James' Submodule Theorem and the Steinberg Module

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Abstract. James' submodule theorem is a fundamental result in the representation theory of the symmetric groups and the finite general linear groups. In this note we consider a version of that theorem for a general finite group with a split BN-pair. This gives rise to a distinguished composition factor of the Steinberg module, first described by Hiss via a somewhat different method. It is a major open problem to determine the dimension of this composition factor.

Key words: groups with a BN-pair; Steinberg representation; modular representations

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1 Introduction

Let G be a finite group of Lie type and St_k be the Steinberg representation of G, defined over an arbitrary field k; see [8]. We shall be concerned here with the case where St_k is reducible. There is only very little general knowledge about the structure of St_k in this case; see, e.g., [3, 4] and the references there. Using the theory of Gelfand–Graev representations of G, Hiss [5] showed that St_k always has a certain distinguished composition factor with multiplicity 1. It appears to be extremely difficult to determine further properties of this composition factor, e.g., its dimension. The purpose of this note is to show that this composition factor can be defined in a somewhat more intrinsic way through a version of James' submodule theorem [6]; see Remark 3.4.

2 Groups with a split BN-pair

Let G be a finite group and $B, N \subseteq G$ be subgroups which satisfy the axioms for an "algebraic group with a split BN-pair" in [1, Section 2.5]. We just recall explicitly those properties of G which will be important for us in the sequel. Firstly, there is a prime number p such that we have a semidirect product decomposition $B = U \rtimes H$ where $H = B \cap N$ is an abelian group of order prime to p and U is a normal p-subgroup of B. The group H is normal in N and $W = N/H$ is a finite Coxeter group with a canonically defined generating set S; let $l\colon W \to \mathbb{N}_0$ be the corresponding length function. For each $w \in W$, let $n_w \in N$ be such that $Hn_w = w$. Let $w_0 \in W$ be the unique element of maximal length; we have $w_0^2 = 1$. Let $n_0 \in N$ be a representative of w_0 and $V := n_0^{-1}Un_0$; then $U \cap V = H$. For $w \in W$, let $U_w := U \cap n_w^{-1}Vn_w$.

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(Note that V, U_w do not depend on the choices of n_0, n_w.) Then we have the following sharp form of the Bruhat decomposition:

$$G = \coprod_{w\in W} Bn_wU_w, \qquad \text{with uniqueness of expressions,}$$

that is, every $g \in Bn_wB$ can be uniquely written as $g = bn_wu$ where $b \in B$ and $u \in U_w$. Note that, since $w_0^2 = 1$, we have $U_{w_0} = U$ and $Bn_0B = Bn_0U = Un_0B$, in both cases with uniqueness of expressions.

Let k be any field and kG be the group algebra of G. All our kG-modules will be finite-dimensional and left modules.

Remark 2.1. Let $\underline{\mathfrak{b}} := \sum_{b\in B} b \in kG$. Then $kG\underline{\mathfrak{b}}$ is a left kG-module which is canonically isomorphic to the induced module $\mathrm{Ind}_B^G(k_B)$; here, k_B denotes the trivial kB-module. Now $kG\underline{\mathfrak{b}}$ carries a natural symmetric bilinear form $\langle\ ,\ \rangle\colon kG\underline{\mathfrak{b}} \times kG\underline{\mathfrak{b}} \to k$ such that, for any $g, g' \in G$, we have

$$\langle g\underline{\mathfrak{b}}, g'\underline{\mathfrak{b}}\rangle = \begin{cases} 1, & \text{if } gB = g'B,\\ 0, & \text{otherwise.}\end{cases}$$

This form is non-singular and G-invariant. For any subset $X \subseteq kG\underline{\mathfrak{b}}$, we denote

$$X^\perp := \{a \in kG\underline{\mathfrak{b}} \mid \langle a, x\rangle = 0 \text{ for all } x \in X\}.$$

If X is a kG-submodule of $kG\underline{\mathfrak{b}}$, then so is $X^\perp$.

