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On the Strong Ratio Limit Property for Discrete-Time Birth-Death Processes

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Abstract. A sufficient condition is obtained for a discrete-time birth-death process to possess the *strong ratio limit property*, directly in terms of the one-step transition probabilities of the process. The condition encompasses all previously known sufficient conditions.

Key words: (a)periodicity; birth-death process; orthogonal polynomials; random walk measure; ratio limit; transition probability

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1 Introduction

In what follows $\mathcal{X} \equiv \{X(n), n = 0, 1, \dots\}$ is a discrete-time birth-death process on $\mathcal{N} \equiv \{0, 1, \dots\}$, with tridiagonal matrix of one-step transition probabilities

$$P := \begin{pmatrix} r_0 & p_0 & 0 & 0 & 0 & \dots \\ q_1 & r_1 & p_1 & 0 & 0 & \dots \\ 0 & q_2 & r_2 & p_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Following Karlin and McGregor [6] we will refer to $\mathcal{X}$ as a *random walk*. We assume throughout that $p_j > 0$, $q_{j+1} > 0$, $r_j \geq 0$, and $p_j + q_j + r_j = 1$ for $j \geq 0$, where $q_0 := 0$. We let

$$\pi_0 := 1, \qquad \pi_n := \frac{p_0 p_1 \cdots p_{n-1}}{q_1 q_2 \cdots q_n}, \qquad n \geq 1, \tag{1.1}$$

and define the polynomials Q_n via the recurrence relation

$$\begin{aligned} &x Q_n(x) = q_n Q_{n-1}(x) + r_n Q_n(x) + p_n Q_{n+1}(x), \qquad n > 1, \\ &Q_0(x) = 1, \qquad p_0 Q_1(x) = x - r_0. \end{aligned} \tag{1.2}$$

Karlin and McGregor [6] have shown that the n-step transition probabilities

$$P_{ij}(n) := \Pr\{X(n) = j \mid X(0) = i\} = (P^n)_{ij}, \qquad n \geq 0, \qquad i, j \in \mathcal{N},$$

may be represented in the form

$$P_{ij}(n) = \pi_j \int_{[-1,1]} x^n Q_i(x) Q_j(x) \psi(dx), \tag{1.3}$$

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where ψ is the (unique) Borel measure on the interval $[-1,1]$, of total mass 1 and with infinite support, with respect to which the polynomials Q_n are orthogonal. Adopting the terminology of [3] we will refer to the measure ψ as a *random walk measure*. Of particular interest to us is $\eta := \sup \operatorname{supp}(\psi)$, the largest point in the support of the random walk measure ψ, which may also be characterized in terms of the polynomials Q_n by

$$x \geq \eta \iff Q_n(x) > 0 \quad \text{for all} \quad n \geq 0 \tag{1.4}$$

(see, for example, Chihara [1, Theorem II.4.1]). We will see in the next section that $\eta > 0$.

The random walk $\mathcal{X}$ is said to have the *strong ratio limit property* (*SRLP*) if the limits

$$\lim_{n\to\infty} \frac{P_{ij}(n)}{P_{kl}(n)}, \qquad i,j,k,l \in \mathcal{N}, \tag{1.5}$$

exist simultaneously. The SRLP was introduced in the more general setting of discrete-time Markov chains on a countable state space by Orey [8] and Pruitt [9], but the problem of finding conditions for the limits (1.5) to exist in the specific setting of random walks had been considered before in [6]. A satisfactory and comprehensive solution to the problem of finding conditions for the SRLP is still lacking, even in the relatively simple setting at hand. So it remains a challenge to find necessary and/or sufficient conditions. For more information on the history of the problem we refer to [5] and [7].

