution is performed in Appendix A. It should be pointed out that two-soliton for one-bright-two-dark soliton case always undertakes elastic collision without shape changing.

## 4 Discussion and conclusion

We conclude the present paper by two comments. First, we comment on a connection of the vector semi-discrete NLS equation to the vector modified Volterra lattice equation studied in [7]. To this end, we consider the two-component semi-discrete NLS equation of focusing type

$$\begin{aligned} i \frac{d}{dt} u_n &= (1 + |u_n|^2 + |v_n|^2)(u_{n+1} + u_{n-1}), \\ i \frac{d}{dt} v_n &= (1 + |u_n|^2 + |v_n|^2)(v_{n+1} + v_{n-1}). \end{aligned} \quad (4.1)$$

Let  $u_n = x_n + iy_n$  and  $v_n = z_n + iw_n$ , then equation (4.1) becomes

$$\begin{aligned} \frac{d}{dt} x_n &= (1 + x_n^2 + y_n^2 + z_n^2 + w_n^2)(y_{n+1} + y_{n-1}), \\ -\frac{d}{dt} y_n &= (1 + x_n^2 + y_n^2 + z_n^2 + w_n^2)(x_{n+1} + x_{n-1}), \\ \frac{d}{dt} z_n &= (1 + x_n^2 + y_n^2 + z_n^2 + w_n^2)(w_{n+1} + w_{n-1}), \\ -\frac{d}{dt} w_n &= (1 + x_n^2 + y_n^2 + z_n^2 + w_n^2)(z_{n+1} + z_{n-1}). \end{aligned}$$

By defining

$$\begin{aligned} U_n^{(1)} &= \begin{cases} x_n, & \text{for } n \text{ even,} \\ y_n, & \text{for } n \text{ odd,} \end{cases} & U_n^{(2)} &= \begin{cases} -y_n, & \text{for } n \text{ even,} \\ x_n, & \text{for } n \text{ odd,} \end{cases} \\ U_n^{(3)} &= \begin{cases} z_n, & \text{for } n \text{ even,} \\ w_n, & \text{for } n \text{ odd,} \end{cases} & U_n^{(4)} &= \begin{cases} -w_n, & \text{for } n \text{ even,} \\ z_n, & \text{for } n \text{ odd,} \end{cases} \\ U_n^{(5)} &= \begin{cases} (-1)^{n/2}, & \text{for } n \text{ even,} \\ (-1)^{(n-1)/2}, & \text{for } n \text{ odd,} \end{cases} \end{aligned}$$

we obtain

$$\frac{d}{dt} U_n^{(1)} = ((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(1)} + U_{n-1}^{(1)}),$$

$$\begin{aligned}\frac{d}{dt}U_n^{(2)} &= ((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(2)} + U_{n-1}^{(2)}), \\ \frac{d}{dt}U_n^{(3)} &= ((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(3)} + U_{n-1}^{(3)}), \\ \frac{d}{dt}U_n^{(4)} &= ((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(4)} + U_{n-1}^{(4)}), \\ \frac{d}{dt}U_n^{(5)} &= ((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(5)} + U_{n-1}^{(5)})\end{aligned}$$

for  $n$  being even, and

$$\begin{aligned}\frac{d}{dt}U_n^{(1)} &= -((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(1)} + U_{n-1}^{(1)}), \\ \frac{d}{dt}U_n^{(2)} &= -((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(2)} + U_{n-1}^{(2)}), \\ \frac{d}{dt}U_n^{(3)} &= -((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(3)} + U_{n-1}^{(3)}), \\ \frac{d}{dt}U_n^{(4)} &= -((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(4)} + U_{n-1}^{(4)}), \\ \frac{d}{dt}U_n^{(5)} &= -((U_n^{(1)})^2 + (U_n^{(2)})^2 + (U_n^{(3)})^2 + (U_n^{(4)})^2 + (U_n^{(5)})^2)(U_{n+1}^{(5)} + U_{n-1}^{(5)})\end{aligned}$$

for  $n$  being odd, in other words,

$$\frac{d}{dt}U_n^{(j)} = (-1)^n \left( \sum_{k=1}^5 (U_n^{(k)})^2 \right) (U_{n+1}^{(j)} + U_{n-1}^{(j)})$$

for  $1 \leq j \leq 5$ . By rewriting  $U_n^{(j)} = i^n V_n^{(j)}$  and  $t = x/i$ , we have

$$\frac{d}{dx}V_n^{(j)} = \left( \sum_{k=1}^5 (V_n^{(k)})^2 \right) (V_{n+1}^{(j)} - V_{n-1}^{(j)}).$$