Remark 2.2. Let $\sigma\colon U \to k^\times$ be a group homomorphism and set

$$\underline{\mathfrak{u}}_\sigma = \sum_{u\in U} \sigma(u)u \in kG.$$

Then $\Gamma_\sigma := kG\underline{\mathfrak{u}}_\sigma$ is a left kG-module which is isomorphic to the induced module $\mathrm{Ind}_U^G(k_\sigma)$, where k_σ denotes the 1-dimensional kU-module corresponding to σ. Note that $\underline{\mathfrak{u}}_\sigma^2 = |U|\underline{\mathfrak{u}}_\sigma$. Hence, if $|U|1_k \neq 0$, then Γ_σ is a projective kG-module.

We say that σ is non-degenerate if the restriction of σ to U_s is non-trivial for every $s \in S$. Note that this can only occur if $|U|1_k \neq 0$. In the case where $G = \mathrm{GL}_n(q)$, the following result is contained in James [6, Theorem 11.7(ii)]; see also Szechtman [9, Note 4.9, Section 10].

Lemma 2.3. *Assume that $|U|1_k \neq 0$ and σ is non-degenerate. Then the subspace $\underline{\mathfrak{u}}_\sigma kG\underline{\mathfrak{b}}$ is 1-dimensional and spanned by $\underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}}$. Furthermore, $\underline{\mathfrak{u}}_\sigma n_w\underline{\mathfrak{b}} = 0$ for all $w \in W$ such that $w \neq w_0$.*

Proof. By the Bruhat decomposition, we can write any $g \in G$ in the form $g = un_wb$ where $u \in U$, $w \in W$ and $b \in B$. Now note that $b\underline{\mathfrak{b}} = \underline{\mathfrak{b}}$ for all $b \in B$ and $\underline{\mathfrak{u}}_\sigma u = \sigma(u)^{-1}\underline{\mathfrak{u}}_\sigma$ for all $u \in U$. Thus, $\underline{\mathfrak{u}}_\sigma kG\underline{\mathfrak{b}}$ is spanned by $\{\underline{\mathfrak{u}}_\sigma n_w\underline{\mathfrak{b}} \mid w \in W\}$. Now let $w \in W$ be such that $w \neq w_0$. We shall show that $\underline{\mathfrak{u}}_\sigma n_w\underline{\mathfrak{b}} = 0$. For this purpose, we use the factorisation $U = U_wU_{w_0w}$ where $U_w \cap U_{w_0w} = \{1\}$; see [1, Proposition 2.5.12]. Since $w \neq w_0$, there exists some $s \in S$ such that $l(ws) > l(w)$ and so $U_s \subseteq U_{w_0w}$; see [1, Propositions 2.5.7(i) and 2.5.10(i)]. (Note that U_s is denoted by X_i in [*loc. cit.*].) Since $n_{w^{-1}} \in n_w^{-1}H$, we obtain

$$\underline{\mathfrak{u}}_\sigma n_{w^{-1}}\underline{\mathfrak{b}} = \underline{\mathfrak{u}}_\sigma n_w^{-1}\underline{\mathfrak{b}} = \Big(\sum_{u_1\in U_w} \sigma(u_1)u_1\Big)n_w^{-1}\Big(\sum_{u_2\in U_{w_0w}} \sigma(u_2)n_wu_2n_w^{-1}\underline{\mathfrak{b}}\Big).$$

By the definition of U_{w_0w}, we have $n_wU_{w_0w}n_w^{-1} \subseteq U$ and so $n_wu_2n_w^{-1}\underline{\mathfrak{b}} = \underline{\mathfrak{b}}$ for every fixed $u_2 \in U_{w_0w}$. Hence, we obtain

$$\sum_{u_2\in U_{w_0w}} \sigma(u_2)n_wu_2n_w^{-1}\underline{\mathfrak{b}} = \Big(\sum_{u_2\in U_{w_0w}} \sigma(u_2)\Big)\underline{\mathfrak{b}}.$$

Finally, since $U_s \subseteq U_{w_0w}$ and the restriction of σ to U_s is non-trivial, the above sum evaluates to 0. Thus, $\underline{\mathfrak{u}}_\sigma n_{w^{-1}}\underline{\mathfrak{b}} = 0$ for all $w \neq w_0$. Since $w_0 = w_0^{-1}$, this also implies that $\underline{\mathfrak{u}}_\sigma n_w\underline{\mathfrak{b}} = 0$ for all $w \neq w_0$, as required.

