A Projective-to-Conformal Fefferman-Type Construction

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Abstract. We study a Fefferman-type construction based on the inclusion of Lie groups $\mathrm{SL}(n+1)$ into $\mathrm{Spin}(n+1,n+1)$. The construction associates a split-signature (n,n)-conformal spin structure to a projective structure of dimension n. We prove the existence of a canonical pure twistor spinor and a light-like conformal Killing field on the constructed conformal space. We obtain a complete characterisation of the constructed conformal spaces in terms of these solutions to overdetermined equations and an integrability condition on the Weyl curvature. The Fefferman-type construction presented here can be understood as an alternative approach to study a conformal version of classical Patterson–Walker metrics as discussed in recent works by Dunajski–Tod and by the authors. The present work therefore gives a complete exposition of conformal Patterson–Walker metrics from the viewpoint of parabolic geometry.

Key words: parabolic geometry; projective structure; conformal structure; Cartan connection; Fefferman spaces; twistor spinors

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1 Introduction

In *conformal geometry* the geometric structure is given by an equivalence class of pseudo-Riemannian metrics: two metrics g and $\hat{g}$ are considered to be equivalent if they differ by a positive smooth rescaling, $\hat{g} = e^{2f} g$. In *projective geometry* the geometric structure is given by an equivalence class of torsion-free affine connections: two connections D and $\hat{D}$ are considered as equivalent if they share the same geodesics (as unparametrised curves). While conformal and projective structures both determine a corresponding class of affine connections, neither of them induces a single distinguished connection on the tangent bundle. Instead, both structures have canonically associated *Cartan connections* that govern the respective geometries and encode prolonged geometric data of the respective structures. It is therefore often useful when studying projective and conformal structures to work in the framework of *Cartan geometries*.

The present paper investigates a geometric construction that produces a conformal class of split-signature metrics on a $2n$-dimensional manifold arising naturally from a projective class of connections on an n-dimensional manifold. Split-signature conformal structures of this type have appeared in several places in the literature before. The projective-to-conformal construction studied in this paper should be understood as a generalisation of the classical *Riemann extensions* of affine spaces by E.M. Patterson and A.G. Walker [26]. One of the main authors motivations for the present study was the article [15] by M. Dunajski and P. Tod, where the Patterson–Walker construction was generalised to a projectively invariant setting in dimension $n = 2$. On the other hand, in [25] conformal structures of signature $(2, 2)$ were constructed using Cartan connections that contain the conformal structures arising from 2-dimensional projective structures as a special case. A generalisation of this Cartan-geometric approach to higher dimensions can be found in [24].

In this paper the construction is studied as an instance of a *Fefferman-type construction*, as formalised in [6, 11], based on an inclusion of the respective Cartan structure groups $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$. We show that in the general situation $n \geq 3$ the induced conformal Cartan geometry is *non-normal*. To obtain information on the conformal structure it is thus important to understand how the normal conformal Cartan connection differs from the induced one, and the main part of the paper concerns the study of this modification. We may summarise the main contributions of the paper as follows:

- A comprehensive treatment of the projective-to-conformal Fefferman-type construction including a discussion of the intermediate Lagrangean contact structure (Section 3) and a comparison with Patterson–Walker metrics (Section 6.1).
- A thorough study of the normalisation process (Section 4) and an explicit formula for the modification needed to obtain the normal conformal Cartan connection (Section 5.2).
- The characterisation of the conformal structures obtained via our Fefferman-type construction (culminating in Theorem 4.14).

Let us comment upon the characterisation in more detail. This is formulated in terms of a conformal Killing field k and a twistor spinor χ on the conformal space together with a (conformally invariant) integrability curvature condition. In Theorem 4.14 the properties of k and χ are specified in terms of corresponding conformal tractors, which nicely reflects the algebraic setup of the Fefferman-type construction in geometric terms.

An alternative equivalent characterisation theorem was obtained by the authors in [20, Theorem 1] by different means, namely, by direct computations based on spin calculus in the spirit of [28, 29]. The conformal properties are given purely in underlying terms and do not refer to tractors. In Section 6.2 (Theorem 6.3) we indicate how this alternative characterisation can be obtained in the current framework.

We remark that, to our knowledge, the present work is the first comprehensive treatment of a non-normal Fefferman-type construction and we expect that the techniques developed should have considerable scope for applications to other similar constructions. A particularly interesting case of this sort is the Fefferman construction for (non-integrable) almost CR-structures. Possible further applications concern relations between solutions of so-called *BGG-equations* and special properties of the induced conformal structures. Several such relationships were already obtained by the authors in [20]. For instance, we can give a full description of Einstein metrics contained in the resulting conformal class in terms of the initial projective structure. Moreover, in [21] we were able to show that the obstruction tensor of the induced conformal structure vanishes.

2 Projective and conformal parabolic geometries

The standard reference for the background material on Cartan and parabolic geometries presented here is [11].

2.1 Cartan and parabolic geometries

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $P \subseteq G$ a closed subgroup with Lie algebra $\mathfrak{p}$. A *Cartan geometry* $(\mathcal{G}, \omega)$ of type (G, P) over a smooth manifold M consists of a P-principal bundle $\mathcal{G} \to M$ together with a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. The canonical principal bundle $G \to G/P$ endowed with the Maurer–Cartan form constitutes the *homogeneous model* for Cartan geometries of type (G, P).

The *curvature* of a Cartan connection ω is the 2-form

$$K \in \Omega^2(\mathcal{G}, \mathfrak{g}), \qquad K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)], \qquad \text{for all } \xi, \eta \in \mathfrak{X}(\mathcal{G}),$$

which is equivalently encoded in the P-equivariant *curvature function*

$$\kappa\colon\ \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}, \qquad \kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) := K\big(\omega^{-1}(u)(X), \omega^{-1}(u)(Y)\big). \tag{2.1}$$

The curvature is a complete obstruction to a local equivalence with the homogeneous model. If the image of κ is contained in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$ the Cartan geometry is called *torsion-free*.

A *parabolic geometry* is a Cartan geometry of type (G, P), where G is a semi-simple Lie group and $P \subseteq G$ is a parabolic subgroup. A subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is parabolic if and only if its maximal nilpotent ideal, called nilradical $\mathfrak{p}_+$, coincides with the orthogonal complement $\mathfrak{p}^\perp$ of $\mathfrak{p} \subseteq \mathfrak{g}$ with respect to the Killing form. In particular, this yields an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of P-modules. The quotient $\mathfrak{g}_0 = \mathfrak{p}/\mathfrak{p}_+$ is called the Levi factor; it is reductive and decomposes into a semi-simple part $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ and the center $\mathfrak{z}(\mathfrak{g}_0)$. The respective Lie groups are $G_0^{ss} \subseteq G_0 \subseteq P$ and $P_+ \subseteq P$ so that $P = G_0 \ltimes P_+$ and $P_+ = \exp(\mathfrak{p}_+)$. An identification of $\mathfrak{g}_0$ with a subalgebra in $\mathfrak{p}$ yields a grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, where $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. We set $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$. If k is the depth of the grading the parabolic geometry is called $|k|$*-graded*.

The grading of $\mathfrak{g}$ induces a grading on $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \cong \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. A parabolic geometry is called *regular* if the curvature function κ takes values only in the components of positive homogeneity. In particular, any torsion-free or $|1|$-graded parabolic geometry is regular.

Given a $\mathfrak{g}$-module V, there is a natural $\mathfrak{p}$-equivariant map, the *Kostant co-differential*,

$$\partial^*\colon\ \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes V \to \Lambda^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes V, \tag{2.2}$$

defining the Lie algebra homology of $\mathfrak{p}_+$ with values in V; see, e.g., [11, Section 3.3.1] for the explicit form. For $V = \mathfrak{g}$, this gives rise to a natural normalisation condition: parabolic geometries satisfying $\partial^*(\kappa) = 0$ are called *normal*. The *harmonic curvature* κ_H of a normal parabolic geometry is the image of κ under the projection $\ker \partial^* \to \ker \partial^* / \operatorname{im} \partial^*$. For regular and normal parabolic geometries, the entire curvature κ is completely determined just by κ_H.

A *Weyl structure* $j\colon \mathcal{G}_0 \hookrightarrow \mathcal{G}$ of a parabolic geometry $(\mathcal{G}, \omega)$ over M is a reduction of the P-principal bundle $\mathcal{G} \to M$ to the Levi subgroup $G_0 \subseteq P$. The class of all Weyl structures, which are parametrised by one-forms on M, includes a particularly important subclass of *exact Weyl structures*, which are parametrised by functions on M: For $|1|$-graded parabolic geometries, these correspond to further reductions of $\mathcal{G}_0 \to M$ just to the semi-simple part G_0^{ss} of G_0 or, equivalently, to sections of the principal $\mathbb{R}_+$-bundle $\mathcal{G}_0/G_0^{ss} \to M$. The latter bundle is called the *bundle of scales* and its sections are the *scales*.

For a Weyl structure $j\colon \mathcal{G}_0 \hookrightarrow \mathcal{G}$, the pullback $j^*\omega = j^*\omega_- + j^*\omega_0 + j^*\omega_+$ of the Cartan connection may be decomposed according to $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$. The $\mathfrak{g}_0$-part $j^*\omega_0$ is a principal connection on the G_0-bundle $\mathcal{G}_0 \to M$; it induces connections on all associated bundles, which are called (*exact*) *Weyl connections*. The $\mathfrak{p}_+$-part $j^*\omega_+$ is the so-called *Schouten tensor*.

2.2 Tractor bundles and BGG operators

Every Cartan connection ω on $\mathcal{G} \to M$ naturally extends to a principal connection $\hat{\omega}$ on the G-principal bundle $\hat{\mathcal{G}} := \mathcal{G} \times_P G \to M$, which further induces a linear connection $\nabla^{\mathcal{V}}$ on any associated vector bundle $\mathcal{V} := \mathcal{G} \times_P V = \hat{\mathcal{G}} \times_G V$ for a G-representation V. Bundles and connections arising in this way are called *tractor bundles* and *tractor connections*. The tractor connections induced by normal Cartan connections are called normal tractor connections.

In particular, for the adjoint representation we obtain the *adjoint tractor bundle* $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$. The canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ and the identification $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ yield a bundle projection $\Pi\colon \mathcal{A}M \to TM$; the inclusion $\mathfrak{p}_+ \subseteq \mathfrak{g}$ and the identification $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ yield a bundle inclusion $T^*M \hookrightarrow \mathcal{A}M$. This allows us to interpret the Cartan curvature κ from (2.1) as a 2-form Ω on M with values in $\mathcal{A}M$.

The holonomy group of the principal connection $\hat{\omega}$ is by definition the *holonomy of the Cartan connection* ω, i.e., $\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega}) \subseteq G$. By the holonomy of a geometric structure we mean the holonomy of the corresponding normal Cartan connection.

In [12], and later in a simplified manner in [4], it was shown that for a tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$ one can associate a sequence of differential operators, which are intrinsic to the given parabolic geometry $(\mathcal{G}, \omega)$,

$$\Gamma(\mathcal{H}_0) \overset{\Theta_0^{\mathcal{V}}}{\to} \Gamma(\mathcal{H}_1) \overset{\Theta_1^{\mathcal{V}}}{\to} \cdots \overset{\Theta_{n-1}^{\mathcal{V}}}{\to} \Gamma(\mathcal{H}_n).$$

The operators $\Theta_k^{\mathcal{V}}$ are the *BGG-operators* and they operate between the sections of subquotients $\mathcal{H}_k = \ker \partial^* / \mathrm{im}\, \partial^*$ of the bundles of $\mathcal{V}$-valued k-forms, where $\partial^*\colon \Lambda^k T^*M \otimes \mathcal{V} \to \Lambda^{k-1} T^*M \otimes \mathcal{V}$ denotes the bundle map induced by the Kostant co-differential (2.2).

The first BGG-operator $\Theta_0^{\mathcal{V}}\colon \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1)$ is constructed as follows. The bundle $\mathcal{H}_0$ is simply the quotient $\mathcal{V}/\mathcal{V}'$, where $\mathcal{V}' \subseteq \mathcal{V}$ is the subbundle corresponding to the largest P-invariant filtration component in the G-representation V. It turns out, there is a distinguished differential operator that splits the projection $\Pi_0\colon \mathcal{V} \to \mathcal{H}_0$, namely, the *splitting operator*, which is the unique map $L_0^{\mathcal{V}}\colon \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{V})$ satisfying

$$\Pi_0(L_0^{\mathcal{V}}(\sigma)) = \sigma, \qquad \partial^*(d^{\nabla^{\mathcal{V}}} L_0^{\mathcal{V}}(\sigma)) = 0, \qquad \text{for all } \sigma \in \Gamma(\mathcal{H}_0).$$

The latter condition allows to define the first BGG-operator by $\Theta_0^{\mathcal{V}} := \Pi_1 \circ d^{\nabla^{\mathcal{V}}} \circ L_0^{\mathcal{V}}$, where $\Pi_1\colon \ker \partial^* \to \Gamma(\mathcal{H}_1)$. The first BGG-operator defines an overdetermined system of differential equations on $\sigma \in \Gamma(\mathcal{H}_0)$, $\Theta_0^{\mathcal{V}}(\sigma) = 0$, which is termed the *first BGG-equation*.

2.3 Further notations and conventions

In order to distinguish various objects related to projective and conformal structures, the symbols referring to conformal data will always be endowed with tildes. To write down explicit formulae, we employ abstract index notation, cf., e.g., [27]. Furthermore, we will use different types of indices for projective and conformal manifolds. E.g., on a projective manifold M we write $\mathbb{E}_A := T^*M$, $\mathbb{E}^A := TM$, and multiple indices denote tensor products, as in $\mathbb{E}_A{}^B := T^*M \otimes TM$. Indices between squared brackets are skew, as in $\mathbb{E}_{[AB]} := \Lambda^2 T^*M$, and indices between round brackets are symmetric, as in $\mathbb{E}^{(AB)} := S^2 TM$. Analogously, on a conformal manifold $\widetilde{M}$ we write $\widetilde{\mathbb{E}}_a := T^*\widetilde{M}$, $\widetilde{\mathbb{E}}^a := T\widetilde{M}$ etc. By $\mathbb{E}(w)$ and $\widetilde{\mathbb{E}}[w]$ we denote the density bundle over M and $\widetilde{M}$, respectively. Tensor products with other natural bundles are denoted as $\mathbb{E}_A(w) := \mathbb{E}_A \otimes \mathbb{E}(w)$, $\widetilde{\mathbb{E}}_{[ab]}[w] := \widetilde{\mathbb{E}}_{[ab]} \otimes \widetilde{\mathbb{E}}[w]$, and the like.

2.4 Projective structures

Let M be a smooth manifold of dimension $n \geq 2$. A *projective structure* on M is given by a class, $\mathbf{p}$, of torsion-free projectively equivalent affine connections: two connections D and $\hat{D}$

are projectively equivalent if they have the same geodesics as unparametrised curves. This is the case if and only if there is a one-form $\Upsilon_A \in \Gamma(\mathbb{E}_A)$ such that, for all $\xi^A \in \Gamma(\mathbb{E}^A)$,

$$\hat{D}_A\xi^B = D_A\xi^B + \Upsilon_A\xi^B + \Upsilon_P\xi^P\delta_A{}^B.$$

An *oriented projective structure* $(M, \mathbf{p})$, which is a projective structure $\mathbf{p}$ on an oriented manifold M, is equivalently encoded as a normal parabolic geometry of type (G, P), where $G = \mathrm{SL}(n+1)$ and $P = \mathrm{GL}_+(n) \ltimes \mathbb{R}^{n*}$ is the stabiliser of a ray in the standard representation $\mathbb{R}^{n+1}$.

Affine connections from the projective class $\mathbf{p}$ are precisely the Weyl connections of the corresponding parabolic geometry. Exact Weyl connections are those $D \in \mathbf{p}$ which preserve a volume form — these are also known as *special* affine connections. In particular, a choice of $D \in \mathbf{p}$ reduces the structure group to $G_0 = \mathrm{GL}_+(n)$, if D is special, the structure group is further reduced to $G_0^{ss} = \mathrm{SL}(n)$.

For later purposes we now give explicit expressions of the main curvature quantities, cf., e.g., [2, 17]. For $D \in \mathbf{p}$, the Schouten tensor is determined by the Ricci curvature of D; if D is special, then the Schouten tensor is $\mathsf{P}_{AB} = \frac{1}{n-1}R_{PA}{}^P{}_B$, in particular, it is symmetric. The projective Weyl curvature and the Cotton tensor are

$$W_{AB}{}^C{}_D = R_{AB}{}^C{}_D + \mathsf{P}_{AD}\delta^C{}_B - \mathsf{P}_{BD}\delta^C{}_A, \qquad Y_{CAB} = 2D_{[A}\mathsf{P}_{B]C}.$$

Henceforth, we use a suitable normalisation of densities so that the line bundle associated to the canonical one-dimensional representation of P has projective weight -1. Hence, comparing with the usual notation, the *density bundle of projective weight* w, denoted by $\mathbb{E}(w)$, is just the bundle of ordinary $\left(\frac{-w}{n+1}\right)$-densities. As an associated bundle to $\mathcal{G} \to M$, $\mathbb{E}(w)$ corresponds to the 1-dimensional representation of P given by

$$\mathrm{GL}_+(n) \ltimes \mathbb{R}^{n*} \to \mathbb{R}_+, \qquad (A, X) \mapsto \det(A)^w. \tag{2.3}$$

The *projective standard tractor bundle* is the tractor bundle associated to the standard representation of $G = \mathrm{SL}(n+1)$. The *projective dual standard tractor bundle* is denoted by $\mathcal{T}^*$, i.e., $\mathcal{T}^* := \mathcal{G} \times_P \mathbb{R}^{n+1*}$. With respect to a choice of $D \in \mathbf{p}$, we write

$$\mathcal{T}^* = \begin{pmatrix} \mathbb{E}_A(1) \\ \mathbb{E}(1) \end{pmatrix}, \qquad \nabla_C^{\mathcal{T}^*} \begin{pmatrix} \varphi_A \\ \sigma \end{pmatrix} = \begin{pmatrix} D_C\varphi_A + \mathsf{P}_{CA}\sigma \\ D_C\sigma - \varphi_C \end{pmatrix}.$$

2.5 Conformal spin structures and tractor formulas

Let $\widetilde{M}$ be a smooth manifold of dimension $2n \geq 4$. A *conformal structure* of signature (n, n) on $\widetilde{M}$ is given by a class, $\mathbf{c}$, of conformally equivalent pseudo-Riemannian metrics of signature (n, n): two metrics g and $\hat{g}$ are conformally equivalent if $\hat{g} = f^2 g$ for a nowhere-vanishing smooth function f on $\widetilde{M}$. It may be equivalently described as a reduction of the frame bundle of $\widetilde{M}$ to the structure group $\mathrm{CO}(n, n) = \mathbb{R}_+ \times \mathrm{SO}(n, n)$. An *oriented conformal structure* of signature (n, n) is a conformal structure of signature (n, n) together with fixed orientations both in time-like and space-like directions, equivalently, a reduction of the frame bundle to the group $\mathrm{CO}_{\mathrm{o}}(n, n) = \mathbb{R}_+ \times \mathrm{SO}_{\mathrm{o}}(n, n)$, the connected component of the identity. An equivariant lift of such a reduction with respect to the 2-fold covering $\mathrm{CSpin}(n, n) = \mathbb{R}_+ \times \mathrm{Spin}(n, n) \to \mathrm{CO}_{\mathrm{o}}(n, n)$ is referred to as a *conformal spin structure* $\big(\widetilde{M}, \mathbf{c}\big)$ of signature (n, n).

