An Infinite-Dimensional $\square_q$-Module Obtained from the q-Shuffle Algebra for Affine $\mathfrak{sl}_2$

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Abstract. Let $\mathbb{F}$ denote a field, and pick a nonzero $q \in \mathbb{F}$ that is not a root of unity. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. Define a unital associative $\mathbb{F}$-algebra $\square_q$ by generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

$$\frac{qx_ix_{i+1} - q^{-1}x_{i+1}x_i}{q - q^{-1}} = 1, \qquad x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0,$$

where $[3]_q = \left(q^3 - q^{-3}\right)/\left(q - q^{-1}\right)$. Let V denote a $\square_q$-module. A vector $\xi \in V$ is called NIL whenever $x_1 \xi = 0$ and $x_3 \xi = 0$ and $\xi \neq 0$. The $\square_q$-module V is called NIL whenever V is generated by a NIL vector. We show that up to isomorphism there exists a unique NIL $\square_q$-module, and it is irreducible and infinite-dimensional. We describe this module from sixteen points of view. In this description an important role is played by the q-shuffle algebra for affine $\mathfrak{sl}_2$.

Key words: quantum group; q-Serre relations; derivation; q-Onsager algebra

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1 Introduction

The algebra $\square_q$ was introduced in [29]. It is associative, infinite-dimensional, and noncommutative. It is defined by generators and relations. There are four generators, and it is natural to identify these with the edges of an oriented four-cycle. The relations are roughly described as follows. Each pair of adjacent edges satisfy a q-Weyl relation. Each pair of opposite edges satisfy the q-Serre relations associated with affine $\mathfrak{sl}_2$; these have degree 3 in one variable and degree 1 in the other variable. The cyclic group $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ acts on the oriented four-cycle as a group of rotational symmetries, and this induces a $\mathbb{Z}_4$-action on $\square_q$ as a group of automorphisms.

In the theory of quantum groups, there is an algebra U_q^+ called the positive part of $U_q(\widehat{\mathfrak{sl}_2})$. The algebra U_q^+ is defined by two generators, subject to the above q-Serre relations [20, Corollary 3.2.6]. The algebras $\square_q$ and U_q^+ are related as follows. In the algebra $\square_q$, each pair of opposite edges generate a subalgebra that is isomorphic to U_q^+ [29, Proposition 5.5]. This gives two subalgebras of $\square_q$ that are isomorphic to U_q^+; consider their tensor product. There is a map from this tensor product to $\square_q$, given by multiplication in $\square_q$. The map is an isomorphism of vector spaces [29, Proposition 5.5]. Thus the vector space $\square_q$ is isomorphic to $U_q^+ \otimes U_q^+$.

Next we discuss how $\square_q$ is related to the q-Onsager algebra $\mathcal{O}_q$. The algebra $\mathcal{O}_q$ originated in algebraic combinatorics [24, Lemma 5.4], [26] and statistical mechanics [1, Section 1], [2, Section 2]. Research on $\mathcal{O}_q$ is presently active in both areas; see [10, 15, 16, 17, 25, 27, 28, 29,

30, 31, 32] and [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12]. The algebra $\mathcal{O}_q$ is defined by two generators, subject to the q-Dolan/Grady relations [29, Definition 4.1]. The q-Dolan/Grady relations resemble the above q-Serre relations, but are slightly more complicated. The algebras $\square_q$ and $\mathcal{O}_q$ are related as follows. By [29, Proposition 5.6] there exists an injective algebra homomorphism $\mathcal{O}_q \to \square_q$ that sends one $\mathcal{O}_q$-generator to a linear combination of two adjacent $\square_q$-generators, and the other $\mathcal{O}_q$-generator to a linear combination of the remaining two $\square_q$-generators. The only constraint on the four coefficients is that the first two are reciprocals and the last two are reciprocals. We just explained how $\square_q$ and $\mathcal{O}_q$ are related. This relationship is our primary motivation for investigating $\square_q$.

The finite-dimensional $\square_q$-modules are investigated in [33, 34]. By [34, Proposition 5.2], each $\square_q$-generator is invertible on every nonzero finite-dimensional $\square_q$-module. This result gets used in [34, Sections 8 and 9] to obtain some remarkable q-exponential formulas. In [33], the finite-dimensional irreducible $\square_q$-modules are classified up to isomorphism. This classification is summarized as follows. There is a family of finite-dimensional irreducible $\square_q$-modules, said to have type 1 [33, Definition 6.8]. Any finite-dimensional irreducible $\square_q$-module can be normalized to have type 1, by twisting it via an appropriate automorphism of $\square_q$ [33, Note 6.9]. Let V denote a finite-dimensional irreducible $\square_q$-module of type 1. In [33, Definition 8.6] V gets attached to a polynomial P_V in one variable z that has constant coefficient 1; this P_V is called the Drinfel'd polynomial of V. By [33, Proposition 1.4], the map $V \mapsto P_V$ induces a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible $\square_q$-modules of type 1; (ii) the polynomials in the variable z that have constant coefficient 1 and do not vanish at $z = 1$.

In the present paper, our topic is a set of infinite-dimensional $\square_q$-modules, said to be NIL. A NIL $\square_q$-module is generated by a vector that is sent to zero by a pair of opposite $\square_q$-generators. We show that up to isomorphism there exists a unique NIL $\square_q$-module, and it is irreducible and infinite-dimensional. We then describe this module from 16 points of view. In this description an important role is played by the q-shuffle algebra for affine $\mathfrak{sl}_2$. This algebra was introduced by Rosso [21] and described further by Green [13].

We now summarize our results in more detail. Let $\mathbb{F}$ denote a field. All vector spaces discussed in this paper are over $\mathbb{F}$. All algebras discussed in this paper are associative, over $\mathbb{F}$, and have a multiplicative identity. Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Define the algebra $\square_q$ by generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

$$\frac{q x_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = 1, \tag{1.1}$$

$$x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0, \tag{1.2}$$

where $[3]_q = (q^3 - q^{-3})/(q - q^{-1})$. The relations (1.1) and (1.2) are called the q-Weyl and q-Serre relations, respectively. Let V denote a $\square_q$-module. A vector $\xi \in V$ is called NIL whenever $x_1 \xi = 0$ and $x_3 \xi = 0$ and $\xi \neq 0$. The $\square_q$-module V is called NIL whenever V is generated by a NIL vector.