In [5, Theorem 3.1] a necessary and sufficient condition for the random walk $\mathcal{X}$ to have the SRLP has been given in terms of the associated random walk measure ψ. Namely, letting

$$C_n(\psi) := \frac{\int_{[-1,0)} (-x)^n \psi(dx)}{\int_{(0,1]} x^n \psi(dx)}, \qquad n \geq 0, \tag{1.6}$$

the limits (1.5) exist simultaneously if and only if

$$\lim_{n\to\infty} C_n(\psi) = 0, \tag{1.7}$$

in which case we have

$$\lim_{n\to\infty} \frac{P_{ij}(n)}{P_{kl}(n)} = \frac{\pi_j Q_i(\eta) Q_j(\eta)}{\pi_l Q_k(\eta) Q_l(\eta)}, \qquad i,j,k,l \in \mathcal{N}.$$

Note that the denominator in (1.6) is positive since $\eta > 0$, so that $C_n(\psi)$ exists and is nonnegative for all n. Some sufficient conditions for (1.7) – and, hence, for $\mathcal{X}$ to possess the SRLP – are also given in [5]. In particular, [5, Theorem 3.2] tells us that

$$\lim_{n\to\infty} |Q_n(-\eta)/Q_n(\eta)| = \infty \;\Rightarrow\; \lim_{n\to\infty} C_n(\psi) = 0. \tag{1.8}$$

The reverse implication is conjectured in [5] to be valid as well.

In this paper we will prove a sufficient condition for $\mathcal{X}$ to have the SRLP directly in terms of the one-step transition probabilities. Concretely, we will establish the following result.

Proposition 1.1. *If the random walk $\mathcal{X}$ satisfies*

$$\sum_{j\geq 0} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} r_k \pi_k = \infty, \tag{1.9}$$

then $\lim_{n\to\infty} |Q_n(-\eta)/Q_n(\eta)| = \infty$.

Together with (1.8) this result immediately leads to the following.

Theorem 1.2. *If the random walk $\mathcal{X}$ satisfies (1.9) then $\mathcal{X}$ possesses the SRLP.*

We will see that Theorem 1.2 encompasses all previously obtained sufficient conditions for the SRLP.

The proof of Proposition 1.1 will be based on three lemmas. Lemma 2.1 and a number of preliminary results related to the polynomials Q_n and the orthogonalizing measure ψ are collected in the next section. Two further auxiliary lemmas are established in Section 3. The actual proof of Proposition 1.1 and some concluding remarks can be found in Section 4, which also contains an example showing that (1.9) is not *necessary* for the SRLP.

2 Preliminaries

Whitehurst [11, Theorem 1.6] has shown that the random walk measure ψ satisfies

$$\int_{[-1,1]} xQ_n^2(x)\psi(dx) \geq 0, \qquad n \geq 0, \tag{2.1}$$

and, conversely, that any Borel measure ψ on the interval $[-1, 1]$, of total mass 1 and with infinite support, is a random walk measure if it satisfies (2.1) (see also [3, Theorem 1.2]). Evidently, (2.1) implies $\eta = \sup \operatorname{supp}(\psi) > 0$, but it can actually be shown (see, for example, [1, Corollary 2 to Theorem IV.2.1]) that

$$\eta > r_j, \qquad j \in \mathcal{N}.$$

By [4, Lemma 2.3] we also have

$$\inf_j \{r_j + r_{j+1}\} \leq \inf \operatorname{supp}(\psi) + \eta \leq \sup_j \{r_j + r_{j+1}\}, \qquad j \in \mathcal{N},$$

so that $\inf \operatorname{supp}(\psi) \geq -\eta$, and hence $\operatorname{supp}(\psi) \subset [-\eta, \eta]$.

The measure ψ is symmetric about 0 if (and only if) the random walk $\mathcal{X}$ is *periodic*, that is, if $r_j = 0$ for all j (see [6, p. 69]). In this case we also have

$$(-1)^n Q_n(-x) = Q_n(x), \qquad n \geq 0,$$

and it follows from (1.3) that $P_{ij}(n) = 0$ if $n + i + j$ is odd. Hence the limits in (1.5) will not exist if $\mathcal{X}$ is periodic, which is also reflected by the fact that $C_n(\psi) = 1$ for all n in this case.

$\mathcal{X}$ is called *aperiodic* if it is not periodic. From Whitehurst [10, Theorem 5.2] we have the subtle result

$$\mathcal{X} \text{ is aperiodic } \Rightarrow \int_{[-\eta,\eta]} \frac{\psi(dx)}{\eta + x} < \infty,$$

so that $\psi(\{-\eta\}) = 0$ if $\mathcal{X}$ is aperiodic.