A conformal spin structure of signature (n, n) is equivalently encoded as a normal parabolic geometry of type $\big(\widetilde{G}, \widetilde{P}\big)$, where $\widetilde{G} = \mathrm{Spin}(n+1, n+1)$ and $\widetilde{P} = \mathrm{CSpin}(n, n) \ltimes \mathbb{R}^{n,n*}$ is the stabiliser of an isotropic ray in the standard representation $\mathbb{R}^{n+1,n+1}$.

A general Weyl connection is a torsion-free affine connection $\widetilde{D}$ such that $\widetilde{D}g \in \mathbf{c}$ for any $g \in \mathbf{c}$. If $\widetilde{D}g = 0$, i.e., $\widetilde{D}$ is the Levi-Civita connection of a metric $g \in \mathbf{c}$, it is an exact Weyl connection. A choice of Weyl connection reduces the structure group to $\widetilde{G}_0 = \mathrm{CSpin}(n,n)$. If the Weyl connection is exact the structure group is further reduced to $\widetilde{G}_0^{ss} = \mathrm{Spin}(n,n)$.

Now we briefly introduce the main curvature quantities of conformal structures, cf., e.g., [16]. For $g \in \mathbf{c}$, the *Schouten tensor*,

$$\widetilde{\mathsf{P}} = \widetilde{\mathsf{P}}(g) = \frac{1}{2n-2}\left(\widetilde{\mathrm{Ric}}(g) - \frac{\widetilde{\mathrm{Sc}}(g)}{2(2n-1)}g\right),$$

is a trace modification of the Ricci curvature $\widetilde{\mathrm{Ric}}(g)$ by a multiple of the scalar curvature $\widetilde{\mathrm{Sc}}(g)$; its trace is denoted $\widetilde{J} = g^{pq}\widetilde{\mathsf{P}}_{pq}$. The conformal Weyl curvature and the Cotton tensors are

$$\widetilde{W}_{ab}{}^{c}{}_{d} = \widetilde{R}_{ab}{}^{c}{}_{d} - 2\delta^{c}_{[a}\widetilde{\mathsf{P}}_{b]d} + 2g_{d[a}\widetilde{\mathsf{P}}_{b]}{}^{c}, \qquad \widetilde{Y}_{cab} = 2\widetilde{D}_{[a}\widetilde{\mathsf{P}}_{b]c}.$$

As for projective structures, we will employ a suitable parametrisation of densities so that the canonical 1-dimensional representation of $\widetilde{P}$ has conformal weight -1. Hence, the *density bundle of conformal weight* w, denoted as $\widetilde{\mathbb{E}}[w]$, is just the bundle of ordinary $\left(\frac{-w}{2n}\right)$-densities. As an associated bundle to the Cartan bundle $\widetilde{\mathcal{G}} \to \widetilde{M}$, it corresponds to the 1-dimensional representation of $\widetilde{P}$ given by

$$(\mathbb{R}_+ \times \mathrm{Spin}(n,n)) \ltimes \mathbb{R}^{2n*} \to \mathbb{R}_+, \qquad (a, A, Z) \mapsto a^{-w}. \tag{2.4}$$

In particular, the conformal structure may be seen as a section of $\widetilde{\mathbb{E}}_{(ab)}[2]$, which is called the *conformal metric* and denoted by $\boldsymbol{g}_{ab}$.

The spin bundles corresponding to the irreducible spin representations of $\mathrm{Spin}(n,n)$ are denoted by $\widetilde{\Sigma}_+$ and $\widetilde{\Sigma}_-$, and $\widetilde{\Sigma} = \widetilde{\Sigma}_+ \oplus \widetilde{\Sigma}_-$. We employ the weighted conformal gamma matrix $\gamma \in \Gamma\big(\widetilde{\mathbb{E}}_a \otimes \big(\mathrm{End}\,\widetilde{\Sigma}\big)[1]\big)$ such that $\gamma_p\gamma_q + \gamma_q\gamma_p = -2\boldsymbol{g}_{pq}$. For $\xi \in \mathfrak{X}\big(\widetilde{M}\big)$ and $\chi \in \Gamma\big(\widetilde{\Sigma}\big)$, the Clifford multiplication of ξ on χ is then written as $\xi \cdot \chi = \xi^p\gamma_p\chi$.

The *conformal standard tractor bundle* is the associated bundle $\widetilde{\mathcal{T}} := \widetilde{\mathcal{G}} \times_{\widetilde{P}} \mathbb{R}^{n+1,n+1}$ with respect to the standard representation. It carries the canonical tractor metric $\mathbf{h}$ and the conformal standard tractor connection $\widetilde{\nabla}^{\widetilde{\mathcal{T}}}$, which preserves $\mathbf{h}$. With respect to a metric $g \in \mathbf{c}$, we have

$$\widetilde{\mathcal{T}} = \begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ \widetilde{\mathbb{E}}_a[1] \\ \widetilde{\mathbb{E}}[1] \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \widetilde{\nabla}^{\widetilde{\mathcal{T}}}_c \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} \widetilde{D}_c\rho - \widetilde{\mathsf{P}}_c{}^b\varphi_b \\ \widetilde{D}_c\varphi_a + \sigma\widetilde{\mathsf{P}}_{ca} + \rho\boldsymbol{g}_{ca} \\ \widetilde{D}_c\sigma - \varphi_c \end{pmatrix}. \tag{2.5}$$

The BGG-splitting operator is given by

$$L_0^{\widetilde{\mathcal{T}}} \colon\ \Gamma\big(\widetilde{\mathbb{E}}[1]\big) \to \Gamma\big(\widetilde{\mathcal{T}}\big), \qquad \sigma \mapsto \begin{pmatrix} \frac{1}{2n}\big(-\widetilde{D}^p\widetilde{D}_p - \widetilde{J}\big)\sigma \\ \widetilde{D}_a\sigma \\ \sigma \end{pmatrix}. \tag{2.6}$$

The *spin tractor bundle* is the associated bundle $\widetilde{\mathcal{S}} := \widetilde{\mathcal{G}} \times_{\widetilde{P}} \Delta^{n+1,n+1}$, where $\Delta^{n+1,n+1}$ is the spin representation of $\widetilde{G} = \mathrm{Spin}(n+1,n+1)$. Since we work in even signature, it decomposes into irreducibles $\Delta^{n+1,n+1} = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$; the corresponding bundles are denoted by $\widetilde{\mathcal{S}}_\pm = \widetilde{\mathcal{G}} \times_{\widetilde{P}} \Delta_\pm^{n+1,n+1}$. Under a choice of $g \in \mathbf{c}$, these decompose as $\widetilde{\mathcal{S}}_\pm = \begin{pmatrix} \widetilde{\Sigma}_\mp[-\frac{1}{2}] \\ \widetilde{\Sigma}_\pm[\frac{1}{2}] \end{pmatrix}$, where $\widetilde{\Sigma}_\pm$

are the natural spin bundles as before. For later use we record the formulas for the Clifford action of $\widetilde{\mathcal{T}}$ on $\widetilde{\mathcal{S}}$ and for the spin tractor connections on $\widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}_+ \oplus \widetilde{\mathcal{S}}_-$,

$$\begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} \tau \\ \chi \end{pmatrix} = \begin{pmatrix} -\varphi_a \gamma^a \tau + \sqrt{2}\rho\chi \\ \varphi_a \gamma^a \chi - \sqrt{2}\sigma\tau \end{pmatrix}, \qquad \widetilde{\nabla}^{\widetilde{\mathcal{S}}}_c \begin{pmatrix} \tau \\ \chi \end{pmatrix} = \begin{pmatrix} \widetilde{D}_c \tau + \frac{1}{\sqrt{2}} \widetilde{\mathsf{P}}_{cp} \gamma^p \chi \\ \widetilde{D}_c \chi + \frac{1}{\sqrt{2}} \gamma_c \tau \end{pmatrix}, \tag{2.7}$$

cf. [19]. The BGG-splitting operator of $\widetilde{\mathcal{S}}_\pm$ is

$$L_0^{\widetilde{\mathcal{S}}_\pm} \colon\ \Gamma\big(\widetilde{\Sigma}_\pm\big[\tfrac{1}{2}\big]\big) \to \Gamma\big(\widetilde{\mathcal{S}}_\pm\big), \qquad \chi \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}n} \not{D} \chi \\ \chi \end{pmatrix}, \tag{2.8}$$

where $\not{D} \colon \Gamma\big(\widetilde{\Sigma}_\pm\big) \to \Gamma\big(\widetilde{\Sigma}_\mp\big)$, $\not{D} := \gamma^p \widetilde{D}_p$, is the *Dirac* operator. The first BGG-operator associated to $\widetilde{\mathcal{S}}_\pm$ is the *twistor operator*

$$\Theta_0^{\widetilde{\mathcal{S}}} \colon\ \Gamma\big(\widetilde{\Sigma}_\pm\big[\tfrac{1}{2}\big]\big) \to \Gamma\big(\widetilde{\mathbb{E}}_a \otimes \widetilde{\Sigma}_\pm\big[\tfrac{1}{2}\big]\big), \qquad \chi \mapsto \widetilde{D}_a \chi + \tfrac{1}{2n} \gamma_a \not{D} \chi,$$

cf., e.g., [3]. Elements in the kernel of $\Theta_0^{\widetilde{\mathcal{S}}}$ are called *twistor spinors*. It is well known that $\Pi_0^{\widetilde{\mathcal{S}}}$ induces an isomorphism between $\widetilde{\nabla}^{\widetilde{\mathcal{S}}}$-parallel sections of $\widetilde{\mathcal{S}}$ with $\ker \Theta_0^{\widetilde{\mathcal{S}}}$.

The *adjoint tractor bundle* is the associated bundle $\mathcal{A}\widetilde{M} := \widetilde{\mathcal{G}} \times_{\widetilde{P}} \tilde{\mathfrak{g}}$ with respect to the adjoint representation of $\widetilde{G}$ on $\tilde{\mathfrak{g}} = \mathfrak{so}(n+1, n+1) \cong \Lambda^2 \mathbb{R}^{n+1,n+1}$. The standard pairing on $\mathcal{A}\widetilde{M}$ induced by the Killing form on $\tilde{\mathfrak{g}}$ is denoted as $\langle \cdot, \cdot \rangle \colon \mathcal{A}\widetilde{M} \times \mathcal{A}\widetilde{M} \to \mathbb{R}$. Henceforth we identify $\mathcal{A}\widetilde{M}$ with $\Lambda^2 \widetilde{\mathcal{T}}$. With respect to a metric $g \in \mathbf{c}$,

$$\mathcal{A}\widetilde{M} = \begin{pmatrix} \widetilde{\mathbb{E}}_a[0] \\ \widetilde{\mathbb{E}}_{[a_0 a_1]}[2] \mid \widetilde{\mathbb{E}}[1] \\ \widetilde{\mathbb{E}}_a[2] \end{pmatrix}.$$

The standard representation of $\tilde{\mathfrak{g}}$ on $\mathbb{R}^{n+1,n+1}$ gives rise to the map

$$\bullet\colon\ \mathcal{A}\widetilde{M} \otimes \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}, \qquad \begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ \beta_a \end{pmatrix} \bullet \begin{pmatrix} \nu \\ \omega_b \\ \sigma \end{pmatrix} = \begin{pmatrix} \rho^r \omega_r - \varphi\nu \\ {\mu_b}^r \omega_r - \sigma\rho_b - \nu\beta_b \\ \beta^r \omega_r + \varphi\sigma \end{pmatrix}. \tag{2.9}$$

The normal tractor connection is given by

$$\widetilde{\nabla}^{\mathcal{A}\widetilde{M}}_c \begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ k_a \end{pmatrix} = \begin{pmatrix} \widetilde{D}_c \rho_a - \widetilde{\mathsf{P}}_c{}^p \mu_{pa} - \widetilde{\mathsf{P}}_{ca}\varphi \\ \begin{pmatrix} \widetilde{D}_c \mu_{a_0 a_1} + 2\boldsymbol{g}_{c[a_0}\rho_{a_1]} \\ +2\widetilde{\mathsf{P}}_{c[a_0} k_{a_1]} \end{pmatrix} \mid \big(\widetilde{D}_c \varphi - \widetilde{\mathsf{P}}_c{}^p k_p + \rho_c\big) \\ \widetilde{D}_c k_a - \mu_{ca} + \boldsymbol{g}_{ca}\varphi \end{pmatrix}. \tag{2.10}$$

Written as a two-form $\widetilde{\Omega}$ with values in $\Lambda^2 \widetilde{\mathcal{T}}$, the curvature of $\widetilde{\nabla}^{\widetilde{\mathcal{T}}}$ is

$$\widetilde{\Omega}_{c_0 c_1} = \begin{pmatrix} -\widetilde{Y}_{a c_0 c_1} \\ \widetilde{W}_{c_0 c_1 a_0 a_1} \mid 0 \\ 0 \end{pmatrix} \in \Gamma\big(\widetilde{\mathbb{E}}_{[c_0 c_1]} \otimes \mathcal{A}\widetilde{M}\big). \tag{2.11}$$

The BGG-splitting operator

$$L_0^{\mathcal{A}\widetilde{M}} \colon\ \Gamma\big(\widetilde{\mathbb{E}}^a\big) = \Gamma\big(\widetilde{\mathbb{E}}_a[2]\big) \to \Gamma\big(\mathcal{A}\widetilde{M}\big), \qquad k_a \mapsto \begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ k_a \end{pmatrix},$$

is determined by

$$\mu_{a_0a_1} = \widetilde{D}_{[a_0}k_{a_1]}, \qquad \varphi = -\frac{1}{2n}\boldsymbol{g}^{pq}\widetilde{D}_p k_q,$$
$$\rho_a = -\frac{1}{4n}\widetilde{D}^p\widetilde{D}_p k_a + \frac{1}{4n}\widetilde{D}^p\widetilde{D}_a k_p + \frac{1}{4n^2}\widetilde{D}_a\widetilde{D}^p k_p + \frac{1}{n}\widetilde{\mathsf{P}}^p{}_a k_p - \frac{1}{2n}\widetilde{J}k_a, \tag{2.12}$$

and the corresponding first BGG-operator of $\mathcal{A}\widetilde{M}$ is computed as

$$\Theta_0^{\mathcal{A}\widetilde{M}}\colon\ \Gamma\big(\widetilde{\mathbb{E}}_a[2]\big) \to \Gamma\big(\widetilde{\mathbb{E}}_{(ab)_0}[2]\big), \qquad \xi_a \mapsto \widetilde{D}_{(c}\xi_{a)_0},$$

where the subscript 0 denotes the trace-free part. Thus $\Theta_0^{\mathcal{A}\widetilde{M}}$ is the conformal Killing operator and solutions to the first BGG-equation are conformal Killing fields. In a prolonged form, the conformal Killing equation is equivalent to

$$\widetilde{\nabla}_b^{\mathcal{A}\widetilde{M}} s = \xi^a\widetilde{\Omega}_{ab}, \tag{2.13}$$

where $s = L_0^{\mathcal{A}\widetilde{M}}(\xi)$, see [7, 18].

3 The Fefferman-type construction

The construction of split-signature conformal structures from projective structures discussed in this section fits into a general scheme relating parabolic geometries of different types. Namely, it is an instance of the so-called Fefferman-type construction, whose name and general procedure is motivated by Fefferman's construction of a canonical conformal structure induced by a CR structure, see [6] and [11] for a detailed discussion.

3.1 General procedure

Suppose we have two pairs of semi-simple Lie groups and parabolic subgroups, (G, P) and $\big(\widetilde{G}, \widetilde{P}\big)$, and a Lie group homomorphism $i\colon G \to \widetilde{G}$ such that the derivative $i'\colon \mathfrak{g} \to \tilde{\mathfrak{g}}$ is injective. Assume further that the G-orbit of the origin in $\widetilde{G}/\widetilde{P}$ is open and that the parabolic $P \subseteq G$ contains $Q := i^{-1}\big(\widetilde{P}\big)$, the preimage of $\widetilde{P} \subseteq \widetilde{G}$.

Given a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), one first forms the *Fefferman space*

$$\widetilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q. \tag{3.1}$$

Then $\big(\mathcal{G} \to \widetilde{M}, \omega\big)$ is automatically a Cartan geometry of type (G, Q). As a next step, one considers the extended bundle $\widetilde{\mathcal{G}} := \mathcal{G} \times_Q \widetilde{P}$ with respect to the homomorphism $Q \to \widetilde{P}$. This is a principal bundle over $\widetilde{M}$ with structure group $\widetilde{P}$ and $j\colon \mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ denotes the natural inclusion. The equivariant extension of $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ yields a unique Cartan connection $\widetilde{\omega}^{\mathrm{ind}} \in \Omega^1\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ of type $\big(\widetilde{G}, \widetilde{P}\big)$ such that $j^*\widetilde{\omega}^{\mathrm{ind}} = i' \circ \omega$. Altogether, one obtains a functor from parabolic geometries $(\mathcal{G} \to M, \omega)$ of type (G, P) to parabolic geometries $\big(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega}^{\mathrm{ind}}\big)$ of type $\big(\widetilde{G}, \widetilde{P}\big)$.

The relation between the corresponding curvatures is as follows: The previous assumptions yield a linear isomorphism $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q}$ and an obvious projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$, where $\mathfrak{q} \subseteq \mathfrak{p}$ is the Lie algebra of $Q \subseteq P$. Composing these two maps one obtains a linear projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/\mathfrak{p}$, whose dual map is denoted as $\varphi\colon (\mathfrak{g}/\mathfrak{p})^* \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$. Since $i'\colon \mathfrak{g} \to \tilde{\mathfrak{g}}$ is a homomorphism of Lie algebras, the curvature function $\widetilde{\kappa}^{\mathrm{ind}}\colon \widetilde{\mathcal{G}} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ is related to $\kappa\colon \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ by $\widetilde{\kappa}^{\mathrm{ind}} \circ j = (\Lambda^2\varphi \otimes i') \circ \kappa$. We note that $\widetilde{\kappa}^{\mathrm{ind}}$ is fully determined by this formula.

Since i' is an embedding, the notation is in most cases simplified such that we write $\mathfrak{g} \subseteq \tilde{\mathfrak{g}}$, $\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}}$, etc.

3.2 Algebraic setup and the homogeneous model

Here we specify the general setup for Fefferman-type constructions from Section 3.1 according to the description of oriented projective and conformal spin structures given in Sections 2.4 and 2.5, respectively. Let $\mathbb{R}^{n+1,n+1}$ be the real vector space $\mathbb{R}^{2n+2}$ with an inner product, h, of split-signature. Let $\Delta_+^{n+1,n+1}$ and $\Delta_-^{n+1,n+1}$ be the irreducible spin representations of

$$\widetilde{G} := \mathrm{Spin}(n+1, n+1)$$

as in Section 2.5. We fix two pure spinors $s_F \in \Delta_-^{n+1,n+1}$ and $s_E \in \Delta_\pm^{n+1,n+1}$ with non-trivial pairing, which is assigned for later use to be $\langle s_E, s_F\rangle = -\frac{1}{2}$. Note that s_E lies in $\Delta_+^{n+1,n+1}$ if n is even or in $\Delta_-^{n+1,n+1}$ if n is odd.