In this paper we obtain the following results. Up to isomorphism, there exists a unique NIL $\square_q$-module, which we denote by $\mathbf{U}$. The $\square_q$-module $\mathbf{U}$ is irreducible, and isomorphic to U_q^+ as a vector space. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. The $\square_q$-module $\mathbf{U}$ has a unique sequence of subspaces $\{\mathbf{U}_n\}_{n \in \mathbb{N}}$ such that (i) $\mathbf{U}_0 \neq 0$; (ii) the sum $\mathbf{U} = \sum_{n \in \mathbb{N}} \mathbf{U}_n$ is direct; (iii) for $n \in \mathbb{N}$,

$$x_0 \mathbf{U}_n \subseteq \mathbf{U}_{n+1}, \qquad x_1 \mathbf{U}_n \subseteq \mathbf{U}_{n-1}, \qquad x_2 \mathbf{U}_n \subseteq \mathbf{U}_{n+1}, \qquad x_3 \mathbf{U}_n \subseteq \mathbf{U}_{n-1},$$

where $\mathbf{U}_{-1} = 0$. The sequence $\{\mathbf{U}_n\}_{n \in \mathbb{N}}$ is described as follows. The subspace $\mathbf{U}_0$ has dimension 1. The nonzero vectors in $\mathbf{U}_0$ are precisely the NIL vectors in $\mathbf{U}$, and each of these vectors

generates $\mathbf{U}$. Let ξ denote a NIL vector in $\mathbf{U}$. Then for $n \in \mathbb{N}$, the subspace $\mathbf{U}_n$ is spanned by the vectors $u_1 u_2 \cdots u_n \xi$ such that $u_i \in \{x_0, x_2\}$ for $1 \leq i \leq n$.

We will state some more results after a few comments. Let $\mathbb{V}$ denote the free algebra on two generators A, B. For $n \in \mathbb{N}$, a word of length n in $\mathbb{V}$ is a product $v_1 v_2 \cdots v_n$ such that $v_i \in \{A, B\}$ for $1 \leq i \leq n$. We interpret the word of length zero to be the multiplicative identity of $\mathbb{V}$; this word is called trivial and denoted by 1. The standard basis for $\mathbb{V}$ consists of the words. There exists a symmetric bilinear form $(\,,\,) : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$ with respect to which the standard basis is orthonormal. The algebra $\mathrm{End}(\mathbb{V})$ consists of the $\mathbb{F}$-linear maps $\mathbb{V} \to \mathbb{V}$ with the following property: the matrix that represents the map with respect to the standard basis for $\mathbb{V}$ has finitely many nonzero entries in each row. We define an invertible $K \in \mathrm{End}(\mathbb{V})$ as follows. The map K is the automorphism of the free algebra $\mathbb{V}$ that sends $A \mapsto q^2 A$ and $B \mapsto q^{-2}B$. For a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$K(v) = vq^{\langle v_1, A\rangle + \langle v_2, A\rangle + \cdots + \langle v_n, A\rangle}, \qquad K^{-1}(v) = vq^{\langle v_1, B\rangle + \langle v_2, B\rangle + \cdots + \langle v_n, B\rangle}$$

where

$\langle\,,\,\rangle$	A	B
A	2	-2
B	-2	2

We define four maps in $\mathrm{End}(\mathbb{V})$, denoted

$$A_L, \quad B_L, \quad A_R, \quad B_R. \tag{1.3}$$

For $v \in \mathbb{V}$,

$$A_L(v) = Av, \qquad B_L(v) = Bv, \qquad A_R(v) = vA, \qquad B_R(v) = vB.$$

We have been discussing the free algebra $\mathbb{V}$. There is another algebra structure on $\mathbb{V}$, called the q-shuffle algebra [13, 21, 22]. We will follow the approach of [13], which is well suited to our purpose. The q-shuffle product will be denoted by $\star$. In the main body of the paper we will describe this product in detail, and for now just make a few points. We have $1 \star v = v \star 1 = v$ for $v \in \mathbb{V}$. For $X \in \{A, B\}$ and a nontrivial word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$X \star v = \sum_{i=0}^{n} v_1 \cdots v_i X v_{i+1} \cdots v_n q^{\langle v_1, X\rangle + \langle v_2, X\rangle + \cdots + \langle v_i, X\rangle},$$

$$v \star X = \sum_{i=0}^{n} v_1 \cdots v_i X v_{i+1} \cdots v_n q^{\langle v_n, X\rangle + \langle v_{n-1}, X\rangle + \cdots + \langle v_{i+1}, X\rangle}.$$

It turns out that K is an automorphism of the q-shuffle algebra $\mathbb{V}$. We define four maps in $\mathrm{End}(\mathbb{V})$, denoted

$$A_\ell, \quad B_\ell, \quad A_r, \quad B_r. \tag{1.4}$$

For $v \in \mathbb{V}$,

$$A_\ell(v) = A \star v, \qquad B_\ell(v) = B \star v, \qquad A_r(v) = v \star A, \qquad B_r(v) = v \star B.$$