Hence, $\underline{\mathfrak{u}}_\sigma kG\underline{\mathfrak{b}}$ is spanned by $\underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}}$. Finally, by the sharp form of the Bruhat decomposition, every element of Bn_0B has a unique expression of the form un_0b where $u \in U$ and $b \in B$. In particular, $\underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}} \neq 0$ and so $\dim \underline{\mathfrak{u}}_\sigma kG\underline{\mathfrak{b}} = 1$. ■

Corollary 2.4. *Assume that $|U|1_k \neq 0$ and σ is non-degenerate. Then the map $\varphi\colon \Gamma_\sigma \to kG\underline{\mathfrak{b}}$, $\gamma \mapsto \gamma n_0\underline{\mathfrak{b}}$, is a non-zero kG-module homomorphism and every homomorphism $\Gamma_\sigma \to kG\underline{\mathfrak{b}}$ is a scalar multiple of φ.*

Proof. The fact that φ, as defined above, is a kG-module homomorphism is clear; it is non-zero since $\varphi(\underline{\mathfrak{u}}_\sigma) = \underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}} \neq 0$ by Lemma 2.3. Since $\underline{\mathfrak{u}}_\sigma$ is a non-zero scalar multiple of an idempotent (see Remark 2.2), we have $\mathrm{Hom}_{kG}(\Gamma_\sigma, kG\underline{\mathfrak{b}}) \cong \underline{\mathfrak{u}}_\sigma kG\underline{\mathfrak{b}}$ and this is 1-dimensional by Lemma 2.3. ■

3 The submodule theorem

We keep the notation of the previous section and assume throughout that $|U|1_k \neq 0$. For any group homomorphism $\sigma\colon U \to k^\times$, we denote by $\sigma^*\colon U \to k^\times$ the group homomorphism given by $\sigma^*(u) = \sigma(u)^{-1}$ for all $u \in U$. Note that, if σ is non-degenerate, then so is σ^*. We can now state the following version of James's "submodule theorem" [6, Theorem 11.2], [6, 11.12(ii)].

Proposition 3.1 (cf. James [6, Theorem 11.2])**.** *Let $\sigma\colon U \to k^\times$ be non-degenerate and consider the submodule $\mathscr{S}_\sigma := kG\underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}} \subseteq kG\underline{\mathfrak{b}}$. Then the following hold.*

(i) *If $M \subseteq kG\underline{\mathfrak{b}}$ is any submodule, then either $\mathscr{S}_\sigma \subseteq M$ or $M \subseteq \mathscr{S}_{\sigma^*}^\perp$.*

(ii) *We have $\mathscr{S}_\sigma \not\subseteq \mathscr{S}_{\sigma^*}^\perp$ and $\mathscr{S}_\sigma \cap \mathscr{S}_{\sigma^*}^\perp \subsetneqq \mathscr{S}_\sigma$ is the unique maximal submodule of $\mathscr{S}_\sigma$.*

(iii) *The kG-module $D_\sigma := \mathscr{S}_\sigma/(\mathscr{S}_\sigma \cap \mathscr{S}_{\sigma^*}^\perp)$ is absolutely irreducible and isomorphic to the contragredient dual of D_{σ^*}.*

(iv) *D_σ occurs with multiplicity 1 as a composition factor of $kG\underline{\mathfrak{b}}$ and of $\mathscr{S}_\sigma$.*

Proof. Having established Lemma 2.3 and Corollary 2.4, this readily follows from the general results in [6, Chapter 11]; the only difference is that James also assumes that $\mathscr{S}_\sigma = \mathscr{S}_{\sigma^*}$. As our notation and setting are somewhat different from those in [6], we recall the most important steps of the argument.