Let us denote by $E, F \subseteq \mathbb{R}^{n+1,n+1}$ the kernels of s_E, s_F with respect to the Clifford multiplication, i.e.,

$$E := \big\{X \in \mathbb{R}^{n+1,n+1} \colon X \cdot s_E = 0\big\}, \qquad F := \big\{X \in \mathbb{R}^{n+1,n+1} \colon X \cdot s_F = 0\big\}.$$

The purity of s_E and s_F means that E and F are maximally isotropic subspaces in $\mathbb{R}^{n+1,n+1}$. The other assumptions guarantee that E and F are complementary and dual each other via the inner product h. Hence we use the decomposition

$$\mathbb{R}^{n+1,n+1} = E \oplus F \cong \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1*} \tag{3.2}$$

to identify the spinor representation $\Delta^{n+1,n+1} = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$ with the exterior power algebra $\Lambda^\bullet E \cong \Lambda^\bullet \mathbb{R}^{n+1}$, whose irreducible subrepresentations are $\Delta_-^{n+1,n+1} \cong \Lambda^{\mathrm{even}}\mathbb{R}^{n+1}$ and $\Delta_+^{n+1,n+1} \cong \Lambda^{\mathrm{odd}}\mathbb{R}^{n+1}$. When n is even, respectively, odd, we can identify $\big(\Delta_-^{n+1,n+1}\big)^* \cong \Delta_+^{n+1,n+1}$, respectively $\big(\Delta_-^{n+1,n+1}\big)^* \cong \Delta_-^{n+1,n+1}$.

Now, let us consider the subgroup in $\widetilde{G}$ defined by

$$G := \{g \in \mathrm{Spin}(n+1, n+1) \colon g \cdot s_E = s_E,\, g \cdot s_F = s_F\}.$$

This subgroup preserves the decomposition (3.2) so that the restriction of the action to F is dual to the restriction to E. It further preserves the volume form on E, respectively $F \cong E^*$, which is determined by s_E and s_F according to the previous identifications. Hence $G \cong \mathrm{SL}(n+1)$ and this defines an embedding $i\colon \mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$.[1]

The G-invariant decomposition (3.2) determines a G-invariant skew-symmetric involution $K \in \mathfrak{so}(n+1, n+1)$ acting by the identity on E and minus the identity on F. The relationship among K, s_E and s_F may be expressed as

$$h(X, K(Y)) = -h(K(X), Y) = 2\langle s_E, (X \wedge Y) \cdot s_F\rangle, \tag{3.3}$$

where

$$(X \wedge Y) \cdot s_F = \frac{1}{2}(X \cdot Y \cdot s_F - Y \cdot X \cdot s_F) = X \cdot Y \cdot s_F + h(X, Y)s_F.$$

The spin action of $\tilde{\mathfrak{g}}$ is denoted by $\bullet$, and thus $A \bullet s = -\frac{1}{4}A \cdot s$, for any $A \in \tilde{\mathfrak{g}}$ and $s \in \Delta$. In particular, $K \bullet s_F = -\frac{1}{2}(n+1)s_F$ and $K \bullet s_E = \frac{1}{2}(n+1)s_E$. Here we identify $\tilde{\mathfrak{g}} = \mathfrak{so}(n+1, n+1)$ with $\Lambda^2\mathbb{R}^{n+1,n+1}$. It is convenient to split $\tilde{\mathfrak{g}}$ in terms of irreducible $\mathfrak{g}$-modules as

$$\tilde{\mathfrak{g}} = \Lambda^2(E \oplus F) = \underbrace{(E \otimes F)_0}_{\mathfrak{g}=\mathfrak{sl}(n+1)} \oplus \underbrace{(E \otimes F)_{Tr} \oplus \Lambda^2 E \oplus \Lambda^2 F}_{\mathfrak{g}^\perp}, \tag{3.4}$$

[1] Instead of the embedding $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$ we could also consider the embedding $\mathrm{SL}(n+1) \hookrightarrow \mathrm{SO}(n+1, n+1)$. The advantage of employing the embedding into the spin group is two-fold: on the one hand, it is then seen directly that the induced conformal structure has a canonical spin structure, and, on the other hand, we can then use convenient spinorial objects for its characterisation.

where $(E\otimes F)_{Tr}=\mathbb{R}K$, and K acts as $[K,\phi]=2\phi$, $[K,\psi]=-2\psi$, $[K,\lambda]=0$, for any $\phi\in\Lambda^2E$, $\psi\in\Lambda^2F$ and $\lambda\in E\otimes F$. Further, the annihilators of s_E and s_F in $\tilde{\mathfrak{g}}$ are the subalgebras $\ker s_E=\mathfrak{sl}(n+1)\oplus\Lambda^2E$ and $\ker s_F=\mathfrak{sl}(n+1)\oplus\Lambda^2F$.

The homogeneous model for conformal spin structures of signature (n,n) is the space of isotropic rays in $\mathbb{R}^{n+1,n+1}$, $\widetilde{G}/\widetilde{P}\cong S^n\times S^n$. The subgroup $G\subseteq\widetilde{G}$ does not act transitively on that space. According to the decomposition (3.2), there are three orbits: the set of rays contained in E, the set of rays contained in F, and the set of isotropic rays that are neither contained in E nor in F. Note that only the last orbit is open in $\widetilde{G}/\widetilde{P}$, which is one of the requirements from Section 3.1. Therefore, we define $\widetilde{P}\subseteq\widetilde{G}$ to be the stabiliser of a ray through a light-like vector $\tilde{v}\in\mathbb{R}^{n+1,n+1}\setminus(E\cup F)$. Denoting by $Q=i^{-1}(\widetilde{P})$ the stabiliser of the ray $\mathbb{R}_+\tilde{v}$ in G, we have the identification of G/Q with the open orbit of the origin in $\widetilde{G}/\widetilde{P}$. The subgroup Q, which is not parabolic, is contained in the parabolic subgroup $P\subseteq G$ defined as the stabiliser in G of the ray through the projection of $\tilde{v}$ to E. In particular, G/P is the standard projective sphere S^n, the homogeneous model of oriented projective structures of dimension n, and $G/Q\to G/P$ is the canonical fibration with the standard fibre P/Q, whose total space is the model Fefferman space.

Let us denote by $L=\mathbb{R}\tilde{v}$ the line spanned by the light-like vector $\tilde{v}$ and let $L^\perp$ be the orthogonal complement in $\mathbb{R}^{n+1,n+1}$ with respect to h. The tangent space of G/Q at the origin can be seen in three different ways, namely,

$$\big(L^\perp/L\big)[1]\cong\mathfrak{g}/\mathfrak{q}\cong\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}.$$

The latter isomorphism is induced by the embedding $\mathfrak{g}\subseteq\tilde{\mathfrak{g}}$, the former one by the standard action of $\mathfrak{g}\subseteq\tilde{\mathfrak{g}}$ on the vector $\tilde{v}\in\mathbb{R}^{n+1,n+1}$. Both these identifications are Q-equivariant.

There are several natural Q-invariant objects that in turn yield distinguished geometric objects on the general Fefferman space. The n-dimensional Q-invariant subspace

$$f:=\big(\big(\bar{F}+L\big)/L\big)[1]\subseteq\big(L^\perp/L\big)[1],\qquad\text{where}\quad\bar{F}:=F\cap L^\perp,$$

which is isomorphic to $\mathfrak{p}/\mathfrak{q}\subseteq\mathfrak{g}/\mathfrak{q}$, the kernel of the projection $\mathfrak{g}/\mathfrak{q}\to\mathfrak{g}/\mathfrak{p}$. Another n-dimensional Q-invariant subspace is

$$e:=\big(\big(\bar{E}+L\big)/L\big)[1]\subseteq\big(L^\perp/L\big)[1],\qquad\text{where}\quad\bar{E}:=E\cap L^\perp.$$

The intersection $e\cap f$ is 1-dimensional with a distinguished Q-invariant generator that corresponds to the G-invariant involution $K\in\tilde{\mathfrak{g}}$,

$$k:=K+\tilde{\mathfrak{p}}\in\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}.$$

Note that all these objects are isotropic with respect to the natural conformal class induced by the restriction of h to $L^\perp\subseteq\mathbb{R}^{n+1.n+1}$. In particular, both e and f are maximally isotropic subspaces such that

$$k\in e\cap f\subseteq k^\perp=e+f.\tag{3.5}$$

In Section 3.1 we introduced a map $\varphi\colon(\mathfrak{g}/\mathfrak{p})^*\to(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$, the dual map to the projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}\cong\mathfrak{g}/\mathfrak{q}\to\mathfrak{g}/\mathfrak{p}$. The kernel of this projection is just f and the image of φ is identified with its annihilator, which will be denoted by f°. Since f is a maximally isotropic subspace in $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}\cong\mathfrak{g}/\mathfrak{q}$,

$$f^\circ\cong f[-2].$$

Since $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*\cong\tilde{\mathfrak{p}}_+$, we may conclude with the help of explicit matrix realisations from Appendix A that $f^\circ=\tilde{\mathfrak{p}}_+\cap\ker s_F$. Moreover, we note that

$$\big(\tilde{\mathfrak{p}}_+\cap\ker s_F\big)_{E\otimes F}=\mathfrak{p}_+,\qquad\big(\tilde{\mathfrak{p}}\cap\ker s_F\big)_{E\otimes F}=\mathfrak{p},\tag{3.6}$$

$$\Lambda^2F\cap\tilde{\mathfrak{p}}=\Lambda^2\bar{F}\subseteq\tilde{\mathfrak{g}}_0,\qquad\big[\tilde{\mathfrak{p}}_+,\Lambda^2\bar{F}\big]=f^\circ,\qquad\big[f^\circ,\Lambda^2\bar{F}\big]=0.\tag{3.7}$$

3.3 The Fefferman space and induced structure

The pairs of Lie groups (G, P) and $(\widetilde{G}, \widetilde{P})$ from the previous subsection satisfy all the properties to launch the Fefferman-type construction.

Proposition 3.1. *The Fefferman-type construction for the pairs of Lie groups (G, P) and $(\widetilde{G}, \widetilde{P})$ yields a natural construction of conformal spin structures $(\widetilde{M}, \mathbf{c})$ of signature (n, n) from n-dimensional oriented projective structures $(M, \mathbf{p})$. The Fefferman space $\widetilde{M}$ is identified with the total space of the weighted cotangent bundle without the zero section $T^*M(2)\backslash\{0\}$.*

Proof. The first part of the statement is obvious from the general setting for Fefferman-type constructions and the Cartan-geometric description of oriented projective and conformal spin structures.

The second part is shown due to two natural identifications: On the one hand, the Fefferman space is by (3.1) equal to the total space of the associated bundle $\widetilde{M} \cong \mathcal{G} \times_P P/Q$ over M. On the other hand, the weighted cotangent bundle to M is identified with the associated bundle $T^*M(2) \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})^*(2)$ with respect to action of P induced by the adjoint action and the representation (2.3) for $w = 2$. Hence it remains to verify that the action of P on $(\mathfrak{g}/\mathfrak{p})^*(2) \setminus \{0\}$ is transitive and Q is a stabiliser of a non-zero element. But this is a purely algebraic task, which may be easily checked in a concrete matrix realisation. ■

From the algebraic setup in Section 3.2 we easily conclude number of specific features of the induced conformal structure on $\widetilde{M}$:

Proposition 3.2. *The conformal spin structure $(\widetilde{M}, \mathbf{c})$ induced from an oriented projective structure $(M, \mathbf{p})$ by the Fefferman-type construction admits the following tractorial objects that are all parallel with respect to the induced tractor connection:*

(a) *pure tractor spinors $\mathbf{s}_E \in \Gamma(\widetilde{\mathcal{S}}_\pm)$ and $\mathbf{s}_F \in \Gamma(\widetilde{\mathcal{S}}_-)$ with non-trivial pairing,*

(b) *a tractor endomorphism $\mathbf{K} \in \Gamma(\mathcal{A}\widetilde{M})$ which is an involution, i.e., $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$, and which acts by the identity, respectively minus the identity on the maximally isotropic complementary subbundles $\widetilde{\mathcal{E}} := \ker \mathbf{s}_E$, respectively $\widetilde{\mathcal{F}} := \ker \mathbf{s}_F$ of $\widetilde{\mathcal{T}}$.*

The corresponding underlying objects $\eta = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_E)$, $\chi = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_F)$ and $k = \Pi_0^{\mathcal{A}\widetilde{M}}(\mathbf{K})$ satisfy:

(c) *$\eta \in \Gamma(\widetilde{\Sigma}_\pm[\frac{1}{2}])$ and $\chi \in \Gamma(\widetilde{\Sigma}_-[\frac{1}{2}])$ are pure spinors, whose kernels $\widetilde{e} := \ker \eta$ and $\widetilde{f} := \ker \chi$ have 1-dimensional intersection and $\widetilde{f}$ coincides with the vertical subbundle of $\widetilde{M} \to M$,*

(d) *$k \in \Gamma(T\widetilde{M})$ is a nowhere-vanishing light-like vector field generating the intersection $\widetilde{e} \cap \widetilde{f}$.*

Proof. The G-invariant spinor $s_E \in \Delta_\pm$ gives rise to the tractor spinor $\mathbf{s}_E \in \Gamma(\widetilde{\mathcal{S}}_\pm = \mathcal{G} \times_Q \Delta_\pm)$ such that it corresponds to the constant (Q-equivariant) map $\mathcal{G} \to \Delta_\pm$. Hence $\mathbf{s}_E$ is automatically parallel with respect to the induced tractor connection on $\widetilde{\mathcal{S}}_\pm$. Similar reasoning for other G-invariant objects and their compatibility described above yield the first part of the statement. In particular, $\widetilde{\mathcal{E}} = \mathcal{G} \times_Q E$, $\widetilde{\mathcal{F}} = \mathcal{G} \times_Q F$ and the decomposition $\widetilde{\mathcal{T}} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}}$ corresponds to the decomposition (3.2).

The filtration $L \subseteq L^\perp \subseteq \mathbb{R}^{n+1,n+1}$ gives rise to the filtration of the standard tractor bundle, which can be written as

$$\begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ 0 \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ \widetilde{\mathbb{E}}_a[1] \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ \widetilde{\mathbb{E}}_a[1] \\ \widetilde{\mathbb{E}}[1] \end{pmatrix} = \widetilde{\mathcal{T}}.$$

In particular, the subbundles associated to $\bar{E}, \bar{F} \subseteq L^\perp$ are distinguished by the middle slot. The corresponding Q-invariant maximally isotropic subspaces $e, f \subseteq \mathfrak{g}/\mathfrak{q}$ determine the distributions $\mathcal{G} \times_Q e$ and $\mathcal{G} \times_Q f$ in $T\widetilde{M} = \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$. According to the tractor Clifford action (2.7) it follows that these are precisely the kernels of the spinors η and χ. Since these subspaces are maximally isotropic, the corresponding spinors are pure. Since $f \cong \mathfrak{p}/\mathfrak{q}$ is the kernel of the projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$, the corresponding subbundle $\widetilde{f}$ is identified with the vertical subbundle of the projection $\widetilde{M} \to M$. The intersection $e \cap f$ is 1-dimensional and it is generated by the projection of $K \in \tilde{\mathfrak{g}}$ to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Indeed, K cannot be contained in $\tilde{\mathfrak{p}}$, since K acts by the identity on E and minus the identity on F and $\tilde{\mathfrak{p}}$ is the stabiliser of a line that is neither contained in E nor in F. Altogether, the corresponding vector field k on $\widetilde{M}$ is a nowhere-vanishing generator of $\widetilde{e} \cap \widetilde{f}$, in particular, it is light-like. ■

3.4 Relating tractors, Weyl structures and scales

As a technical preliminary for further study we now relate natural objects associated to the original projective Cartan geometry $(\mathcal{G}, \omega)$ on M and the induced conformal geometry $(\widetilde{\mathcal{G}}, \widetilde{\omega}^{\mathrm{ind}})$ on the Fefferman space $\widetilde{M}$.

Since $G \subseteq \widetilde{G}$, any $\widetilde{G}$-representation V is also a G-representation, which yields compatible tractor bundles over M and $\widetilde{M}$ with compatible tractor connections: $\mathcal{V} = \mathcal{G} \times_P V \to M$ with the tractor connection ∇ induced by ω and $\widetilde{\mathcal{V}} = \widetilde{\mathcal{G}} \times_{\widetilde{P}} V = \mathcal{G} \times_Q V \to \widetilde{M}$ with the tractor connection $\widetilde{\nabla}^{\mathrm{ind}}$ induced by $\widetilde{\omega}^{\mathrm{ind}}$. Sections of $\mathcal{V}$ bijectively correspond to P-equivariant functions $\varphi\colon \mathcal{G} \to V$, while sections of $\widetilde{\mathcal{V}}$ correspond to Q-equivariant functions $\varphi\colon \mathcal{G} \to V$. Since $Q \subseteq P$, every section of $\mathcal{V}$ gives rise to a section of $\widetilde{\mathcal{V}}$, and we can view $\Gamma(\mathcal{V}) \subseteq \Gamma(\widetilde{\mathcal{V}})$. Now, Proposition 3.2 in [8] admits a straightforward generalisation to Fefferman-type constructions for which P/Q is connected and thus, in particular, to the one studied in this article:

Proposition 3.3.

(a) *A section $s \in \Gamma(\widetilde{\mathcal{V}})$ is contained in $\Gamma(\mathcal{V})$ (i.e., the corresponding Q-equivariant function φ is indeed P-equivariant) if and only if $\widetilde{\nabla}^{\mathrm{ind}} s$ is strictly horizontal (i.e., $v^a \widetilde{\nabla}^{\mathrm{ind}}_a s = 0$ for all $v^a \in \Gamma(\widetilde{f})$).*

(b) *The restriction of $\widetilde{\nabla}^{\mathrm{ind}}$ to $\Gamma(\mathcal{V}) \subseteq \Gamma(\widetilde{\mathcal{V}})$ coincides with the tractor connection ∇.*

Remark 3.4. Another instance of compatible bundles over M and $\widetilde{M}$ is provided by the density bundles $\mathbb{E}(w)$ and $\widetilde{\mathbb{E}}[w]$, which are defined via the representation of P and $\widetilde{P}$ as in (2.3) and (2.4), respectively. Restricting these representations to Q, it easily follows that the notation is indeed compatible so that we can view $\Gamma(\mathbb{E}(w)) \subseteq \Gamma(\widetilde{\mathbb{E}}[w])$.

Both projective and conformal density bundles can be described as associated bundles to the respective bundles of scales. Hence everywhere positive sections of density bundles are considered as scales. In particular, the inclusion $\Gamma(\mathbb{E}_+(1)) \subseteq \Gamma(\widetilde{\mathbb{E}}_+[1])$ may be interpreted so that any projective scale induces a conformal one. Such conformal scales will be called *reduced scales*. An intrinsic characterisation of reduced scales among all conformal ones is formulated in Proposition 5.2.

The previous remark yields that any projective exact Weyl structure on M induces a conformal exact Weyl structure on $\widetilde{M}$. This fact can be generalised as follows:

Proposition 3.5. *Any projective (exact) Weyl structure on M induces a conformal (exact) Weyl structure on the Fefferman space $\widetilde{M}$.*

Proof. A version of this result in a more general context was proved in [1, Proposition 6.1]: any Weyl structure for ω induces a Weyl structure for $\widetilde{\omega}^{\mathrm{ind}}$ if $P_+ \subseteq \widetilde{P}$ and $(G_0 \cap \widetilde{P}) \subseteq \widetilde{G}_0$. But both these conditions are satisfied as follows from the setup in Section 3.2 and explicit realisations in Appendix A. ■

Conformal Weyl structures induced by projective ones as above will be called *reduced Weyl structures.*

3.5 Normality

Here we show that our Fefferman-type construction does not preserve the normality in general, see Proposition 3.8. This can be shown directly as we did in a previous version of the article, see arXiv:1510.03337v2. Alternatively, we can treat the construction as the composition of two other constructions via a natural intermediate Lagrangean contact structure.