Consequently,

$$\begin{aligned}u_n &= \begin{cases} U_n^{(1)} - iU_n^{(2)}, & n \text{ even,} \\ U_n^{(2)} + iU_n^{(1)}, & n \text{ odd} \end{cases} = \begin{cases} i^{n-1}(V_n^{(2)} + iV_n^{(1)}), & n \text{ even,} \\ i^n(V_n^{(2)} + iV_n^{(1)}), & n \text{ odd,} \end{cases} \\ v_n &= \begin{cases} U_n^{(3)} - iU_n^{(4)}, & n \text{ even,} \\ U_n^{(4)} + iU_n^{(3)}, & n \text{ odd} \end{cases} = \begin{cases} i^{n-1}(V_n^{(4)} + iV_n^{(3)}), & n \text{ even,} \\ i^n(V_n^{(4)} + iV_n^{(3)}), & n \text{ odd.} \end{cases}\end{aligned}$$

$u_n$  and  $u_n^*$  correspond to  $U_n^{(1)}$  and  $U_n^{(2)}$ , and  $v_n$  and  $v_n^*$  correspond to  $U_n^{(3)}$  and  $U_n^{(4)}$ , with even-odd parity depending gauge factor  $i^{(n \text{ or } n-1)}$ .

The correspondence between coupled defocusing-defocusing and focusing-defocusing coupled Ablowitz–Ladik equation and the vector modified Volterra lattice equation can be constructed by similar variable transformations. In all cases, we need 5-components in the vector modified Volterra lattice, one of which is a trivial wave  $V_n^{(5)} = i^{(0 \text{ or } 1)}$ , in order to recover 2-component coupled Ablowitz–Ladik equation. In principle any solutions of coupled Ablowitz–Ladik equation can be derived from those of vector modified Volterra lattice and vice versa. However it is not so easy to make exact matching.

Second, we give a comparison between the NLS-type equations and the sine/sinh-Gordon-type equations since their belong to the simplest positive and negative flows of the AKNS hierarchy, respectively. In a series papers by Barashenlov, Getmanov et al. [9, 10, 11, 12],

a generic integrable relativistic system associated with the  $\mathfrak{sl}(2, \mathbb{C})$  was systematically investigated, from which the massive Thirring model, the complex sine-Gordon equation in Euclidean and Minkowski spaces etc. are produced by different reductions. Especially an  $O(1, 1)$  sine-Gordon equation with the Lagrangian

$$L = \frac{u_1 \xi u_{1\eta} - u_2 \xi u_{2\eta}}{1 - (u_1^2 - u_2^2)} + (u_1^2 - u_2^2 - 1),$$

exhibits nontrivial interaction of solitons such as decay and fusion [10]. Here  $u_1$  and  $u_2$  are real variables. In parallel to the  $O(1, 1)$  sine-Gordon equation, we can have a NLS equation of  $O(1, 1)$  type, whose Lagrangian is

$$L = u_1 u_{2t} - u_{1t} u_2 + (u_{1x}^2 - u_{2x}^2) - \sigma (u_1^2 - u_2^2)^2.$$

On the contrary, in parallel to the  $U(1, 1)$  coupled NLS system, e.g., system (1.1) with  $\sigma = 1$  and  $\sigma = -1$ , whose Lagrangian can be written by

$$L = \frac{i}{2} (u_t u^* - u^* u_t - v_t v^* + v^* v_t) + (|u_x|^2 - |v_x|^2) - \sigma (|u|^2 - |v|^2)^2,$$

where  $u$  and  $v$  are complex variable and  $*$  represents complex conjugate, we could propose a  $U(1, 1)$  coupled complex sine-Gordon system with Lagrangian

$$L = \frac{u_\eta u_\xi^* - v_\eta v_\xi^*}{1 - (|u|^2 - |v|^2)} - (|u|^2 - |v|^2).$$

Then several natural questions arise: does the NLS equation of  $O(1, 1)$  type exhibit nontrivial interaction of solitons such as decay and fusion? Are there any nontrivial interaction of solitons for the  $U(1, 1)$  coupled NLS system and the  $U(1, 1)$  complex sine-Gordon system? Unfortunately, the answers to these questions are not clear at this moment. We expect above questions could be answered by the authors or others in the near future.