(i) Since $M \subseteq kG\underline{\mathfrak{b}}$, it follows from Lemma 2.3 that, for any $m \in M$, there exists some $c_m \in k$ such that $\underline{\mathfrak{u}}_\sigma m = c_m\underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}}$. If there exists some $m \in M$ with $c_m \neq 0$, then $\underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}} = c_m^{-1}\underline{\mathfrak{u}}_\sigma m \in M$ and, hence, we have $\mathscr{S}_\sigma \subseteq M$ in this case. Now assume that $c_m = 0$ for all $m \in M$; that is, we have $\underline{\mathfrak{u}}_\sigma M = \{0\}$. Let $m \in M$ and $g \in G$. Using the definition of $\underline{\mathfrak{u}}_\sigma$, $\underline{\mathfrak{u}}_{\sigma^*}$ and the G-invariance of $\langle\ ,\ \rangle$, we obtain

$$\langle m, g\underline{\mathfrak{u}}_{\sigma^*}n_0\underline{\mathfrak{b}}\rangle = \langle g^{-1}m, \underline{\mathfrak{u}}_{\sigma^*}n_0\underline{\mathfrak{b}}\rangle = \langle \underline{\mathfrak{u}}_\sigma(g^{-1}m), n_0\underline{\mathfrak{b}}\rangle = 0,$$

where the last equality holds since $\underline{\mathfrak{u}}_\sigma M = \{0\}$. Thus, $M \subseteq \mathscr{S}_{\sigma^*}^\perp$ in this case.

(ii) For $u, u' \in U$, we have $un_0B = u'n_0B$ if and only if $u = u'$, by the sharp form of the Bruhat decomposition. Thus, we obtain

$$\langle \underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}}, \underline{\mathfrak{u}}_{\sigma^*} n_0\underline{\mathfrak{b}}\rangle = \sum_{u,u'\in U} \sigma(u)\sigma^*(u')\langle un_0\underline{\mathfrak{b}}, u'n_0\underline{\mathfrak{b}}\rangle = |U|1_k \neq 0,$$

which means that $\mathscr{S}_\sigma \not\subseteq \mathscr{S}_{\sigma^*}^\perp$ and so $\mathscr{S}_\sigma \cap \mathscr{S}_{\sigma^*}^\perp \subsetneqq \mathscr{S}_\sigma$. Now let $M \subsetneqq \mathscr{S}_\sigma$ be any maximal submodule. Then (i) immediately implies that $M = \mathscr{S}_\sigma \cap \mathscr{S}_{\sigma^*}^\perp$.

(iii) By (ii), we already know that D_σ is irreducible. The remaining assertions then follow exactly as in the proof of [6, Theorem 11.2].

(iv) By construction, D_σ occurs at least once in $kG\underline{\mathfrak{b}}$ and in $\mathscr{S}_\sigma$. Let $\varphi\colon \Gamma_\sigma \to kG\underline{\mathfrak{b}}$ be as in Corollary 2.4. Since Γ_σ is projective and $\varphi(\Gamma_\sigma) = \mathscr{S}_\sigma$, some indecomposable direct summand of Γ_σ is a projective cover of D_σ and so the desired multiplicity of D_σ is at most $\dim \mathrm{Hom}_{kG}(\Gamma_\sigma, kG\underline{\mathfrak{b}}) = 1$, where the last equality holds by Corollary 2.4. ■

We now relate the modules $\mathscr{S}_\sigma$, D_σ in Proposition 3.1 to the Steinberg module St_k of G, as defined in [8]. Recall that

$$\mathrm{St}_k = kG\mathfrak{e} \subseteq kG\underline{\mathfrak{b}}, \qquad \text{where} \quad \mathfrak{e} := \sum_{w\in W} (-1)^{l(w)} n_w\underline{\mathfrak{b}}.$$

We have $\dim \mathrm{St}_k = |U|$; a basis of St_k is given by $\{u\mathfrak{e} \mid u \in U\}$.

Proposition 3.2. *We have $\mathscr{S}_\sigma = kG\underline{\mathfrak{u}}_\sigma\mathfrak{e} \subseteq \mathrm{St}_k$. Consequently, D_σ is a composition factor* (*with multiplicity* 1) *of* St_k.