A *Lagrangean contact structure* on M' consists of a contact distribution $\mathcal{H} \subseteq TM'$ together with a decomposition $\mathcal{H} = e' \oplus f'$ into two subbundles that are maximally isotropic with respect to the Levi form $\mathcal{H} \times \mathcal{H} \to TM'/\mathcal{H}$. Such structure on a manifold M' of dimension $2n-1$ is equivalently encoded as a normal parabolic geometry of type (G, P'), where $G = \mathrm{SL}(n+1)$ and $P' \subseteq G$ is the stabiliser of a flag of type line-hyperplane in the standard representation $\mathbb{R}^{n+1}$. For $n > 2$ there are three harmonic curvatures, two of which are torsions whose vanishing is equivalent to the integrability of the respective subbundles $e', f' \subseteq \mathcal{H}$. For $n = 2$ there are two harmonic curvatures of homogeneity 4, hence the Cartan connection is torsion-free. In that case both e' and f' are 1-dimensional and thus automatically integrable.

On the one hand, P' is contained in P, where $P \subseteq G$ is the stabiliser of a ray in $\mathbb{R}^{n+1}$. For suitable choices as in Appendix A, the Lie algebra to P' consists of matrices of the form

$$\mathfrak{p}' = \begin{pmatrix} a & U^t & w \\ 0 & B & V \\ 0 & 0 & c \end{pmatrix}.$$

Given a projective Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), it turns out that the correspondence space $M' := \mathcal{G}/P'$ can be identified with the projectivised cotangent bundle $\mathcal{P}(T^*M)$. The Cartan geometry $(\mathcal{G} \to M', \omega)$ of type (G, P') is regular and thus it covers a natural Lagrangean contact structure on M'. In particular, the canonical contact distribution on $\mathcal{P}(T^*M)$ coincides with $\mathcal{H}$ and the vertical subbundle of the projection $M' \to M$ coincides with one of the two distinguished subbundles, say $f' \subseteq \mathcal{H}$. As in general, this construction preserves normality. In accord with [5], respectively [11, Section 4.4.2] we may state:

Proposition 3.6. *Let $(\mathcal{G} \to M, \omega)$ be a normal projective parabolic geometry and let $(\mathcal{G} \to M', \omega)$ be the corresponding normal Lagrangean contact parabolic geometry. The latter geometry is torsion-free if and only if $n = 2$ or it is flat, i.e., the initial projective structure is flat.*

On the other hand, P' contains Q, where $Q = G \cap \widetilde{P}$ as before. This allows us to consider the Fefferman-type construction for the pairs (G, P') and $(\widetilde{G}, \widetilde{P})$. Given a Lagrangean contact structure on M', it induces a conformal spin structure on $\widetilde{M} = \mathcal{G}/Q$. This construction is indeed very similar to the original Fefferman construction; one deals with different real forms of the same complex Lie groups in the two cases. That is why the following statement and its proof is analogous to the one for the CR case. Following [8], respectively [11, Section 4.5.2] we may state:

Proposition 3.7. *Let $(\mathcal{G} \to M', \omega)$ be the normal Lagrangean contact parabolic geometry and let $\left(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega}^{\mathrm{ind}}\right)$ be the conformal parabolic geometry obtained by the Fefferman-type construction. Then $\widetilde{\omega}^{\mathrm{ind}}$ is normal if and only if ω is torsion-free.*

Altogether, composing the two previous steps we obtain our projective-to-conformal Fefferman-type construction with the desired control of the normality. Note that from (3.5) and the respective matrix realisations it follows that the induced objects on $\widetilde{M} = T^*M(2) \setminus \{0\}$ from Proposition 3.2 correspond to the induced objects on $M' = \mathcal{P}(T^*M)$. In particular, the vertical subbundle of the projection $\widetilde{M} \to M'$ is spanned by k and the decomposition $k^\perp = \widetilde{e} \oplus \widetilde{f} \subseteq T\widetilde{M}$ descends to the decomposition $\mathcal{H} = e' \oplus f' \subseteq TM'$

$$
\begin{tikzcd}
 & & \widetilde{\mathcal{G}} \arrow[d, "\widetilde{P}"] \\
\mathcal{G} \arrow[urr, hook] \arrow[rr, "Q"] \arrow[dr, "P'"] \arrow[dd, "P"'] & & \widetilde{M} \arrow[dl] \\
 & M' \arrow[dl] & \\
M. & &
\end{tikzcd}
$$

Proposition 3.8. *Let $(\mathcal{G} \to M, \omega)$ be a normal projective parabolic geometry and let $\big(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega}^{\rm ind}\big)$ be the conformal parabolic geometry obtained by the Fefferman-type construction.*

(*a*) *If* $\dim M = 2$ *then* $\widetilde{\omega}^{\rm ind}$ *is normal.*

(*b*) *If* $\dim M > 2$ *then* $\widetilde{\omega}^{\rm ind}$ *is normal if and only if* ω *is flat.*

Moreover, independently of the dimension of M, $\widetilde{\omega}^{\rm ind}$ is flat if and only if ω is flat.

3.6 Remarks on torsion-free Lagrangean contact structures

At this stage it is easy to formulate a local characterisation of split-signature conformal structures arising from torsion-free Lagrangean contact structures, see Proposition 3.10. As before, the results and their proofs are very analogous to those in the CR case, therefore we just quickly indicate the reasoning and point to differences.

As in Proposition 3.2, the G-invariant algebraic objects induce the tractor fields $\mathbf{s}_E$, $\mathbf{s}_F$ and $\mathbf{K}$ on the conformal Fefferman space that are parallel with respect to the induced tractor connection and have the required compatibility properties. But, starting with a torsion-free Lagrangean contact structure, the induced connection is already normal. In particular, the corresponding underlying objects χ, η and k are pure twistor spinors and a light-like conformal Killing field, respectively.

The existence of parallel tractors $\mathbf{s}_E$, $\mathbf{s}_F$ and $\mathbf{K}$ with the algebraic properties as in Proposition 3.2 are by no means independent conditions:

Proposition 3.9. *Let $\big(\widetilde{M}, \mathbf{c}\big)$ be a conformal spin structure of split-signature (n,n). Then the following conditions are locally equivalent:*

(*a*) *The spin tractor bundle admits two pure parallel tractor spinors $\mathbf{s}_E \in \Gamma\big(\widetilde{\mathcal{S}}_\pm\big)$ and $\mathbf{s}_F \in \Gamma\big(\widetilde{\mathcal{S}}_-\big)$ with non-trivial pairing.*

(*b*) *The conformal holonomy* $\mathrm{Hol}(\mathbf{c})$ *reduces to* $\mathrm{SL}(n+1) \subseteq \mathrm{Spin}(n+1, n+1)$ *preserving a decomposition into maximally isotropic subspaces $E \oplus F = \mathbb{R}^{n+1,n+1}$.*

(*c*) *The adjoint tractor bundle admits a parallel involution $\mathbf{K} \in \Gamma\big(\mathcal{A}\widetilde{M}\big)$, i.e., $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$.*

The only subtle point within the proof concerns the consequences of property (c). The existence of a parallel skew-symmetric involution $\mathbf{K}$ on the standard tractor bundle immediately implies that the conformal holonomy $\mathrm{Hol}(\mathbf{c})$ is reduced to $\mathrm{GL}(n+1)$. But, analogously to the corresponding discussion for the CR case in [9] or [23], one can show that $\mathrm{Hol}(\mathbf{c})$ is actually contained in $\mathrm{SL}(n+1)$. The rest follows easily.

It turns out that conformal spin structures induced by torsion-free Lagrangean contact structures are locally characterised by any of the three equivalent conditions above. Indeed, according to results from [10], the holonomy reduction of the conformal structure to $G = \mathrm{SL}(n+1) \subseteq \mathrm{Spin}(n+1, n+1) = \widetilde{G}$ yields the so-called curved orbit decomposition of $\widetilde{M}$, which corresponds to the decomposition of the homogeneous model $\widetilde{G}/\widetilde{P}$ with respect to the action of G. Each subset from the decomposition of $\widetilde{M}$, provided it is non-empty, further carries a geometry of the same type as its counterpart in the homogeneous model. From Section 3.2 we know there is one open and two closed n-dimensional orbits. The closed n-dimensional orbits carry Cartan geometries of type (G, P), and thus inherit projective structures, the open orbit carries a Cartan geometry of type (G, Q). Note that the two closed orbits coincide with the zero sets of χ and η, the open subset is the one where both spinors, and thus k, are non-vanishing. Since k is the conformal Killing field corresponding to the parallel adjoint tractor $\mathbf{K}$, it inserts trivially into the curvature of the normal Cartan connection, cf. (2.13). Hence, according to [5], the Cartan geometry of type (G, Q) on the open orbit of $\widetilde{M}$ descends to a Cartan geometry of type (G, P') on the local leaf space M' determined by k. It follows that this Cartan geometry is torsion-free and thus determines a torsion-free Lagrangean contact structure. Altogether, following [9] we may state the following characterisation:

Proposition 3.10. *A split-signature conformal spin structure is locally induced by a torsion-free Lagrangean contact structure via the Fefferman-type construction if and only if any of the equivalent conditions from Proposition 3.9 holds and the underlying twistor spinors χ and η and the conformal Killing field k are nowhere-vanishing.*

3.7 The exceptional case: dimension $n = 2$

From Section 3.5 we know that the intermediate 3-dimensional Lagrangean contact structure on M' induced by a 2-dimensional projective structure on M is torsion-free. Hence the induced conformal Cartan geometry on $\widetilde{M}$ is normal and thus all the equivalent conditions from Proposition 3.9 are satisfied. Moreover, the fact that it comes from a projective structure implies that any vertical vector of the projection $\widetilde{M} \to M$ inserts trivially into the Cartan curvature, i.e.,

$$i_X \widetilde{\kappa}(u) = 0, \qquad \text{for all } X \in f,\, u \in \widetilde{\mathcal{G}}. \tag{3.8}$$

Analogously to the discussion before Proposition 3.10 we may conclude:

Proposition 3.11. *A conformal spin structure of signature $(2, 2)$ is locally induced by a 2-dimensional projective structure via the Fefferman-type construction if and only if any of the equivalent conditions from Proposition 3.9 holds, the underlying twistor spinors χ and η and the conformal Killing field k are nowhere-vanishing and the curvature of the normal conformal Cartan connection satisfies (3.8).*

Remark 3.12. Conformal structures induced from 2-dimensional projective structures are well-studied, see, e.g., [14, 15, 25]. Notably, the intermediate 3-dimensional Lagrangean contact structure can be equivalently viewed as a path geometry (or the geometry associated to second order ODEs modulo point transformations). Such structure is induced by a projective structure (i.e., the paths are the unparametrised geodesics of the projective class of connections) if and only if one of the two harmonic curvatures vanishes. It follows from [25] that this is equivalent to vanishing of the self-dual, respectively anti-self-dual part of the Weyl curvature of the induced conformal structure. In particular, the condition (3.8) in the previous proposition can be replaced by the condition that the conformal structure is half-flat.

4 Normalisation and characterisation

By Proposition 3.8, for $n \geq 3$, the induced conformal Cartan connection associated to a non-flat n-dimensional projective structure differs from the normal conformal Cartan connection for the induced conformal structure. In this section we will analyse the form of the difference and thus derive properties of the induced conformal structures. Furthermore, we will show that any split-signature conformal manifold having these properties is locally equivalent to the conformal structure on the Fefferman space over a projective manifold.

4.1 The normalisation process

We are going to normalise the conformal Cartan connection $\widetilde{\omega}^{\mathrm{ind}} \in \Omega^1\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ that is induced by a normal projective Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Any other conformal Cartan connection $\widetilde{\omega}'$ differs from $\widetilde{\omega}^{\mathrm{ind}}$ by some $\Psi \in \Omega^1\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ so that $\widetilde{\omega}' = \widetilde{\omega}^{\mathrm{ind}} + \Psi$. This Ψ must vanish on vertical fields and be $\widetilde{P}$-equivariant. The condition on $\widetilde{\omega}'$ to induce the same conformal structure on $\widetilde{M}$ as $\widetilde{\omega}^{\mathrm{ind}}$ is that Ψ has values in $\tilde{\mathfrak{p}} \subseteq \tilde{\mathfrak{g}}$. One can therefore regard Ψ as a $\widetilde{P}$-equivariant function $\Psi\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. According to the general theory as outlined in [11, Section 3.1.13] there is a unique such Ψ such that the curvature function $\widetilde{\kappa}'$ of $\widetilde{\omega}'$ satisfies $\widetilde{\partial}^*\widetilde{\kappa}' = 0$, and then $\widetilde{\omega}'$ is the normal conformal Cartan connection $\widetilde{\omega}^{\mathrm{nor}}$.

The failure of $\widetilde{\omega}^{\mathrm{ind}}$ to be normal is given by $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. The normalisation of $\widetilde{\omega}^{\mathrm{ind}}$ proceeds by homogeneity of $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$, which decomposes into two homogeneous components according to the decomposition $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{p}}_+$. In the first step of normalisation one looks for a Ψ^1 such that $\widetilde{\omega}^1 = \widetilde{\omega} + \Psi^1$ has $\widetilde{\partial}^*\widetilde{\kappa}^1$ taking values in the highest homogeneity, i.e., $\widetilde{\partial}^*\widetilde{\kappa}^1\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}_+$.

To write down this first normalisation we employ Weyl structures $\widetilde{\mathcal{G}}_0 \hookrightarrow \widetilde{\mathcal{G}}$. By Proposition 3.5 we can take a reduced Weyl structure, i.e., one that is induced by a reduction $\mathcal{G}_0 \hookrightarrow \mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ with respect to the structure group $Q_0 := Q \cap G_0$. This allows us to project $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}$ to $\big(\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}\big)_0\colon \mathcal{G}_0 \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$ and to employ the $\widetilde{G}_0$-equivariant Kostant Laplacian $\widetilde{\square}\colon (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0 \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$, $\widetilde{\square} := \widetilde{\partial} \circ \widetilde{\partial}^* + \widetilde{\partial}^* \circ \widetilde{\partial}$. For the first normalisation step we need to form a map $\Psi^1\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$ that agrees with $-\widetilde{\square}^{-1}\big(\widetilde{\partial^*\widetilde{\kappa}^{\mathrm{ind}}}\big)_0$ in the $\tilde{\mathfrak{g}}_0$-component. If we have formed any such Ψ^1 along $\mathcal{G}_0 \hookrightarrow \widetilde{\mathcal{G}}$ we can just equivariantly extend this to all of $\widetilde{\mathcal{G}}$.

To proceed with the analysis of the normalisation we need to establish a couple of technical lemmas. As before, we denote by $f^\circ \subset \tilde{\mathfrak{p}}_+ \cong (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$ the annihilator of $f = \mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Recall that $f^\circ = \varphi(\mathfrak{p}_+) \cong f[-2]$.

Lemma 4.1. *Let V be a $\mathfrak{g}$-representation contained in a $\tilde{\mathfrak{g}}$-representation $\widetilde{V}$ and denote by $\phi \mapsto \widetilde{\phi}$ the inclusion $\Lambda^k\mathfrak{p}_+ \otimes V \hookrightarrow \Lambda^k\tilde{\mathfrak{p}}_+ \otimes \widetilde{V}$ induced by $\varphi\colon \mathfrak{p}_+ \to \tilde{\mathfrak{p}}_+$ and $V \hookrightarrow \widetilde{V}$. Then, for any $\phi \in \Lambda^k\mathfrak{p}_+ \otimes V$,*

$$\widetilde{\partial^*\phi} - \widetilde{\partial}^*\widetilde{\phi} \in \Lambda^{k-1} f^\circ \otimes \big(\Lambda^2\bar{F} \bullet V\big) \subseteq \Lambda^{k-1}\tilde{\mathfrak{p}}_+ \otimes \widetilde{V}.$$

In particular, for the adjoint representations, $\partial^*\phi = 0$ if and only if $\widetilde{\partial}^*\widetilde{\phi} \in \Lambda^{k-1} f^\circ \otimes \Lambda^2\bar{F}$.

Proof. For the sake of presentation, assume that ϕ is decomposable, i.e., of the form $\phi = Z_1 \wedge \cdots \wedge Z_k \otimes v$, where $Z_i \in \mathfrak{p}_+$ and $v \in V$. Let us denote by the same symbols also the images of these elements under the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ and $V \hookrightarrow \widetilde{V}$, i.e., $Z_i \in \tilde{\mathfrak{p}}$ and $v \in \widetilde{V}$, respectively. Let $\widetilde{Z}_i \in f^\circ$ be the images of Z_i under the inclusion $\varphi\colon \mathfrak{p}_+ \to \tilde{\mathfrak{p}}_+$. Now, by definition of the

Kostant co-differential, the difference $\widetilde{\partial^*\phi} - \widetilde{\partial}^*\widetilde{\phi}$ evaluated on any $k-1$ elements from $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ is a linear combination of terms of the form

$$\big(Z_i - \widetilde{Z}_i\big) \bullet v. \tag{4.1}$$

However, the differences $Z_i - \widetilde{Z}_i \in \tilde{\mathfrak{p}}$ are represented by the matrices as in (A.1) in the Appendix where only the Z-entries are non-vanishing and hence contained in $\Lambda^2 F \cap \tilde{\mathfrak{p}} = \Lambda^2 \bar{F}$. Thus (4.1) belong to the image of $\bullet\colon \Lambda^2\bar{F} \times V \to \widetilde{V}$ and the first claim follows.

For the second claim we use that $\Lambda^2\bar{F} \bullet \mathfrak{g} = \big[\Lambda^2\bar{F}, \mathfrak{g}\big] \subseteq \Lambda^2 F$ and $\Lambda^2 F \cap \mathfrak{g} = 0$: since $\widetilde{\partial^*\phi}$ (evaluated on any $k-1$ elements from $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$) has values in $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, vanishing of $\partial^*\phi$ is equivalent to $\widetilde{\partial}^*\widetilde{\phi}$ having values in $\Lambda^2 F$. But $\widetilde{\partial}^*\widetilde{\phi}$ has generally values in $\tilde{\mathfrak{p}}$ and $\Lambda^2 F \cap \tilde{\mathfrak{p}} = \Lambda^2\bar{F}$, hence the claim follows. ■

Lemma 4.2. *If $\psi \in \tilde{\mathfrak{p}}_+ \wedge f^\circ \otimes \Lambda^2\bar{F} \subseteq \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}$ then $\widetilde{\partial}^*\psi \in f^\circ \otimes f^\circ \subseteq \tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}_+$.*

Proof. ψ is a sum of terms of the form $Z_1 \wedge Z_2 \otimes A$, where $Z_1 \in \tilde{\mathfrak{p}}_+$, $Z_2 \in f^\circ$ and $A \in \Lambda^2\bar{F}$. Applying the Kostant co-differential gives

$$\widetilde{\partial}^*(Z_1 \wedge Z_2 \otimes A) = Z_1 \otimes [Z_2, A] - Z_2 \otimes [Z_1, A].$$

Now $[Z_2, A]$ belongs to $\big[f^\circ, \Lambda^2\bar{F}\big] = 0$ and $[Z_1, A]$ belongs to $\big[\tilde{\mathfrak{p}}_+, \Lambda^2\bar{F}\big] = f^\circ$, hence the claim follows. ■

The following lemma contains the crucial information which is necessary to perform our normalisation. We are going to specify the curvature function $\widetilde{\kappa}^{\mathrm{ind}}$ (later also $\widetilde{\kappa}^{\mathrm{nor}}$) by describing its values along the natural Q-reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ over $\widetilde{M}$. Recall from Section 3.2 that $\Lambda^2\bar{F}$ is a Q-invariant subspace in $\tilde{\mathfrak{g}}_0$, which can be identified with $(\Lambda^2 f)[-2]$.