## A Appendix

By taking  $N = 1$  we get the tau functions for one-soliton solution,

$$\begin{aligned} f_n &= \text{Pf}(a_1, a_2, b_1, b_2) = -1 - c_{1\bar{1}} E_1 \bar{E}_1, \\ g_n^{(j)} &= \text{Pf}(d_0, \beta_j, a_1, a_2, b_1, b_2) = -\alpha_1^{(1)} E_1, \quad j = 1, \dots, m, \\ h_n^{(l)} &= \text{Pf}(c_1^{(l)}, c_2^{(l)}, b_1, b_2) = -1 - d_{1\bar{1}}^{(l)} E_1 \bar{E}_1, \quad l = 1, \dots, M - m, \end{aligned}$$

where

$$c_{1\bar{1}} = \frac{\sum_{k=1}^m \bar{\alpha}_1^{(k)} \sigma_k \alpha_1^{(k)}}{(p_1 \bar{p}_1 - 1)^2 \left( \frac{s}{|p_1|^2} + \sum_{l=1}^{M-m} \frac{|\rho_l|^2 (a_l - \bar{a}_l)^2}{|1 - a_l p_1|^2 |1 - \bar{a}_l \bar{p}_1|^2} \right)}, \quad d_{1\bar{1}}^{(l)} = -\frac{(p_1 - a_l)(\bar{p}_1 - \bar{a}_l)}{(1 - a_l p_1)(1 - \bar{a}_l \bar{p}_1)} c_{1\bar{1}}.$$

Based on the  $N$ -soliton solution of the vector discrete NLS equation, the tau-functions for two-soliton solution can be expanded for  $N = 2$

$$\begin{aligned} f &= \text{Pf}(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \\ &= 1 + c_{1\bar{1}} E_1 \bar{E}_1 + c_{2\bar{1}} E_2 \bar{E}_1 + c_{1\bar{2}} E_1 \bar{E}_2 + c_{2\bar{2}} E_2 \bar{E}_2 + c_{12\bar{1}\bar{2}} E_1 E_2 \bar{E}_1 \bar{E}_2, \end{aligned}$$

$$\begin{aligned}
 g_n^{(j)} &= \text{Pf}(d_0, \beta_j, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \\
 &= \alpha_1^{(j)} E_1 + \alpha_2^{(j)} E_2 + c_{12\bar{1}}^{(j)} E_1 E_2 \bar{E}_1 + c_{12\bar{2}}^{(j)} E_1 E_2 \bar{E}_2, \quad j = 1, \dots, m, \\
 h_n^{(l)} &= \text{Pf}(c_1^{(l)}, c_2^{(l)}, c_3^{(l)}, c_4^{(l)}, b_1, b_2, b_3, b_4) = 1 + d_{1\bar{1}}^{(l)} E_1 \bar{E}_1 \\
 &\quad + d_{2\bar{1}}^{(l)} E_2 \bar{E}_1 + d_{1\bar{2}}^{(l)} E_1 \bar{E}_2 + d_{2\bar{2}}^{(l)} E_2 \bar{E}_2 + d_{12\bar{1}\bar{2}}^{(l)} E_1 E_2 \bar{E}_1 \bar{E}_2, \quad l = 1, \dots, M - m,
 \end{aligned}$$

where

$$\begin{aligned}
 c_{i\bar{j}} &= \frac{\sum_{k=1}^m \bar{\alpha}_j^{(k)} \sigma_k \alpha_i^{(k)}}{(p_i \bar{p}_j - 1)^2 \left( \frac{s}{p_i \bar{p}_j} + \sum_{l=1}^{M-m} \frac{|\rho_l|^2 (a_l - \bar{a}_l)^2}{|1 - a_l p_i|^2 |1 - \bar{a}_l \bar{p}_j|^2} \right)}, \quad d_{i\bar{j}}^{(l)} = -\frac{(p_i - a_l)(\bar{p}_j - a_l)}{(1 - a_l p_i)(1 - a_l \bar{p}_j)} c_{i\bar{j}}, \\
 c_{12\bar{1}\bar{2}} &= |P_{12}|^2 (P_{1\bar{1}} P_{2\bar{2}} c_{1\bar{2}} c_{2\bar{1}} - P_{1\bar{2}} P_{2\bar{1}} c_{1\bar{1}} c_{2\bar{2}}), \quad c_{12\bar{j}}^{(k)} = P_{12} (\alpha_1^{(k)} P_{1\bar{1}} c_{2\bar{j}} - \alpha_2^{(k)} P_{2\bar{1}} c_{1\bar{j}}), \\
 d_{12\bar{1}\bar{2}}^{(l)} &= \frac{(p_1 - a_l)(\bar{p}_1 - a_l)(p_2 - a_l)(\bar{p}_2 - a_l)}{(1 - a_l p_1)(1 - a_l \bar{p}_1)(1 - a_l p_2)(1 - a_l \bar{p}_2)} c_{12\bar{1}\bar{2}}, \\
 P_{ij} &= \frac{p_i - p_j}{p_i p_j - 1}, \quad P_{i\bar{j}} = \frac{p_i - \bar{p}_j}{p_i \bar{p}_j - 1}.
 \end{aligned}$$