We recall the concept of an adjoint. For $X \in \mathrm{End}(\mathbb{V})$ there exists a unique $X^* \in \mathrm{End}(\mathbb{V})$ such that $(Xu, v) = (u, X^*v)$ for all $u, v \in \mathbb{V}$. The element X^* is called the adjoint of X with respect to $(\,,\,)$. For example $K^* = K$. We will consider

$$A_L^*, \quad B_L^*, \quad A_R^*, \quad B_R^* \tag{1.5}$$

and

$$A_\ell^*, \quad B_\ell^*, \quad A_r^*, \quad B_r^*. \tag{1.6}$$

We acknowledge that the maps (1.3), (1.4) and (1.5), (1.6) are well known in the literature on quantum groups and q-shuffle algebras. For instance, in [18, Section 3.4] the maps

$$e_0' = K^{-1}A_r^*, \qquad e_1' = KB_r^*, \qquad e_0'' = A_\ell^*, \qquad e_1'' = B_\ell^*$$

give the Kashiwara operators for the negative part of $U_q(\widehat{\mathfrak{sl}}_2)$. In [18, Lemma 3.4.2] the above maps e_0', e_1' and $f_0 = A_L$, $f_1 = B_L$ are used to obtain a module for the reduced q-analog $\mathcal{B}_q(\widehat{\mathfrak{sl}}_2)$. The maps A_r^*, B_r^* are discussed in [13, Definition 4.2], where they are called Δ_0, Δ_1. The map A_R^* (resp. B_R^*) is called ∂_0 (resp. ∂_1) in [13, Section 3.1] and $\mathbf{e_0'}$ (resp. $\mathbf{e_1'}$) in [19, p. 696]. In the present paper, we will put the well known maps (1.3), (1.4) and (1.5), (1.6) to a new use.

Let J denote the 2-sided ideal of the free algebra $\mathbb{V}$ generated by

$$J^+ = A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3,$$
$$J^- = B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3.$$

As we will see, the ideal J is invariant under $K^{\pm 1}$ and (1.3), (1.6). The quotient algebra $\mathbb{V}/J$ is often denoted by U_q^+ and called the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$; see for example [14, p. 40] or [20, Corollary 3.2.6]. Let U denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by A, B. As we will see, the algebra U is invariant under $K^{\pm 1}$ and (1.4), (1.5). It is well known that the algebra U is isomorphic to U_q^+; see [22, Theorem 15] or [19, p. 696].

We are now ready to state some more results. For notational convenience define $Q = 1 - q^2$.

Theorem 1.1. *For each row in the tables below, the vector space $\mathbb{V}/J$ becomes a $\square_q$-module on which the generators $\{x_i\}_{i\in\mathbb{Z}_4}$ act as indicated.*

module label	x_0	x_1	x_2	x_3
I	A_L	$Q(A_\ell^* - B_r^*K)$	B_L	$Q(B_\ell^* - A_r^*K^{-1})$
IS	A_R	$Q(A_r^* - B_\ell^*K)$	B_R	$Q(B_r^* - A_\ell^*K^{-1})$
IT	B_L	$Q(B_\ell^* - A_r^*K^{-1})$	A_L	$Q(A_\ell^* - B_r^*K)$
IST	B_R	$Q(B_r^* - A_\ell^*K^{-1})$	A_R	$Q(A_r^* - B_\ell^*K)$

module label	x_0	x_1	x_2	x_3
II	$Q(A_L - KB_R)$	A_ℓ^*	$Q(B_L - K^{-1}A_R)$	B_ℓ^*
IIS	$Q(A_R - KB_L)$	A_r^*	$Q(B_R - K^{-1}A_L)$	B_r^*
IIT	$Q(B_L - K^{-1}A_R)$	B_ℓ^*	$Q(A_L - KB_R)$	A_ℓ^*
IIST	$Q(B_R - K^{-1}A_L)$	B_r^*	$Q(A_R - KB_L)$	A_r^*

Each $\square_q$-module in the tables is isomorphic to $\mathbf{U}$.

Theorem 1.2. *For each row in the tables below, the vector space U becomes a $\square_q$-module on which the generators $\{x_i\}_{i\in\mathbb{Z}_4}$ act as indicated.*

module label	x_0	x_1	x_2	x_3
III	A_ℓ	$Q(A_L^* - B_R^*K)$	B_ℓ	$Q(B_L^* - A_R^*K^{-1})$
IIIS	A_r	$Q(A_R^* - B_L^*K)$	B_r	$Q(B_R^* - A_L^*K^{-1})$
IIIT	B_ℓ	$Q(B_L^* - A_R^*K^{-1})$	A_ℓ	$Q(A_L^* - B_R^*K)$
IIIST	B_r	$Q(B_R^* - A_L^*K^{-1})$	A_r	$Q(A_R^* - B_L^*K)$

module label	x_0	x_1	x_2	x_3
IV	$Q(A_\ell - KB_r)$	A_L^*	$Q(B_\ell - K^{-1}A_r)$	B_L^*
IVS	$Q(A_r - KB_\ell)$	A_R^*	$Q(B_r - K^{-1}A_\ell)$	B_R^*
IVT	$Q(B_\ell - K^{-1}A_r)$	B_L^*	$Q(A_\ell - KB_r)$	A_L^*
IVST	$Q(B_r - K^{-1}A_\ell)$	B_R^*	$Q(A_r - KB_\ell)$	A_R^*

Each $\square_q$-module in the tables is isomorphic to $\mathbf{U}$.

We will state some additional results shortly.

We recall the concept of a derivation. Let $\mathcal{A}$ denote an algebra, and let φ, ϕ denote automorphisms of $\mathcal{A}$. By a (φ, ϕ)-derivation of $\mathcal{A}$ we mean an $\mathbb{F}$-linear map $\delta \colon \mathcal{A} \to \mathcal{A}$ such that $\delta(uv) = \varphi(u)\delta(v) + \delta(u)\phi(v)$ for all $u, v \in \mathcal{A}$.

The derivation concept is well known in the theory of quantum groups and q-shuffle algebras. For instance, by [13, Theorem 3.2] the maps A_R^* and B_R^* act on the q-shuffle algebra $\mathbb{V}$ as a (I, K)-derivation and (I, K^{-1})-derivation, respectively. By [13, Section 4.2] the maps A_r^* and B_r^* act on the free algebra $\mathbb{V}$ as a (I, K)-derivation and (I, K^{-1})-derivation, respectively. Concerning $\square_q$ we have the following results.