Lemma 4.3. *For any $u \in \mathcal{G}$, we have*

$$\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}(u) \in f^\circ \otimes \Lambda^2\bar{F} \subseteq \tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}_0.$$

Identifying $\Lambda^2\bar{F} \cong \big(\Lambda^2 f\big)[-2]$ and $f^\circ \cong f[-2]$, we have in fact $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}(u) \in \big(f \odot \Lambda^2 f\big)[-4]$, i.e., $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}(u)$ is contained in the kernel of the alternation map

$$\mathrm{alt}\colon\ \big(f \otimes \Lambda^2 f\big)[-4] \to \big(\Lambda^3 f\big)[-4].$$

Proof. It is a general assumption that $\widetilde{\omega}^{\mathrm{ind}}$ is induced by a normal projective Cartan connection on $\mathcal{G}$, i.e., $\partial^*\kappa(u) = 0$, for any $u \in \mathcal{G}$. Hence it follows from Lemma 4.1 that $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}(u)$ belongs to $f^\circ \otimes \Lambda^2\bar{F} \cong \big(f \otimes \Lambda^2 f\big)[-4]$.

Further we need a finer discussion involving the properties of $\kappa\colon \mathcal{G} \to \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ to show that $\kappa(u)$ belongs to the kernel of the Q-equivariant map $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \to \big(\Lambda^3 f\big)[-4]$ given by

$$\phi \mapsto \mathrm{alt}\,\big(\widetilde{\partial}^*\widetilde{\phi}\big). \tag{4.2}$$

Note that any element $\phi \in \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{p}_+$ for which $\partial^*\phi = 0$ is mapped to zero: since $\widetilde{\phi} \in \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}$ and $[\tilde{\mathfrak{p}}_+, \tilde{\mathfrak{p}}] = \tilde{\mathfrak{p}}_+$, the co-differential $\widetilde{\partial}^*\widetilde{\phi}$ has values in $f^\circ \otimes \tilde{\mathfrak{p}}_+$. But, by Lemma 4.1, it also has values in $f^\circ \otimes \Lambda^2\bar{F}$ and $\tilde{\mathfrak{p}}_+ \cap \Lambda^2\bar{F} = 0$.

Thus it suffices to consider the harmonic elements from $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}_0$, i.e., the ones corresponding to the projective Weyl tensor. For that purpose we consider the simple part of $Q_0 = Q \cap G_0$ which is isomorphic to $\mathrm{SL}(n-1)$, cf. the matrix realisation (A.2) in the Appendix where it corresponds to the A-block. Considering both $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}_0 \cong \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n$ and $\big(\Lambda^3 f\big)[-4] \cong \Lambda^3\mathbb{R}^{n*}$ as representations of $\mathrm{SL}(n-1)$, the map (4.2) is either trivial or an isomorphism on each $\mathrm{SL}(n-1)$-irreducible component.

One can check that there is only one $\mathrm{SL}(n-1)$-irreducible component that occurs in both spaces, and it is isomorphic to $\Lambda^2\mathbb{R}^{n-1*}$. Hence it suffices to compute (4.2) on one element contained in such component. Let $X_n \in \mathfrak{g}_-$ and $Z_n \in \mathfrak{p}_+$ be the two dual basis vectors stabilised by $\mathrm{SL}(n-1)$ and consider an element

$$\phi = Z_1 \wedge Z_2 \otimes X_n \otimes Z_n - Z_1 \wedge Z_2 \otimes X_1 \otimes Z_1 + Z_n \wedge Z_2 \otimes X_n \otimes Z_1.$$

Indeed ϕ is completely trace-free, satisfies the algebraic Bianchi identity and the $\mathrm{SL}(n-1)$-orbit of ϕ is isomorphic to $\Lambda^2\mathbb{R}^{n-1*}$. Now,

$$\widetilde{\partial}^*\widetilde{\phi} = -\widetilde{Z}_1 \otimes \widetilde{Z}_n \wedge \widetilde{Z}_2 - \widetilde{Z}_n \otimes \widetilde{Z}_1 \wedge \widetilde{Z}_2,$$

which indeed lies in the kernel of the alternation map. Hence the statement follows. ∎

We can now determine the form of the normal conformal Cartan connection:

Proposition 4.4. *The normal conformal Cartan connection is of the form*

$$\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \Psi^1 + \Psi^2,$$

where $\Psi^1 = -\frac{1}{2}\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}} \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \widetilde{\mathfrak{p}}\big)$ *and* $\Psi^2 \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \widetilde{\mathfrak{p}}_+\big)$. *Furthermore, along the reduction* $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ *we have* $\Psi^1 \in \Omega^1_{\mathrm{hor}}\big(\mathcal{G}, \Lambda^2\bar{F}\big)$, $\Psi^2 \in \Omega^1_{\mathrm{hor}}(\mathcal{G}, f^\circ)$.

Remark 4.5. Since Ψ^1 and Ψ^2 are horizontal, they may equivalently be regarded as bundle-valued 1-forms on $\widetilde{M}$. Denoting by $\Lambda^2\widetilde{\bar{\mathcal{F}}}$ the associated bundle $\mathcal{G} \times_Q \Lambda^2\bar{F}$ over $\widetilde{M}$ and by $\widetilde{f}^\circ \subseteq T^*\widetilde{M}$ the annihilator of $\widetilde{f} = \ker\chi \subseteq T\widetilde{M}$, Proposition 4.4 says

$$\Psi^1 \in \Omega^1\big(\widetilde{M}, \Lambda^2\widetilde{\bar{\mathcal{F}}}\big), \qquad \Psi^2 \in \Omega^1\big(\widetilde{M}, \widetilde{f}^\circ\big), \qquad \Psi^1(v) = \Psi^2(v) = 0, \qquad \text{for all } v \in \Gamma(\ker\chi).$$

Below we also use the corresponding frame forms, i.e., the $\widetilde{P}$-equivariant functions $\phi^1\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$ and $\phi^2\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}_+$ such that, for any $u \in \widetilde{\mathcal{G}}$, $\Psi^1 = \phi^1(u) \circ \widetilde{\omega}^{\mathrm{ind}}$ and $\Psi^2 = \phi^2(u) \circ \widetilde{\omega}^{\mathrm{ind}}$. In these terms, the proposition means that along the reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ these maps restrict to Q-equivariant functions

$$\phi^1\colon\ \mathcal{G} \to f^\circ \otimes \Lambda^2\bar{F}, \qquad \phi^2\colon\ \mathcal{G} \to f^\circ \otimes f^\circ.$$

Further we put $\Psi = \Psi^1 + \Psi^2$ and $\phi = \phi^1 + \phi^2$.

Proof. The Kostant Laplacian $\widetilde{\square}$ restricts to an invertible endomorphism of $((\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0) \cap \operatorname{im}\widetilde{\partial}^*$ that acts by scalar multiplication on each of the $\widetilde{G}_0$-irreducible components. Now, restricting to $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ and suppressing all arguments $u \in \mathcal{G}$, it was shown in Lemma 4.3 that $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}$ is contained in one of the irreducible components, namely in $\big(f \odot \Lambda^2 f\big)[-4]$. On this component $\widetilde{\square}$ acts by multiplication by 2. Thus, the modification map accomplishing the first normalisation step is

$$\phi^1\colon\ \mathcal{G} \to f^\circ \otimes \Lambda^2\bar{F}, \qquad \phi^1 := -\frac{1}{2}\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}} = -\widetilde{\square}^{-1}\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}.$$

Now, let $\widetilde{\omega}^1 := \widetilde{\omega}^{\mathrm{ind}} + \phi^1 \circ \widetilde{\omega}^{\mathrm{ind}}$ be the modified Cartan connection. The corresponding curvature function $\widetilde{\kappa}^1$ can be expressed in terms of $\widetilde{\kappa}^{\mathrm{ind}}$, ϕ^1 and its differential $d\phi^1$ so that

$$\begin{aligned}\widetilde{\kappa}^1(X,Y) = \widetilde{\kappa}^{\mathrm{ind}}(X,Y) &+ \big[X, \phi^1(Y)\big] - \big[Y, \phi^1(X)\big]\\ &+ d\phi^1(\xi)(Y) - d\phi^1(\eta)(X) - \phi^1([X,Y]) + \big[\phi^1(X), \phi^1(Y)\big],\end{aligned} \tag{4.3}$$

where $X, Y \in \mathfrak{g}$ and $\xi = \big(\widetilde{\omega}^{\mathrm{ind}}\big)^{-1}(X)$, $\eta = \big(\widetilde{\omega}^{\mathrm{ind}}\big)^{-1}(Y)$, cf. [11, formula (3.1)]. For the last term we have $\big[\phi^1(X), \phi^1(Y)\big] = 0$ since $\phi^1(X)$ has values in $\Lambda^2 \bar{F}$. The first three terms are

$$\widetilde{\kappa}^{\mathrm{ind}}(X,Y) + \big[X, \phi^1(Y)\big] - \big[Y, \phi^1(X)\big] = \widetilde{\kappa}^{\mathrm{ind}}(X,Y) + \widetilde{\partial}\phi^1(X,Y),$$

which by construction vanishes upon application of the Kostant co-differential, i.e., $\widetilde{\partial}^*\big(\widetilde{\kappa}^{\mathrm{ind}} + \widetilde{\partial}\phi^1\big) = 0$. The remaining terms in (4.3) can be combined into a map $\Lambda^2\mathfrak{g} \to \Lambda^2 \bar{F}$,

$$(X,Y) \mapsto d\phi^1(\xi)(Y) - d\phi^1(\eta)(X) - \phi^1([X,Y]),$$

which vanishes upon insertion of two elements $X, Y \in \mathfrak{p}$. Therefore, applying Lemma 4.2, we conclude that $\widetilde{\partial}^*\widetilde{\kappa}^1$ has values in $f^\circ \otimes f^\circ$. Thus the second modification map is

$$\phi^2\colon\ \mathcal{G} \to f^\circ \otimes f^\circ, \qquad \phi^2 := -\widetilde{\Box}^{-1}\widetilde{\partial}^*\widetilde{\kappa}^1. \qquad \blacksquare$$

4.2 Properties

The information provided in the previous proposition allows us to determine the properties satisfied by the normal conformal Cartan curvature:

Proposition 4.6. *The normal conformal Cartan curvature $\widetilde{\kappa}^{\mathrm{nor}}$ restricts to a map*

$$\widetilde{\kappa}^{\mathrm{nor}}\colon\ \mathcal{G} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \big(\mathfrak{sl}(n+1) \oplus \Lambda^2 F\big). \tag{4.4}$$

Moreover, the following integrability condition holds:

$$i_X\widetilde{\kappa}^{\mathrm{nor}}(u) \in f^\circ \otimes \big(\Lambda^2\bar{F} \oplus f^\circ\big), \qquad \textit{for all } X \in f,\ u \in \mathcal{G}. \tag{4.5}$$

Proof. Let $\widetilde{\kappa}^{\mathrm{nor}}$ be the curvature function of the normal Cartan connection $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \phi \circ \widetilde{\omega}^{\mathrm{ind}}$, where $\phi = \phi^1 + \phi^2$. With the same conventions as in the proof of Proposition 4.4, [11, formula (3.1)] yields

$$\begin{aligned}\widetilde{\kappa}^{\mathrm{nor}}(X,Y) = \widetilde{\kappa}^{\mathrm{ind}}(X,Y) &+ [X, \phi(Y)] - [Y, \phi(X)]\\ &+ d\phi(\xi)(Y) - d\phi(\eta)(X) - \phi([X,Y]) + [\phi(X), \phi(Y)].\end{aligned}$$

Clearly, $\widetilde{\kappa}^{\mathrm{ind}}(X,Y)$ has values in $\mathfrak{sl}(n+1)$ and vanishes upon insertion of $X \in \mathfrak{p}$. A term of the form $[X, \phi(Y)]$ vanishes if $Y \in \mathfrak{p}$ and has values in $\big[\mathfrak{p}, \Lambda^2\bar{F} \oplus f^\circ\big] \subseteq \Lambda^2\bar{F} \oplus f^\circ$ for $X \in \mathfrak{p}$. A term of the form $d\phi(\xi)(Y)$ has values in $\Lambda^2\bar{F} \oplus f^\circ$ and vanishes for $Y \in \mathfrak{p}$. The term $\phi([X,Y])$ has values in $\Lambda^2\bar{F} \oplus f^\circ$ and vanishes for $X, Y \in \mathfrak{p}$. The last term $[\phi(X), \phi(Y)]$ vanishes for all $X, Y \in \mathfrak{g}$ since $\phi(X)$ has values in $\Lambda^2\bar{F} \oplus f^\circ$. Altogether, we obtain (4.4) and (4.5). $\blacksquare$

We observe here that it follows directly from (4.4) that the pairing of $\widetilde{\kappa}^{\mathrm{nor}}$ with the involution K vanishes, $\langle \widetilde{\kappa}^{\mathrm{nor}}, K\rangle = 0$.

To derive properties of induced tractorial and underlying objects on the conformal structure we will need the following preparatory lemma.

Lemma 4.7. *Let V be a $\widetilde{G}$-representation and $v \in V$ an element which is stabilised under $G \subseteq \widetilde{G}$. Let $\mathbf{v} \in \Gamma\big(\widetilde{\mathcal{V}}\big)$ be the section of the associated tractor bundle $\widetilde{\mathcal{G}} \times_{\widetilde{P}} V$ corresponding to the constant function $\mathcal{G} \to V$, $u \mapsto v$, along $\mathcal{G}$. Then the covariant derivative $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{v}$ corresponds to the Q-equivariant function*

$$\mathcal{G} \to f^\circ \otimes V, \qquad u \mapsto \phi^1(u) \bullet v + \phi^2(u) \bullet v.$$

Proof. The covariant derivative $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{v}$ corresponds to the map

$$X \in \mathfrak{g} \mapsto \big(\widetilde{\omega}^{\mathrm{nor}}\big)^{-1}(X) \cdot v + X \bullet v. \tag{4.6}$$

The first term in (4.6) vanishes since it is the directional derivative of the constant function v. Now $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \phi^1 + \phi^2$, and since $X \bullet v = 0$ the claim follows. ■

We now show that the distinguished tractors $\mathbf{s}_E$, $\mathbf{s}_F$ and $\mathbf{K}$ on the Fefferman space are all given as BGG-splittings from their underlying objects. Moreover, several stronger properties hold:

Proposition 4.8. *Let $\mathbf{s}_E \in \Gamma\big(\widetilde{\mathcal{S}}_\pm\big)$, $\mathbf{s}_F \in \Gamma\big(\widetilde{\mathcal{S}}_-\big)$ and $\mathbf{K} \in \Gamma\big(\mathcal{A}\widetilde{M}\big)$ be the tractor spinors and the adjoint tractor, respectively, and let $\eta = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_E)$, $\chi = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_F)$ and $k = \Pi_0^{\mathcal{A}\widetilde{M}}(\mathbf{K})$ be the corresponding underlying objects as in Proposition 3.2.*

(*a*) *The tractor spinor $\mathbf{s}_F$ is parallel, i.e., $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{s}_F = 0$. In particular, χ is a pure twistor spinor, $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}_-}(\chi)$ and $\mathrm{Hol}(\mathbf{c}) \subseteq \mathrm{SL}(n+1) \ltimes \Lambda^2\big(\mathbb{R}^{n+1}\big)^* \subseteq \mathrm{Spin}(n+1, n+1)$.*

(*b*) *The tractor spinor $\mathbf{s}_E$ is a BGG-splitting, i.e., $\widetilde{\partial}^*\big(\widetilde{\nabla}^{\mathrm{nor}}\mathbf{s}_E\big) = 0$ and $\mathbf{s}_E = L_0^{\widetilde{\mathcal{S}}_\pm}(\eta)$.*

(*c*) *The adjoint tractor $\mathbf{K}$ is a BGG-splitting and k is a conformal Killing field, i.e., $\widetilde{\partial}^*\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} = 0$, $\mathbf{K} = L_0^{\mathcal{A}\widetilde{M}}(k)$ and $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} = i_k\widetilde{\Omega}^{\mathrm{nor}}$. Moreover, we have*

$$i_k\widetilde{\kappa}^{\mathrm{nor}} = 2\phi_{\Lambda^2 F}. \tag{4.7}$$

Proof. (a) Since ϕ^1, ϕ^2 have values in $\ker s_F$ we have $\phi^1 \bullet s_F + \phi^2 \bullet s_F = 0$. Thus, according to Lemma 4.7, we have $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{s}_F = 0$ and the rest is obvious.

(b) The spinor s_E is of the form $s_E = \binom{*}{\eta}$. According to Lemmas 4.3 and 4.7, ϕ^1 has values in $\big(f \odot \Lambda^2 f\big)[-4]$ and $\widetilde{\partial}^*\big(\widetilde{\nabla}^{\mathrm{nor}}\mathbf{s}_E\big)$ corresponds to

$$\widetilde{\partial}^*\big(\phi^1 \bullet s_E\big) = \begin{pmatrix} \big(\phi^1 \bullet \eta\big)_{\widetilde{\Sigma}_\mp[-\frac{1}{2}]} \\ 0 \end{pmatrix}.$$

The projection $\big(\phi^1 \bullet \eta\big)_{\widetilde{\Sigma}_\mp[-\frac{1}{2}]}$ can be realised as the full (triple) Clifford action on $\phi^1(u) \in \big(\bigotimes^3 f\big)[-4]$, where $u \in \mathcal{G}$. Now it is easy to see that this action must vanish for a $\phi^1(u) \in \big(f \odot \Lambda^2 f\big)[-4]$: We realise $\phi^1(u)$ equivalently in $\big(S^2 f \otimes f\big)[-4]$ by symmetrisation in the first two slots, then the complete Clifford action on η vanishes because the action of the first two slots is just a (trivial) trace multiplication.

(c) According to Lemma 4.7, $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K}$ corresponds to $\phi^1 \bullet K + \phi^2 \bullet K$. Since $K/\tilde{\mathfrak{p}} = k \in f$, the previous element lies in $\tilde{\mathfrak{p}}$. In particular, $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K}$ has trivial projecting slot, and thus $k = \Pi_0(\mathbf{K})$ is a conformal Killing field. Since $\phi^2 \bullet K \in \tilde{\mathfrak{p}}_+$, we have that $\widetilde{\partial}^*\big(\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K}\big)$ corresponds to $\widetilde{\partial}^*\big(\phi^1 \bullet K\big)$. Now $\phi^1 \bullet K = -K \bullet \phi^1 = 2\phi^1$, since K acts by multiplication with -2 on $\Lambda^2 \bar{F}$. But $\phi^1 \in \operatorname{im} \widetilde{\partial}^* \subseteq \ker \widetilde{\partial}^*$, and the expression $\widetilde{\partial}^*\big(\phi^1 \bullet K\big)$ therefore vanishes. The equality $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} = i_k\widetilde{\Omega}^{\mathrm{nor}}$ is just (2.13) for the conformal Killing field k with its BGG-splitting $\mathbf{K}$. In terms of the Q-equivariant functions $\phi = \phi^1 + \phi^2$ and $\widetilde{\kappa}^{\mathrm{nor}}$ along $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$, this can be expressed as $\phi \bullet K = i_k\widetilde{\kappa}^{\mathrm{nor}}$, which yields (4.7). ■

We now collect the essential information about the induced conformal structure $(\widetilde{M}, \mathbf{c})$ which we derived:

Proposition 4.9. *Let $\big(\widetilde{M}, \mathbf{c}\big)$ be the conformal spin structure induced from an oriented projective structure $(M, \mathbf{p})$ via the Fefferman-type construction. Then the following properties are satisfied:*

(a) $\big(\widetilde{M},\mathbf{c}\big)$ *admits a nowhere-vanishing light-like conformal Killing field k such that the corresponding tractor endomorphism $\mathbf{K} = L_0^{\mathcal{A}\widetilde{M}}(k)$ is an involution, i.e., $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$.*

(b) $\big(\widetilde{M},\mathbf{c}\big)$ *admits a pure twistor spinor $\chi \in \Gamma\big(\widetilde{\Sigma}_-\big[\tfrac{1}{2}\big]\big)$ with $k \in \Gamma(\ker\chi)$ such that the corresponding parallel tractor spinor $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}_-}(\chi)$ is pure.*

(c) $\mathbf{K}$ *acts by minus the identity on* $\ker \mathbf{s}_F$.

