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Structure Relations of Classical Orthogonal Polynomials in the Quadratic and q-Quadratic Variable

Maurice KENFACK NANGHO [†‡] *and Kerstin JORDAAN* [§]

[†] *Department of Mathematics and Applied Mathematics, University of Pretoria, Private bag X20 Hatfield, 0028 Pretoria, South Africa*

[‡] *Department of Mathematics and Computer Science, University of Dschang, Cameroon*
E-mail: *kenfnang@gmail.com*

[§] *Department of Decision Sciences, University of South Africa, PO Box 392, Pretoria, 0003, South Africa*
E-mail: *jordakh@unisa.ac.za*

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Abstract. We prove an equivalence between the existence of the first structure relation satisfied by a sequence of monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$, the orthogonality of the second derivatives $\{\mathbb{D}_x^2 P_n\}_{n=2}^{\infty}$ and a generalized Sturm–Liouville type equation. Our treatment of the generalized Bochner theorem leads to explicit solutions of the difference equation [Vinet L., Zhedanov A., *J. Comput. Appl. Math.* **211** (2008), 45–56], which proves that the only monic orthogonal polynomials that satisfy the first structure relation are Wilson polynomials, continuous dual Hahn polynomials, Askey–Wilson polynomials and their special or limiting cases as one or more parameters tend to ∞. This work extends our previous result [arXiv:1711.03349] concerning a conjecture due to Ismail. We also derive a second structure relation for polynomials satisfying the first structure relation.

Key words: classical orthogonal polynomials; classical q-orthogonal polynomials; Askey–Wilson polynomials; Wilson polynomials; structure relations; characterization theorems

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1 Introduction

A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$, $\deg(P_n(x)) = n$, is orthogonal with respect to a positive measure μ on the real numbers $\mathbb{R}$, if

$$\int_S P_m(x)P_n(x)\mathrm{d}\mu(x) = d_n\delta_{m,n}, \qquad m,n \in \mathbb{N},$$

where S is the support of μ, $d_n > 0$ and $\delta_{m,n}$ the Kronecker delta. A sequence $\{P_n(x)\}$ of monic polynomials orthogonal with respect to a positive measure satisfies a three-term recurrence relation

$$P_{n+1}(x) = (x - a_n)P_n(x) - b_nP_{n-1}(x), \qquad n = 0,1,2,\dots, \tag{1.1}$$

with initial conditions $P_{-1}(x) = 0$, $P_0(x) = 1$, and recurrence coefficients

$$a_n \in \mathbb{R}, \qquad n = 0,1,2,\dots, \qquad b_n > 0, \qquad n = 1,2,\dots.$$

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A sequence of monic orthogonal polynomials is classical if the sequence $\{P_n(x)\}$ as well as $D^m P_{n+m}(x)$, $m \in \mathbb{N}$, where D is the usual derivative or one of its extensions (difference operator, q-difference operator or divided-difference operator), satisfies a three-term recurrence of the form (1.1).

The classical orthogonal polynomials of Jacobi, Laguerre and Hermite are known to be the only polynomials satisfying

1) the *first structure relation* (cf. [2, 25])

$$\pi(x)DP_n(x) = \sum_{j=-1}^{1} a_{n,n+j}P_{n+j}(x), \qquad n = 1, 2, \dots, \qquad a_{n,n-1} \neq 0,$$

where $\pi(x)$ is a polynomial of degree at most 2;

2) the *second structure relation* (cf. [24, 25])

$$P_n(x) = \sum_{j=-1}^{1} b_{n,n+j}DP_{n+j}(x), \qquad n = 0, 1, \dots, \qquad b_{n,n+1} = \frac{1}{(n+1)} \neq 0; \tag{1.2}$$

3) the orthogonality of the sequence of derivatives $\{DP_{n+1}\}_{n=0}^{\infty}$ with respect to $\pi(x)w(x)$ (cf. [1]), where $\pi(x)$ is a polynomial of degree at most 2, and $w(x)$ denotes the weight function corresponding to $\{P_n\}_{n=0}^{\infty}$;

4) a Sturm–Liouville differential equation of the form (cf. [7])

$$\phi(x)D^2P_n(x) + \psi(x)DP_n(x) + \lambda_n P_n = 0,$$

where $\phi(x)$, $\psi(x)$ are polynomials with $\deg(\phi(x)) \leq 2$, $\deg \psi(x) = 1$ and λ_n is constant. This result is known as *Bochner's theorem* (cf. [7]).

The first structure relation, second structure relation and Bochner's theorem have been generalized to orthogonal polynomials involving the difference and q-difference operator (cf. [3, 9, 11, 19, 20]) and play an important role when studying properties of zeros or connection and linearization problems involving polynomials (see, for example, [15, 19]).

Askey–Wilson polynomials [5, equation (1.15)]

$$\frac{a^n p_n(x; a, b, c, d|q)}{(ab, ac, ad; q)_n} = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{-i\theta}, ae^{i\theta} \\ ab, ac, ad \end{matrix}; q, q\right), \qquad x = \cos\theta, \tag{1.3}$$

and Wilson polynomials [18, equation (9.1.1)]

$$\frac{W_n\big(x^2; a, b, c, d\big)}{(a+b)_n(a+c)_n(a+d)_n} = {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a-ix, a+ix \\ a+b, a+c, a+d \end{matrix}; 1\right) \tag{1.4}$$

do not satisfy structure relations of the type mentioned above but they do satisfy the shift relations (cf. [18, equations (14.1.9) and (9.1.8)])

$$\mathcal{D}_q p_n(x; a, b, c, d\,|\,q) = \frac{2q^{\frac{1-n}{2}}\big(1-q^n\big)\big(1-abcdq^{n-1}\big)}{1-q} p_{n-1}\big(x; aq^{\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}}\,|\,q\big),$$

$$\frac{\delta W_n(x^2; a, b, c, d)}{\delta x^2} = -n(n+a+b+c+d-1)W_{n-1}\left(x^2; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right),$$

where $\mathcal{D}_q$ is the Askey–Wilson operator (cf. [5, p. 33], [18, equation (1.16.4)], [14, equation (12.1.9)])

$$\mathcal{D}_q f(x) = \frac{f\big(q^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\theta}\big) - f\big(q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\theta}\big)}{\big(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\big)\big(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\big)/2}, \qquad x = \cos\theta, \tag{1.5}$$

and δ is the Wilson operator

$$\delta f(x^2) = f\left(\left(x + \frac{\mathrm{i}}{2}\right)^2\right) - f\left(\left(x - \frac{\mathrm{i}}{2}\right)^2\right). \tag{1.6}$$

Here

$${}_{s+1}\phi_s\left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_{s+1};q)_k}{(b_1;q)_k \cdots (b_s;q)_k} \frac{z^k}{(q;q)_k},$$

with

$$(a;q)_0 = 1, \qquad (a;q)_k = \prod_{j=0}^{k-1} \big(1 - aq^j\big), \qquad k = 1, 2, \dots,$$

and

$${}_{s+1}F_s\left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{s+1})_k}{(b_1)_k \cdots (b_s)} \frac{z^k}{k!},$$

with $(a)_0 = 1$ and $(a)_k = \prod\limits_{j=0}^{k-1} (a + j)$, $k = 1, 2, \dots$.

Since the appearance of Askey–Wilson and Wilson polynomials in the early 1980's (cf. [4, 5]), many authors have studied these polynomials (see, for example, [6, 8, 13, 14, 21, 22, 28]). Ismail considered the problem of the first structure relation for Askey–Wilson polynomials in the conjecture [14, Conjecture 24.7.9]. In [17, Corollary 3.3], we completed the conjecture by proving that a sequence of monic orthogonal polynomials satisfies the first structure relation

$$\pi(x)\mathcal{D}_q^2 P_n(x) = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}(x), \qquad a_{n,n-2} \neq 0, \qquad x = \cos\theta, \tag{1.7}$$

where π is a polynomial of degree at most 4, if and only if $P_n(x)$ is an Askey–Wilson polynomial up to a multiplicative constant or a subcase of Askey–Wilson polynomials, including limiting cases as one or more of the parameters tend to ∞ (cf. [13]). This result holds for orthogonal polynomials of the variable $x = \frac{\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta}}{2} = \cos\theta$ which can also be written as $x(s) = \frac{q^{-s} + q^s}{2}$, $\mathrm{e}^{\mathrm{i}\theta} = q^s$.

Even though Askey–Wilson polynomials (1.3) are a basic hypergeometric analog of the Wilson polynomials (1.4) (cf. [6, p. 188]), the coefficients in the analog of (1.7) for the Wilson operator

$$\pi(x)\frac{\delta^2 P_n(x)}{\delta^2 x^2} = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}(x), \qquad a_{n,n-2} \neq 0, \tag{1.8}$$

as well as its solutions can not easily be deduced from those of Askey–Wilson polynomials. It therefore is necessary to consider the Ismail conjecture for the Wilson variable $x(z) = z^2$ ($z = \mathrm{i}s$, $\mathrm{i}^2 = -1$), or, more generally, for the quadratic and q-quadratic variable (cf. [26])

$$x(s) = \begin{cases} c_1 q^{-s} + c_2 q^s + c_3 & \text{if } q \neq 1, \\ c_4 s^2 + c_5 s + c_6 & \text{if } q = 1, \end{cases} \tag{1.9}$$

where $c_1 \neq 0$ and $c_4 \neq 0$. This problem of characterizing the orthogonal polynomials of the variable $x(s)$ whose derivatives satisfy a generalized first structural relation is a generalization of the Askey problem (cf. [2, p. 69]).

The aim of this paper is to use generalizations of Bochner's theorem in [13, 28] (see also [12]) for classical orthogonal polynomials of the quadratic and q-quadratic variable $x(s)$ defined in (1.9) to obtain a generalized first structure relation for classical orthogonal polynomials of the variable $x(s)$ of the form

$$\pi(x)\mathbb{D}_x^2 P_n(x) = \sum_{j=-r}^{t} a_{n,n+j} P_{n+j}(x), \qquad n = 1, 2, \dots,$$

where $\mathbb{D}_x$ is the divided-difference operator (cf. [10])

$$\mathbb{D}_x f(x(s)) = \frac{f\bigl(x\bigl(s+\frac{1}{2}\bigr)\bigr) - f\bigl(x\bigl(s-\frac{1}{2}\bigr)\bigr)}{x\bigl(s+\frac{1}{2}\bigr) - x\bigl(s-\frac{1}{2}\bigr)}. \tag{1.10}$$

This work is organized as follows: In Section 3, we derive explicit solutions for the second-order difference equation [28, equation (1.3)]

$$A(s)P_n(x(s+1)) + B(s)P_n(x(s)) + C(s)P_n(x(s-1)) = \lambda_n P_n(x(s)), \tag{1.11}$$

where $A(s)$, $B(s)$, $C(s)$ are some functions of the discrete argument s, and $P_0 = 1$, shown to characterize polynomials of the variable $x(s)$ in [28] by Vinet and Zhedanov. In Section 4, we will show that this generalized Bochner theorem (cf. [28]) is related to the generalized Askey problem and we will characterize Wilson and Askey–Wilson polynomials, and subcases, including limiting cases, as the only monic orthogonal polynomials satisfying the first structure relation

$$\pi(x(s))\mathbb{D}_x^2 P_n(x(s)) = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}(x(s)), \qquad a_{n,n-2} \neq 0, \qquad n = 2, 3, \dots,$$

where $\pi(x)$ is a polynomial of degree at most four and $x(s)$ is given by (1.9). We then derive the second structure relation

$$P_n(x(s)) = \sum_{j=-2}^{2} b_{n,n+j} \mathbb{D}_x^2 P_{n+j}(x(s)), \tag{1.12}$$

for classical orthogonal polynomials of the variable $x(s)$ and conclude the section by connecting the second structure relation (1.12) to that of Costas-Santos and Marcellán [8, p. 118]

$$\mathcal{M}P_n(x(s)) = e_n \mathbb{D}_x P_{n+1}(x(s)) + f_n \mathbb{D}_x P_n(x(s)) + g_n \mathbb{D}_x P_{n-1}(x(s)), \tag{1.13}$$
$$\mathcal{M}f(s) = \frac{f\bigl(s+\frac{1}{2}\bigr) + f\bigl(s-\frac{1}{2}\bigr)}{2}.$$

In Section 5 we compute coefficients of (1.8) for the Wilson polynomials as well as those of the second structure relation (1.12) for the Wilson polynomials and Askey–Wilson polynomials.