Theorem 1.3. *For each $\square_q$-module in Theorem 1.1, the elements x_1 and x_3 act on the algebra $\mathbb{V}/J$ as a derivation of the following sort:*

module label	x_1	x_3
I, II	(K, I)-derivation	$\left(K^{-1}, I\right)$-derivation
IS, IIS	(I, K)-derivation	$\left(I, K^{-1}\right)$-derivation
IT, IIT	$\left(K^{-1}, I\right)$-derivation	(K, I)-derivation
IST, IIST	$\left(I, K^{-1}\right)$-derivation	(I, K)-derivation

Theorem 1.4. *For each $\square_q$-module in Theorem 1.2, the elements x_1 and x_3 act on the algebra U as a derivation of the following sort:*

module label	x_1	x_3
III, IV	(K, I)-derivation	$\left(K^{-1}, I\right)$-derivation
IIIS, IVS	(I, K)-derivation	$\left(I, K^{-1}\right)$-derivation
IIIT, IVT	$\left(K^{-1}, I\right)$-derivation	(K, I)-derivation
IIIST, IVST	$\left(I, K^{-1}\right)$-derivation	(I, K)-derivation

This paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we recall the free algebra $\mathbb{V}$ on two generators. In Section 4, we endow $\mathbb{V}$ with a bilinear form and discuss the corresponding adjoint map. In Section 5, we describe two automorphisms and one antiautomorphism of $\mathbb{V}$ that will play a role in our main results. In Section 6, we recall the q-shuffle product on $\mathbb{V}$, and embed U_q^+ into the q-shuffle algebra $\mathbb{V}$. In Sections 7–9, we give a detailed description of how the free algebra $\mathbb{V}$ is related to the q-shuffle algebra $\mathbb{V}$. In Section 10 we introduce an algebra $\square_q^\vee$ that is a homomorphic preimage of $\square_q$. In Section 11 we give sixteen $\square_q^\vee$-module structures on $\mathbb{V}$. In Section 12 we describe many homomorphisms between our sixteen $\square_q^\vee$-modules. In Section 13 we use our sixteen $\square_q^\vee$-modules to obtain sixteen $\square_q$-modules. In Section 14 we show that these sixteen $\square_q$-modules are mutually isomorphic and irreducible. In Section 15 we characterize these $\square_q$-modules using the notion of a NIL $\square_q$-module. Appendix A contains some data on the q-shuffle product. Appendix B gives some matrix representations of the maps used in our main results.

2   Preliminaries

We now begin our formal argument. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{F}$ denote a field. We will be discussing vector spaces, algebras, and tensor products. Every vector space discussed is over $\mathbb{F}$. Every algebra discussed is over $\mathbb{F}$, associative, and has a multiplicative identity. Every tensor product discussed is over $\mathbb{F}$. Let $\mathcal{A}$ denote an algebra. By an *automorphism* (resp. *antiautomorphism*) of $\mathcal{A}$ we mean an $\mathbb{F}$-linear bijection $\gamma \colon \mathcal{A} \to \mathcal{A}$ such that $\gamma(uv) = \gamma(u)\gamma(v)$ (resp. $\gamma(uv) = \gamma(v)\gamma(u)$) for all $u, v \in \mathcal{A}$. Let φ, ϕ denote automorphisms of $\mathcal{A}$. By a (φ, ϕ)-*derivation* of $\mathcal{A}$ we mean an $\mathbb{F}$-linear map $\delta \colon \mathcal{A} \to \mathcal{A}$ such that $\delta(uv) = \varphi(u)\delta(v) + \delta(u)\phi(v)$ for all $u, v \in \mathcal{A}$. The set of all (φ, ϕ)-derivations of $\mathcal{A}$ is closed under addition and scalar multiplication, and is therefore a vector

space over $\mathbb{F}$. Let δ denote a (φ, ϕ)-derivation of $\mathcal{A}$. Then $\delta(1) = 0$. For an automorphism σ of $\mathcal{A}$, the composition $\delta\sigma$ is a $(\varphi\sigma, \phi\sigma)$-derivation of $\mathcal{A}$, and $\sigma\delta$ is a $(\sigma\varphi, \sigma\phi)$-derivation of $\mathcal{A}$. Let J denote a 2-sided ideal of $\mathcal{A}$ with $J \neq \mathcal{A}$, and consider the quotient algebra $\overline{\mathcal{A}} = \mathcal{A}/J$. Assume that $\varphi(J) = J$ and $\phi(J) = J$. Then there exist automorphisms $\overline{\varphi}$ and $\overline{\phi}$ of $\overline{\mathcal{A}}$ such that $\overline{\varphi}(a + J) = \varphi(a) + J$ and $\overline{\phi}(a + J) = \phi(a) + J$ for all $a \in \mathcal{A}$. Assume further that $\delta(J) \subseteq J$. Then there exists a $(\overline{\varphi}, \overline{\phi})$-derivation $\overline{\delta}$ of $\overline{\mathcal{A}}$ such that $\overline{\delta}(a + J) = \delta(a) + J$ for all $a \in \mathcal{A}$. In the main body of the paper we will suppress the overline notation.

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n \in \mathbb{Z}.$$

3 The free algebra $\mathbb{V}$ with generators A, B

Let A, B denote noncommuting indeterminates, and let $\mathbb{V}$ denote the free algebra with generators A, B. For $n \in \mathbb{N}$, a *word of length n* in $\mathbb{V}$ is a product $v_1 v_2 \cdots v_n$ such that $v_i \in \{A, B\}$ for $1 \leq i \leq n$. We interpret the word of length zero to be the multiplicative identity in $\mathbb{V}$; this word is called *trivial* and denoted by 1. The vector space $\mathbb{V}$ has a basis consisting of its words; this basis is called *standard*.

Definition 3.1. For $n \in \mathbb{N}$, let $\mathbb{V}_n$ denote the subspace of $\mathbb{V}$ spanned by the words of length n. For notational convenience define $\mathbb{V}_{-1} = 0$.