We continue with some useful observations from the recurrence relations (1.2). The first one is the *Christoffel–Darboux* identity

$$p_n\pi_n(Q_n(x)Q_{n+1}(y) - Q_n(y)Q_{n+1}(x)) = (y - x)\sum_{j=0}^{n} \pi_j Q_j(x)Q_j(y)$$

(see, for example, [1, Theorem I.4.5]). Hence, by (1.4),

$$\eta \leq x < y \ \Rightarrow \ Q_n(x)Q_{n+1}(y) > Q_n(y)Q_{n+1}(x) > 0 \qquad \text{for all } \ n \geq 0. \tag{2.2}$$

Since $p_j + q_j + r_j = 1$ for all j it follows readily from (1.2) that $Q_n(1) = 1$ for all n, so (2.2) leads to

$$\eta \le x < 1 \;\Rightarrow\; 0 < Q_{n+1}(x) < Q_n(x) < Q_0(x) = 1 \qquad \text{for all} \;\; n \ge 1. \tag{2.3}$$

Next, writing $\bar{Q}_n(x) := (-1)^n Q_n(x)$, we see from (1.2) that

$$\begin{aligned} p_n\pi_n(\bar{Q}_{n+1}(x) - \bar{Q}_n(x)) &= p_{n-1}\pi_{n-1}(\bar{Q}_n(x) - \bar{Q}_{n-1}(x)) \\ &\quad + (2r_n - 1 - x)\pi_n\bar{Q}_n(x), \qquad n \ge 1, \\ p_0\pi_0(\bar{Q}_1(x) - \bar{Q}_0(x)) &= (2r_0 - 1 - x)\pi_0\bar{Q}_0(x), \end{aligned}$$

from which we readily obtain

$$\bar{Q}_{n+1}(x) = 1 + \sum_{j=0}^{n} \frac{1}{p_j\pi_j} \sum_{k=0}^{j} (2r_k - 1 - x)\pi_k\bar{Q}_k(x), \qquad n \ge 0,$$

and hence

$$\bar{Q}_{n+1}(-1) = 1 + 2\sum_{j=0}^{n} \frac{1}{p_j\pi_j} \sum_{k=0}^{j} r_k\pi_k\bar{Q}_k(-1), \qquad n \ge 0. \tag{2.4}$$

This equation, observed already by Karlin and McGregor [6, p. 76], leads to the first of our three lemmas.

Lemma 2.1. *The sequence $\{(-1)^n Q_n(-1)\}_n$ is increasing, and strictly increasing for n sufficiently large, if (and only if) $\mathcal{X}$ is aperiodic. Moreover,*

$$\sum_{j\ge 0} \frac{1}{p_j\pi_j} \sum_{k=0}^{j} r_k\pi_k = \infty \iff \lim_{n\to\infty} (-1)^n Q_n(-1) = \infty. \tag{2.5}$$

Proof. Since $\bar{Q}_0(-1) = 1$, while, by (2.4),

$$\bar{Q}_{n+1}(-1) = \bar{Q}_n(-1) + \frac{2}{p_n\pi_n} \sum_{k=0}^{n} r_k\pi_k\bar{Q}_k(-1), \qquad n \ge 0, \tag{2.6}$$

the first statement is obviously true. So we have $\bar{Q}_n(-1) \ge 1$, which, in view of (2.4) implies the necessity in the second statement. To prove the sufficiency we let

$$\beta_j := \frac{2}{p_j\pi_j} \sum_{k=0}^{j} r_k\pi_k, \qquad j \ge 0,$$

and assume that $\sum_j \beta_j$ converges. By (2.6) we then have

$$\bar{Q}_{n+1}(-1) \le \bar{Q}_n(-1)(1 + \beta_n), \qquad n \ge 0,$$

since $\bar{Q}_n(-1)$ is increasing in n. It follows that

$$\bar{Q}_{n+1}(-1) \le \prod_{j=0}^{n} (1 + \beta_j), \qquad n \ge 0.$$

But since $\prod_j (1 + \beta_j)$ and $\sum_j \beta_j$ converge together, we must have $\lim_{n\to\infty} \bar{Q}_n(-1) < \infty$. ■

The above lemma also plays a central role in [2], where the conditions in (2.5) are shown to be equivalent to *asymptotic aperiodicity* of the random walk. For completeness' sake we have included the proof.