2 Preliminaries and notation

Let us recall some basic results and notations. $x(s)$ given by (1.9) satisfies (cf. [6])

$$x(s+n) - x(s) = \gamma_n \nabla x_{n+1}(s), \qquad \frac{x(s+n) + x(s)}{2} = \alpha_n x_n(s) + \beta_n,$$

for $n = 0, 1, \ldots$, with

$$x_\mu(s) = x\left(s + \frac{\mu}{2}\right), \qquad \mu \in \mathbb{C},$$

where $\mathbb{C}$ is the set of complex numbers and ∇ is the backward difference operator $\nabla f(s) := f(s) - f(s-1)$. The sequences (α_n), (β_n), (γ_n) are given explicitly by (cf. [6]), $\alpha_1 = \alpha$, $\beta_1 = \beta$,

$$\alpha_n = 1, \qquad \beta_n = \beta n^2, \qquad \gamma_n = n, \qquad \alpha = 1, \qquad \beta = \frac{c_4}{4} \qquad \text{for} \quad q = 1,$$

and

$$\alpha_n = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2}, \qquad \beta_n = \frac{\beta(1-\alpha_n)}{1-\alpha}, \qquad \gamma_n = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$
$$\alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}, \qquad \beta = -c_3 \frac{\left(\sqrt{q}-1\right)^2}{2\sqrt{q}}, \qquad \text{for} \quad q \neq 1.$$

The following hold (cf. [10]):

$$\mathbb{D}_x(fg) = \mathbb{D}_x(f)\mathbb{S}_x(g) + \mathbb{S}_x(f)\mathbb{D}_x(g), \tag{2.1}$$
$$\mathbb{S}_x(fg) = \mathbb{S}_x(f)\mathbb{S}_x(g) + U_2\mathbb{D}_x(f)\mathbb{D}_x(g), \tag{2.2}$$
$$\mathbb{D}_x\mathbb{S}_x = \alpha\mathbb{S}_x\mathbb{D}_x + U_1\mathbb{D}_x^2, \tag{2.3}$$
$$\mathbb{S}_x^2 = U_1\mathbb{S}_x\mathbb{D}_x + \alpha U_2\mathbb{D}_x^2 + \mathbb{I}, \tag{2.4}$$

where

$$U_1(x(s)) = \left(\alpha^2 - 1\right)x(s) + \beta(\alpha+1),$$
$$U_2(x(s)) = \left(\frac{x\left(s+\frac{1}{2}\right) - x\left(s-\frac{1}{2}\right)}{2}\right)^2 = \left(\alpha^2-1\right)x(s)^2 + 2\beta(\alpha+1)x(s) + C_x,$$

with

$$C_x = \frac{c_5^2}{4} - c_4c_6, \qquad \text{for} \quad q = 1 \qquad \text{and} \qquad C_x = \frac{(q-1)^2\left(c_3^2 - 4c_1c_2\right)}{4q}, \qquad \text{for} \quad q \neq 1.$$

Note that $\mathbb{I}(f) = f$ and $\mathbb{S}_x$ the averaging operator

$$\mathbb{S}_x f(x(s)) = \frac{f\left(x\left(s+\frac{1}{2}\right)\right) + f\left(x\left(s-\frac{1}{2}\right)\right)}{2},$$

which is a generalization of [14, equation (12.1.21)]. Taking $\mathrm{e}^{\mathrm{i}\theta} = q^s$, the Askey–Wilson operator (1.5) reads as

$$\mathcal{D}_q f(x(s)) = \frac{f\left(x\left(s+\frac{1}{2}\right)\right) - f\left(x\left(s-\frac{1}{2}\right)\right)}{x\left(s+\frac{1}{2}\right) - x\left(s-\frac{1}{2}\right)} = \mathbb{D}_x f(x(s)), \qquad x(s) = \frac{q^{-s}+q^s}{2}.$$

The Wilson operator (1.6) is connected to the divided-difference operator (1.10) as follows:

$$\frac{\delta f\left(s^2\right)}{\delta s^2} = \frac{f\left(-\left(\mathrm{i}s-\frac{1}{2}\right)^2\right) - f\left(-\left(\mathrm{i}s+\frac{1}{2}\right)^2\right)}{-\left(\mathrm{i}s-\frac{1}{2}\right)^2 + \left(\mathrm{i}s+\frac{1}{2}\right)^2} = -\mathbb{D}_x f(-x(\mathrm{i}s)), \qquad x(z) = z^2, \qquad z = \mathrm{i}s.$$

3 Generalized Bochner theorem

In this section, using properties of the divided-difference operator $\mathbb{D}_x$ and the averaging operator $\mathbb{S}_x$, we discuss generalized versions of Bochner's theorem in [13] and [28] and derive explicit expressions for the polynomial solutions characterized by the results.

Lemma 3.1. *Polynomial solutions $P_n(x(s))$, $\deg(P_n(x(s))) = n$, of the Sturm–Liouville type equation*

$$\phi(x)\mathbb{D}_x^2 y(x) + \psi(x)\mathbb{S}_x\mathbb{D}_x y(x) + \lambda y(x) = 0, \tag{3.1}$$

where $\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0$ and $\psi(x) = \psi_1 x + \psi_0$ are polynomials of degree at most two and one, can be expanded as

$$P_n(x(s)) = \sum_{k=0}^{n} d_k \prod_{j=0}^{k-1} [x(s) - x(\eta + j)], \tag{3.2}$$

where η is a complex number such that $\sigma(x(\eta)) = 0$ where

$$\sigma(x(s)) = \phi(x(s)) - \frac{\nabla x_1(s)}{2}\psi(x(s)), \tag{3.3}$$

and d_k is solution to the first-order recurrence relation

$$\begin{aligned}&(\lambda + \gamma_k\gamma_{k-1}\phi_2 + \gamma_k\alpha_{k-1}\phi_1)\, d_k \\ &\quad + \left(\gamma_k\gamma_{k-1}\left(\phi_2\,(x(\eta+k) + x(\eta)) + \phi_1 - \frac{\psi_1 \nabla x_1(\eta)}{2}\right) + \alpha_k\gamma_{k+1}\psi(x(\eta+k))\right) d_{k+1} = 0\end{aligned} \tag{3.4}$$

with $\lambda = -\gamma_n\gamma_{n-1}\phi_2 - \gamma_n\alpha_{n-1}\psi_1$.

Proof. Write

$$w_k(x(s), \eta) = \prod_{j=0}^{k-1} [x(s) - x(\eta + j)], \qquad k > 1 \qquad \text{and} \qquad w_0(x(s)) \equiv 1, \tag{3.5}$$

and obtain by direct computation

$$\mathbb{D}_x w_k(x(s), \eta) = \gamma_k w_{k-1}\left(x(s), \eta + \frac{1}{2}\right), \tag{3.6}$$

$$\mathbb{S}_x w_k(x(s), \eta) = \alpha_k w_k\left(x(s), \eta - \frac{1}{2}\right) - \frac{\gamma_k \nabla x(\eta)}{2} w_{k-1}\left(x(s), \eta + \frac{1}{2}\right), \tag{3.7}$$

$$x(s) w_k(x(s), \eta) = w_{k+1}(x(s), \eta - 1) + x(\eta - 1) w_k(x(s), \eta), \tag{3.8}$$

$$w_k(x(s), \eta) = w_k(x(s), \eta + 1) + (x(\eta + k) - x(\eta)) w_{k-1}(x(s), \eta + 1). \tag{3.9}$$

Next, take $P_n(x(s)) = \sum_{k=0}^{\infty} d_k w_k(x(s), \eta)$ with $\phi(x(s)) = \phi_2 x(s)^2 + \phi_1 x(s) + \phi_0$ and $\psi(x(s)) = \psi_1 x(s) + \psi_0$ in (3.1) and use (3.6)–(3.8) for simplification to derive

$$\begin{aligned}\sum_{k=0}^{\infty} \Bigg\{ &\gamma_k(\gamma_{k-1}\,(\phi_2\,(x(\eta-1) + x(\eta)) + \phi_1) + \psi_1\left(\alpha_{k-1} x(\eta - 1) - \frac{\gamma_{k-1}\nabla x_1(\eta)}{2} + \alpha_{k-1}\psi_0\right) \\ &+ \alpha_{k-1}\psi_0) w_k(x(s), \eta) + (\lambda + \gamma_k\gamma_{k-1}\phi_2 + \gamma_k\alpha_{k-1}\psi_1)\, w_k(x(s), \eta - 1) + \sigma(x(\eta)) \\ &\times w_k(x(s), \eta + 1)\Bigg\} d_k = 0.\end{aligned}$$

Now, use the fact that $\sigma(x(\eta))=0$ and the relation (3.9) to obtain

$$\sum_{k=0}^{\infty}\left(A_k d_k+B_k d_{k+1}\right)w_k(x(s),\eta)=0,$$

and equate coefficients of w_k to obtain the two-term recurrence relation

$$A_k d_k+B_k d_{k+1}=0,\qquad k\geq 0,$$

where

$$A_k=\lambda+\gamma_k\gamma_{k-1}\phi_2+\gamma_k\alpha_{k-1}\phi_1,$$
$$B_k=\gamma_k\gamma_{k-1}\left[\phi_2\left(x(\eta+k)+x(\eta)\right)+\phi_1-\frac{\psi_1\nabla x_1(\eta)}{2}\right]+\alpha_k\gamma_{k+1}\psi(x(\eta+k)).$$

By assumption, P_n is a polynomial of degree n, that is $d_k=0$, $k\geq n+1$. Hence taking $k=n$ we obtain $\lambda=-\phi_2\gamma_n\gamma_{n-1}-\psi_1\gamma_n\alpha_{n-1}$. Taking into account this expression of λ the required relation is obtained. ■

Remark 3.2. The explicit expressions of $w_k(x(s),\eta)$ in (3.5) for the corresponding lattices $x(s)$ are provided in the following table:

Representation of $w_k(x(s),\eta)$	On the lattice $x(s)$
$\left(-\frac{q^{-\eta}}{2}\right)^k q^{-\binom{k}{2}}\left(q^\eta q^{-s};q\right)_k\left(q^\eta q^s;q\right)_k$	$x(s)=\frac{q^{-s}+q^s}{2}$
$\left(-c_1q^{-\eta}\right)^k q^{-\binom{k}{2}}\left(q^\eta q^{-s};q\right)_k\left(\frac{c_2}{c_1}q^\eta q^s;q\right)_k$	$x(s)=c_1q^{-s}+c_2q^s+c_3$, $c_1\neq 0$
$\left(-c_1q^{-\eta}\right)^k q^{-\binom{k}{2}}\left(q^\eta q^{-s};q\right)_k$	$x(s)=c_1q^{-s}+c_3$
$(-c_4)^k\left(s+\frac{c_5}{c_4}+\eta\right)_k(-s+\eta)_k$	$x(s)=c_4s^2+c_5s+c_6$, $c_4\neq 0$
$(-c_5)^k(-s+\eta)_k$	$x(s)=c_5s+c_6$

When the function $\sigma(x(s))$ (with $x(s)=c_1q^{-s}+c_2q^s$, $c_1c_2\neq 0$) happens to be of the form $C(q^s)^m$, $m=0,1,\ldots$, it has no zeros and therefore Lemma 3.1 can no longer be used for expanding polynomial solutions of the Sturm–Liouville type equation (3.1). We will see later that this problem arises for the special case of (3.1) when

$$\phi(x)=\phi_2x^2+\phi_0\qquad\text{and}\qquad\psi(x)=\psi_1x. \tag{3.10}$$

In [16], a method for solving (3.1), when ϕ and ψ are of the form (3.10), was developed using the generalized form of the basis $\rho_n(x)=\left(1+e^{2i\theta}\right)\left(-q^{2-n}e^{2i\theta};q^2\right)_{n-1}e^{-in\theta}$, $x=\cos\theta$ (cf. [14, equation (20.3.8)]) on the lattice $x(s)=c_1q^{-s}+c_2q^s$. This result can be written as:

Lemma 3.3 ([16, Theorem 13])**.** *On q-quadratic lattices $x(s)=c_1q^{-s}+c_2q^s$, polynomial solutions P_n of (3.1) when ϕ and ψ are of the form*

$$\phi(x(s))=\phi_2x(s)^2+\phi_0,\qquad\psi(x(s))=\psi_1x(s),$$

can be expanded as

$$P_n(x(s))=\sum_{k=0}^{[\frac{n}{2}]}d_{n-2k}K_{n-2k}(x(s)),$$

where

$$K_j(x(s)) = \left(c_1q^{-s}\right)^j \left(1+\frac{c_2}{c_1}q^{2s}\right)\left(-\frac{c_2}{c_1}q^{-j+2}q^{2s};q^2\right)_{j-1}, \qquad K_0(x(s))=1, \qquad j\geq 1,$$

and d_j is given by the two-term recurrence relation

$$\begin{aligned}&\left(\gamma_j\left(\phi_2\gamma_{j-1}+(\gamma_j-\alpha\gamma_{j-1})\psi_1\right)+\lambda\right)d_j+\gamma_{j+2}\gamma_{j+1}\phi\left(\mathrm{i}\sqrt{c_1c_2}\left(q^{\frac{j}{2}}-q^{-\frac{j}{2}}\right)\right)d_{j+2}\\&\quad+\psi_1\gamma_{j+2}\left(\gamma_j\left(\mathrm{i}\sqrt{c_1c_2}\left(q^{\frac{j+1}{2}}-q^{-\frac{j+1}{2}}\right)\right)^2-\alpha\gamma_{j+1}\left(\mathrm{i}\sqrt{c_1c_2}\left(q^{\frac{j}{2}}-q^{-\frac{j}{2}}\right)\right)^2\right)d_{j+2}=0,\end{aligned}$$

with the coefficient $\lambda=-\gamma_n\gamma_{n-1}\phi_2-\gamma_n\alpha_{n-1}\psi_1$.