(d) *The following integrability condition holds:*

$$v^a w^c \widetilde{W}_{abcd} = 0, \qquad \textit{for all } v, w \in \Gamma(\ker\chi). \tag{W}$$

The only thing left to show for Proposition 4.9 is that the integrability condition (4.5) is equivalent to the condition (W) on the Weyl tensor:

Lemma 4.10. *Let $\big(\widetilde{M},\mathbf{c}\big)$ be a split-signature conformal spin structure endowed with tractors $\mathbf{s}_E$, $\mathbf{s}_F$ and $\mathbf{K}$ satisfying conditions (a) and (b) from Proposition 4.9. Then condition (4.5) is equivalent to (W).*

Proof. The implication (4.5) $\Longrightarrow$ (W) is obvious. It remains to prove the converse implication (W) $\Longrightarrow$ (4.5).

By (W), one has that $\big(i_X\widetilde{\kappa}^{\mathrm{nor}}\big)_{\tilde{\mathfrak{g}}_0}(u) \in \big(f \otimes \Lambda^2 f\big)[-4] \subseteq f^\circ \otimes \Lambda^2 \bar{F}$ for $X \in f$, $u \in \mathcal{G}$. Since $\mathbf{s}_F$ is parallel with respect to $\widetilde{\nabla}^{\mathrm{nor}}$, we have $\widetilde{\kappa}^{\mathrm{nor}}(u) \in \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \big(\tilde{\mathfrak{p}} \cap \ker s_F\big)$. The projection of $\tilde{\mathfrak{p}} \cap \ker s_F$ to $\tilde{\mathfrak{p}}_+$ is precisely f°, hence it follows that $\big(i_X\widetilde{\kappa}^{\mathrm{nor}}\big)_{\tilde{\mathfrak{p}}_+}(u) \in (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes f^\circ$, and we obtain $i_X\widetilde{\kappa}^{\mathrm{nor}}(u) \in (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \big(\Lambda^2\bar{F} \oplus f^\circ\big)$.

We now prove that $i_{X_1} i_{X_2}\widetilde{\kappa}^{\mathrm{nor}} = 0$ for all $X_1, X_2 \in f$. For this purpose it will be useful to work with the curvature form $\widetilde{\Omega}^{\mathrm{nor}}$, which we can represent as in (2.11). By (W) and the algebraic Bianchi identity, $\widetilde{W}_{abcd}$ vanishes upon insertion of $v, w \in \Gamma(\ker\chi)$ into any two slots, and in particular $v^a w^b \widetilde{W}_{abcd} = 0$. Thus, it remains to check that $v^a w^b \widetilde{Y}_{dab} = 0$. As in the proof of Proposition 3.2, a vector field $w \in \Gamma(\ker\chi)$ corresponds to a section $\left(\begin{smallmatrix} * \\ w^d \\ 0 \end{smallmatrix}\right) \in \Gamma\big(\widetilde{\widetilde{\mathcal{F}}}\big)$. According to (2.9),

$$v^a \widetilde{\Omega}_{ab}^{\mathrm{nor}} \bullet \begin{pmatrix} * \\ w^d \\ 0 \end{pmatrix} = \begin{pmatrix} -v^a \widetilde{Y}_{rab} w^r \\ v^a \widetilde{W}_{ab}{}^c{}_d w^d \\ 0 \end{pmatrix} \in \Gamma\big(\widetilde{\mathbb{E}}_b \otimes \widetilde{\mathcal{T}}\big).$$

Since $i_v\widetilde{\Omega}^{\mathrm{nor}}$ annihilates $\widetilde{\widetilde{\mathcal{F}}}$, it follows that $v^a w^r \widetilde{Y}_{rab} = 0$. Using $\widetilde{Y}_{rab} = -\widetilde{Y}_{bra} - \widetilde{Y}_{abr}$, we obtain also $v^a w^b \widetilde{Y}_{rab} = 0$. ■

4.3 Characterisation

We are now going to characterise the induced conformal structures. For this purpose we will introduce the following ('intermediate') Cartan connection form:

$$\widetilde{\omega}' := \widetilde{\omega}^{\mathrm{nor}} - \frac{1}{2} i_k \widetilde{\kappa}^{\mathrm{nor}}. \tag{4.8}$$

The following observation then follows immediately from Proposition 4.4 and formula (4.7):

Lemma 4.11. *The pullbacks of the Cartan connection forms $\widetilde{\omega}' \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$, $\widetilde{\omega}^{\mathrm{nor}} \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ and $\widetilde{\omega}^{\mathrm{ind}} \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ to $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ agree modulo forms with values in $\mathfrak{p}_+ \subseteq \mathfrak{sl}(n+1) \subseteq \tilde{\mathfrak{g}}$.*

For the rest of this section, we will start with a given split-signature conformal spin structure $(\widetilde{M}, \mathbf{c})$ satisfying all the properties of Proposition 4.9. In particular, $\widetilde{M}$ is endowed with a conformal Killing field $k \in \Gamma(T\widetilde{M})$, and we can still use formula (4.8) to define a Cartan connection $\widetilde{\omega}'$. The corresponding tractor connection will be denoted by $\widetilde{\nabla}'$ and the curvature by $\widetilde{\Omega}'$ or $\widetilde{\kappa}'$. The following proposition now shows that the so constructed Cartan connection $\widetilde{\omega}'$ is in fact an $\mathrm{SL}(n+1)$-connection.

Proposition 4.12. *Let $(\widetilde{M}, \mathbf{c})$ be a split-signature conformal (spin) structure satisfying all the properties of Proposition 4.9. Then the sections $\mathbf{s}_F$ and $\mathbf{K}$ are parallel with respect to the tractor connection $\widetilde{\nabla}'$, i.e., $\widetilde{\nabla}'\mathbf{s}_F = 0$ and $\widetilde{\nabla}'\mathbf{K} = 0$.*

In particular, $\mathrm{Hol}(\widetilde{\omega}') \subseteq \mathrm{SL}(n+1) \subseteq \mathrm{Spin}(n+1, n+1)$ and $\widetilde{\omega}'$ pulls back to a Cartan connection of type $(\mathrm{SL}(n+1), Q)$ with respect to the Q-reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$. Along that reduction, the curvature functions $\widetilde{\kappa}'$ and $\widetilde{\kappa}^{\mathrm{nor}}$ are related according to $\widetilde{\kappa}' = \big(\widetilde{\kappa}^{\mathrm{nor}}\big)_{\mathfrak{sl}(n+1)}$ and $\widetilde{\kappa}'$ satisfies the following integrability condition:

$$i_X\widetilde{\kappa}'(u) \in f^\circ \otimes \mathfrak{p}_+, \qquad \text{for all } X \in f,\ u \in \mathcal{G}. \tag{4.9}$$

Proof. A tractor connection induced by $\widetilde{\omega}'$ can be written as $\widetilde{\nabla}' = \widetilde{\nabla}^{\mathrm{nor}} + \Psi$ with $\Psi = -\frac{1}{2}i_k\widetilde{\Omega}^1$. That $\widetilde{\nabla}'\mathbf{s}_F = 0$ follows immediately from the fact that $\widetilde{\nabla}' - \widetilde{\nabla}^{\mathrm{nor}} = -\frac{1}{2}i_k\widetilde{\Omega}^{\mathrm{nor}}$ has values in $\Lambda^2\widetilde{\mathcal{F}}$. Since k is a conformal Killing field we have $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} = i_k\widetilde{\Omega}^{\mathrm{nor}}$. By definition

$$\widetilde{\nabla}'\mathbf{K} = \widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} - \frac{1}{2}i_k\widetilde{\Omega}^{\mathrm{nor}} \bullet \mathbf{K},$$

which vanishes, since $i_k\widetilde{\Omega}^{\mathrm{nor}}$ has values in $\Lambda^2\widetilde{\mathcal{F}}$ and therefore $\frac{1}{2}i_k\widetilde{\Omega}^{\mathrm{nor}} \bullet \mathbf{K} = i_k\widetilde{\Omega}^{\mathrm{nor}}$. As in Proposition 4.9, we write the decomposition of $\widetilde{\mathcal{T}}$ into maximally isotropic eigenspaces of $\mathbf{K}$ with eigenvalues ± 1 as $\widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}}$. Since $\mathbf{K}$ is $\widetilde{\nabla}'$-parallel, it follows that this decomposition is preserved by $\widetilde{\nabla}'$. Moreover, since $\widetilde{\mathcal{F}}$ is the kernel of the pure tractor spinor $\mathbf{s}_F$ it follows that $\mathrm{Hol}(\widetilde{\omega}') \subseteq \mathrm{SL}(n+1)$. In particular, $\widetilde{\omega}'$ reduces to a Cartan connection of type $(\mathrm{SL}(n+1), Q)$ on a Q-principal bundle $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$.

We further compute that

$$\begin{aligned}\widetilde{\Omega}' &= \widetilde{\Omega}^{\mathrm{nor}} - \frac{1}{2}d^{\widetilde{\nabla}^{\mathrm{nor}}}i_k\widetilde{\Omega}^{\mathrm{nor}} = \widetilde{\Omega}^{\mathrm{nor}} - \frac{1}{2}d^{\widetilde{\nabla}^{\mathrm{nor}}}\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K}\\ &= \widetilde{\Omega}^{\mathrm{nor}} - \frac{1}{2}\widetilde{\Omega}^{\mathrm{nor}} \bullet \mathbf{K} = \widetilde{\Omega}^{\mathrm{nor}} + \frac{1}{2}\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}} = \big(\widetilde{\Omega}^{\mathrm{nor}}\big)_{(\widetilde{\mathcal{E}}\otimes\widetilde{\mathcal{F}})_0},\end{aligned}$$

where we are again using $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} = i_k\widetilde{\Omega}^{\mathrm{nor}}$ for the conformal Killing field k and that $\widetilde{\Omega}^{\mathrm{nor}}$ has values in $\widetilde{\mathcal{E}} \otimes \widetilde{\mathcal{F}} \oplus \Lambda^2\widetilde{\mathcal{F}}$. Stated for the corresponding curvature functions, this yields $\widetilde{\kappa}' = \big(\widetilde{\kappa}^{\mathrm{nor}}\big)_{\mathfrak{sl}(n+1)}$. Moreover, since $\widetilde{\kappa}^{\mathrm{nor}}$ has values in $\ker s_F \cap \widetilde{\mathfrak{p}}$, it follows from (3.6) that $(\widetilde{\mathfrak{p}} \cap \ker s_F)_{\mathfrak{sl}(n+1)} = \mathfrak{p}$, thus $\widetilde{\kappa}'$ has values in $\mathfrak{p}$.

We know from (4.5) that $i_X\widetilde{\kappa}^{\mathrm{nor}}$ has values in $\Lambda^2\bar{F} \oplus f^\circ$ for $X \in f$. But since $\big(\Lambda^2\bar{F}\big)_{\mathfrak{sl}(n+1)} = 0$ and $(\widetilde{\mathfrak{p}}_+ \cap \ker s_F)_{\mathfrak{sl}(n+1)} = \mathfrak{p}_+$, we obtain that $(i_X\widetilde{\kappa}^{\mathrm{nor}})_{\mathfrak{sl}(n+1)}$ has values in $\mathfrak{p}_+$. Finally, $\big(i_{X_1}i_{X_2}\widetilde{\kappa}^{\mathrm{nor}}\big)_{\mathfrak{sl}(n+1)} = 0$ for $X_1, X_2 \in f$ follows immediately from (4.5), and altogether we obtain (4.9). ■

Next, before proving the main characterisation Theorem 4.14, we will show the following proposition on factorisations of particular Cartan geometries. This proposition can be understood as an adapted variant of [5, Theorem 2.7].

Proposition 4.13. *Let $(\mathcal{G} \to \widetilde{M}, \omega)$ be a Cartan geometry of type $(\mathrm{SL}(n+1), Q)$ with curvature $\kappa\colon \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{q})^* \otimes \mathfrak{g}$ and let the following conditions be satisfied:*

$$i_{X_1}i_{X_2}\kappa(u) \in \mathfrak{p}, \qquad \text{for all } X_1, X_2 \in \mathfrak{g}/\mathfrak{q},\ u \in \mathcal{G},$$

$$i_{X_1} i_{X_2} \kappa(u) \in \mathfrak{p}_+, \qquad \text{for all } X_1 \in \mathfrak{p}/\mathfrak{q},\ X_2 \in \mathfrak{g}/\mathfrak{q},\ u \in \mathcal{G},$$
$$i_{X_1} i_{X_2} \kappa(u) = 0, \qquad \text{for all } X_1, X_2 \in \mathfrak{p}/\mathfrak{q},\ u \in \mathcal{G}.$$

Then $\mathcal{G}$ is locally a P-bundle over $M = \mathcal{G}/P$ and ω defines a canonical projective structure on M.

Proof. The third of the above listed conditions implies that $\mathcal{G}$ is locally a P-bundle $\mathcal{G} \to M$ by [5]. We will restrict $\mathcal{G}$ to assume this globally. We define $M = \mathcal{G}/P$ and $\mathcal{G}_0 = \mathcal{G}/P_+$.

Let $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$ be a G_0-equivariant splitting. It follows from the second of the above listed conditions that

$$\mathcal{L}_{\zeta_X} \omega = -\mathrm{ad}(X) \circ \omega \mod \mathfrak{p}_+, \qquad \text{for all } X \in \mathfrak{p}.$$

Now define $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_-)$, $\gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ and $\rho \in \Omega^1(\mathcal{G}_0, \mathfrak{p}_+)$ via the decomposition $\sigma^*\omega = \theta \oplus \gamma \oplus \rho$. Since σ is G_0-equivariant and the Lie derivative is compatible with pullbacks it follows that

$$\mathcal{L}_{\bar{\zeta}_X} (\theta \oplus \gamma) = -\mathrm{ad}(X) \circ (\theta \oplus \gamma), \qquad \text{for all } X \in \mathfrak{g}_0.$$

In particular, θ and γ are G_0-equivariant and define a (reductive) Cartan geometry $(\mathcal{G}_0 \to M, \theta \oplus \gamma)$ of type $(\mathbb{R}^n \rtimes \mathrm{SL}(n), \mathrm{SL}(n))$, i.e., an affine connection on M. Since by assumption Ω has values in $\mathfrak{p}$, $\theta \oplus \gamma$ is torsion-free and so is the affine connection.

Now take another splitting $\sigma' = \sigma \cdot \exp \Upsilon$, for some $\Upsilon\colon \mathcal{G} \to \mathfrak{p}_+$. Since $\mathrm{Ad}(\exp \Upsilon)$ acts by the identity on $\mathfrak{g}_- = \mathfrak{g}/\mathfrak{p}$, one has $(r^{\exp \Upsilon})^*\omega = \omega \mod \mathfrak{p}$, and thus θ is independent of the choice of splitting. Then $\sigma'^*\omega = \theta \oplus \gamma' \oplus \rho'$ and $\theta \oplus \gamma' = \mathrm{Ad}(\exp \Upsilon) \circ (\theta \oplus \gamma)$, projected to $\mathfrak{g}_- \oplus \mathfrak{g}_0$. But since $\exp \Upsilon \in P_+$, this shows that γ' is projectively equivalent to γ. We thus obtain a well-defined projective structure on M.

Since ω is P-torsion-free and P-equivariant modulo $\mathfrak{p}_+$, it can be (uniquely) modified to a normal Cartan connection $\omega^{\mathrm{nor}} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with $\omega^{\mathrm{nor}} - \omega \in \Omega^1(\mathcal{G}, \mathfrak{p}_+)$. In particular, each splitting $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$ is in fact a Weyl structure of the projective structure on M. ■

Theorem 4.14. *A split-signature (n, n) conformal spin structure $\mathbf{c}$ on a manifold $\widetilde{M}$ is (locally) induced by an n-dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:*

(a) *$(\widetilde{M}, \mathbf{c})$ admits a nowhere-vanishing light-like conformal Killing field k such that the corresponding tractor endomorphism $\mathbf{K} = L_0^{\mathcal{A}\widetilde{M}}(k)$ is an involution, i.e., $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$.*

(b) *$(\widetilde{M}, \mathbf{c})$ admits a pure twistor spinor $\chi \in \Gamma\big(\widetilde{\Sigma}_-\big[\tfrac{1}{2}\big]\big)$ with $k \in \Gamma(\ker \chi)$ such that the corresponding parallel tractor spinor $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}_-}(\chi)$ is pure.*

(c) *$\mathbf{K}$ acts by minus the identity on $\ker \mathbf{s}_F$.*

(d) *The following integrability condition holds:*

$$v^a w^c \widetilde{W}_{abcd} = 0, \qquad \text{for all } v, w \in \Gamma(\ker \chi).$$

Proof. Starting with a projective structure $(M, \mathbf{p})$, it follows from Proposition 4.9 that the induced conformal structure $(\widetilde{M}, \mathbf{c})$ has all the stated properties. On the other hand, let $(\widetilde{M}, \mathbf{c})$ be a conformal structure with the stated properties. Then, by Proposition 4.12, $\widetilde{\omega}'$ restricts to a Q-equivariant Cartan connection form with values in $\mathfrak{sl}(n+1)$ on the reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$. The corresponding curvature $\widetilde{\kappa}'$ takes values in $\mathfrak{p}$ and for $X \in f$ we have that $i_X \widetilde{\kappa}'$ takes values in $\mathfrak{p}_+$. It follows from Proposition 4.13 that $\widetilde{\omega}'$ factorises to a projective structure $\mathbf{p}$ on the leaf space M.

Let us now show that the two constructions are inverse to each other. Assume first that a conformal structure $(\widetilde{M}, \mathbf{c})$ is induced by a projective structure $(M, \mathbf{p})$. Then according to Lemma 4.11 $\widetilde{\omega}'$ and $\widetilde{\omega}^{\mathrm{ind}}$ agree modulo $\mathfrak{p}_+$. This implies that the projective structure defined by $\widetilde{\omega}'$ is equal to the original projective structure. Conversely, assume now that $(M, \mathbf{p})$ is a projective structure with associated Cartan geometry $(\mathcal{G}, \omega')$ that is induced from a conformal structure $(\widetilde{M}, \mathbf{c})$ with associated Cartan geometry $(\widetilde{\mathcal{G}}, \widetilde{\omega}^{\mathrm{nor}})$. Since $\widetilde{\omega}'$ is not normal, but torsion-free, there is $\varphi \in \Omega^1_{\mathrm{hor}}(\mathcal{G}, \mathfrak{p}_+)$ such that $\omega' + \varphi$ is the normal projective Cartan connection. Since $\mathfrak{p}_+ \subseteq \widetilde{\mathfrak{p}}$ the induced conformal structure on $\widetilde{M}$ agrees with the original conformal structure. Thus, the Fefferman-type construction (with normalisation) and the described factorisation are (locally) inverse to each other. ■

For a reformulation of the characterisation theorem in terms of underlying objects, see Section 6.2.