We recall from [6] that $\mathcal{X}$ is *recurrent*, that is, the probability, for any state, of returning to that state is one, if and only if

$$L := \sum_{j\geq 0} \frac{1}{p_j \pi_j} = \infty. \tag{2.7}$$

$\mathcal{X}$ is called *transient* if it is not recurrent. It has been shown in [6] that

$$\int_{[-\eta,\eta]} \frac{\psi(dx)}{1-x} = L,$$

so we must have $\eta = 1$ if $\mathcal{X}$ is recurrent. From Lemma 2.1 we now obtain

$$\mathcal{X} \text{ is aperiodic and recurrent } \Rightarrow \lim_{n\to\infty} (-1)^n Q_n(-1) = \infty, \tag{2.8}$$

a result noted earlier by Karlin and McGregor [6, p. 76]. Considering (1.8) and the fact that $\eta = 1$ if $\mathcal{X}$ is recurrent, the conclusion in (2.8) implies the SRLP, so that we have regained [6, Theorem 2]. (This result was later generalized to *symmetrizable* Markov chains by Orey [8, Theorem 2].) For later use we also note that

$$\sum_{j\geq 0} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} r_k \pi_k \geq \sum_{j\geq 0} \frac{r_j}{p_j}, \tag{2.9}$$

so that, by Lemma 2.1,

$$\sum_{j\geq 0} \frac{r_j}{p_j} = \infty \Rightarrow \lim_{n\to\infty} (-1)^n Q_n(-1) = \infty. \tag{2.10}$$

3 Two auxiliary lemmas

Throughout this section θ is a fixed number satisfying $\theta \geq \eta$. Defining $q_0(\theta) := 0$ and

$$p_j(\theta) := \frac{Q_{j+1}(\theta)}{Q_j(\theta)} \frac{p_j}{\theta}, \qquad r_j(\theta) := \frac{r_j}{\theta}, \qquad q_{j+1}(\theta) := \frac{Q_j(\theta)}{Q_{j+1}(\theta)} \frac{q_{j+1}}{\theta}, \qquad j \in \mathcal{N}, \tag{3.1}$$

the parameters $p_j(\theta)$, $q_j(\theta)$ and $r_j(\theta)$ satisfy $p_j(\theta) > 0$, $q_{j+1}(\theta) > 0$, $r_j(\theta) \geq 0$, and $p_j(\theta) + q_j(\theta) + r_j(\theta) = 1$, so that they may be interpreted as the one-step transition probabilities of a random walk $\mathcal{X}_\theta$ on $\mathcal{N}$. Denoting the corresponding polynomials by $Q_n(\cdot;\theta)$ it follows readily that

$$Q_n(x;\theta) = \frac{Q_n(\theta x)}{Q_n(\theta)}, \qquad n \geq 0, \tag{3.2}$$

so that the associated measure ψ_θ satisfies

$$\psi_\theta([-1,x]) = \psi([-\theta, x\theta]), \qquad -1 \leq x \leq 1.$$

Evidently, we have

$$\eta(\theta) := \sup \mathrm{supp}(\psi_\theta) = \eta\theta^{-1} \leq 1,$$

while the analogues $\pi_n(\theta)$ of the constants π_n of (1.1) are easily seen to satisfy

$$\pi_n(\theta) = \pi_n Q_n^2(\theta), \qquad n \geq 0. \tag{3.3}$$

(In [4, Appendix 2]) the special case $\theta = \eta$ is considered.) Obviously, $\mathcal{X}_\theta$ is periodic if and only if $\mathcal{X}$ is periodic. Note that by choosing $\theta = 1$ we return to the setting of the previous sections.