Referring to Definition 3.1, the dimension of $\mathbb{V}_n$ is 2^n. We have

$$\mathbb{V} = \sum_{n \in \mathbb{N}} \mathbb{V}_n \qquad \text{(direct sum)}. \tag{3.1}$$

The vector 1 is a basis for $\mathbb{V}_0$. For $r, s \in \mathbb{N}$ we have $\mathbb{V}_r \mathbb{V}_s \subseteq \mathbb{V}_{r+s}$. By these comments the sum (3.1) is a grading of $\mathbb{V}$ in the sense of [23, p. 704].

Let $\text{End}(\mathbb{V})$ denote the algebra consisting of the $\mathbb{F}$-linear maps $\mathbb{V} \to \mathbb{V}$ with the following property: the matrix that represents the map with respect to the standard basis for $\mathbb{V}$ has finitely many nonzero entries in each row.

Definition 3.2. We define four maps in $\text{End}(\mathbb{V})$, denoted

$$A_L, \quad B_L, \quad A_R, \quad B_R. \tag{3.2}$$

For $v \in \mathbb{V}$,

$$A_L(v) = Av, \qquad B_L(v) = Bv, \qquad A_R(v) = vA, \qquad B_R(v) = vB.$$

Lemma 3.3. *For $n \in \mathbb{N}$,*

$$A_L \mathbb{V}_n \subseteq \mathbb{V}_{n+1}, \qquad B_L \mathbb{V}_n \subseteq \mathbb{V}_{n+1}, \qquad A_R \mathbb{V}_n \subseteq \mathbb{V}_{n+1}, \qquad B_R \mathbb{V}_n \subseteq \mathbb{V}_{n+1}.$$

The following lemma is about A_L and B_L; a similar result holds for A_R and B_R.

Lemma 3.4. *Let W denote a subspace of $\mathbb{V}$ that is closed under A_L and B_L. Assume that $1 \in W$. Then $W = \mathbb{V}$.*

Proof. The free algebra $\mathbb{V}$ is generated by A, B. $\blacksquare$

Definition 3.5. Let J denote the 2-sided ideal of the free algebra $\mathbb{V}$ generated by

$$J^+ = A^3 B - [3]_q A^2 B A + [3]_q A B A^2 - B A^3, \tag{3.3}$$
$$J^- = B^3 A - [3]_q B^2 A B + [3]_q B A B^2 - A B^3. \tag{3.4}$$

Definition 3.6. The quotient algebra $\mathbb{V}/J$ is often denoted by U_q^+ and called the *positive part of $U_q(\widehat{\mathfrak{sl}}_2)$*; see for example [14, p. 40] or [20, Corollary 3.2.6].

4   A bilinear form on $\mathbb{V}$

We continue to discuss the free algebra $\mathbb{V}$ with generators A, B. In this section we endow $\mathbb{V}$ with a symmetric nondegenerate bilinear form. We describe the corresponding adjoints of the four maps listed in (3.2).

Definition 4.1. Define a bilinear form $(\, , \,) \colon \mathbb{V} \times \mathbb{V} \to \mathbb{F}$ as follows. Recall that the standard basis for $\mathbb{V}$ consists of the words in A, B. This basis is orthonormal with respect to $(\, , \,)$.

The bilinear form $(\, , \,)$ is symmetric and nondegenerate. The summands in (3.1) are mutually orthogonal with respect to $(\, , \,)$.

We now recall the adjoint map. By linear algebra, for $X \in \mathrm{End}(\mathbb{V})$ there exists a unique $X^* \in \mathrm{End}(\mathbb{V})$ such that $(Xu, v) = (u, X^*v)$ for all $u, v \in \mathbb{V}$. With respect to the standard basis for $\mathbb{V}$, the matrices representing X and X^* are transposes. The element X^* is called the *adjoint* of X with respect to $(\, , \,)$. The adjoint map $\mathrm{End}(\mathbb{V}) \to \mathrm{End}(\mathbb{V})$, $X \mapsto X^*$ is an antiautomorphism of $\mathrm{End}(\mathbb{V})$. We now consider

$$A_L^*, \quad B_L^*, \quad A_R^*, \quad B_R^*. \tag{4.1}$$

Lemma 4.2. *We have*

$$A_L^*(1) = 0, \qquad B_L^*(1) = 0, \qquad A_R^*(1) = 0, \qquad B_R^*(1) = 0. \tag{4.2}$$

Moreover for $v \in \mathbb{V}$,

$$
\begin{aligned}
A_L^*(Av) &= v, & A_L^*(Bv) &= 0, & B_L^*(Av) &= 0, & B_L^*(Bv) &= v, \\
A_R^*(vA) &= v, & A_R^*(vB) &= 0, & B_R^*(vA) &= 0, & B_R^*(vB) &= v.
\end{aligned}
$$

Proof. This follows from Definition 4.1 and the meaning of the adjoint. We illustrate with a detailed proof of $A_L^*(Av) = v$. For $u \in \mathbb{V}$,

$$(u, A_L^*(Av)) = (A_L(u), Av) = (Au, Av) = (u, v).$$

Therefore $A_L^*(Av) = v$ since $(\, , \,)$ is nondegenerate.   ∎

We now describe how the maps (4.1) act on the standard basis for $\mathbb{V}$. In view of (4.2), we focus on the nontrivial basis elements. Recall that the Kronecker delta $\delta_{r,s}$ is equal to 1 if $r = s$, and 0 if $r \neq s$.

Lemma 4.3. *For an integer $n \geq 1$ and a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,*

$$
\begin{aligned}
A_L^*(v) &= v_2 \cdots v_n \delta_{v_1,A}, & B_L^*(v) &= v_2 \cdots v_n \delta_{v_1,B}, \\
A_R^*(v) &= v_1 \cdots v_{n-1} \delta_{v_n,A}, & B_R^*(v) &= v_1 \cdots v_{n-1} \delta_{v_n,B}.
\end{aligned}
$$

Proof. Use Lemma 4.2.   ∎

Lemma 4.4. *For $n \in \mathbb{N}$,*

$$A_L^* \mathbb{V}_n \subseteq \mathbb{V}_{n-1}, \qquad B_L^* \mathbb{V}_n \subseteq \mathbb{V}_{n-1}, \qquad A_R^* \mathbb{V}_n \subseteq \mathbb{V}_{n-1}, \qquad B_R^* \mathbb{V}_n \subseteq \mathbb{V}_{n-1}.$$

Proof. Use Lemma 4.3.   ∎

The next two lemmas are about A_L^* and B_L^*; similar results hold for A_R^* and B_R^*.