Ismail [13] gave the following generalization of Bochner's theorem for Askey–Wilson polynomials where $\mathcal{S}_q$ (cf. [14, equation (12.1.21)]) is the restriction of the averaging operator

$$\mathbb{S}_x f(x(s)) = \frac{f\left(x\left(s+\frac{1}{2}\right)\right)+f\left(x\left(s-\frac{1}{2}\right)\right)}{2}$$

to functions of the variable $x=\cos\theta=\frac{q^{-s}+q^s}{2}$, $q^s=\mathrm{e}^{\mathrm{i}\theta}$.

Theorem 3.4 ([13, Theorem 3.1])**.** *The Sturm–Liouville type equation*

$$\phi(x)\mathcal{D}_q^2y(x)+\psi(x)\mathcal{S}_q\mathcal{D}_qy(x)+\lambda_ny(x)=0, \qquad x=\cos\theta, \tag{3.11}$$

where ϕ and ψ are polynomials of degree at most 2 and 1, has a polynomial solution $P_n(x)$ of degree $n=1,2,3,\dots$ if and only if $P_n(x)$ is a multiple of the Askey–Wilson polynomial $p_n(x;a,b,c,d\,|\,q)$ for some parameters a, b, c, d, including limiting cases as one or more of the parameters tend to ∞.

In order to obtain all the explicit solutions characterized by (3.11) we use the following scheme:

1. Since $\phi(x(s))$ is at most quadratic and $\psi(x(s))$ is linear, we write

$$\phi(x(s))=\phi_2x(s)^2+\phi_1x(s)+\phi_0, \qquad \psi(x(s)=\psi_1x(x(s))+\psi_0.$$

Substituting $\phi(x(s))$, $\psi(x(s))$ and $x(s)=\frac{q^{-s}+q^s}{2}$ into (3.3) we obtain, for $X=q^s$, $\sigma(x(s))=\frac{P(X)}{X^2}$ where

$$\begin{aligned}8\sqrt{q}P(X)&=\left(2\phi_2\sqrt{q}-q\psi_1+\psi_1\right)X^4+\left(4\sqrt{q}\phi_1-2q\psi_0+2\psi_0\right)X^3\\&\quad+\left(8\sqrt{q}\phi_0+4\phi_2\sqrt{q}\right)X^2+\left(4\sqrt{q}\phi_1+2q\psi_0-2\psi_0\right)X+2\phi_2\sqrt{q}+q\psi_1-\psi_1.\end{aligned} \tag{3.12}$$

2. Suppose $P(X)$ is of degree 4, that is $\psi_1\neq\frac{2\phi_2\sqrt{q}}{q-1}$, and write

$$P(X)=C(X-a)(X-b)(X-c)(X-d), \qquad a,b,c,d\in\mathbb{C}.$$

3. Expand the factorized form of $P(X)$ and identify coefficients of X^i, $i=0,1,2,3,4$ with those in (3.12) to obtain a system of five equations.
4. Solve the system with unknowns ϕ_2, ϕ_1, ϕ_0, ψ_1 and ψ_0 to obtain polynomial coefficients of (3.11).

5. If one of the a, b, c, d, say a, is different from 0, then use Lemma 3.1, with $q^\eta = a$, to obtain polynomial solution of (3.11) of the form

$$\sum_{k=0}^{n} d_k w_k(x,\eta), \tag{3.13}$$

where $\frac{d_{k+1}}{d_k}$ is given by (3.4).

6. Iterate (3.4) to obtain d_k and use it as well as the representation of w_k, for the lattice $x(s) = \frac{q^s+q^{\bar{s}}}{2}$ (see Remark 3.2), to obtain the basic hypergeometric representation of (3.13).
7. If none of a, b, c and d, is different from zero (which corresponds to $\phi(x) = \phi_2 x^2 + \phi_0$ and $\psi(x) = \psi_1 x$), use Lemma 3.3 to solve (3.11).

At the end of the day, one has the following:

1. If P(X) is of degree 4,

- If $P(X) = CX^4$, then polynomial coefficients of (3.11) are up to a multiplicative factor equal to

$$\phi(x(s)) = 2x(s)^2 - 1, \qquad \psi(x(s)) = -\frac{4\sqrt{q}}{q-1}x(s),$$

and the corresponding polynomial, with $q^s = \mathrm{e}^{\mathrm{i}\theta}$, is up to a multiplicative factor equal to

$$\sum_{k=0}^{[\frac{n}{2}]} d_{n-2k} K_{n-2k}(x),$$

with

$$\frac{d_{n-2k}}{d_{n-2(k-1)}} = -\frac{1}{4}\frac{\left(1-q^{-n-1}q^{2k}\right)\left(1-q^{-n-2}q^{2k}\right)}{1-q^{2k}}q^{n+1},$$
$$d_{n-2k} = d_n \frac{\left(q^{-n};q\right)_{2k}}{\left(q^2;q^2\right)_k}\left(-\frac{1}{4}q^{n+1}\right)^k.$$

Taking $d_n = 1$, we obtain after straightforward computation

$$\sum_{k=0}^{[\frac{n}{2}]} \frac{\left(q^{-n},q\right)_{2k}}{\left(q^2,q^2\right)_k}\left(-\frac{q^{n+1}}{4}\right)^k K_{n-2k}(x) = 2^{-n} H_n(x\,|\,q),$$

where $H_n(x\,|\,q)$ is the continuous q-Hermite polynomial [18, equation (14.26.1)].

- If $P(X) = C(X-a)X^3$, $a \neq 0$, then coefficients of (3.11) are up to a multiplicative factor equal to

$$\phi(x(s)) = 2x(s)^2 - ax(s) - 1, \qquad \psi(x(s)) = -\frac{4\sqrt{q}}{q-1}x(s) + \frac{2a\sqrt{q}}{q-1}.$$

So,

$$\frac{d_{k+1}}{d_k} = -2\frac{q^k qa\left(1-q^k q^{-n}\right)}{\left(1-q^k q\right)}, d_k = \frac{\left(q^{-n};q\right)_k(-2qa)^k q^{\binom{k}{2}}}{(q;q)_k}d_0,$$

and the basic hypergeometric representation of the corresponding polynomial, with $q^s = \mathrm{e}^{\mathrm{i}\theta}$, is

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, a\mathrm{e}^{-\mathrm{i}\theta}, a\mathrm{e}^{\mathrm{i}\theta} \\ 0, 0 \end{matrix}; q, q\right) = a^n H_n(x; a \,|\, q), $$

where $H_n(x;a|q)$ is the continuous big q-Hermite polynomial [18, equation (14.18.1)].

- If $P(X) = C(X-a)(X-b)X^2$, $ab \neq 0$,

$$ \phi(x(s)) = 2x(s)^2 - (a+b)x(s) + ab - 1, $$
$$ \psi(x(s)) = -\frac{4\sqrt{q}}{q-1}x(s) + \frac{2\sqrt{q}(a+b)}{q-1}, $$
$$ \frac{d_{k+1}}{d_k} = -2\frac{q^k qa\left(1-q^{-n}q^k\right)}{\left(1-q^k q\right)\left(1-q^k ab\right)}, \qquad d_k = \frac{\left(q^{-n};q\right)_k(-2qa)^k q^{\binom{k}{2}}}{(q;q)_k(ab;q)_k}d_0, $$

and the basic hypergeometric representation of the corresponding polynomial, with $q^s = \mathrm{e}^{\mathrm{i}\theta}$, is

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, a\mathrm{e}^{-\mathrm{i}\theta}, a\mathrm{e}^{\mathrm{i}\theta} \\ ab, 0 \end{matrix}; q, q\right) = \frac{a^n}{(ab;q)_n} Q_n(x; a, b \,|\, q), $$

where $Q_n(x;a,b\,|\,q)$ is the Al-Salam Chihara polynomial [18, equation (14.8.1)].

- If $P(X) = C(X-a)(X-b)(X-c)X$, $abc \neq 0$,

$$ \phi(x(s)) = 2x(s)^2 - (abc+a+b+c)x(s) + ab + ac + bc - 1, $$
$$ \psi(x(s)) = -\frac{4\sqrt{q}}{q-1}x(s) - \frac{2\sqrt{q}(abc-a-b-c)}{q-1}, $$
$$ \frac{d_{k+1}}{d_k} = -2\frac{q^k qa\left(1-q^{-n}q^k\right)}{\left(1-q^k q\right)\left(1-q^k ac\right)\left(1-q^k ab\right)}, \qquad d_k = \frac{\left(q^{-n};q\right)_k(-2qa)^k q^{\binom{k}{2}}}{(q;q)_k(ab;q)_k(ac;q)_k}d_0, $$

and the basic hypergeometric representation of the corresponding polynomial is

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, a\mathrm{e}^{-\mathrm{i}\theta}, a\mathrm{e}^{\mathrm{i}\theta} \\ ab, ac \end{matrix}; q, q\right) = \frac{a^n p_n(x; a, b, c \,|\, q)}{(ab, ac; q)_n}, $$

where $p_n(x;a,b,c|q)$ is the continuous dual q-Hahn polynomial [18, equation (14.3.1)].

- If $P(X) = C(X-a)(X-b)(X-c)(X-d)$, $abcd \neq 0$,

$$ \begin{aligned} \phi(x(s)) &= 2(abcd+1)x(s)^2 - (abc+abd+acd+bcd+a+b+c+d)x(s) \\ &\quad - abcd + ab + ac + ad + bc + bd + cd - 1, \end{aligned} $$
$$ \psi(x(s)) = 4\frac{\sqrt{q}(abcd-1)}{q-1}x(s) - 2\frac{\sqrt{q}(abc+abd+acd+bcd-a-b-c-d)}{q-1}, $$
$$ \frac{d_{k+1}}{d_k} = -2\frac{q^k qa\left(1-q^k q^{-n}\right)\left(1-q^{n-1}abcdq^k\right)}{\left(1-q^k q\right)\left(1-q^k ad\right)\left(1-q^k ac\right)\left(1-q^k ab\right)}, $$
$$ d_k = \frac{(-2aq)^k\left(q^{-n};q\right)_k q^{\binom{k}{2}}\left(q^{n-1}abcd;q\right)_k}{(q;q)_k(ad;q)_k(ac;q)_k(ab;q)_k}d_0, $$

and the basic hypergeometric representation of the corresponding polynomial is

$$ {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, a\mathrm{e}^{-\mathrm{i}\theta}, a\mathrm{e}^{\mathrm{i}\theta} \\ ab, ac, ad \end{matrix}; q, q\right) = \frac{a^n p_n(x; a, b, c, d; q)}{(ab, ac, ad; q)_n}, $$

where $p_n(x;a,b,c,d\,|\,q)$ is the Askey–Wilson polynomial [18, equation (14.1.1)].