We have seen in Lemma 2.1 that $(-1)^n Q_n(-1;\theta)$ is increasing, and strictly increasing for n sufficiently large, if $\mathcal{X}_\theta$ is aperiodic, or, equivalently, $\mathcal{X}$ is aperiodic. It thus follows from (3.2) that $|Q_n(\theta)/Q_n(-\theta)|$ is decreasing, and strictly decreasing for n sufficiently large, if $\mathcal{X}$ is aperiodic. Since $Q_n(-x;\theta) = (-1)^n Q_n(x;\theta)$ if $\mathcal{X}_\theta$ is periodic, we conclude the following.

Lemma 3.1. *Let $\theta \geq \eta$. If $\mathcal{X}$ is periodic then $|Q_n(\theta)/Q_n(-\theta)| = 1$ for all n. If $\mathcal{X}$ is aperiodic then $|Q_n(\theta)/Q_n(-\theta)|$ is decreasing and tends to a limit satisfying*

$$0 \leq \lim_{n\to\infty} |Q_n(\theta)/Q_n(-\theta)| < 1.$$

In what follows we let

$$M_n(\theta) := \sum_{j=0}^{n} \frac{1}{p_j(\theta)\pi_j(\theta)} \sum_{k=0}^{j} r_k(\theta)\pi_k(\theta), \qquad 0 \leq n \leq \infty, \tag{3.4}$$

so that in particular $M_\infty(1)$ equals the left-hand side of (1.9). In combination with Lemma 2.1, interpreted in terms of $\mathcal{X}_\theta$, Lemma 3.1 gives us the next result.

Corollary 3.2. *For $\theta \geq \eta$ we have*

$$M_\infty(\theta) = \infty \iff \lim_{n\to\infty} |Q_n(\theta)/Q_n(-\theta)| = 0. \tag{3.5}$$

In view of (1.8) it follows in particular that the random walk $\mathcal{X}$ possesses the SRLP if $M_\infty(\eta) = \infty$, which readily leads to some further sufficient conditions. Indeed, choosing $\theta = \eta$ and defining $L(\eta)$ in analogy with (2.7) we have

$$L(\eta) = \sum_{j\geq 0} \frac{1}{p_j \pi_j Q_j(\eta) Q_{j+1}(\eta)},$$

so, in analogy with (2.8), Corollary 3.2 leads to

$$\mathcal{X} \text{ is aperiodic and } L(\eta) = \infty \;\Rightarrow\; \lim_{n\to\infty} |Q_n(\eta)/Q_n(-\eta)| = 0. \tag{3.6}$$

By (2.3) we have $L(\eta) \geq L(1) \equiv L$ so the premise in (3.6) certainly prevails if $\mathcal{X}$ is aperiodic and recurrent. When $L(\eta) = \infty$ the random walk $\mathcal{X}$ is called *η-recurrent* (see [4] for more information). The conclusion that η-recurrence is sufficient for an aperiodic random walk to possess the SRLP is not surprising, since Pruitt [9, Theorem 2] already established this result in the more general setting of symmetrizable Markov chains.

Another sufficient condition for the conclusion in (3.5) is obtained in analogy with (2.10), namely

$$\sum_{j\geq 0} \frac{r_j Q_j(\eta)}{p_j Q_{j+1}(\eta)} = \infty \;\Rightarrow\; \lim_{n\to\infty} |Q_n(\eta)/Q_n(-\eta)| = 0.$$

Since, by (2.3), $Q_{j+1}(\eta) \leq Q_j(\eta)$ it follows in particular that

$$\sum_{j\geq 0} \frac{r_j}{p_j} = \infty \;\Rightarrow\; \lim_{n\to\infty} |Q_n(\eta)/Q_n(-\eta)| = 0. \tag{3.7}$$

Interestingly, we have thus verified a passing remark by Karlin and McGregor [6, p. 77] to the effect that the premise in (3.7) is sufficient for the SRLP.

We now turn to the third lemma needed for the proof of Proposition 1.1, which concerns the behaviour of $M_n(\theta)$ as a function of θ.