Proof. By Lemma 2.3, we have the identity

$$\underline{\mathfrak{u}}_\sigma\mathfrak{e} = \sum_{w\in W} (-1)^{l(w)}\underline{\mathfrak{u}}_\sigma n_w\underline{\mathfrak{b}} = (-1)^{l(w_0)}\underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}}$$

and so $\mathscr{S}_\sigma = kG\underline{\mathfrak{u}}_\sigma n_0\underline{\mathfrak{b}} = kG\underline{\mathfrak{u}}_\sigma\mathfrak{e} \subseteq kG\mathfrak{e} = \mathrm{St}_k$. The statement about D_σ then follows from Proposition 3.1(iv). ■

Remark 3.3. Gow [4, Section 2] gives an explicit formula for the restriction of the bilinear form $\langle\ ,\ \rangle\colon kG\underline{\mathfrak{b}} \times kG\underline{\mathfrak{b}} \to k$ (see Remark 2.1) to St_k. For this purpose, he first works over $\mathbb{Z}$ and then rescales to obtain a non-zero form over k. One can also proceed directly, as follows. We have

$$\langle u_1\mathfrak{e}, u_2\mathfrak{e}\rangle = c_W\big(u_1^{-1}u_2\big)1_k \qquad \text{for any } u_1, u_2 \in U,$$

where $c_W(u) := |\{w \in W \mid n_w^{-1}un_w \in U\}|$ for $u \in U$. Indeed, since $\langle\ ,\ \rangle$ is G-invariant, it is enough to show that $\langle \mathfrak{e}, u\mathfrak{e}\rangle = c_W(u)1_k$ for $u \in U$. Now, we have

$$\langle \mathfrak{e}, u\mathfrak{e}\rangle = \sum_{w,w'\in W} (-1)^{l(w)+l(w')}\langle n_w\underline{\mathfrak{b}}, un_{w'}\underline{\mathfrak{b}}\rangle = \sum_{w\in W} \langle n_w\underline{\mathfrak{b}}, un_w\underline{\mathfrak{b}}\rangle,$$

where the second equality holds since $n_wB = un_{w'}B$ if and only if $w = w'$. Furthermore, $n_wB = un_wB$ if and only if $n_w^{-1}un_w \in B$. Since $n_w^{-1}un_w$ is a p-element, the latter condition is equivalent to $n_w^{-1}un_w \in U$. This yields the desired formula.

Remark 3.4. Since $|U|1_k \neq 0$, the module Γ_σ in Remark 2.2 is projective. Also note that $\mathscr{S}_\sigma = \varphi(\Gamma_\sigma)$ where φ is defined in Corollary 2.4. Using also Proposition 3.2, we conclude that $\dim \mathrm{Hom}_{kG}(\Gamma_\sigma, \mathrm{St}_k) = \dim \mathrm{Hom}_{kG}(\Gamma_\sigma, kG\underline{\mathfrak{b}}) = 1$. So there is a unique indecomposable direct summand P_{St} of Γ_σ such that

$$\mathrm{Hom}_{kG}(P_{\mathrm{St}}, \mathrm{St}_k) \neq 0.$$

Being projective indecomposable, P_{St} has a unique simple quotient whose Brauer character is denoted by σ_G by Hiss [5, Section 6]. By Proposition 3.1(iv) (and its proof), we now see that σ_G is the Brauer character of D_σ.

Remark 3.5. It is known that the socle of St_k is simple; see [3, 10]. We claim that this simple socle is contained in $\mathscr{S}_\sigma \cap \mathscr{S}_{\sigma^*}^\perp$, unless St_k is irreducible. Indeed, assume that $\mathscr{S}_\sigma \cap \mathscr{S}_{\sigma^*}^\perp = \{0\}$. Then $D_\sigma \subseteq \mathrm{St}_k \subseteq kG\underline{\mathfrak{b}}$ and, hence, D_σ belongs to the Harish-Chandra series defined by the pair $(\varnothing, k_H)$ (notation of Hiss [5, Theorem 5.8]). By Remark 3.4 and the argument in [5, Lemma 6.2], it then follows that $[G:B]1_k \neq 0$. So St_k is irreducible by [8].