In the following items, we are going to consider cases for which degree of $P(X) < 4$. For each of them, after giving the factorized form of P(X), we will follow steps 3 and 4 of the scheme to look for ϕ_2, ϕ_1, ϕ_0, ψ_1 and ψ_0. Then we will follow steps 5 and 6 for degree of $P(X) = 1, 2, 3$ and steps 5 and 7 for degree $P(X) = 0$ to obtain the corresponding polynomials system.

2. If $P(X)$ is of degree 3 and $P(X) = C(X-a)(X-b)(X-c)$ with none of a, b and c equal to zero, then

$$\phi(x(s)) = -2abcx(s)^2 + (ab+ac+bc+1)x(s) + abc - a - b - c,$$
$$\psi(x(s)) = -\frac{4abc\sqrt{q}}{q-1}x(s) + 2\frac{\sqrt{q}(ab+ac+bc-1)}{q-1},$$
$$\frac{d_{k+1}}{d_k} = 2\frac{q^k abc\left(q^n - q^k\right)d(k)}{\left(q^k q - 1\right)\left(q^k ac - 1\right)\left(q^k ab - 1\right)}, \qquad d_k = \frac{\left(q^{-n}, q\right)_k \left(-2abcq^n\right)^k q^{\binom{k}{2}}}{(q,q)_k (ac;q)_k (ab;q)_k} d_0,$$

and the basic hypergeometric representation of the corresponding polynomial is

$$\sum_{k=0}^{n} \frac{\left(q^{-n};q\right)_k \left(aq^s, aq^{-s};q\right)_k}{(ac;q)_k (ab;q)_k} \frac{\left(bcq^n\right)^k}{(q;q)_k} = \lim_{d\to\infty} \frac{a^n p_n(x;a,b,c,d\,|\,q)}{(ab;q)_n (ac;q)_n (ad;q)_n}.$$

3. If $P(X)$ is of degree 2 and $P(X) = C(X-a)(X-b)$, $a, b \neq 0$, then

$$\phi(x) = 2abcx^2 - (a+b)x + 1 - ab, \qquad \psi(x) = \frac{4ab\sqrt{q}}{q-1}x - \frac{2\sqrt{q}(a+b)}{q-1},$$
$$\frac{d_{k+1}}{d_k} = 2\frac{s_2 q^n \left(1 - q^k q^{-n}\right)}{\left(1 - q^k q\right)\left(1 - q^k ab\right)}, \qquad d_k = \frac{\left(q^{-n}, q\right)_k (2bq^n)^k}{(q;q)_k (ab;q)_k} d_0,$$

and the basic hypergeometric representation of the corresponding polynomial is

$$\sum_{k=0}^{n} \frac{\left(q^{-n};q\right)_k \left(aq^s, aq^{-s};q\right)_k q^{-\binom{k}{2}}}{(ab;q)_k} \frac{\left(\frac{bq^n}{a}\right)^k}{(q;q)_k} = \lim_{c,d\to\infty} \frac{a^n p_n(x;a,b,c,d\,|\,q)}{(ab;q)_n (ac;q)_n (ad;q)_n}.$$

4. If $P(X)$ is of degree 1 and $P(X) = C(X-a)$, $a \neq 0$, then

$$\phi(x) = -2ax(s)^2 + x(s) + a, \qquad \psi(x) = -\frac{4\sqrt{q}a}{q-1}x(s) + \frac{2\sqrt{q}}{q-1},$$
$$\frac{d_{k+1}}{d_k} = 2\frac{q^n\left(1 - q^k q^{-n}\right)}{q^k a\left(q^k q - 1\right)}, \qquad d_k = \frac{\left(q^{-n};q\right)_k q^{-\binom{k}{2}}}{(q;q)_k} \left(-2\frac{q^n}{a}\right)^k d_0,$$

and the basic hypergeometric representation of the corresponding polynomial is

$$\sum_{k=0}^{n} \left(q^{-n};q\right)_k \left(aq^s, aq^{-s};q\right)_k q^{-2\binom{k}{2}} \frac{\left(\frac{q^n}{a^2}\right)^k}{(q;q)_k} = \lim_{b,c,d\to\infty} \frac{a^n p_n(x;a,b,c,d\,|\,q)}{(ab;q)_n (ac;q)_n (ad;q)_n}.$$

5. If $P(X)$ is a constant, that is $P(X) = C$, then

$$\phi(x) = 2x(s)^2 - 1, \qquad \psi(x) = \frac{4\sqrt{q}}{q-1}x(s),$$

and the corresponding polynomial is up to a multiplicative factor equal to

$$\sum_{k=0}^{[\frac{n}{2}]} d_{n-2k} K_{n-2k}(x(s)),$$

where

$$\frac{d_{n-2k}}{d_{n-2(k-1)}} = \frac{1}{4}\frac{\left(1-q^{-n-1}q^{2k}\right)\left(1-q^{-n-2}q^{2k}\right)q^{n+2}q^{-2k}}{1-q^{2k}},$$

$$\frac{d_{n-2k}}{d_n} = \frac{\left(q^{-n};q\right)_{2k}}{\left(q^2;q^2\right)_k}\left(\frac{q^n}{4}\right)^k q^{-2\binom{k}{2}}.$$

Take $d_n = 1$ to obtain

$$\sum_{k=0}^{[\frac{n}{2}]} \frac{\left(q^{-n};q\right)_{2k}}{\left(q^2;q^2\right)_k}\left(\frac{q^n}{4}\right)^k q^{-2\binom{k}{2}} K_{n-2k}(x(s)) = 2^n H_n\left(x \,|\, q^{-1}\right)$$
$$= \lim_{a,b,c,d\to\infty} \frac{2^n a^{2n} p_n(x;a,b,c,d\,|\,q)}{(ab;q)_n(ac;q)_n(ad;q)_n},$$

where $H(x\,|\,q)$ is the continuous q-Hermite polynomials.

Remark 3.5. It is important to note that in each of the cases of items 2 to 4 above, if one of the a, b, c, d is zero, then $\psi_1 = 0$, which is impossible because the degree of ψ is equal to 1.

In order to expand on the generalization of Bochner's theorem in [28], we need to connect the second-order difference equation (1.11) used in [28] to the Sturm–Liouville type equation (3.1). Let $P_n(x(s))$ be a polynomial solution to the second-order difference equation (1.11). From the definition of $\mathbb{D}_x$ and $\mathbb{S}_x$ we have

$$\nabla x_1(s)\mathbb{D}_x^2 P_n(x(s)) = \frac{P_n(x(s+1)) - P_n(x(s))}{x(s+1)-x(s)} - \frac{P_n(x(s)) - P_n(x(s-1))}{x(s)-x(s-1)},$$

$$2\mathbb{S}_x\mathbb{D}_x P_n(x(s)) = \frac{P_n(x(s+1)) - P_n(x(s))}{x(s+1)-x(s)} + \frac{P_n(x(s)) - P_n(x(s-1))}{x(s)-x(s-1)}.$$

Solve the system with unknowns $\{P_n(x(s+1)), P_n(x(s-1))\}$ and substitute the solution into (1.11) to obtain a Sturm–Liouville type equation of the form (3.1) where the coefficient of $P_n(x(s))$ is $A(s)+B(s)+C(s)-\lambda_n$. Taking $n=0$ in (1.11) and using the fact that $\lambda_0 = 0$ (see [28, equation (3.1)]) and $P_0 = 1$, we obtain $A(s)+B(s)+C(s) = 0$. Therefore P_n is a polynomial solution of (3.1). This leads us to the following restatement of the generalization of Bochner's theorem in [28]:

Theorem 3.6. *The Sturm–Liouville type equation*

$$\phi(x)\mathbb{D}_x^2 y(x) + \psi(x)\mathbb{S}_x\mathbb{D}_x y(x) + \lambda_n y(x) = 0,$$

where ϕ and ψ are polynomials of degree at most 2 and 1 respectively, and λ_n is a constant, has a polynomial solution $P_n(x)$ of degree $n = 0,1,2,3,\dots$ if and only if

(1) *On $x(s) = c_1 q^{-s} + c_2 q^s$, $c_1 \neq 0$, $P_n(x)$ is, up to a multiplicative constant, equal to*

$${}_4\phi_3\left(\begin{matrix} q^{-n}, u^2abcdq^{n-1}, aq^{-s}, auq^s \\ abu, acu, adu \end{matrix}; q, q\right)$$

$$= \frac{\left(u^{\frac{1}{2}}a\right)^n p_n\left(\frac{u^{\frac{1}{2}}q^s+u^{-\frac{1}{2}}q^{-s}}{2}; u^{\frac{1}{2}}a, u^{\frac{1}{2}}b, u^{\frac{1}{2}}c, u^{\frac{1}{2}}d \,|\, q\right)}{(abu, acu, adu; q)_n},$$

as well as subcases including limiting cases as one or more parameters a, b, c, d, tend to ∞. Here, $p_n(x; a, b, c, d \,|\, q)$ denotes Askey–Wilson (1.3) polynomials and $u = \frac{c_2}{c_1}$.

(2) *On $x(s) = c_4 s^2 + c_5 s$, $c_4 \neq 0$, $P_n(x)$ is, up to a multiplicative constant, equal to*

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} -n, a+b+c+d+2u+n-1, a-s, a+u+s \\ a+b+u, a+c+u, a+d+u \end{matrix}; 1\right) \\ &\quad = \frac{W_n\left(-\left(s+\frac{u}{2}\right)^2; a+\frac{u}{2}, b+\frac{u}{2}, c+\frac{u}{2}, d+\frac{u}{2}\right)}{(a+b+u)_n(a+c+u)_n(a+d+u)_n}, \end{aligned} \tag{3.14}$$

or the polynomial

$${}_3F_2\left(\begin{matrix} -n, a-s, a+u+s \\ a+b+u, a+c+u \end{matrix}; 1\right) = \frac{S_n\left(-\left(s+\frac{u}{2}\right)^2; a+\frac{u}{2}, b+\frac{u}{2}, c+\frac{u}{2}\right)}{(a+b+u)_n(a+c+u)_n(a+d+u)_n}, \tag{3.15}$$

where $u = \frac{c_5}{c_4}$, W_n denotes Wilson polynomials [18, equation (9.1.1)] *and S_n denotes continuous dual Hahn polynomials* [18, equation (9.3.1)].

Proof. For the proof of Theorem 3.6(1), follow the scheme described for Theorem 3.4 to obtain the result. For the proof of Theorem 3.6(2):

1. Take $\phi(x(s)) = \phi_2 x(s)^2 + \phi_1 x(s) + \phi_0$, $\psi(x(s) = \psi_1 x(x(s)) + \psi_0$ and $x(s) = c_4 s^2 + c_5 s$ in $\sigma(x(s))$, with $X = s$, to obtain the polynomial

$$\begin{aligned} P(X) = {}& X^4\phi_2{c_4}^2 + \left(2u\phi_2{c_4}^2 - \psi_1{c_4}^2\right)X^3 + \left(u^2\phi_2{c_4}^2 + \phi_1 c_4 - 3/2u\psi_1{c_4}^2\right)X^2 \\ &+ \left(u\phi_1 c_4 - \psi_0 c_4 - 1/2u^2\psi_1{c_4}^2\right)X + \phi_0 - 1/2u\psi_0 c_4, \qquad u = \frac{c_5}{c_4}. \end{aligned} \tag{3.16}$$

2. Suppose P is of degree 4, that is $\phi_2 \neq 0$:

   - Write
   $$P(X) = C(X-a)(X-b)(X-c)(X-d), \qquad a, b, c, d \in \mathbb{C},$$
   expand it and identify the coefficients of X^i, $i = 0, 1, 2, 3, 4$ with those of P in (3.16) to obtain a system of five equations;
   - Solve the system with unknowns ϕ_2, ϕ_1, ϕ_0, ψ_1 and ψ_0 to obtain the corresponding polynomial coefficients of (3.1)

$$\begin{aligned} \phi(x(s)) = {}& \frac{x(s)^2}{c_4^2} \\ &+ \frac{u(4u+3a+3b+3c+3d) + 2(ab+ac+ad+bc+bd+cd)}{2c_4}x(s) \\ &+ abcd + \frac{uabc + u^2bc + ubcd + uabd + uacd + u^3b + u^3c}{2} \\ &+ \frac{u^3d + u^3a + u^4 + u^2bd + u^2cd + u^2ac + u^2ab + u^2ad}{2}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \psi(x(s)) = {}& \frac{b+c+d+a+2u}{{c_4}^2}x(s) \\ &+ \frac{abc + ubc + bcd + abd + acd + u^2b + u^2c + u^2d}{c_4} \\ &+ \frac{u^2a + u^3 + ubd + ucd + uac + uab + uad}{c_4}. \end{aligned} \tag{3.18}$$

- Use Lemma 3.1 with $\eta = a$ and take into account the fact that $w_k(x(s), a) = (-c_4)^k (a+s)_k (a+u-s)_k$ on $x(s) = c_4 s^2 + c_5 s$, (see Remark (3.2)), to obtain

$$\sum_{k=0}^{n} \frac{(-n)_k (a+b+c+d+2u+n-1)_k}{(a+b+u)_k (a+c+u)_k (a+d+u)_k k!} \left(\frac{-1}{c_4}\right)^k w_k(x(s), a) = {}_4F_3\left(\begin{matrix} -n, a+b+c+d+2u+n-1, a-s, a+u+s \\ a+b+u, a+c+u, a+d+u \end{matrix}; 1\right),$$

  where $u = \frac{c_5}{c_4}$.