Lemma 3.3. *Let $\eta \le \theta_1 \le \theta_2$, then, for all n, $M_n(\theta_1) \ge M_n(\theta_2)$.*

Proof. First consider an arbitrary random walk with parameters p_j, q_j, and r_j, $j \in \mathcal{N}$. Let n be fixed and write

$$M_n = \sum_{j=0}^{n} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} r_k \pi_k, \qquad n = 0, 1, \ldots.$$

Suppose that in the single state ℓ, $0 \le \ell \le n$, the transition probabilities p_ℓ, q_ℓ and r_ℓ are changed into the one-step (random walk) transition probabilities p'_ℓ, q'_ℓ and r'_ℓ satisfying, besides the usual requirements,

$$p'_\ell \le p_\ell, \qquad q'_\ell \ge q_\ell \qquad \text{and} \qquad r'_\ell \ge r_\ell. \tag{3.8}$$

Let M'_n denote the value of M_n after the change. A somewhat tedious but straightforward calculation then yields that

$$M'_n = M_n + \left\{ (c_1 - 1) \sum_{k=0}^{\ell-1} r_k \pi_k + (c_1 c_2 - 1) r_\ell \pi_\ell \right\} \sum_{j=\ell}^{n} \frac{1}{p_j \pi_j},$$

where c_1 and c_2 are constants satisfying

$$q_\ell c_1 = \frac{p_\ell q'_\ell}{p'_\ell} \qquad \text{and} \qquad r_\ell c_2 = \frac{p_\ell r'_\ell}{p'_\ell}.$$

The values of c_1 when $\ell = 0$ and c_2 when $r_\ell = 0$ are clearly irrelevant, but let us choose $c_1 = 1$ and $c_2 = 1$ in these cases. Then, under the given circumstances, we always have $c_1 \ge 1$ and $c_2 \ge 1$, and hence $M'_n \ge M_n$.

Back to the setting of the lemma we note that if $\eta \le \theta_1 < \theta_2$, then $r_j(\theta_1) \ge r_j(\theta_2)$, and, by (2.2),

$$q_j(\theta_1) = \frac{Q_{j-1}(\theta_1)}{Q_j(\theta_1)} \frac{q_j}{\theta_1} > \frac{Q_{j-1}(\theta_2)}{Q_j(\theta_2)} \frac{q_j}{\theta_1} > \frac{Q_{j-1}(\theta_2)}{Q_j(\theta_2)} \frac{q_j}{\theta_2} = q_j(\theta_2), \qquad j > 0.$$

Since $p_j(\theta) + q_j(\theta) + r_j(\theta) = 1$, it follows that

$$p_j(\theta_1) < p_j(\theta_2). \tag{3.9}$$

Now let $p_j = p_j(\theta_2)$, $q_j = q_j(\theta_2)$, $r_j = r_j(\theta_2)$ for all $j \in \mathcal{N}$ and suppose we perform the change operation with $p'_\ell = p_\ell(\theta_1)$, $q'_\ell = q_\ell(\theta_1)$ and $r'_\ell = r_\ell(\theta_1)$ (so that (3.8) is satisfied) successively for $\ell = 0, 1, \ldots, n$. Letting $M^{(\ell)}$ be the value into which $M^{(0)} := M_n(\theta_2)$ has been transformed after the ℓth change operation, we then obviously have

$$M_n(\theta_1) = M^{(n)} \ge M^{(n-1)} \ge \cdots \ge M^{(1)} \ge M^{(0)} = M_n(\theta_2),$$

which was to be proven. ■

We have now gathered sufficient information to draw our conclusions in the final section, after noting as an aside that (3.9) leads to a strengthening of (2.2), namely

$$\eta \le x < y \;\Rightarrow\; x Q_n(x) Q_{n+1}(y) > y Q_n(y) Q_{n+1}(x).$$

4 Proof of Theorem 1.2 and concluding remarks

Choosing $\theta_1 = \eta$ and $\theta_2 = 1$ in Lemma 3.3 we conclude that $M_n(\eta) \geq M_n(1)$ for all n. Hence $M_\infty(\eta) \geq M_\infty(1)$, so that

$$M_\infty(1) = \sum_{j\geq 0} \frac{1}{p_j\pi_j} \sum_{k=0}^{j} r_k\pi_k = \infty \;\Rightarrow\; M_\infty(\eta) = \infty,$$

which, by Corollary 3.2, leads to Proposition 1.1.