Lemma 4.5. *For $v \in \mathbb{V}$ the following are equivalent:*

(i) $v \in \mathbb{V}_0$;

(ii) $A_L^* v = 0$ and $B_L^* v = 0$.

Proof. (i) $\Rightarrow$ (ii): By (4.2).

(ii) $\Rightarrow$ (i): Consider the orthogonal direct sum $\mathbb{V} = \mathbb{V}_0 + A\mathbb{V} + B\mathbb{V}$. We have $0 = (A_L^* v, \mathbb{V}) = (v, A\mathbb{V})$ and $0 = (B_L^* v, \mathbb{V}) = (v, B\mathbb{V})$. The vector v is orthogonal to both $A\mathbb{V}$ and $B\mathbb{V}$, and is therefore contained in $\mathbb{V}_0$. ∎

Lemma 4.6. *Let W denote a nonzero subspace of $\mathbb{V}$ that is closed under A_L^* and B_L^*. Then* $1 \in W$.

Proof. There exists $0 \neq w \in W$. Write $w = \sum_{i=0}^n w_i$ with $w_i \in \mathbb{V}_i$ for $0 \leq i \leq n$ and $w_n \neq 0$. Call n the degree of w. Choose w such that this degree is minimal. By assumption $A_L^* w \in W$. By Lemma 4.4 the vector $A_L^* w$ is either 0 or has degree less than n. So $A_L^* w = 0$ by the minimality of n. Similarly $B_L^* w = 0$. Now $w \in \mathbb{V}_0$ by Lemma 4.5. The result follows. ∎

5   Some automorphisms and antiautomorphisms

We continue to discuss the free algebra $\mathbb{V}$ with generators A, B. In this section we introduce the automorphisms K, T of $\mathbb{V}$ and the antiautomorphism S of $\mathbb{V}$.

Definition 5.1. Let K denote the automorphism of the free algebra $\mathbb{V}$ that sends $A \mapsto q^2 A$ and $B \mapsto q^{-2} B$.

The automorphism K is described as follows. For a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$K(v) = v q^{\langle v_1, A \rangle + \langle v_2, A \rangle + \cdots + \langle v_n, A \rangle}, \qquad K^{-1}(v) = v q^{\langle v_1, B \rangle + \langle v_2, B \rangle + \cdots + \langle v_n, B \rangle},$$

where

$\langle\, ,\, \rangle$	A	B
A	2	-2
B	-2	2

We have $K\mathbb{V}_n = \mathbb{V}_n$ for $n \in \mathbb{N}$, and also $K^* = K$.

Definition 5.2. Define $S \in \operatorname{End}(\mathbb{V})$ such that for each word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$S(v) = v_n \cdots v_2 v_1.$$

The map S is described as follows. It is the unique antiautomorphism of the free algebra $\mathbb{V}$ that fixes A and B. We have $S\mathbb{V}_n = \mathbb{V}_n$ for $n \in \mathbb{N}$. We have $(S(u), S(v)) = (u, v)$ for $u, v \in \mathbb{V}$. By this and $S^2 = I$ we obtain $(S(u), v) = (u, S(v))$ for all $u, v \in \mathbb{V}$. Therefore $S^* = S$.

Lemma 5.3. *We have $KS = SK$ and*

$$A_L S = S A_R, \qquad A_R S = S A_L, \qquad B_L S = S B_R, \qquad B_R S = S B_L, \tag{5.1}$$
$$A_L^* S = S A_R^*, \qquad A_R^* S = S A_L^*, \qquad B_L^* S = S B_R^*, \qquad B_R^* S = S B_L^*. \tag{5.2}$$

Proof. The first five equations are readily verified by applying both sides to a word in $\mathbb{V}$. We illustrate with a detailed proof of $A_L S = S A_R$. For a word v in $\mathbb{V}$,

$$A_L S(v) = A S(v) = S(A) S(v) = S(vA) = S A_R(v).$$

Therefore $A_L S = S A_R$. To obtain the equations (5.2), apply the adjoint map to each equation in (5.1). ∎

Definition 5.4. Let T denote the automorphism of the free algebra $\mathbb{V}$ that swaps A and B.

The map T is described as follows. We have $T\mathbb{V}_n = \mathbb{V}_n$ for $n \in \mathbb{N}$. We have $(T(u), T(v)) = (u, v)$ for $u, v \in \mathbb{V}$. By this and $T^2 = I$ we obtain $(T(u), v) = (u, T(v))$ for $u, v \in \mathbb{V}$. Therefore $T^* = T$. We have $ST = TS$.

Lemma 5.5. *We have $KT = TK^{-1}$ and*

$$A_L T = T B_L, \qquad A_R T = T B_R, \qquad B_L T = T A_L, \qquad B_R T = T A_R, \tag{5.3}$$
$$A_L^* T = T B_L^*, \qquad A_R^* T = T B_R^*, \qquad B_L^* T = T A_L^*, \qquad B_R^* T = T A_R^*. \tag{5.4}$$

Proof. The first five equations are readily verified by applying both sides to a word in $\mathbb{V}$. We illustrate with a detailed proof of $A_L T = T B_L$. For a word v in $\mathbb{V}$,

$$A_L T(v) = A T(v) = T(B) T(v) = T(Bv) = T B_L(v).$$

Therefore $A_L T = T B_L$. To obtain the equations (5.4), apply the adjoint map to each equation in (5.3). $\blacksquare$

Recall $J^\pm$, J from Definition 3.5.