3. If P is of degree 3, that is $\phi_2 = 0$,

$$P(X) = -\psi_1 {c_4}^2 X^3 + \left(\phi_1 c_4 - \frac{3u\psi_1 {c_4}^2}{2}\right) X^2 + \left(u\phi_1 c_4 - \phi_0 c_4 - \frac{u^2 \psi_1 {c_4}^2}{2}\right) X + \phi_0 - \frac{u\psi_0 c_4}{2},$$

write

$$P(X) = C(X-a)(X-b)(X-c), \qquad a, b, c \in \mathbb{C},$$

and use an algorithm analogous to the one described above to obtain the following polynomial coefficients of (3.1)

$$\phi(x(s)) = -\frac{(3u+2a+2b+2c)}{2c_4} x(s) - abc - \frac{u\left(u^2+ua+ub+uc+ab+ac+bc\right)}{2},$$
$$\psi(x(s)) = -\frac{x(s)}{c_4^2} - \frac{\left(u^2+ua+ub+uc+ab+ac+bc\right)}{c_4},$$

and (3.15) as the corresponding polynomial system. Since $\psi_1 \neq 0$, P can not be of degree less than 3. ∎

Remark 3.7.

1. Taking $c_4 \to 1$, $c_5 \to 0$ and $s \to \mathrm{i}s$, with $\mathrm{i}^2 = -1$, (3.14) reads as

$$\frac{W_n\left(s^2; a, b, c, d\right)}{(a+b)_n (a+c)_n (a+d)_n} = \frac{W_n(-x(\mathrm{i}s); a, b, c, d)}{(a+b)_n (a+c)_n (a+d)_n},$$

   where $W_n\left(s^2, a, b, c, d\right)$ is the Wilson polynomial [18, equation (9.1.1)] and $x(z) = z^2$.

2. If $c_1 = 1$, $c_2 = \gamma + \delta + 1$, $a = 0$, $b = \alpha - \gamma - \delta$, $c = \beta - \gamma$ and $d = -\gamma$, the polynomial in (3.14) is the Racah polynomial [18, equation (9.2.1)] and S_n is the continuous dual Hahn polynomial [18, equation (9.3.1)].

3. Taking $c_4 \to 1$, $c_5 \to 0$ and $s \to \mathrm{i}s$, with $\mathrm{i}^2 = -1$, (3.15) reads as

$$\frac{S_n\left(s^2; a, b, c\right)}{(a+b)_n (a+c)_n} = \frac{S_n(-x(\mathrm{i}s); a, b, c)}{(a+b)_n (a+c)_n},$$

   where $S_n\left(s^2; a, b, c\right)$ is the continuous dual Hahn polynomial [18, equation (9.3.1)] and $x(z) = z^2$.

Corollary 3.8.

(1) *Wilson polynomials satisfy the Sturm–Liouville type equation* (3.1) *with*

$$\phi(x(z)) = x(z)^2 + (ab + ac + ad + cd + cb + bd)x(z) + abcd,$$
$$\psi(x(z)) = (a + b + c + d)x(z) + abc + abd + acd + bcd,$$
$$\lambda_n = -n(a + b + c + d + n - 1).$$

(2) *Continuous dual Hahn polynomials satisfy the Sturm–Liouville type equation* (3.1) *with*

$$\phi(x(z)) = (a + b + c)x(z) + abc, \qquad \psi(x(z)) = x(z) + ab + ac + bc, \qquad \lambda_n = -n.$$

In both cases, $x(z) = z^2$ $(z = \mathrm{i}s,\ \mathrm{i}^2 = -1)$.

Proof. (1) Take $c_4 \to 1$, $c_5 \to 0$ and $s \to z$, with $z = \mathrm{i}s$, $\mathrm{i}^2 = -1$ in (3.17) and (3.18) to obtain polynomial coefficients of (3.1), $x(z) = z^2$, then use (3.2) to get $\lambda_n = -n(a+b+c+d+n-1)$. (2) is obtained in a similar way. ∎

Corollary 3.9. *The Sturm–Liouville type equation* (3.1) *has a polynomial solution* $P_n(x)$ *of degree* $n = 1, 2, 3, \dots$ *if and only if* $P_n(x)$ *is a multiple of a Wilson polynomial, continuous dual Hahn polynomial or Askey–Wilson polynomial* $p_n(x; a, b, c, d \,|\, q)$ *and subcases, including limiting cases as one or more parameters* a*,* b*,* c*,* d*, tend to* ∞*.*

Remark 3.10. In [26], Nikiforov et al. classified orthogonal polynomials of the quadratic and q-quadratic variable by solving the Pearson type equation [26, equation (3.2.9)] and obtained Racah polynomials, dual Hahn polynomials and their q-analogs. Our approach, which is based on the Sturm–Liouville type equation (see Theorem 3.6) and uses the polynomial P (appearing in the proof of Theorems 3.4 and 3.6), Lemmas 3.1 and 3.3, and Remark 3.2, leads, in addition to Racah polynomials and dual Hahn polynomials, to Askey–Wilson polynomials, subcases and limiting cases. This completes the result in Nikiforov et al. [26], generalizes [13, Theorem 3.1] and provides explicit solutions to [28, equation (1.3)]. To the best of our knowledge, our treatment of the generalized Bochner Theorem is new.

4 Structure relations of orthogonal polynomials of the quadratic and *q*-quadratic variable

In this section, for a family $\{P_n(x(s))\}_{n\geq 0}$ of classical orthogonal polynomials of the quadratic and q-quadratic variable, we prove equivalence between the Sturm–Liouville type equation (3.1), the orthogonality of the second derivatives $\left\{\mathbb{D}_x^2 P_n\right\}_{n\geq 2}$ and a first structure relation that generalizes (1.7). This will enable us to derive, from Theorem 3.6, the solution to the Askey problem related to (1.11) and a second structure relation for classical orthogonal polynomials of the quadratic and q-quadratic variable.

We begin by generalizing [17, Lemma 3.1]

Lemma 4.1. *Let* $\{P_n\}_{n=0}^{\infty}$ *be a sequence of monic orthogonal polynomials. If there exist two sequences* (a_n') *and* (b_n') *of numbers such that*

$$\frac{1}{\gamma_{n+1}}\mathbb{D}_x P_{n+1}(x) = (x - a_n')\frac{1}{\gamma_n}\mathbb{D}_x P_n(x) - \frac{b_n'}{\gamma_{n-1}}\mathbb{D}_x P_{n-1}(x) + c_n, \qquad c_n \in \mathbb{C},$$

then, there exist two polynomials $\phi(x)$ *and* $\psi(x)$ *of degree at most two and of degree one respectively, and* λ_n *a constant such that* $P_n(x)$ *satisfies the divided-difference equation*

$$\phi(x)\mathbb{D}_x^2 P_n(x) + \psi(x)\mathbb{S}_x\mathbb{D}_x P_n(x) + \lambda_n P_n(x) = 0, \qquad n \geq 5. \tag{4.1}$$

Proof. The proof follows exactly the same argument of the proof of [17, Lemma 3.1], replacing the averaging operator $\mathcal{S}_q$ by $\mathbb{S}_x$, the Askey–Wilson operator $\mathcal{D}_q$ by $\mathbb{D}_x$ and where the polynomials U_1 and U_2 are those appearing in (2.3) and (2.2). ■

Theorem 4.2. *Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal with respect to a positive weight function $w(x)$. The following properties are equivalent:*

a) There exists a polynomial $\pi(x)$ of degree at most 4 and sequences of five elements $\{a_{n,n+k}\}_{n\geq 2}$, $-2 \leq k \leq 2$, $a_{n,n-2} \neq 0$ such that P_n satisfies the structure relation

$$\pi(x)\mathbb{D}_x^2 P_n(x) = \sum_{k=-2}^{2} a_{n,n+k} P_{n+k}(x). \tag{4.2}$$

b) There exists a polynomial $\pi(x)$ of degree at most four such that $\left\{\mathbb{D}_x^2 P_n\right\}_{n=2}^{\infty}$ is orthogonal with respect to $\pi(x)w(x)$.

c) There exist two polynomials $\phi(x)$ and $\psi(x)$ of degree at most two and of degree one respectively, and a constant λ_n such that

$$\phi(x)\mathbb{D}_x^2 P_n(x) + \psi(x)\mathbb{S}_x\mathbb{D}_x P_n(x) + \lambda_n P_n(x) = 0, \qquad n \geq 5. \tag{4.3}$$

Proof. The proof is organized as follows:

Step 1. $(a) \Rightarrow (b) \Rightarrow (a)$ which is equivalent to $(a) \Leftrightarrow (b)$.

Step 2. $(b) \Rightarrow (c) \Rightarrow (a)$ which, taking into account Step 1, is equivalent to $(b) \Leftrightarrow (c)$.

Step 1: We assume that (a) is satisfied and we prove (b). Let $m \geq 2$ and $n \geq 2$ be two integers, and assume that $m \leq n$. From (a), there exists a polynomial π of degree at most four and there exist sequences of five elements $\{a_{n,n+j}\}_n$, $j = -2, -1, 0, 1, 2$ such that

$$\pi\mathbb{D}_x^2 P_n = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}, \qquad \text{with} \quad a_{n,n-2} \neq 0. \tag{4.4}$$

Since $m \leq n$, $m-2 \leq n-2 \leq n+j \leq n+2$ (for $j = -2, -1, 0, 1, 2$). So, multiplying both sides of (4.4) by $W\mathbb{D}_x^2 P_m$, integrating on (a,b) and then taking into account the fact that (P_n) is orthogonal on the interval (a,b) with respect to the weight function W, we obtain

$$\int_a^b \mathbb{D}_x^2 P_m(x)\mathbb{D}_x^2 P_n(x)\pi(x)W(x)\mathrm{d}x \begin{cases} = 0 & \text{if } m < n, \\ \neq 0 & \text{if } m = n. \end{cases}$$

If $n < m$, we substitute in (4.4), n by m and by a similar way, we obtain

$$\int_a^b \pi(x)W(x)\mathbb{D}_x^2 P_n(x)\mathbb{D}_x^2 P_m(x)\mathrm{d}x = 0.$$

Now we assume (b) and we prove (a). Since $\pi(x)\mathbb{D}_x^2 P_n$ is a polynomial of degree less or equal to $n+2$, $\pi(x)\mathbb{D}_x^2 P_n$ can be expanded in the orthogonal basis $\{P_j\}_{j=0}^{\infty}$ as

$$\pi(x)\mathbb{D}_x^2 P_n = \sum_{j=0}^{n+2} a_{n,j} P_j,$$

where $a_{n,j}$, $j = 0, \dots, n+2$ is given by

$$a_{n,j}\int_a^b W(x)\left(\mathbb{D}_x^2 P_n(x)\right)^2\mathrm{d}x = \int_a^b \pi(x)W(x)P_j(x)\mathbb{D}_x^2 P_n(x)\mathrm{d}x.$$

Since $\mathbb{D}_x^2 P_n(x)$ is of degree $n-2$ we deduce from the hypothesis that $a_{n,j}=0$ for $j=0,\dots,n-2$ and $a_{n,n-2}\neq 0$.