4 Examples

We keep the setting of the previous section. We also assume now that G is a true finite group of Lie type, as in [1, Section 1.18]. Thus, using the notation in [*loc. cit.*], we have $G = \mathbf{G}^F$ where $\mathbf{G}$ is a connected reductive algebraic group $\mathbf{G}$ over $\overline{\mathbb{F}}_p$ and $F\colon \mathbf{G} \to \mathbf{G}$ is an endomorphism such that some power of F is a Frobenius map. Then the ingredients of the BN-pair in G will also be derived from $\mathbf{G}$: we have $B = \mathbf{B}^F$ where $\mathbf{B}$ is an F-stable Borel subgroup of $\mathbf{G}$ and $H = \mathbf{T}^F$ where $\mathbf{T}$ is an F-stable maximal torus contained in $\mathbf{B}$; furthermore, $N = N_{\mathbf{G}}(\mathbf{T})^F$ and $U = \mathbf{U}^F$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. In this setting, one can single out a certain class of non-degenerate group homomorphisms $\sigma\colon U \to k^\times$, as follows.

The commutator subgroup $[\mathbf{U}, \mathbf{U}]$ is an F-stable closed connected normal subgroup of $\mathbf{U}$. We define the subgroup $U^* := [\mathbf{U}, \mathbf{U}]^F \subseteq U$; then $[U, U] \subseteq U^*$. Furthermore, we shall fix a group homomorphism $\sigma\colon U \to k^\times$ which is a *regular character*, that is, we have $U^* \subseteq \ker(\sigma)$ and the restriction of σ to U_s is non-trivial for all $s \in S$. (Such characters always exist; see [1, Section 8.1] and [2, Definition 14.27].) Then the corresponding module $\Gamma_\sigma = kG\underline{\mathfrak{u}}_\sigma$ is called a *Gelfand–Graev module* for G. Let $h \in H$ and $\sigma^h\colon U \to k^\times$ be defined by $\sigma^h(u) := \sigma(h^{-1}uh)$ for $u \in U$. Then σ^h also is a regular character and

$$\underline{\mathfrak{u}}_{\sigma^h} = \sum_{u \in U} \sigma^h(u)u = h\underline{\mathfrak{u}}_\sigma h^{-1} \qquad \text{for all } h \in H.$$

Hence, right multiplication by h^{-1} defines an isomorphism between the corresponding Gelfand–Graev modules Γ_σ and Γ_{σ^h}.

Remark 4.1. Let $Z(G)$ be the center of G. Then $Z(G) \subseteq H$ and $Z(G) = Z(\mathbf{G})^F$; see [1, Proposition 3.6.8]. Assume now that $Z(\mathbf{G})$ is connected. Then there are precisely $|H/Z(G)|$ regular characters and they are all conjugate under the action of H; see [1, Proposition 8.1.2]. For any $h \in H$ we have

$$\underline{\mathfrak{u}}_{\sigma^h} n_0 \underline{\mathfrak{b}} = h\underline{\mathfrak{u}}_\sigma h^{-1} n_0 \underline{\mathfrak{b}} = h\underline{\mathfrak{u}}_\sigma n_0 h^{-1} \underline{\mathfrak{b}} = h\underline{\mathfrak{u}}_\sigma n_0 \underline{\mathfrak{b}}$$

and so $\mathscr{S}_\sigma = \mathscr{S}_{\sigma^h}$. It follows that $\mathscr{S}_\sigma = \mathscr{S}_{\sigma^*} = \mathscr{S}_{\sigma'}$ for all regular characters σ, σ' of U, and we can denote this submodule simply by $\mathscr{S}_0$. By Proposition 3.1(iii), the simple module $D_0 := \mathscr{S}_0/(\mathscr{S} \cap \mathscr{S}_0^\perp)$ is now self-dual. Furthermore, we have

$$\dim \mathscr{S}_0 \geqslant |H/Z(G)|.$$

Indeed, we have $\underline{\mathfrak{u}}_\sigma n_0 \underline{\mathfrak{b}} \in \mathscr{S}_0$ for all regular characters σ. Since pairwise distinct group homomorphisms $U \to k^\times$ are linearly independent, the elements $\underline{\mathfrak{u}}_\sigma n_0 \underline{\mathfrak{b}}$ (where σ runs over all regular characters of U) form a set of $|H/Z(G)|$ linearly independent elements in $\mathscr{S}_0$.