Lemma 5.6. *We have*

$$K(J^\pm) = q^{\pm 4} J^\pm, \qquad S(J^\pm) = -J^\pm, \qquad T(J^\pm) = J^\mp. \tag{5.5}$$

Moreover

$$K(J) = J, \qquad S(J) = J, \qquad T(J) = J. \tag{5.6}$$

Proof. The relations (5.5) are routinely checked using (3.3) and (3.4). Line (5.6) follows from (5.5). $\blacksquare$

6 The q-shuffle algebra $\mathbb{V}$ and the map θ

We have been discussing the free algebra $\mathbb{V}$. There is another algebra structure on $\mathbb{V}$, called the q-shuffle algebra [13, 21, 22]. For this algebra the product is denoted by $\star$. To describe this product, we start with some special cases. We have $1 \star v = v \star 1 = v$ for $v \in \mathbb{V}$. For $X \in \{A, B\}$ and a nontrivial word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$X \star v = \sum_{i=0}^{n} v_1 \cdots v_i X v_{i+1} \cdots v_n q^{\langle v_1, X \rangle + \langle v_2, X \rangle + \cdots + \langle v_i, X \rangle}, \tag{6.1}$$

$$v \star X = \sum_{i=0}^{n} v_1 \cdots v_i X v_{i+1} \cdots v_n q^{\langle v_n, X \rangle + \langle v_{n-1}, X \rangle + \cdots + \langle v_{i+1}, X \rangle}. \tag{6.2}$$

For nontrivial words $u = u_1 u_2 \cdots u_r$ and $v = v_1 v_2 \cdots v_s$ in $\mathbb{V}$,

$$u \star v = u_1 \big((u_2 \cdots u_r) \star v \big) + v_1 \big(u \star (v_2 \cdots v_s) \big) q^{\langle u_1, v_1 \rangle + \langle u_2, v_1 \rangle + \cdots + \langle u_r, v_1 \rangle}, \tag{6.3}$$

$$u \star v = \big(u \star (v_1 \cdots v_{s-1}) \big) v_s + \big((u_1 \cdots u_{r-1}) \star v \big) u_r q^{\langle u_r, v_1 \rangle + \langle u_r, v_2 \rangle + \cdots + \langle u_r, v_s \rangle}. \tag{6.4}$$

For $r, s \in \mathbb{N}$ we have $\mathbb{V}_r \star \mathbb{V}_s \subseteq \mathbb{V}_{r+s}$. Therefore the sum (3.1) is a grading for the q-shuffle algebra $\mathbb{V}$. The following examples illustrate the shuffle product.

Example 6.1. We have

$$A \star A = (1 + q^2)AA, \qquad A \star B = AB + q^{-2}BA,$$
$$B \star A = BA + q^{-2}AB, \qquad B \star B = (1 + q^2)BB.$$

Appendix A contains additional examples. Using these examples (or by [13, p. 10]) one obtains

$$A \star A \star A \star B - [3]_q A \star A \star B \star A + [3]_q A \star B \star A \star A - B \star A \star A \star A = 0, \qquad (6.5)$$
$$B \star B \star B \star A - [3]_q B \star B \star A \star B + [3]_q B \star A \star B \star B - A \star B \star B \star B = 0. \qquad (6.6)$$

For the q-shuffle algebra $\mathbb{V}$ let U denote the subalgebra generated by A, B. The algebra U is described as follows. There exists an algebra homomorphism θ from the free algebra $\mathbb{V}$ to the q-shuffle algebra $\mathbb{V}$, that sends $A \mapsto A$ and $B \mapsto B$. By construction $\theta(\mathbb{V}) = U$. Comparing (3.3), (3.4) with (6.5), (6.6) we obtain $\theta(J^{\pm}) = 0$. Recall that $J^{\pm}$ generate the 2-sided ideal J of the free algebra $\mathbb{V}$. Consequently $\theta(J) = 0$, so the kernel $\ker(\theta)$ contains J. By [22, Theorem 15] we have $\ker(\theta) = J$, so θ induces an algebra isomorphism $\mathbb{V}/J \to U$. By this and $\mathbb{V}/J = U_q^+$, we get an algebra isomorphism $U_q^+ \to U$. This isomorphism is discussed around [22, Theorem 15] and also [19, p. 696].

Our next goal is to describe how θ acts on the standard basis for $\mathbb{V}$. By construction $\theta(1) = 1$. Pick an integer $n \geq 1$. We view the symmetric group S_n as the group of permutations of the set $\{1, 2, \ldots, n\}$. For $\sigma \in S_n$, by an *inversion for σ* we mean an ordered pair (i, j) of integers such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$. Let $\mathrm{Inv}(\sigma)$ denote the set of inversions for σ.

Lemma 6.2. *For an integer $n \geq 1$ and a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,*

$$\theta(v) = \sum_{\sigma \in S_n} v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(n)} \prod_{(i,j) \in \mathrm{Inv}(\sigma)} q^{\langle v_{\sigma(i)}, v_{\sigma(j)} \rangle}.$$

Proof. By induction on n, using (6.1) or (6.2). ∎

By Lemma 6.2 we have $\theta \mathbb{V}_n \subseteq \mathbb{V}_n$ for $n \in \mathbb{N}$.

Lemma 6.3. *For words $u = u_1 u_2 \cdots u_r$ and $v = v_1 v_2 \cdots v_s$ in $\mathbb{V}$,*

$$(\theta(u), v) = (u, \theta(v)). \qquad (6.7)$$

For $r \neq s$ the common value (6.7) is 0. For $r = s = 0$ the common value (6.7) is 1. For $r = s \geq 1$ the common value (6.7) is equal to

$$\sum_{\sigma} \prod_{(i,j) \in \mathrm{Inv}(\sigma)} q^{\langle v_i, v_j \rangle},$$

where the sum is over all $\sigma \in S_r$ such that $v_k = u_{\sigma(k)}$ for $1 \leq k \leq r$.