Step 2: We suppose (b) and we prove (c). Firstly, we prove that $\{P_n\}_{n=0}^{\infty}$ satisfies an equation of type (4.1). Since $\frac{x}{\gamma_n}\mathbb{D}_x P_n$ is a monic polynomial of degree n, it can be expanded as

$$x\frac{1}{\gamma_n}\mathbb{D}_x P_n = \frac{1}{\gamma_{n+1}}\mathbb{D}_x P_{n+1} + \sum_{j=1}^{n}\frac{e_{n,j}}{\gamma_j}\mathbb{D}_x P_j, \qquad e_{n,j}\in\mathbb{R}. \tag{4.5}$$

Applying $\mathbb{D}_x$ to both sides, we obtain

$$(\alpha x+\beta)\frac{1}{\gamma_n}\mathbb{D}_x^2 P_n + \frac{1}{\gamma_n}\mathbb{S}_x\mathbb{D}_x P_n = \frac{1}{\gamma_{n+1}}\mathbb{D}_x^2 P_{n+1} + \sum_{j=2}^{n}\frac{e_{n,j}}{\gamma_j}\mathbb{D}_x^2 P_j. \tag{4.6}$$

Apply $\mathbb{D}_x$ to both sides of the three-term recurrence relation (1.1), use the product rule (2.1) and the fact $\mathbb{S}_x(x(s))=\alpha x(s)+\beta$ to obtain

$$\mathbb{D}_x P_{n+1} = (\alpha x+\beta-a_n)\mathbb{D}_x P_n + \mathbb{S}_x P_n - b_n\mathbb{D}_x P_{n-1}. \tag{4.7}$$

Apply $\mathbb{D}_x$ to both sides and use (2.3) as well as the expression of U_1, $U_1(x)=\big(\alpha^2-1\big)x+\beta(\alpha+1)$ to obtain

$$\mathbb{D}_x^2 P_{n+1} = \big[\big(2\alpha^2-1\big)x + 2\beta(\alpha+1) - a_n\big]\mathbb{D}_x^2 P_n + 2\alpha\mathbb{S}_x\mathbb{D}_x P_n - b_n\mathbb{D}_x^2 P_{n-1}. \tag{4.8}$$

By using this relation to eliminate $\mathbb{S}_x\mathbb{D}_x P_n$ in (4.6) we obtain

$$\begin{aligned}\frac{1}{\gamma_n}x\mathbb{D}_x^2 P_n - \frac{(2\beta+a_n)}{\gamma_n}\mathbb{D}_x^2 P_n + \frac{b_n}{\gamma_n}\mathbb{D}_x^2 P_{n-1}&\\ = \left(\frac{2\alpha}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right)\mathbb{D}_x^2 P_{n+1} + \sum_{j=2}^{n}\frac{2\alpha e_{n,j}}{\gamma_j}\mathbb{D}_x^2 P_j.&\end{aligned} \tag{4.9}$$

Since $\left\{\frac{\mathbb{D}_x^2 P_n}{\gamma_n\gamma_{n-1}}\right\}$ is orthogonal, there exist a_n'' and b_n'' such that

$$x\frac{\mathbb{D}_x^2 P_n}{\gamma_n} = \frac{\gamma_{n-1}}{\gamma_{n+1}\gamma_n}\mathbb{D}_x^2 P_{n+1} + a_n''\mathbb{D}_x^2 P_n + b_n''\mathbb{D}_x^2 P_{n-1}. \tag{4.10}$$

So, using the relation $\gamma_{n+1}-2\alpha\gamma_n+\gamma_{n-1}=0$, obtained by direct computation, (4.9) becomes

$$\left(a_n'' - \frac{2\beta+a_n}{\gamma_n}\right)\mathbb{D}_x^2 P_n + \left(b_n'' + \frac{b_n}{\gamma_n}\right)\mathbb{D}_x^2 P_{n-1} = \sum_{j=2}^{n}\frac{2\alpha e_{n,j}}{\gamma_j}\mathbb{D}_x^2 P_j.$$

Therefore, $e_{n,j}=0$ (for $j=2,3,\dots,n-2$) and (4.5) reads as

$$\frac{x}{\gamma_n}\mathbb{D}_x P_n = \frac{1}{\gamma_{n+1}}\mathbb{D}_x P_{n+1} + \frac{e_{n,n}}{\gamma_n}\mathbb{D}_x P_n + \frac{e_{n,n-1}}{\gamma_{n-1}}\mathbb{D}_x P_{n-1} + e_{n,1}.$$

Then, from Lemma 4.1, there exist two polynomials ϕ of degree at most 2 and ψ of degree 1, and a constant λ_n such that P_n satisfies

$$\phi\mathbb{D}_x^2 P_n + \psi\mathbb{S}_x\mathbb{D}_x P_n + \lambda_n P_n = 0, \qquad n\geq 5.$$

Let us prove that $(c)\Rightarrow(a)$. Note that P_n, $n=1,2,3,4$, satisfies (4.3). In fact, it follows from the algorithm described in the proof of Theorem 3.6, that, for $n\geq 5$, P_n is up to a multiplicative factor equal to Askey–Wilson polynomial and Wilson polynomial, and subcases, including limiting cases, denoted by p_n. Since $\{p_n\}_{n=0}^{\infty}$ is orthogonal (cf. [17, 18]) both families are orthogonal with respect to the same measure. Therefore P_n, $n = 1, 2, 3, 4$, is up to a multiplicative factor equal to p_n which satisfies (4.3) by Theorem 3.6).

Apply $\mathbb{S}_x$ to both sides of (4.7) and use (2.1) as well as (2.4) to obtain

$$\mathbb{S}_x\mathbb{D}_x P_{n+1} = \big(\alpha^2 x + U_1(x) + \beta(\alpha+1) - a_n\big)\mathbb{S}_x\mathbb{D}_x P_n + 2\alpha U_2 \mathbb{D}_x^2 P_n + P_n - b_n \mathbb{S}_x\mathbb{D}_x P_{n-1}.$$

Adding ψ times the previous equation and ϕ times (4.8), and then using the assumption (c), we obtain

$$\begin{aligned}\lambda_{n+1}P_{n+1} &= \lambda_n\big(\alpha^2 x + U_1(x) + \beta(\alpha+1) - a_n\big)P_n \\ &\quad - 2\alpha\big(\phi\mathbb{S}_x\mathbb{D}_x P_n + U_2\psi\mathbb{D}_x^2 P_n\big) - \psi P_n - b_n\lambda_{n-1}P_{n-1}.\end{aligned}$$

Multiplying the latter equation by ψ and using the relation $\psi\mathbb{S}_x\mathbb{D}_x P_n = -\phi\mathbb{D}_x^2 P_n - \lambda_n P_n$, obtained from the assumption, and then substituting U_1 by $\big(2\alpha^2-1\big)x + \beta(\alpha+1)$, we obtain

$$\begin{aligned}2\alpha\big(\phi^2 - U_2\psi^2\big)\mathbb{D}_x^2 P_n &= \lambda_{n+1}\psi P_{n+1} \\ &\quad + \big[\psi^2 - 2\alpha\lambda_n\phi - \lambda_n\big(\big(\alpha^2-1\big)x + 2\beta(\alpha+1) - a_n\big)\psi\big]P_n + \lambda_{n-1}b_n\psi P_{n-1}.\end{aligned}$$

Taking $\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0$ and $\psi(x) = \psi_1 x + \psi_0$ and using the three-term recurrence relation (4.10), we transform the above equation into

$$\big(\phi^2 - U_2\psi^2\big)\mathbb{D}_x^2 P_n = \sum_{j=-2}^{2} a_{n,n+j}P_{n+j}, \tag{4.11}$$

where

$$a_{n,n-2} = \frac{\big[\psi_1^2 - \lambda_n\big(2\alpha\phi_2 + \big(2\alpha^2-1\big)\psi_1\big)\big]b_n b_{n-1} + \psi_1 b_{n-1}b_n\lambda_{n-1}}{2\alpha}.$$

$a_{n,n-2} \neq 0$, for $b_n > 0$, $n = 0, 1, \dots$, and ψ_1 does not depend on n. ■

Corollary 4.3. *A family of monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ satisfies the relation (4.2) if and only if $P_n(x)$ is a multiple of the Wilson polynomial, continuous dual Hahn polynomial or Askey–Wilson polynomial $p_n(x; a, b, c, d \,|\, q)$ and subcases, including limiting cases as one or more parameters a, b, c, d tend to infinity.*

Proof. The proof is deduced from Theorem 4.2, Corollary 3.9 and the fact that limiting cases of Askey–Wilson polynomials as one or more parameters a, b, c, d tend to ∞ are orthogonal polynomials families (cf. [13, Remark 3.2]), see also [17]. ■

Next, we turn our attention to the second structure relation.

Proposition 4.4. *Let $\{P_n\}$ be a sequence of polynomials orthogonal with respect to a weight function W on (a, b). If $\big\{\mathbb{D}_x^2 P_n\big\}$ is orthogonal with respect to the weight function πW where π is a polynomial of degree at most 4, then there exist sequences of five elements $\{b_{n,n+j}\}$, $j = -2, -1, 0, 1, 2$ such that*

$$P_n = \sum_{j=-2}^{2} b_{n,n+j}\mathbb{D}_x^2 P_{n+j}, \qquad b_{n,n+2} \neq 0, \qquad n = 2, 3, \dots.$$

Proof. Replace $\mathcal{D}_q$ by $\mathbb{D}_x$ in the proof of [17, Proposition 3.8]. ■

Corollary 4.5. *Wilson polynomials, continuous dual Hahn polynomials, Askey–Wilson polynomials, special cases and limiting cases as one or more parameters a, b, c, d tend to ∞, satisfy the structure relation*

$$P_n(x) = \sum_{j=-2}^{2} b_{n,n+j} \mathbb{D}_x^2 P_{n+j}(x). \tag{4.12}$$

Proof. Since those polynomials are orthogonal and they are the only solutions to (3.1), the result is obtained by using Theorem 4.2 and Proposition 4.4. ■

In the following proposition we show the connection of the structure relation (cf. [8, p. 118]) given by (1.13) to (4.12).

Proposition 4.6. *The structure relation* (1.13) *is connected to our second structure relation* (4.12) *as follows:*

$$\begin{aligned}\mathbb{S}_x \mathcal{M} P_n(x(s)) &= P_n(x(s)) + \alpha^{-1}\mathbb{D}_x^2 P_{n+1}(x(s)) + \alpha^{-1} c_n \mathbb{D}_x^2 P_{n-1}(x(s)) \\ &\quad + \alpha^{-1}\big(b_n - \alpha^2 x(s) - \beta(\alpha+1) - U_1(x(s)) + \alpha^2 U_2(x(s))\big)\mathbb{D}_x^2 P_n(x(s)),\end{aligned}$$

where b_n and c_n are coefficients of the three-term recurrence relation (1.1)*.*

Proof. Observe that $\mathbb{S}_x \mathcal{M} P_n(x(s)) = \mathbb{S}_x^2 P_n(x(s))$, then take into account (2.4) to obtain

$$\mathbb{S}_x^2 P_n(x(s)) = P_n(x(s)) + U_1(x(s))\mathbb{S}_x\mathbb{D}_x P_n(x(s)) + \alpha U_2(x(s))\mathbb{D}_x^2 P_n(x(s)).$$

Then use (4.8) to eliminate $\mathbb{S}_x\mathbb{D}_x$. ■

Remark 4.7. From (4.11), $\{\mathbb{D}_x^2 P_n\}_{n\geq 2}$ is orthogonal with respect to

$$\big(\phi^2(x(s)) - U_2(x(s))\psi^2(x(s))\big)W(x(s)).$$

So, there exists a constant $c > 0$ such that

$$\begin{aligned}\pi(x(s)) &= c\big(\phi^2(x(s)) - U_2(x(s))\psi^2(x(s))\big) \\ &= c\left(\phi(x(s)) - \frac{\nabla x_1(s)}{2}\psi(x(s))\right)\left(\phi(x) + \frac{\nabla x_1(s)}{2}\psi(x(s))\right) \\ &\qquad \text{for} \quad U_2(x(s)) = \left(\frac{\nabla x_1(s)}{2}\right)^2 \\ &= c\sigma(x(s))\tau(x(s)),\end{aligned}$$

where $\sigma(x(s))$ and $\tau(x(s))$ are functions defined by

$$\sigma(x(s)) = \phi(x(s)) - \frac{\nabla x_1(s)}{2}\psi(x(s)), \qquad \tau(x(s)) = \phi(x(s)) + \frac{\nabla x_1(s)}{2}\psi(x(s)).$$

So, without loss of generality, we can take

$$\pi(x(s)) = \phi^2(x(s)) - U_2(x(s))\psi^2(x(s)).$$

Let us mention that the function $\sigma(x(s))$ is the one defined by (3.3) and also appearing in the proof of Theorem 3.4 and that of Theorem 3.6.