Next, we investigate the asymptotic behavior of two-soliton solution. To this end, we assume  $(\text{Re} \ln(ip_2)) > \text{Re}(\ln(ip_1)) > 0$ ,  $s \text{Re}(p_2 - p_2^{-1}) / \text{Re}(\ln(ip_2)) > s \text{Re}(p_1 - p_1^{-1}) / \text{Re}(\ln(ip_1))$  without loss of generality. For the above choice of parameters, we have (i)  $\eta_{1R} \approx 0$ ,  $\eta_{2R} \rightarrow \mp\infty$  as  $t \rightarrow \mp\infty$  for soliton 1 and (ii)  $\eta_{2R} \approx 0$ ,  $\eta_{1R} \rightarrow \pm\infty$  as  $t \rightarrow \mp\infty$  for soliton 2. This leads to the following asymptotic forms for two-soliton solution.

(i) Before collision ( $t \rightarrow -\infty$ ). Soliton 1 ( $\eta_{1R} \approx 0$ ,  $\eta_{2R} \rightarrow -\infty$ ):

$$\begin{aligned}
 q_n^{(j)} &\rightarrow \alpha_1^{(j)} \frac{i^n E_1}{1 + c_{1\bar{1}} E_1 \bar{E}_1} \rightarrow A_j^{1-} e^{i\xi_{1l}} \text{sech}(\xi_{1R} + \xi_0^{1-}), \\
 q_n^{(l+m)} &\rightarrow \frac{1 + d_{1\bar{1}}^{(l)} E_1 \bar{E}_1}{1 + c_{1\bar{1}} E_1 \bar{E}_1} \rho_l (i a_l)^n e^{\omega_l t} \\
 &\rightarrow \frac{1}{2} \rho_l e^{i\zeta_l} (1 + e^{2i\phi_l^{(1)}} + (e^{2i\phi_l^{(1)}} - 1) \tanh(\xi_{1R} + \xi_0^{1-})),
 \end{aligned}$$

where

$$\begin{aligned}
 A_j^{1-} &= \frac{\alpha_1^{(j)}}{2\sqrt{c_{1\bar{1}}}}, \quad e^{2\xi_0^{1-}} = c_{1\bar{1}}, \\
 e^{2i\phi_l^{(1)}} &= -\frac{(p_1 - a_l)(\bar{p}_1 - a_l)}{(1 - a_l p_1)(1 - a_l \bar{p}_1)}, \quad \zeta_l = n\varphi_l + n\pi/2 + \omega_l t, \\
 \xi_1 &= \xi_{1R} + i\xi_{1l} = n \ln(ip_1) + s(p_1 - p_1^{-1})t, \quad e^{i\varphi_l} = a_l.
 \end{aligned}$$

Soliton 2 ( $\eta_{2R} \approx 0$ ,  $\eta_{1R} \rightarrow \infty$ ):

$$\begin{aligned}
 q_n^{(j)} &\rightarrow \frac{i^n c_{12\bar{1}}^{(j)} E_2}{c_{1\bar{1}} + c_{12\bar{1}\bar{2}} E_2 \bar{E}_2} \rightarrow A_j^{2-} e^{i\xi_{2l}} \text{sech}(\xi_{2R} + \xi_0^{2-}), \\
 q_n^{(l+m)} &\rightarrow \frac{d_{1\bar{1}}^{(l)} + d_{12\bar{1}\bar{2}}^{(l)} E_2 \bar{E}_2}{c_{1\bar{1}} + c_{12\bar{1}\bar{2}} E_2 \bar{E}_2} \rho_l (i a_l)^n e^{\omega_l t} \\
 &\rightarrow \frac{1}{2} \rho_l e^{i(\zeta_l + 2\phi_l^{(1)})} (1 + e^{2i\phi_l^{(2)}} + (e^{2i\phi_l^{(2)}} - 1) \tanh(\xi_{2R} + \xi_0^{2-})),
 \end{aligned}$$