Example 4.2. Let $G = \mathrm{GL}_n(q)$ where q is a prime power. Then our module $\mathscr{S}_0$ is S_λ in James' notation [6, Definition 11.11], where λ is the partition of n with all parts equal to 1. We claim that $\mathscr{S}_0 = \mathrm{St}_k$ in this case.

Indeed, by [6, Theorem 16.5], $\dim \mathscr{S}_0$ is independent of the field k, as long as $\mathrm{char}(k) \neq p$. Since $\mathrm{St}_\mathbb{Q}$ is irreducible, we conclude that $\dim \mathscr{S}_0 = \dim \mathrm{St}_k$ and the claim follows. Consequently, by Proposition 3.1(ii), D_0 is the unique simple quotient of St_k. (The facts that $\mathscr{S}_0 = \mathrm{St}_k$ and that this module has a unique simple quotient are also shown by Szechtman [9, Section 4].) However, $\dim D_0$ may certainly vary as the field k varies; see the tables in [6, p. 107].

See [3, 4.14] for further examples where $\mathscr{S}_0 = \mathrm{St}_k$. On the other hand, Gow [4, Section 5] gives examples (where $G = \mathrm{Sp}_4(q)$) where St_k does not have a unique simple quotient, and so $\mathscr{S}_0 \subsetneq \mathrm{St}_k$. Here is a further example.

Example 4.3. Let $G = \mathrm{Ree}\left(q^2\right)$ be the Ree group of type ${}^2G_2$, where q is an odd power of $\sqrt{3}$. Then G has a BN-pair of rank 1 and so $[G : B] = \dim \mathrm{St}_k + 1$. Let k be a field of characteristic 2. Then $kG\underline{\mathfrak{b}}$ and St_k have socle series as follows:

$$kG\underline{\mathfrak{b}}\colon \quad \begin{matrix} & k_G & \\ & \varphi_2 & \\ \varphi_4 & \varphi_3 & \varphi_5, \\ & \varphi_2 & \\ & k_G & \end{matrix} \qquad\qquad \mathrm{St}_k\colon \quad \begin{matrix} & \varphi_2 & \\ \varphi_4 & \varphi_3 & \varphi_5. \\ & \varphi_2 & \\ & k_G & \end{matrix}$$

Here, φ_i $(i = 1, 2, 3, 4, 5)$ are simple kG-modules and φ_4 is the contragredient dual of φ_5; see Landrock–Michler [7, Proposition 3.8(b)]. By Proposition 3.1, we have $D_0 \cong \varphi_3$ and $\mathscr{S}_0$ is the uniserial submodule with composition factors k_G, φ_2, φ_3.

References

[1] Carter R.W., Finite groups of Lie type. Conjugacy classes and complex characters, *Wiley Classics Library*, John Wiley & Sons, Ltd., Chichester, 1993.

[2] Digne F., Michel J., Representations of finite groups of Lie type, *London Mathematical Society Student Texts*, Vol. 21, Cambridge University Press, Cambridge, 1991.

[3] Geck M., On the modular composition factors of the Steinberg representation, *J. Algebra* **475** (2017), 370–391.

[4] Gow R., The Steinberg lattice of a finite Chevalley group and its modular reduction, *J. London Math. Soc.* **67** (2003), 593–608.

[5] Hiss G., Harish-Chandra series of Brauer characters in a finite group with a split BN-pair, *J. London Math. Soc.* **48** (1993), 219–228.

[6] James G.D., Representations of general linear groups, *London Mathematical Society Lecture Note Series*, Vol. 94, Cambridge University Press, Cambridge, 1984.

[7] Landrock P., Michler G.O., Principal 2-blocks of the simple groups of Ree type, *Trans. Amer. Math. Soc.* **260** (1980), 83–111.

[8] Steinberg R., Prime power representations of finite linear groups. II, *Canad. J. Math.* **9** (1957), 347–351.

[9] Szechtman F., Steinberg lattice of the general linear group and its modular reduction, *J. Group Theory* **14** (2011), 603–635, arXiv:0812.2232.

[10] Tinberg N.B., The Steinberg component of a finite group with a split (B, N)-pair, *J. Algebra* **104** (1986), 126–134.