5 Coefficients of the structure relations

5.1 The Wilson polynomials

Proposition 5.1. *The first structure relation* (4.2) *for monic Wilson polynomials* $P_n\big(s^2;a,b,c,d\big)$ *is*

$$\big(s^2+a^2\big)\big(s^2+b^2\big)\big(s^2+c^2\big)\big(s^2+d^2\big)\frac{\delta^2 P_n(s^2;a,b,c,d)}{\delta^2 s^2}=\sum_{j=-2}^{2}a_{n,n+j}P_{n+j}\big(s^2;a,b,c,d\big),$$

where

$$\begin{aligned}
a_{n,n+2}&=n(n-1),\\
\frac{a_{n,n+1}}{n(n-1)}&=A_n(c,b,a,d+1)+A_{n-1}(b,a,c+1,d+1)\\
&\quad+A_{n-2}(a,b+1,c+1,d+1)+A_{n+1}(d,b,c,a),\\
\frac{a_{n,n}}{n(n-1)}&=\big(A_n(c,b,a,d+1)+A_{n-1}(b,a,c+1,d+1)\\
&\quad+A_{n-2}(a,b+1,c+1,d+1)\big)A_n(d,b,c,a)+\big(A_{n-1}(b,a,c+1,d+1)\\
&\quad+A_{n-2}(a,b+1,c+1,d+1)\big)A_{n-1}(c,b,a,d+1)\\
&\quad+A_{n-2}(a,b+1,c+1,d+1)A_{n-2}(b,a,c+1,d+1),\\
\frac{a_{n,n-1}}{n(n-1)}&=\big[\big(A_{n-1}(b,a,c+1,d+1)+A_{n-2}(a,b+1,c+1,d+1)\big)A_{n-1}(c,b,a,d+1)\\
&\quad+A_{n-2}(a,b+1,c+1,d+1)A_{n-2}(b,a,c+1,d+1)\big]A_{n-1}(d,b,c,a)\\
&\quad+A_{n-2}(a,b+1,c+1,d+1)A_{n-2}(b,a,c+1,d+1)A_{n-2}(c,b,a,d+1),\\
\frac{a_{n,n-2}}{n(n-1)}&=A_{n-2}(a,b+1,c+1,d+1)A_{n-2}(b,a,c+1,d+1)\\
&\quad\times A_{n-2}(c,b,a,d+1)A_{n-2}(d,b,c,a),
\end{aligned}$$

and $A_n(a,b,c,d)$ *is defined in Lemma* A.1*.*

Proof. Substitute the polynomial coefficients $\phi(x(\mathrm{i}s))$ and $\psi(x(\mathrm{i}s))$, $x(z)=z^2$ of (4.3) given in Corollary 3.8 and take into account the fact that

$$U_2(x(\mathrm{i}s))=\left(\frac{x\big(\mathrm{i}s+\frac12\big)-x\big(\mathrm{i}s-\frac12\big)}{2}\right)^2$$

to obtain $\pi(x(\mathrm{i}s))=\big(s^2+a^2\big)\big(s^2+b^2\big)\big(s^2+c^2\big)\big(s^2+d^2\big)$. Since

$$\mathbb{D}_x^2 P_n\big(s^2;a,b,c,d\big)=n(n-1)P_{n-2}\big(s^2;a,b,c,d\big)=\frac{\delta^2 P_n\big(s^2;a,b,c,d\big)}{\delta^2 s}\tag{5.1}$$

(see (A.4)), the structure relation (4.2) becomes

$$\begin{aligned}
&\big(s^2+a^2\big)\big(s^2+b^2\big)\big(s^2+c^2\big)\big(s^2+d^2\big)P_{n-2}\big(s^2;a+1,b+1,c+1,d+1\big)\\
&\qquad=\sum_{j=-2}^{2}\frac{a_{n,n+j}}{n(n-1)}P_{n+j}\big(s^2;a,b,c,d\big).
\end{aligned}$$

Replace n by $n-2$ is the first equation of Lemma A.1, multiply the obtained equation by $\big(s^2+b^2\big)\big(s^2+c^2\big)\big(s^2+d^2\big)$ and use the second, third and the fourth relation of this lemma to get $a_{n,n+j}$, $j\in\{-2,\dots,2\}$ in terms of A_{n+j}. ■

Proposition 5.2. *The second structure relation (4.12) for monic Wilson polynomials* $P_n\big(s^2;a,b,c,d\big)$ *is*

$$P_n\big(s^2;a,b,c,d\big)=\sum_{j=-2}^{2}b_{n,n+j}\frac{\delta^2P_{n+j}\big(s^2;a,b,c,d\big)}{\delta^2s^2},$$

where

$$\begin{aligned}
&(n+2)(n+1)b_{n,n+2}=1,\\
&(n+1)(n)b_{n,n-1}=C_n(b,a+1,c,d)+C_n(a,b,c,d)+C_n(d,c+1,a+1,b+1)\\
&\qquad\qquad+C_n(c,d,a+1,b+1),\\
&(n)(n-1)b_{n,n-2}=\big(C_n(b,a+1,c,d)+C_n(a,b,c,d)\big)\big(C_{n-1}(d,c+1,a+1,b+1)\\
&\qquad\qquad+C_{n-1}(c,d,a+1,b+1)\big)+C_n(c,d,a+1,b+1)\\
&\qquad\qquad\times C_{n-1}(d,c+1,a+1,b+1)+C_n(a,b,c,d)C_{n-1}(b,a+1,c,d),\\
&(n-1)(n-2)b_{n,n-3}=\big(C_n(b,a+1,c,d)+C_n(a,b,c,d)\big)C_{n-1}(c,d,a+1,b+1)\\
&\qquad\qquad\times C_{n-2}(d,c+1,a+1,b+1)+C_n(a,b,c,d)C_{n-1}(b,a+1,c,d)\\
&\qquad\qquad\times\big(C_n(n-2,d,c+1,a+1,b+1)+C_{n-2}(c,d,a+1,b+1)\big),\\
&(n-2)(n-3)b_{n,n-4}=C_n(b,a,c,d)C_{n-1}(a,b+1,c,d)C_{n-2}(d,c,a+1,b+1)\\
&\qquad\qquad\times C_{n-3}(c,d+1,a+1,b+1),
\end{aligned}$$

and $C_n(a,b,c,d)$ *is given in Lemma A.1.*

Proof. From Corollary 4.5 and the relation (5.1), we obtain

$$P_n\big(s^2;a,b,c,d\big)=\sum_{j=-2}^{2}(n+j)(n+j-1)b_{n,n+j}P_{n-2+j}\big(s^2;a+1,b+1,c+1,d+1\big).$$

Take into account (A.2) to obtain

$$\begin{aligned}
P_n\big(s^2;a,b,c,d\big)&=P_n\big(s^2;a+1,b+1,c,d\big)+\big(C_n(b,a+1,c,d)+C_n(a,b,c,d)\big)\\
&\quad\times P_{n-1}\big(s^2;a+1,b+1,c,d\big)\\
&\quad+C_n(a,b,c,d)C_{n-1}(b,a+1,c,d)P_{n-2}(a+1,b+1,c,d).
\end{aligned}\tag{5.2}$$

Substitute c by $c+1$ and d by $d+1$ to obtain

$$\begin{aligned}
&P_n\big(s^2;a,b,c+1,d+1\big)=P_n\big(s^2;d+1,c+1,a+1,b+1\big)\\
&\quad+\big(C_n(d,c+1,a+1,b+1)+C_n(c,d,a+1,b+1)\big)P_{n-1}\big(s^2;d+1,c+1,a+1,b+1\big)\\
&\quad+C_n(c,d,a+1,b+1)C_{n-1}(d,c+1,a+1,b+1)P_{n-2}\big(s^2;d+1,c+1,a+1,b+1\big).
\end{aligned}$$

Permute a and c, b and d and use the fact that P_n is symmetric with respect to its parameters to obtain

$$\begin{aligned}
&P_n\big(s^2;a+1,b+1,c,d\big)=P_n\big(s^2;a+1,b+1,c+1,d+1\big)\\
&\quad+\big(C_n(d,c+1,a+1,b+1)+C_n(c,d,a+1,b+1)\big)P_{n-1}\big(s^2;a+1,b+1,c+1,d+1\big)\\
&\quad+C_n(c,d,a+1,b+1)C_{n-1}(d,c+1,a+1,b+1)P_{n-2}\big(s^2;a+1,b+1,c+1,d+1\big).
\end{aligned}$$

Take this relation into account in (5.2) to obtain the result. ∎

5.2 The Askey–Wilson polynomials

Coefficients of the first structure relation (4.2) for the Askey–Wilson polynomials have been given in [17]. As for those of the second structure relation we have the following:

Proposition 5.3. *The Askey–Wilson polynomials satisfy the second structure relation*

$$P_n(x;a,b,c,d\,|\,q) = \sum_{j=-2}^{2} b_{n,n+j}\mathcal{D}_q^2 P_{n+j}(x;a,b,c,d\,|\,q), \qquad x=\cos\theta,$$

where

$$\begin{aligned}
&\gamma_{n+2}\gamma_{n+1}b_{n,n+2} = 1,\\
&-2\gamma_n\gamma_{n-1}b_{n,n+1} = C_n(b,aq,c,d) + C_n(a,b,c,d) + C_n(d,cq,aq,bq) + C_n(c,d,aq,bq),\\
&4\gamma_{n+1}\gamma_n b_{n,n} = (C_n(b,aq,c,d) + C_n(a,b,c,d))(C_{n-1}(d,cq,aq,bq) + C_{n-1}(c,d,aq,bq))\\
&\qquad + C_n(c,d,aq,bq)C_{n-1}(d,cq,aq,bq) + C_n(a,b,c,d)C_{n-1}(b,aq,c,d),\\
&-8\gamma_n\gamma_{n-1}b_{n,n-1} = (C_n(b,aq,c,d) + C_n(a,b,c,d))C_{n-1}(c,d,aq,bq)C_{n-2}(d,cq,aq,bq)\\
&\qquad + C_n(a,b,c,d)C_{n-1}(b,aq,c,d)(C_{n-2}(d,cq,aq,bq) + C_{n-2}(c,d,aq,bq)),\\
&16\gamma_{n-1}\gamma_{n-2}b_{n,n-2} = C_n(a,b,c,d)C_{n-1}(b,aq,c,d)C_{n-2}(c,d,aq,bq)C_{n-3}(d,cq,aq,bq),
\end{aligned}$$

and

$$C_n(a,b,c,d) = \frac{a\big(1-q^n\big)\big(1-bcq^{n-1}\big)\big(1-bdq^{n-1}\big)\big(1-dcq^{n-1}\big)}{\big(1-abcdq^{2n-2}\big)\big(1-abcdq^{2n-1}\big)}$$

is the coefficient appearing in Lemma A.2.