where

$$\begin{aligned} A_j^{2-} &= \frac{c_{12\bar{1}}^{(j)}}{2\sqrt{c_{1\bar{1}}}\sqrt{c_{12\bar{1}\bar{2}}}}, & e^{2\xi_0^{2-}} &= \frac{c_{12\bar{1}\bar{2}}}{c_{1\bar{1}}}, \\ e^{2i\phi_l^{(2)}} &= -\frac{(p_2 - a_l)(\bar{p}_2 - a_l)}{(1 - a_l p_2)(1 - a_l \bar{p}_2)}, & \zeta_0^{2-} &= n\varphi_l + n\pi/2 + \omega_l t, \\ \xi_2 &= \xi_{2R} + i\xi_{2I} = n \ln(ip_2) + s(p_2 - p_2^{-1})t. \end{aligned}$$

(ii) After the collision ( $t \rightarrow \infty$ ). Soliton 1 ( $\eta_{1R} \approx 0$ ,  $\eta_{2R} \rightarrow +\infty$ ):

$$\begin{aligned} q_n^{(j)} &\rightarrow \frac{c_{12\bar{2}}^{(j)} i^n E_1}{c_{2\bar{2}} + c_{12\bar{1}\bar{2}} E_1 \bar{E}_1} \rightarrow A_j^{1+} e^{i\xi_{1I}} \operatorname{sech}(\xi_{1R} + \xi_0^{1+}), \\ q_n^{(l+m)} &\rightarrow \frac{d_{2\bar{2}}^{(l)} + d_{12\bar{1}\bar{2}}^{(l)} E_1 \bar{E}_1}{c_{2\bar{2}} + c_{12\bar{1}\bar{2}} E_1 \bar{E}_1} \rho_l (ia_l)^n e^{\omega_l t} \\ &\rightarrow \frac{1}{2} \rho_l e^{i(\zeta_l + 2i\phi_l^{(2)})} (1 + e^{2i\phi_l^{(1)}} + (e^{2i\phi_l^{(1)}} - 1) \tanh(\xi_{1R} + \xi_0^{1+})), \end{aligned}$$

where

$$\begin{aligned} A_j^{1+} &= \frac{c_{12\bar{2}}^{(j)}}{2\sqrt{c_{1\bar{1}}}\sqrt{c_{12\bar{1}\bar{2}}}}, & e^{2\xi_0^{1+}} &= \frac{c_{12\bar{1}\bar{2}}}{c_{2\bar{2}}}, \\ e^{2i\phi_l^{(2)}} &= -\frac{(p_2 - a_l)(\bar{p}_2 - a_l)}{(1 - a_l p_2)(1 - a_l \bar{p}_2)}, & \zeta_l &= n\varphi_l + n\pi/2 + 2\phi_l^{(2)} + \omega_l t, \end{aligned}$$

Soliton 2 ( $\eta_{2R} \approx 0$ ,  $\eta_{1R} \rightarrow -\infty$ ):

$$\begin{aligned} q_n^{(j)} &\rightarrow \frac{\alpha_2^{(j)} i^n E_2}{1 + c_{2\bar{2}} E_2 \bar{E}_2} \rightarrow A_j^{2+} e^{i\xi_{2I}} \operatorname{sech}(\xi_{2R} + \xi_0^{2+}), \\ q_n^{(l+m)} &\rightarrow \frac{1 + d_{2\bar{2}}^{(l)} E_2 \bar{E}_2}{1 + c_{2\bar{2}} E_2 \bar{E}_2} \rho_l (ia_l)^n e^{\omega_l t} \rightarrow \frac{1}{2} \rho_l e^{i\zeta_l} (1 + e^{2i\phi_l^{(2)}} + (e^{2i\phi_l^{(2)}} - 1) \tanh(\xi_{2R} + \xi_0^{2+})), \end{aligned}$$

where

$$A_j^{2+} = \frac{\alpha_2^{(j)}}{2\sqrt{c_{2\bar{2}}}}, \quad e^{2\xi_0^{2+}} = c_{2\bar{2}}.$$

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