5 Reduced scales and explicit normalisation

Although we obtained the desired characterisation in Theorem 4.14, we do not yet know the explicit relationship between the induced Cartan connection form $\widetilde{\omega}^{\mathrm{ind}}$ and the normal conformal Cartan connection form $\widetilde{\omega}^{\mathrm{nor}}$. One of the aims of the present section is to obtain a formula for this difference, which is achieved in Theorem 5.7. As a consequence, we also obtain an explicit formula for the curvature $\widetilde{\Omega}^{\mathrm{ind}}$ in terms of the normal conformal Cartan curvature $\widetilde{\Omega}^{\mathrm{nor}}$ in Corollary 5.8. In this more refined analysis, reduced scales will play an important role.

5.1 Characterisation of reduced scales

The notion of reduced Weyl structures and reduced scales is introduced in Section 3.4. Here we shall find an intrinsic characterisation (i.e., using the conformal structure only) of reduced scales and discuss their properties.

As the scale bundle on the projective manifold M we may consider the positive elements in the density bundle $\mathbb{E}(1)$, which is the projecting part of the dual standard tractor bundle $\mathcal{T}^*$, see Section 2.4. Similarly, on the Fefferman space $\widetilde{M}$ we take the positive elements in the density bundle $\widetilde{\mathbb{E}}(1)$, the projecting part of the conformal standard tractor bundle $\widetilde{\mathcal{T}}$. Hence for a projective scale $\rho \in \Gamma(\mathbb{E}_+(1))$ we have the tractor $L_0^{\mathcal{T}^*}(\rho) \in \Gamma(\mathcal{T}^*)$; similarly, for a conformal scale $\sigma \in \Gamma\big(\widetilde{\mathbb{E}}_+(1)\big)$ we have the tractor $L_0^{\widetilde{\mathcal{T}}}(\sigma) \in \Gamma\big(\widetilde{\mathcal{T}}\big)$. These will be termed *scale tractors*.

On the one hand, reduced scales correspond to the sections of $\mathbb{E}_+(1) \to M$ seen as a subset of all sections of $\widetilde{\mathbb{E}}_+(1) \to \widetilde{M}$, see Remark 3.4. On the other hand, sections of $\mathcal{T}^* \to M$ are understood as specific sections of the bundle $\widetilde{\mathcal{F}} \to \widetilde{M}$, which is a subbundle in $\widetilde{\mathcal{T}} \to \widetilde{M}$, see the generalities in Section 3.4 and the setup of our construction in Section 3.2. It follows that these two natural inclusions commute with the BGG-splitting operators.

Lemma 5.1. *The full arrows in the following diagram commute:*

$$\begin{tikzcd}
\Gamma(\mathcal{T}^*) \arrow[r, hook] \arrow[d, dotted, "\Pi_0"] & \Gamma\big(\widetilde{\mathcal{T}}\big) \arrow[d, dotted, "\widetilde{\Pi}_0"'] \\
\Gamma(\mathbb{E}_+(1)) \arrow[r, hook] \arrow[u, bend left, "L_0^{\mathcal{T}^*}"] & \Gamma\big(\widetilde{\mathbb{E}}_+[1]\big). \arrow[u, bend right, "L_0^{\widetilde{\mathcal{T}}}"']
\end{tikzcd}$$

Proof. Consider a projective density $\rho \in \Gamma(\mathbb{E}_+(1))$ on M, the corresponding tractor $L_0^{\mathcal{T}^*}(\rho) \in \Gamma(\mathcal{T}^*)$, and its extension to $\widetilde{\mathcal{F}} \subseteq \widetilde{\mathcal{T}}$, which is denoted by s'. The extension of $\rho \in \Gamma(\mathbb{E}_+(1))$ to $\widetilde{\mathbb{E}}_+[1]$ obviously coincides with the projection $\widetilde{\Pi}_0(s')$, and it is denoted by σ. We need to show that $s' = L_0^{\widetilde{\mathcal{T}}}(\sigma)$, i.e., that $\widetilde{\partial}^* \widetilde{\nabla}^{\mathrm{nor}} s' = 0$. According to Proposition 4.4, $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \Psi^1 + \Psi^2$ with $\Psi^1 \in \Omega^1\big(\widetilde{M}, \Lambda^2 \widetilde{\widetilde{\mathcal{F}}}\big)$, $\Psi^2 \in \Omega^1\big(\widetilde{M}, \widetilde{f}^\circ\big)$, hence we have

$$\widetilde{\nabla}^{\mathrm{nor}} s' = \widetilde{\nabla}^{\mathrm{ind}} s' + \Psi^1 \bullet s' + \Psi^2 \bullet s'.$$

Since $\Lambda^2 \widetilde{\widetilde{\mathcal{F}}}$ acts trivially on $\widetilde{\mathcal{F}} \subseteq \widetilde{\mathcal{T}}$, we have $\Psi^1 \bullet s' = 0$. Since $\widetilde{f}^\circ \subseteq T^* \widetilde{M}$, it follows that $\widetilde{\partial}^* \big(\Psi^2 \bullet s'\big) = 0$. It thus follows that $\widetilde{\partial}^* \big(\widetilde{\nabla}^{\mathrm{nor}} s'\big) = \widetilde{\partial}^* \big(\widetilde{\nabla}^{\mathrm{ind}} s'\big)$. Let ϕ be the frame form of $\nabla L_0^{\mathcal{T}^*}(\rho)$. Then, according to Lemma 4.1, we have that $\widetilde{\partial}^* \widetilde{\phi} = 0$ since $\Lambda^2 F \bullet F = 0$, and in particular $\widetilde{\partial}^* \big(\widetilde{\nabla}^{\mathrm{nor}} s'\big) = 0$. ■

We can now characterise reduced scales in terms of the corresponding scale tractors:

Proposition 5.2. *Let $\big(\widetilde{M}, \mathbf{c}\big)$ be a conformal spin structure associated to an oriented projective structure $(M, \mathbf{p})$ via the Fefferman-type construction. Let $\sigma \in \Gamma\big(\widetilde{\mathbb{E}}_+[1]\big)$ be a conformal scale and let $s := L_0^{\widetilde{\mathcal{T}}}(\sigma) \in \Gamma\big(\widetilde{\mathcal{T}}\big)$ be the corresponding scale tractor. Then the following statements are equivalent:*

- *(a) The scale σ is reduced.*
- *(b) The tractor s is a section of $\widetilde{\mathcal{F}} \subseteq \widetilde{\mathcal{T}}$.*
- *(c) The twistor spinor χ is parallel with respect to the Levi-Civita connection $\widetilde{D}$ of the metric corresponding to the scale σ.*

Furthermore, in a reduced scale, the Schouten tensor is strictly horizontal, i.e., it satisfies

$$v^a \widetilde{\mathsf{P}}_{ab} = 0, \qquad \text{for all } v^a \in \Gamma(\ker \chi), \tag{5.1}$$

and the scalar curvature $\widetilde{J}$ vanishes.

Proof. (a) $\Longrightarrow$ (b): This follows from definitions and Lemma 5.1.

(b) $\Longrightarrow$ (c): The condition (b) means that $s \cdot \mathbf{s}_F = 0$. According to (2.6), (2.8) and (2.7), this condition expanded in slots yields

$$s \cdot \mathbf{s}_F = L_0^{\widetilde{\mathcal{T}}}(\sigma) \cdot L_0^{\widetilde{\mathcal{S}}}(\chi) = \begin{pmatrix} -\frac{1}{2n} \widetilde{J} \sigma \\ 0 \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{n\sqrt{2}} \not{D} \chi \\ \chi \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2n} \widetilde{J} \chi \sigma \\ -\frac{1}{n} \not{D} \chi \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where we use the Levi-Civita connection $\widetilde{D}$ corresponding to σ. In particular, $\not{D}\chi = 0$ and, since χ is a twistor spinor, the condition (c) follows.

(c) $\Longrightarrow$ (b): The condition (c) yields $\mathbf{s}_F = \binom{0}{\chi}$ according to (2.8). The fact that $\widetilde{\nabla}_a \mathbf{s}_F = 0$ yields $\widetilde{\mathsf{P}}_{ac} \gamma^c \chi = 0$ according to (2.7). Hence (5.1) holds, which in particular means $\widetilde{J} = 0$. Summarising, we have

$$s = L_0^{\widetilde{\mathcal{T}}}(\sigma) = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix} \qquad \text{and} \qquad \mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}}(\chi) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}. \tag{5.2}$$

Hence $s \cdot \mathbf{s}_F = 0$ and the condition (b) follows.

(b) and (c) $\Longrightarrow$ (a): According to the previous reasoning and (2.5), we have

$$\widetilde{\nabla}_a^{\mathrm{nor}} s = \begin{pmatrix} 0 \\ \sigma \widetilde{\mathsf{P}}_{ab} \\ 0 \end{pmatrix}.$$

Hence $\widetilde{\nabla}^{\mathrm{nor}} s$ is strictly horizontal, i.e., $v^a \widetilde{\nabla}_a^{\mathrm{nor}} s = 0$ for every $v^a \in \Gamma(\ker \chi)$. Since $\widetilde{\nabla}^{\mathrm{nor}} = \widetilde{\nabla}^{\mathrm{ind}} + \Psi$ and Ψ is horizontal, the horizontality of $\widetilde{\nabla}^{\mathrm{nor}} s$ is equivalent to the horizontality of $\widetilde{\nabla}^{\mathrm{ind}} s$. Altogether, the condition (a) follows from Proposition 3.3 and Lemma 5.1. ■

We will need some finer discussion on the slots of the distinguished tractor

$$\mathbf{K} = \begin{pmatrix} \rho_a \\ \mu_{ab} \,|\, \varphi \\ k_a \end{pmatrix} \in \Gamma\big(\mathcal{A}\widetilde{M}\big) \tag{5.3}$$

in reduced scales. From Proposition 4.8 we know that $\mathbf{K}$ is the BGG-splitting $L_0^{\mathcal{A}\widetilde{M}}(k)$, which in particular means that $\mu_{ab} = \widetilde{D}_{[a}k_{b]}$ and $\varphi = -\frac{1}{2n}\widetilde{D}^r k_r$ according to (2.12). However, in the following statement we only exploit the algebraic properties of $\mathbf{K}$, namely, that it acts by minus and plus the identity on $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{E}}$, respectively.

Lemma 5.3. *Let us fix a reduced scale. Then the expression of* $\mathbf{K}$ *as in* (5.3) *satisfy* $\rho_a = 0$, $\varphi = -1$, $\mu_a{}^r v_r = -v_a$ *for every* $v^a \in \Gamma(\ker\chi)$ *and* $\mu_a{}^r \mu_{rb} = g_{ab}$. *Further we have* $\mu_{ab} = \langle \gamma_{[a}\gamma_{b]}\chi, \bar\eta\rangle$ *for some* $\bar\eta \in \Gamma\big(\widetilde{\Sigma}_{\mp}\big[-\frac{1}{2}\big]\big)$.

Proof. Firstly, we use $\mathbf{K} \bullet s = -s$ for any $s \in \Gamma\big(\widetilde{\mathcal{F}}\big)$. The scale tractor $s = L_0^{\widetilde{\mathcal{T}}}(\sigma)$ of a reduced scale σ is a section of $\widetilde{\mathcal{F}}$ and it has the form as in (5.2). Thus it follows from (2.9) that $\rho_a = 0$ and $\varphi = -1$. Next, for every $v^a \in \Gamma(\ker\chi)$, the tractor $s = \left(\begin{smallmatrix} 0 \\ v_a \\ 0 \end{smallmatrix}\right)$ is clearly a section of $\widetilde{\mathcal{F}}$, since $s \cdot \mathbf{s}_F = 0$. Thus it follows from (2.9) that $\mu_a{}^r v_r = -v_a$.

Secondly, we use $\mathbf{K} \bullet s = s$ for any $s \in \Gamma\big(\widetilde{\mathcal{E}}\big)$. Considering the tractor $s = \left(\begin{smallmatrix} 0 \\ \omega_a \\ 0 \end{smallmatrix}\right)$ with arbitrary $\omega^a \in \Gamma\big(T\widetilde{M}\big)$, the tractor $\bar s := s + \mathbf{K} \bullet s$ is a section of $\widetilde{\mathcal{E}}$, whose middle slot is $\omega_a + \mu_a{}^r\omega_r$. It follows again from (2.9) that $\mu_a{}^r\mu_{rb} = g_{ab}$.

Thirdly, we use (3.3) which shows how $\mathbf{K}$ is built from $\mathbf{s}_E$ and $\mathbf{s}_F$. Since the top slot of $\mathbf{s}_F$ vanishes, middle slots of $\mathbf{K}$ are given by a suitable tensor product of χ and the top slot of $\mathbf{s}_E$. ■

We will also need more properties of conformal curvature quantities in reduced scales.

Lemma 5.4. *In a reduced scale,*

$$\widetilde{W}_{abcd}\mu^{cd} = 0, \qquad \textit{where } \mu_{ab} = \widetilde{D}_{[a}k_{b]}, \tag{5.4}$$

$$v^c\widetilde{Y}_{abc} = 0, \qquad \textit{for all } v^a \in \Gamma(\ker\chi). \tag{5.5}$$

Proof. In slots, the condition $\widetilde{\Omega}^{\mathrm{nor}}_{ab} \bullet \mathbf{s}_F = 0$ implies that $\widetilde{W}_{abcd}\gamma^c\gamma^d\chi = 0$. Pairing both sides of the latter equality with a spinor $\bar\eta$ from Lemma 5.3 yields (5.4).

Consider two arbitrary sections v^a, w^b of $\widetilde{f} = \ker\chi$. Conditions (W) and (5.1) imply $\widetilde{R}_{abcd}v^a w^d = 0$. Now, inserting v^a and w^e into the Bianchi identity $\widetilde{D}_{[a}\widetilde{R}_{bc]de} = 0$, we obtain $v^a w^e \widetilde{D}_a\widetilde{R}_{bcde} = 0$, where we used the fact that $\widetilde{f}$ is parallel. Since we can always regard g^{ab} as a section of $\widetilde{f} \otimes \widetilde{f}^*$, this implies $0 = g^{ec}v^a\widetilde{D}_a\widetilde{R}_{bcde} = v^a\widetilde{D}_a\widetilde{\mathrm{Ric}}_{bc}$ where $\widetilde{\mathrm{Ric}}_{bc}$ is the Ricci tensor of $\widetilde{D}_a$. Since $\widetilde{J} = 0$, we have that $\widetilde{\mathsf{P}}_{ab}$ is proportional to $\widetilde{\mathrm{Ric}}_{ab}$ by a constant factor. Thus $v^a\widetilde{D}_a\widetilde{\mathsf{P}}_{bc} = 0$. From (5.1) and since $\widetilde{f}$ is parallel we also have $v^b\widetilde{D}_a\widetilde{\mathsf{P}}_{bc} = 0$. Altogether, (5.5) follows by the definition of the Cotton tensor. ■

5.2 Explicit normalisation formula

So far we discussed three Cartan connections on the Fefferman space $\widetilde{M}$: the induced one $\widetilde{\omega}^{\mathrm{ind}}$ (Section 3.3), the corresponding normal one $\widetilde{\omega}^{\mathrm{nor}}$ (Section 4.1) and the modified auxiliary one $\widetilde{\omega}'$ (Section 4.3). Various properties of these and derived objects are enumerated in Propositions 4.9 and 4.12. The following proposition refines the integrability conditions included there.

Proposition 5.5. *Let $(\widetilde{M}, \mathbf{c})$ be the conformal spin structure induced from an oriented projective structure $(M, \mathbf{p})$ via the Fefferman-type construction. Then, along the reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$,*

$$i_X \widetilde{\kappa}^{\mathrm{nor}}(u) \in f^\circ \otimes \Lambda^2 \bar{F}, \qquad \text{for all } X \in f,\ u \in \mathcal{G}, \tag{5.6}$$
$$i_X \widetilde{\kappa}'(u) = 0, \qquad \text{for all } X \in f,\ u \in \mathcal{G}. \tag{5.7}$$

Proof. From (4.5) we already know that $i_X \widetilde{\kappa}^{\mathrm{nor}}$ has values in $f^\circ \otimes (\Lambda^2 \bar{F} \oplus f^\circ)$. We note that the top slot of sections of $\Lambda^2 \widetilde{\bar{\mathcal{F}}}$ vanishes in reduced scales, cf. (3.7). Thus the part in f° corresponds to $v^r \widetilde{Y}_{abr}$ for a $v \in \Gamma(\widetilde{f})$, which however has to vanish by (5.5). Hence (5.6) follows. The last condition (5.7) follows from $\widetilde{\kappa}' = (\widetilde{\kappa}^{\mathrm{nor}})_{\mathfrak{sl}(n+1)}$, cf. Proposition 4.12. ∎

Since $\widetilde{\omega}'$ is an $\mathrm{SL}(n+1)$-connection on $\widetilde{\mathcal{G}} \to \widetilde{M}$, it is the extension of a Cartan connection ω', on $\mathcal{G} \to \widetilde{M}$. Now, due to (5.7), any section $v \in \Gamma(\ker \chi)$ inserts trivially into its curvature. But this is the standard condition on the connection ω' to be a Cartan connection also on the bundle $\mathcal{G} \to M$, i.e., to be a projective Cartan connection, cf. [5].

Furthermore, we will show that the descended Cartan connection is normal, i.e., $\omega' = \omega$. To do this, we first compute $\widetilde{\partial}^* \widetilde{\kappa}'$ and then use the relation between the co-differentials ∂^* on M and $\widetilde{\partial}^*$ on $\widetilde{M}$ discussed in Lemma 4.1.

Proposition 5.6. *The curvature $\widetilde{\kappa}'$ satisfies*

$$\widetilde{\partial}^* \widetilde{\kappa}'(u) = i_k \widetilde{\kappa}^{\mathrm{nor}}(u) \in f^\circ \otimes \Lambda^2 \bar{F}, \qquad \text{for all } u \in \mathcal{G}. \tag{5.8}$$

Proof. We shall compute $\widetilde{\partial}^* \widetilde{\Omega}'$ directly. First observe that using Proposition 4.12 we have $\widetilde{\Omega}' = (\widetilde{\Omega}^{\mathrm{nor}})_{\mathfrak{sl}(n+1)} = \widetilde{\Omega}^{\mathrm{nor}} + \frac{1}{2} \mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}$. Hence $\widetilde{\partial}^* \widetilde{\Omega}' = \frac{1}{2} \widetilde{\partial}^* (\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}})$, since $\widetilde{\partial}^* \widetilde{\Omega}^{\mathrm{nor}} = 0$. Using (5.3) and (2.11), we compute $\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}_{ab}$ as

$$\begin{pmatrix} \rho_c \\ \mu_{c_0 c_1} \mid \varphi \\ k_c \end{pmatrix} \bullet \begin{pmatrix} -\widetilde{Y}_{dab} \\ \widetilde{W}_{abd_0 d_1} \mid 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \rho^r \widetilde{W}_{abrc} - \mu_c{}^r \widetilde{Y}_{rab} + \varphi \widetilde{Y}_{cab} \\ -2 \widetilde{W}_{ab}{}^r{}_{[c_0} \mu_{c_1]r} + 2 k_{[c_0} \widetilde{Y}_{c_1]ab} \mid k^r \widetilde{Y}_{rab} \\ k^r \widetilde{W}_{abrc} \end{pmatrix}.$$

In a reduced scale, from the previous display together with Lemmas 5.3 and 5.4 we compute

$$\widetilde{\partial}^* (\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}_{ab}) = \begin{pmatrix} 0 \\ 2 k^r \widetilde{W}_{rac_0 c_1} \mid 0 \\ 0 \end{pmatrix} = 2 k^r \widetilde{\Omega}^{\mathrm{nor}}_{ra},$$

which yields (5.8). ∎

Theorem 5.7. *Let $(\mathcal{G}, \omega)$ be a projective normal Cartan geometry over M and let $(\widetilde{\mathcal{G}}, \widetilde{\omega}^{\mathrm{ind}})$ be the conformal Cartan geometry over $\widetilde{M}$ induced via the Fefferman-type construction. Then*

(*a*) $\widetilde{\omega}^{\mathrm{ind}} = \widetilde{\omega}' = \widetilde{\omega}^{\mathrm{nor}} - \frac{1}{2} i_k \widetilde{\kappa}^{\mathrm{nor}}$,

(*b*) $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \Psi^1$, *where* $\Psi^1 = -\frac{1}{2} \widetilde{\partial}^* \widetilde{\kappa}^{\mathrm{ind}} = \frac{1}{2} i_k \widetilde{\kappa}^{\mathrm{nor}}$.