It seems unlikely that there are values of θ_1 and θ_2 such that $\eta < \theta_1 < \theta_2$ and $M_\infty(\theta_1) = \infty$, but $M_\infty(\theta_2) < \infty$, since there do not seem to be values of $x > \eta$ that are "special" in any sense. So we conjecture that $M_\infty(\theta_1)$ and $M_\infty(\theta_2)$ converge or diverge together. It is tempting to go one step further by extending this conjecture to $\eta \leq \theta_1 < \theta_2$. Maintaining the conjecture in [5] that also the reverse implication in (1.8) is valid, we would then arrive at the conjecture that (1.9) is not only sufficient but also necessary for $\mathcal{X}$ to possess the SRLP. However, this not correct, since it is possible to have $M_\infty(\eta) = \infty$ and $M(1) < \infty$ simultaneously, as the next example shows.

Example 4.1. Consider a random walk $\tilde{\mathcal{X}}$ determined by one-step transition probabilities $\tilde{p}_j$, $\tilde{q}_j$ and $\tilde{r}_j$ with $\tilde{r}_0 > 0$ and $\tilde{r}_j = 0$ for $j > 0$. Quantities associated with $\tilde{\mathcal{X}}$ will be indicated by a tilde. We will assume that $\tilde{\mathcal{X}}$ is *recurrent*, so that $\tilde{\eta} = 1$. Now let $\alpha > 1$ and define

$$p_j := \frac{\tilde{Q}_{j+1}(\alpha)}{\tilde{Q}_j(\alpha)} \frac{\tilde{p}_j}{\alpha}, \qquad r_j := \frac{\tilde{r}_j}{\alpha}, \qquad q_{j+1} := \frac{\tilde{Q}_j(\alpha)}{\tilde{Q}_{j+1}(\alpha)} \frac{\tilde{q}_{j+1}}{\alpha}, \qquad j \in \mathcal{N}. \tag{4.1}$$

These quantities, like those in (3.1), can be interpreted as the one-step transition probabilities of a new random walk $\mathcal{X}$, say. In what follows we associate quantities without tilde with $\mathcal{X}$. In analogy with (3.2) and (3.3) we thus have $Q_n(x) = \tilde{Q}_n(\alpha x)/\tilde{Q}(\alpha)$ and $\pi_n = \tilde{\pi}_n\tilde{Q}_n^2(\alpha)$. Also, $\eta = \tilde{\eta}\alpha^{-1} = \alpha^{-1} < 1$, so that $\mathcal{X}$ must be transient. Next, letting $M_n(\theta)$ be defined as in (3.4) and (3.1) where p_j, q_j and r_j are given by (4.1), we have

$$M_\infty(1) = r_0 \sum_{j\geq 0} \frac{1}{p_j\pi_j} < \infty,$$

since $\mathcal{X}$ is transient. But on the other hand

$$M_\infty(\eta) = M_\infty\big(\alpha^{-1}\big) = r_0 \sum_{j\geq 0} \frac{1}{p_j\pi_j Q_j\big(\alpha^{-1}\big)Q_{j+1}\big(\alpha^{-1}\big)} = \tilde{r}_0 \sum_{j\geq 0} \frac{1}{\tilde{p}_j\tilde{\pi}_j} = \infty,$$

since $\tilde{\mathcal{X}}$ is recurrent.

We have already encountered several known sufficient conditions for the random walk $\mathcal{X}$ to possess the SRLP. In particular, η-recurrence – and thus recurrence, which is simply 1-recurrence – was shown to be sufficient in (3.6). Also, in view of (2.9) we regain directly from Theorem 1 Karlin and McGregor's claim on [6, p. 77]

$$\sum_{j\geq 0} \frac{r_j}{p_j} = \infty \;\Rightarrow\; \mathcal{X} \text{ possesses the SRLP},$$

referred to after (3.7). Several authors (see [6, p. 77], [5, Corollary 3.2]) have shown that for the SRLP to prevail it is sufficient that $r_j > \delta > 0$ for j sufficiently large, but this condition is evidently weaker than the previous one.

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