Proof. Use Corollary 4.5, the relation (A.5) with $k=2$, as well as the fact that $\mathbb{D}_x P_n(x;a,b,c,d\,|\,q) = \mathcal{D}_q P_n(x;a,b,c,d\,|\,q)$ to obtain

$$P_n(x;a,b,c,d\,|\,q) = \sum_{j=-2}^{2} \gamma_{n+j}\gamma_{n-1+j}b_{n,n+j}P_{n-2+j}(x;aq,bq,cq,dq\,|\,q).$$

Substitute the second relation of Lemma A.2 into the first to obtain

$$\begin{aligned}
&4P_n(x;a,b,c,d\,|\,q) = C_n(a,b,c,d)C_{n-1}(b,aq,c,d)P_{n-2}(x;bq,aq,c,d\,|\,q)\\
&\quad + 4P_n(x;bq,aq,c,d\,|\,q) + -2(C_n(b,aq,c,d) + C_n(a,b,c,d))P_{n-1}(x;bq,aq,c,d\,|\,q).
\end{aligned}$$

Substitute c by cq, d by dq. Permute a and c, b and d and use the symmetric property of P_n with respect to its parameters to obtain

$$\begin{aligned}
&4P_n(x;aq,bq,c,d\,|\,q) = 4P_n(x;dq,cq,aq,bq\,|\,q) - 2(C_n(d,cq,aq,bq) + C_n(c,d,aq,bq))\\
&\quad\times P_{n-1}(x;dq,cq,aq,bq\,|\,q) + C_n(c,d,aq,bq)C_{n-1}(d,cq,aq,bq)P_{n-2}(x;dq,cq,aq,bq|q).
\end{aligned}$$

Take into account this relation in the previous one to obtain the result. ∎

Remark 5.4. For the continuous q-Hermite polynomials $2^nP_n(x) = H_n(x\,|\,q)$ (cf. [18, equation (14.26.1)]), the *second structure relation* (4.12) reads as

$$P_n(x) = \sum_{j=-2}^{2} b_{n,n+j}\gamma_{n+j}\gamma_{n-1+j}P_{n-2+j}(x),$$

for $\mathbb{D}_xP_n(x) = \mathcal{D}_qP_n(x) = \gamma_nP_{n-1}(x)$ (cf. [18, equation (14.26.7)]). Therefore $\gamma_{n+2}\gamma_{n+1}b_{n,n+2} = 1$ and $b_{n,n+j}=0, j=-2,\dots,1$. Thus, for the monic continuous q-Hermite polynomials P_n, the right hand side of (4.12) is P_n. This result is analogous to that of monic Hermite polynomials for the second structure relation (1.2) (cf. [23, Table VI]).

6 Conclusion

We present another treatment of generalized Bochner theorem and develop two structure relations for classical orthogonal polynomials of the quadratic and q-quadratic variable: a first structure relation that we use to characterize Wilson polynomials, continuous dual Hahn polynomials, Askey–Wilson polynomials and subcases, including limiting cases when one or more parameters tend to ∞, as the family of classical orthogonal polynomials of the quadratic and q-quadratic variable; a second structure relation involving only the divided-difference operator $\mathbb{D}_x$, that generalizes the Wilson operator and the Askey–Wilson operator. Our treatment of the generalized Bochner theorem leads to explicit solutions of the difference equation [28, equation (1.3)]. This work generalizes the result in [13, Theorem 3.1] as well as our previous work (cf. [17]), where we completed and proved the conjecture by Ismail (cf. [14, equation (24.7.9)]). Moreover, by completing the work of Koornwinder (cf. [21]) as shown in [17] and that of Costas-Santos and Marcellán (cf. [8]) in the present paper, we have illustrated that polynomials appearing in the Askey scheme and q-Askey scheme [18] can be effectively studied by using only the operator $\mathbb{D}_x$.

A Appendix

In this section, we state and prove some contiguous relations for Wilson polynomials and Askey–Wilson polynomials. To the best of our knowledge, those for Wilson polynomials are new.

Lemma A.1. *The monic Wilson polynomials*

$$P_n\big(s^2;a,b,c,d\big)=\frac{W_n\big(s^2;a,b,c,d\big)}{(-1)^n(a+b+c+d+n-1)_n},$$

where $W_n\big(s^2;a,b,c,d\big)$ is given by (1.4), satisfy the contiguous relations

$$\begin{aligned}
&\big(s^2+a^2\big)P_n\big(s^2;a+1,b,c,d\big)=P_{n+1}\big(s^2;a,b,c,d\big)+A_n(a,b,c,d)P_n\big(s^2;a,b,c,d\big),\\
&\big(s^2+b^2\big)P_n\big(s^2;a,b+1,c,d\big)=P_{n+1}\big(s^2;a,b,c,d\big)+A_n(b,a,c,d)P_n\big(s^2;a,b,c,d\big),\\
&\big(s^2+c^2\big)P_n\big(s^2;a,b,c+1,d\big)=P_{n+1}\big(s^2;a,b,c,d\big)+A_n(c,b,a,d)P_n\big(s^2;a,b,c,d\big),\\
&\big(s^2+d^2\big)P_n\big(s^2;a,b,c,d+1\big)=P_{n+1}\big(s^2;a,b,c,d\big)+A_n(d,b,c,a)P_n\big(s^2;a,b,c,d\big),\\
&P_n\big(s^2;a,b,c,d\big)=P_n\big(s^2;a+1,b,c,d\big)+C_n(a,b,c,d)P_{n-1}\big(s^2;a+1,b,c,d\big), \qquad \text{(A.1)}\\
&P_n\big(s^2;a+1,b,c,d\big)=P_n\big(s^2;a+1,b+1,c,d\big)\\
&\qquad\qquad +C_n(b,a+1,c,d)P_{n-1}\big(s^2;a+1,b+1,c,d\big), \qquad \text{(A.2)}
\end{aligned}$$

where

$$\begin{aligned}
A_n(a,b,c,d)&=\frac{(a+b+c+d+n-1)(a+b+n)(a+c+n)(a+d+n)}{(a+b+c+d+2n-1)(a+b+c+d+2n)},\\
C_n(a,b,c,d)&=\frac{n(b+c+n-1)(b+d+n-1)(c+d+n-1)}{(a+b+c+d+2n-2)(a+b+c+d+2n-1)}
\end{aligned}$$

are the coefficients appearing in the three-term recurrence relation [18, equation (9.1.5)] *of* $P_n\big(s^2;a,b,c,d\big)$.

Proof. For the first relation, write

$$\big(s^2+a^2\big)P_n\big(s^2;a+1,b,c,d\big)=\sum_{j=0}^{n+1}l_jP_j\big(s^2;a,b,c,d\big),$$

and use the fact that $\{P_n(s^2;a,b,c,d)\}_{n=0}^{\infty}$ is orthogonal with respect to the weight function

$$w(s^2;a,b,c,d) = \left|\frac{\Gamma(a+\mathrm{i}s)\Gamma(b+\mathrm{i}s)\Gamma(c+\mathrm{i}s)\Gamma(d+\mathrm{i}s)}{\Gamma(2\mathrm{i}s)}\right|$$

on the interval $(0;\infty)$ (cf. [18, equation (9.1.2)]), where Γ is the gamma function, to obtain

$$\begin{aligned} l_j \int_0^\infty & w(s^2;a,b,c,d)P_j^2(s^2;a,b,c,d)\mathrm{d}s \\ &= \int_0^\infty w(s^2;a,b,c,d)(s^2+a^2)P_n(s^2;a+1,b,c,d)P_j(s^2;a,b,c,d)\mathrm{d}s. \end{aligned}$$

Use the relation

$$w(s^2;a+1,b,c,d) = (s^2+a^2)w(s^2;a,b,c,d), \tag{A.3}$$

obtained from the property $\Gamma(z+1) = z\Gamma(z)$, to obtain

$$\begin{aligned} l_j \int_0^\infty & w(s^2;a,b,c,d)P_j^2(s^2;a,b,c,d)\mathrm{d}s \\ &= \int_0^\infty w(s^2;a+1,b,c,d)P_n(s^2;a+1,b,c,d)P_j(s^2;a,b,c,d)\mathrm{d}s = 0, \end{aligned}$$

for $j < n$. That is

$$(s^2+a^2)P_n(s^2;a+1,b,c,d) = P_{n+1}(s^2;a+1,b,c,d) + l_nP_n(s^2;a,b,c,d).$$

Let $s^2 = -a^2$ and solve the equation to obtain $l_n = A_n(a,b,c,d)$. Since $w(s^2;a,b,c,d)$ is symmetric with respect to a, b, c and d, and $P_n(s^2;a,b,c,d)$ is monic, $P_n(s^2;a,b,c,d)$ is symmetric with respect to a, b, c and d. Using this property, the second, third and fourth relation are deduced from the first. Note that, since the family $\{P_n(s^2;a+1,b,c,d)\}_{n=0}^{\infty}$ is orthogonal with respect to $(s^2+a^2)w(s^2;a,b,c,d)$ (see (A.3)), the polynomial (s^2+a^2) is nonnegative on $(0,\infty)$ for $\operatorname{Re}(a,b,c,d) > 0$ and non-real parameters occur in conjugate pairs (see [18, p. 186]), the first relation of the lemma can be also deduced from [14, Theorem 2.7.1].

For (A.1), expand $P_n(s^2;a+1,b,c,d)$ in the basis $P_j(s^2;a,b,c,d)$; use the fact that $\{P_n(s^2;a,b,c,d)\}_{n=0}^{\infty}$ is orthogonal with respect to $w(s^2;a,b,c,d)$ as well as the relation (A.3) to obtain

$$P_n(s^2;a,b,c,d) = P_n(s^2;a+1,b,c,d) + m_{n-1}P_{n-1}(s^2;a+1,b,c,d).$$

Apply $\mathbb{D}_x$ $n-1$ time to both sides then use the relation, with $k = n-1$,

$$\mathbb{D}_x^k P_n(s^2;a,b,c,d) = (-n)_k P_{n-k}\left(s^2;a+\frac{k}{2},b+\frac{k}{2},c+\frac{k}{2},d+\frac{k}{2}\right), \tag{A.4}$$

obtained by iterating [18, equation (9.1.8)], with

$$W_n(s^2;a,b,c,d) = (-1)^n(a+b+c+d+n-1)_nP_n(s^2;a,b,c,d),$$

and solve the equation with unknown m_{n-1} to obtain $m_{n-1} = C_n(a,b,c,d)$. For the last relation, permute a and b in (A.1) then substitute a for $a+1$ and use the fact that $P_n(s^2;b,a+1,c,d) = P_n(s^2;a+1,b,c,d)$, for P_n is symmetric with respect to its parameters, to obtain the result. ∎

Lemma A.2 ([27]). *The Askey–Wilson polynomials satisfy the contiguous relations*

$$P_n(x;a,b,c,d\,|\,q) = P_n(x;aq,b,c,d\,|\,q) - \frac{C_n(a,b,c,d;q)}{2}P_{n-1}(x;aq,b,c,d\,|\,q),$$
$$P_n(x;aq,b,c,d\,|\,q) = P_n(x;aq,bq,c,d\,|\,q) - \frac{C_n(b,aq,c,d)}{2}P_{n-1}(x;aq,bq,c,d\,|\,q),$$

where

$$C_n(a,b,c,d) = \frac{a\big(1-q^n\big)\big(1-bcq^{n-1}\big)\big(1-bdq^{n-1}\big)\big(1-dcq^{n-1}\big)}{\big(1-abcdq^{2n-2}\big)\big(1-abcdq^{2n-1}\big)}$$

is the coefficient C_n appearing in the three-term recurrence relation [18, equation (14.1.5)].

Proof. For the first relation, expand $P_n(x;a,b,c,d\,|\,q)$ in the basis $\{P_j(x;aq,b,c,d\,|\,q)\}$. Use the orthogonality relation [5, equation (2.3)] as well as the relation $w(x;aq,b,c,d\,|\,q) = \big(1-2ax+a^2\big)w(x;a,b,c,d\,|\,q)$ (cf. [5, p. 16]) to obtain

$$P_n(x;a,b,c,d\,|\,q) = P_n(x;aq,b,c,d\,|\,q) + t_{n-1}P_{n-1}(x;aq,b,c,d\,|\,q).$$

Apply $\mathcal{D}_q^{n-1}$ to both sides and take into account the relation

$$\mathcal{D}_q^k P_{n-1}(x;a,b,c,d\,|\,q) = \gamma_n\gamma_{n-1}\cdots\gamma_{n-k-1}P_{n-k}\left(x;a+\frac{k}{2},b+\frac{k}{2},c+\frac{k}{2},d+\frac{k}{2}\,|\,q\right), \quad \text{(A.5)}$$

deduced from [18, equation (14.1.9)]. Solve the equation obtained for the unknown t_{n-1} to get the result. For the second relation, permute a and b in the first one, substitute a by aq and use the fact that P_n is symmetric with respect to its parameters, that is $P_n(x;a,b,c,d\,|\,q) = P_n(x;b,a,c,d\,|\,q)$ (cf. [5, p. 15]), to obtain the result. ∎

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