Proof. (a) We use that $i_X \widetilde{\kappa}' = 0$ for all $X \in f$ according to (5.7). Then Proposition 5.6 together with Lemma 4.1 imply that $\partial^* \kappa' = 0$. Thus ω' is projectively normal, and therefore we obtain $\widetilde{\omega}' = \widetilde{\omega}^{\mathrm{ind}}$.

(b) The normalisation process of Proposition 4.4 provides $\Psi = \Psi^1 + \Psi^2$ such that $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \Psi$, where Ψ^1, Ψ^2 are the first and second normalisation steps. However since $\widetilde{\omega}' = \widetilde{\omega}^{\mathrm{ind}}$, it follows from Proposition 5.6 and (4.8) that $\widetilde{\partial}^* \widetilde{\kappa}' = \widetilde{\partial}^* \widetilde{\kappa}^{\mathrm{ind}}$ is, up to a constant multiple, the difference between $\widetilde{\omega}^{\mathrm{nor}}$ and $\widetilde{\omega}^{\mathrm{ind}}$. Therefore already the first normalisation step completes the normalisation, i.e., $\Psi^2 = 0$. ∎

Using the explicit relationship provided in Theorem 5.7 we can also obtain a detailed description of the difference between the induced and the normal Cartan curvatures:

Corollary 5.8. *In a reduced scale, we have the following relation between the curvatures of the induced and the normal conformal Cartan connection:*

$$\widetilde{\Omega}^{\mathrm{ind}}_{ab} = \widetilde{\Omega}^{\mathrm{nor}}_{ab} + \frac{1}{2}\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}_{ab} = \begin{pmatrix} -\widetilde{Y}_{cab} \\ \widetilde{W}_{abc_0c_1} - \widetilde{W}_{ab}{}^{r}{}_{[c_0}\mu_{c_1]r} + k_{[c_0}\widetilde{Y}_{c_1]ab} \mid 0 \\ \frac{1}{2}k^r\widetilde{W}_{abrc} \end{pmatrix}. \tag{5.9}$$

In particular, $\frac{1}{2}i_k\widetilde{W}$ is the torsion of the induced Cartan connection $\widetilde{\omega}^{\mathrm{ind}}$.

Proof. We obtained the concrete expression of $\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}$ in the proof of Proposition 5.6. Now Lemmas 5.3 and 5.4, and a short computation yields (5.9). ■

6 Comparison with Patterson–Walker metrics and alternative characterisation

In this section we will show that the Fefferman-type construction studied in this article is closely related to the construction of so-called *Patterson–Walker metrics*. These are the Riemann extensions of affine connected spaces, firstly described in [26]. A conformal version of this construction was obtained by [15] for dimension $n = 2$, and was treated by the authors of the present article in general dimension in [20].

6.1 Comparison

Let M be a smooth manifold and $p\colon T^*M \to M$ its cotangent bundle. The vertical subbundle $V \subseteq T(T^*M)$ of this projection is canonically isomorphic to T^*M. An affine connection D on M determines a complementary horizontal distribution $H \subseteq T(T^*M)$ that is isomorphic to TM via the tangent map of p.

Definition 6.1. The *Riemann extension* or the *Patterson–Walker metric* associated to a torsion-free affine connection D on M is the pseudo-Riemannian metric g on T^*M fully determined by the following conditions:

(a) both V and H are isotropic with respect to g,

(b) the value of g with one entry from V and another entry from H is given by the natural pairing between $V \cong T^*M$ and $H \cong TM$.

It follows that V is parallel with respect to the Levi-Civita connection of the just constructed metric. Hence Patterson–Walker metrics are special cases of Walker metrics, i.e., metrics admitting a parallel isotropic distribution. The explicit description of the metric g in terms of the Christoffel symbols of D can be found in [20, 26].

The previous definition can be adapted to weighted cotangent bundles $T^*M(w) = T^*M \otimes \mathbb{E}(w)$, provided that M is oriented and D is special, i.e., preserving a volume form on M, which allows a trivialisation of $\mathbb{E}(w)$. It turns out that Patterson–Walker metrics induced by projectively equivalent connections are conformally equivalent if and only if $w = 2$ (interpreted as a projective weight according to the conventions from Section 2.4). Altogether, we have a natural split-signature conformal structure on $T^*M(2)$ induced by an oriented projective structure $(M, \mathbf{p})$.

From Section 3.3 we know that $\widetilde{M} = T^*M(2) \setminus \{0\}$ is the Fefferman space. Special affine connections from $\mathbf{p}$ are just the exact Weyl connections of the corresponding parabolic geometry.

The corresponding objects on $\widetilde{M}$ are the reduced Weyl connections, respectively reduced scales, which correspond to distinguished metrics in the conformal class, see Section 3.4. We are going to show that these metrics are just the Patterson–Walker metrics.

Proposition 6.2. *Let $(\widetilde{M}, \mathbf{c})$ be the conformal structure of signature (n, n) associated to an n-dimensional projective structure $(M, \mathbf{p})$ via the Fefferman-type construction. Then any metric in $\mathbf{c}$ corresponding to a reduced scale is a Patterson–Walker metric.*

Proof. Within the proof we refer to the notation and explicit matrix realisations from Appendix A. By definition, the Fefferman space is $\widetilde{M} = \mathcal{G}/Q$, which yields $T\widetilde{M} \cong \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$. Conformally invariant objects on $\widetilde{M}$, respectively objects related to a choice of reduced scale, correspond to data on $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ that are invariant under the action of Q, respectively $G_0^{ss} \cap Q$. Elements in $\mathfrak{g}/\mathfrak{q}$ will be represented by matrices of the form

$$\begin{pmatrix} -\frac{z}{2} & * & * \\ X & * & * \\ w & Y^t & -\frac{z}{2} \end{pmatrix},$$

where $z, w \in \mathbb{R}$ and $X, Y \in \mathbb{R}^{n-1}$. Firstly, one verifies that

$$Y^t X - zw, \tag{6.1}$$

is the only quadratic form that is invariant under $G_0^{ss} \cap Q$. Hence any reduced-scale metric in $\mathbf{c}$ corresponds to the quadratic form (6.1) in a suitable frame. Secondly, the vertical subbundle $V \subseteq T\widetilde{M}$ corresponds to the Q-invariant subspace $f = \mathfrak{p}/\mathfrak{q} \subseteq \mathfrak{g}/\mathfrak{q}$ given by $X = 0$ and $w = 0$. The horizontal distribution $H \subseteq T\widetilde{M}$ induced by a linear connection from $\mathbf{p}$ corresponds to the unique $(G_0^{ss} \cap Q)$-invariant subspace $h \subseteq \mathfrak{g}/\mathfrak{q}$ complementary to f, which is given by $Y = 0$ and $z = 0$. Obviously, both f and h are isotropic with respect to (6.1). Hence any reduced-scale metric in $\mathbf{c}$ satisfies the condition (a) from Definition 6.1. Thirdly, the canonical identification $V \cong T^*M(2)$ corresponds to an isomorphism $f \cong (\mathfrak{g}/\mathfrak{p})^*(2)$ of Q-modules. Identifying $(\mathfrak{g}/\mathfrak{p})^*(2)$ with $\mathfrak{p}_+(2)$, it turns out to be given by

$$\begin{pmatrix} -\frac{z}{2} & * & * \\ 0 & * & * \\ 0 & Y^t & -\frac{z}{2} \end{pmatrix} \mapsto \begin{pmatrix} 0 & Y^t & -z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, the inner product of any $v \in f$ and $u \in h$ coincides with the pairing of the corresponding elements $v \in \mathfrak{p}_+(2)$ and $u \in \mathfrak{g}/\mathfrak{p}$. Hence any reduced-scale metric in $\mathbf{c}$ satisfies also the condition (b) from Definition 6.1 and so it is a Patterson–Walker metric. ■

6.2 Alternative characterisation

We have characterised split-signature (n, n) conformal structures $\mathbf{c}$ on $\widetilde{M}$ induced by an n-dimensional projective structure via the Fefferman-type construction in Theorem 4.14. Now we know these structures correspond to conformal Patterson–Walker metrics discussed in [20]. There we found the following characterisation in terms of underlying objects by direct computations and spin calculus. Our aim here is to indicate how to reach the same result in the current framework.

Theorem 6.3. *A split-signature (n, n) conformal spin structure $\mathbf{c}$ on a manifold $\widetilde{M}$ is (locally) induced by an n-dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:*

(a) *$(\widetilde{M}, \mathbf{c})$ admits a nowhere-vanishing light-like conformal Killing field k.*

(*b*) $(\widetilde{M}, \mathbf{c})$ *admits a pure twistor spinor* χ *such that* $\widetilde{f} = \ker\chi$ *is integrable and* $k \in \Gamma(\widetilde{f})$.

(*c*) *The Lie derivative of* χ *with respect to the conformal Killing field* k *is* $\mathcal{L}_k\chi = -\frac{1}{2}(n+1)\chi$.

(*d*) *The following integrability condition holds:*

$$v^r w^s \widetilde{W}_{arbs} = 0, \qquad \text{for all } v^r, w^s \in \Gamma(\ker\chi). \tag{W}$$

We now express the conditions from Proposition 4.9 in underlying terms:

(i) For a conformal Killing field k with the splitting $\mathbf{K}$ as in (5.3), a straightforward computation shows that the condition $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$ is equivalent to

$$\begin{aligned}
&k^a k_a = 0, && \rho^a\rho_a = 0,\\
&\mu^a{}_b k^b = \varphi k^a, && \mu^a{}_b\rho^b = -\varphi\rho^a,\\
&k^a\rho_a = \varphi^2 - 1, && \mu_a{}^c\mu_{cb} = g_{ab} + 2k_{(a}\rho_{b)}.
\end{aligned} \tag{6.2}$$

(ii) For a twistor spinor χ, the corresponding tractor spinor $\mathbf{s}_F = \binom{\bar\chi}{\chi} \in \Gamma(\widetilde{\mathcal{S}}_-)$ is parallel with respect to $\widetilde{\nabla}^{\mathrm{nor}}$. In particular, purity of $\mathbf{s}_F$ can be checked at one point. If $\chi = 0$, respectively $\bar\chi = 0$, this tractor spinor is pure whenever $\bar\chi$, respectively χ, is pure. If $\chi \neq 0$ and $\bar\chi \neq 0$, the purity of $\binom{\bar\chi}{\chi}$ is equivalent to χ and $\bar\chi$ being pure and their kernels having $(n-1)$-dimensional intersection, cf. [13, Proposition III-1.12] or [22, 30].

(iii) Let k be a conformal Killing field which splits to $\mathbf{K}$ and χ a twistor spinor which splits to $\mathbf{s}_F$. Then the condition $\mathcal{L}_k\chi = -\frac{1}{2}(n+1)\chi$ is equivalent to $\mathbf{K}\bullet\mathbf{s}_F = -\frac{1}{2}(n+1)\mathbf{s}_F$. If the tractor spinor $\mathbf{s}_F$ is pure it has an $(n+1)$-dimensional maximally isotropic kernel $\ker\mathbf{s}_F$. Then $\mathbf{K}\bullet\mathbf{s}_F = -\frac{1}{2}(n+1)\mathbf{s}_F$ is equivalent to $\mathbf{K}$ acting by minus the identity on $\ker\mathbf{s}_F$, which therefore coincides with the eigenspace of $\mathbf{K}$ corresponding to -1.

The assumption on the pure twistor spinor χ in Theorem 6.3 guarantees the existence of a suitable compatible metric for which χ is parallel. This result is proved in [20, Proposition 4.2]. Henceforth we shall assume $\widetilde{D}\chi = 0$ where $\widetilde{D}$ is the corresponding Levi-Civita connection. In particular, we have $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}_-}(\chi) = \binom{0}{\chi}$, which is pure and parallel by observation (ii). Expanding the latter condition according to (2.7) yields

$$v^a\widetilde{\mathsf{P}}_{ab} = 0, \qquad \text{for all } v^a \in \Gamma(\ker\chi), \tag{6.3}$$

For $\mathbf{K} = L_0^{\mathcal{A}\widetilde{M}}(k)$, we know from observation (iii) that $\mathbf{K}$ acts by $-\,\mathrm{id}$ on $\ker\mathbf{s}_F$. By the very same reasoning as in the first part of the proof of Lemma 5.3 it follows that

$$\rho_a = 0, \qquad \varphi = -1, \qquad \mu_a{}^r v_r = -v_a, \qquad \text{for all } v^a \in \Gamma(\ker\chi). \tag{6.4}$$

Now we are prepared to prove the theorem:

Proof of Theorem 6.3. If $(\widetilde{M}, \mathbf{c})$ is induced by a projective structure, the stated properties hold according to Proposition 4.9 and previous observations. For the converse direction, it remains to show that $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$, which is equivalently characterised by the identities (6.2). Then the properties (a)–(d) of Proposition 4.9 will be satisfied and the result will follow from the characterisation Theorem 4.14.

The expansion of the prolonged conformal Killing equation (2.13) for k gives

$$\widetilde{D}_a k_b = \mu_{ab} + g_{ab}, \tag{6.5}$$

$$\widetilde{D}_a\mu_{br} = -2\widetilde{\mathsf{P}}_{a[b}k_{r]} - W_{bras}k^s, \tag{6.6}$$

according to (2.10) and (2.11). Next, from (6.4) we especially have ${\mu_b}^r k_r = -k_b$. Applying $\widetilde{D}_a$ to both sides of this equality and using (6.5) we obtain

$$\big(\widetilde{D}_a \mu_{br}\big) k^r + {\mu_b}^r \mu_{ar} + \mu_{ba} = -(\mu_{ab} + g_{ab}).$$

From (6.6), (W) and (6.3) we have $\big(\widetilde{D}_a \mu_{br}\big) k^r = 0$, hence the previous display shows ${\mu_a}^r \mu_{rb} = g_{ab}$. This together with (6.4) implies that all identities from (6.2) are satisfied, hence $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$. ■

A Explicit matrix realisations

Here we provide explicit realisations of the Lie algebras introduced in Section 3.2 in terms of matrices. We will consider the inner product h and the involution K on $\mathbb{R}^{n+1,n+1}$ given by the block matrices

$$h := \begin{pmatrix} 0 & I_{n+1} \\ I_{n+1} & 0 \end{pmatrix} \qquad \text{and} \qquad K := \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_{n+1} \end{pmatrix}$$

with respect to the standard basis $(e_1, \dots, e_{2n+2})$. Then $E = \langle e_1, \dots, e_{n+1} \rangle$ and $F = \langle e_{n+2}, \dots, e_{2n+2} \rangle$ and the decomposition (3.4) can be written as

$$\tilde{\mathfrak{g}} = \Lambda^2(E \oplus F) = \begin{pmatrix} E \otimes F & \Lambda^2 E \\ \Lambda^2 F & E \otimes F \end{pmatrix}.$$

For $\tilde{v} := e_1 + e_{2n+2}$, the Lie algebra $\tilde{\mathfrak{p}}$ of the parabolic subgroup $\widetilde{P} \subseteq \widetilde{G}$ is of the following form

$$\tilde{\mathfrak{p}} = \left(\begin{array}{ccc|ccc} a & U^t & w & 0 & -W^t & -b \\ X & B & V & W & C & -X \\ 0 & Y^t & c & b & X^t & 0 \\ \hline 0 & -Y^t & -d & -a & -X^t & 0 \\ Y & D & -Z & -U & -B^t & -Y \\ d & Z^t & 0 & -w & -V^t & -c \end{array}\right), \tag{A.1}$$

where $a, b, c, d, w \in \mathbb{R}$ with $a - b = d - c$, $U, V, W, X, Y, Z \in \mathbb{R}^{n-1}$, $B \in \mathfrak{gl}(n-1)$ and $C, D \in \mathfrak{so}(n-1)$. The nilradical $\tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{p}}^\perp$ is then of the form

$$\tilde{\mathfrak{p}}_+ = \left(\begin{array}{ccc|ccc} a & U^t & w & 0 & -V^t & -a \\ 0 & 0 & V & V & 0 & 0 \\ 0 & 0 & a & a & 0 & 0 \\ \hline 0 & 0 & -a & -a & 0 & 0 \\ 0 & 0 & -U & -U & 0 & 0 \\ a & U^t & 0 & -w & -V^t & -a \end{array}\right).$$

A choice of Levi subalgebra $\tilde{\mathfrak{g}}_0 \subseteq \tilde{\mathfrak{p}}$ determines a grading $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_- \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{p}}_+$. We shall choose $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{p}} \cap \tilde{\mathfrak{p}}_{\mathrm{op}}$, where $\tilde{\mathfrak{p}}_{\mathrm{op}} \subseteq \tilde{\mathfrak{g}}$ is the stabiliser of the light-like vector e_{n+2}. Explicitly,

$$\tilde{\mathfrak{g}}_0 = \left(\begin{array}{ccc|ccc} a & 0 & 0 & 0 & 0 & 0 \\ X & B & V & 0 & C & -X \\ 0 & Y^t & c & 0 & X^t & 0 \\ \hline 0 & -Y^t & -a-c & -a & -X^t & 0 \\ Y & D & -Z & 0 & -B^t & -Y \\ a+c & Z^t & 0 & 0 & -V^t & -c \end{array}\right).$$

The embedding $i' \colon \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ of Lie algebras has the form $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & -A^t \end{smallmatrix}\right)$. The subgroup $Q = i^{-1}(\widetilde{P})$ is contained in P, the stabiliser in G of $v = (\tilde{v})_E = e_1$; the inclusion of corresponding Lie algebras is

$$\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}} = \begin{pmatrix} a & U^t & w \\ 0 & A & V \\ 0 & 0 & -a \end{pmatrix} \subseteq \begin{pmatrix} a & U^t & w \\ 0 & B & V \\ 0 & X^t & c \end{pmatrix} = \mathfrak{p},$$

where $\operatorname{tr}(A) = 0$ and $a + \operatorname{tr}(B) + c = 0$. The standard projective grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$,

$$\mathfrak{g}_- = \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ y & 0 & 0 \end{pmatrix}, \qquad \mathfrak{g}_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & B & V \\ 0 & X^t & c \end{pmatrix}, \qquad \mathfrak{p}_+ = \begin{pmatrix} 0 & U^t & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is compatible with the previous conformal grading so that the reduced Lie subalgebra $\mathfrak{q}_0 := \mathfrak{q} \cap \mathfrak{g}_0$ coincides with the intersection of $\mathfrak{g}_0 \cap \tilde{\mathfrak{g}}_0$. Explicitly,

$$\mathfrak{q}_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & A & V \\ 0 & 0 & -a \end{pmatrix}, \tag{A.2}$$

where $\operatorname{tr}(A) = 0$.

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