Proof. By Lemma 6.2 and since the standard basis for $\mathbb{V}$ is orthonormal with respect to (,). ∎

The following result is a reformulation of [13, Theorem 4.5].

Corollary 6.4. *We have $\theta^* = \theta$.*

Proof. By Lemma 6.3 we have $(\theta(u), v) = (u, \theta(v))$ for $u, v \in \mathbb{V}$. ∎

Let the set $U^{\perp}$ consist of the vectors in $\mathbb{V}$ that are orthogonal to U with respect to (,).

Lemma 6.5. *We have $J = U^{\perp}$.*

Proof. Use $J = \ker(\theta)$ and Corollary 6.4, along with the fact that $(\,,\,)$ is nondegenerate. Here are the details. For $v \in \mathbb{V}$,

$$v \in J \;\Leftrightarrow\; \theta(v) = 0 \;\Leftrightarrow\; (\theta(v), \mathbb{V}) = 0 \;\Leftrightarrow\; (v, \theta(\mathbb{V})) = 0 \;\Leftrightarrow\; (v, U) = 0 \;\Leftrightarrow\; v \in U^\perp. \qquad \blacksquare$$

Note that $\theta(U) \subseteq U$ since $U \subseteq \mathbb{V}$ and $\theta(\mathbb{V}) = U$.

Lemma 6.6. *For the q-shuffle algebra $\mathbb{V}$, the maps K and T are automorphisms and S is an antiautomorphism.*

Proof. Use the definition of the q-shuffle product. $\qquad \blacksquare$

Lemma 6.7. *We have*

$$K\theta = \theta K, \qquad S\theta = \theta S, \qquad T\theta = \theta T.$$

Proof. Apply each side to a word in $\mathbb{V}$, and evaluate the result using Lemma 6.6. $\qquad \blacksquare$

Lemma 6.8. *We have*

$$K(U) = U, \qquad S(U) = U, \qquad T(U) = U.$$

Proof. By Lemma 6.7 and since each of K, S, T is invertible. We give the details for the equation involving K. We have

$$K(U) = K\theta(\mathbb{V}) = \theta K(\mathbb{V}) = \theta(\mathbb{V}) = U. \qquad \blacksquare$$

7   The maps A_ℓ, B_ℓ, A_r, B_r

In Definition 3.2 we used the free algebra $\mathbb{V}$ to obtain the four maps listed in (3.2). In this section we use the q-shuffle algebra $\mathbb{V}$ to obtain four analogous maps. We investigate how these maps and their adjoints are related to S, T, θ.

Definition 7.1. We define four maps in $\operatorname{End}(\mathbb{V})$, denoted

$$A_\ell, \quad B_\ell, \quad A_r, \quad B_r. \tag{7.1}$$

For $v \in \mathbb{V}$,

$$A_\ell(v) = A \star v, \qquad B_\ell(v) = B \star v, \qquad A_r(v) = v \star A, \qquad B_r(v) = v \star B.$$

Lemma 7.2. *For $n \in \mathbb{N}$,*

$$A_\ell \mathbb{V}_n \subseteq \mathbb{V}_{n+1}, \qquad B_\ell \mathbb{V}_n \subseteq \mathbb{V}_{n+1}, \qquad A_r \mathbb{V}_n \subseteq \mathbb{V}_{n+1}, \qquad B_r \mathbb{V}_n \subseteq \mathbb{V}_{n+1}.$$

Proof. By (6.1), (6.2) and Definition 7.1. $\qquad \blacksquare$

The following lemma is about A_ℓ and B_ℓ; a similar result holds for A_r and B_r.

Lemma 7.3. *Let W denote a subspace of U that is closed under A_ℓ, B_ℓ. Assume that $1 \in W$. Then $W = U$.*

Proof. By Definition 7.1, and since U is the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by A, B. $\qquad \blacksquare$

We now consider

$$A_\ell^*, \quad B_\ell^*, \quad A_r^*, \quad B_r^*. \tag{7.2}$$

Lemma 7.4. *For a word* $v = v_1 v_2 \cdots v_n$ *in* $\mathbb{V}$,

$$A_\ell^*(v) = \sum_{i=0}^{n} v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \delta_{v_i,A} q^{\langle v_1,A \rangle + \langle v_2,A \rangle + \cdots + \langle v_{i-1},A \rangle},$$

$$B_\ell^*(v) = \sum_{i=0}^{n} v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \delta_{v_i,B} q^{\langle v_1,B \rangle + \langle v_2,B \rangle + \cdots + \langle v_{i-1},B \rangle},$$

$$A_r^*(v) = \sum_{i=0}^{n} v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \delta_{v_i,A} q^{\langle v_n,A \rangle + \langle v_{n-1},A \rangle + \cdots + \langle v_{i+1},A \rangle},$$

$$B_r^*(v) = \sum_{i=0}^{n} v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \delta_{v_i,B} q^{\langle v_n,B \rangle + \langle v_{n-1},B \rangle + \cdots + \langle v_{i+1},B \rangle}.$$

Proof. For each map X in (7.1), use (6.1), (6.2) to compute the matrix representing X with respect to the standard basis for $\mathbb{V}$. The transpose of this matrix represents X^* with respect to the standard basis for $\mathbb{V}$. The result follows. ∎

Lemma 7.5. *For* $n \in \mathbb{N}$,

$$A_\ell^* \mathbb{V}_n \subseteq \mathbb{V}_{n-1}, \qquad B_\ell^* \mathbb{V}_n \subseteq \mathbb{V}_{n-1}, \qquad A_r^* \mathbb{V}_n \subseteq \mathbb{V}_{n-1}, \qquad B_r^* \mathbb{V}_n \subseteq \mathbb{V}_{n-1}.$$

Proof. Use Lemma 7.4. ∎

We now consider how the maps (7.1), (7.2) are related to S, T, θ.

Lemma 7.6. *We have*

$$SA_\ell = A_r S, \qquad SA_r = A_\ell S, \qquad SB_\ell = B_r S, \qquad SB_r = B_\ell S, \tag{7.3}$$

$$SA_\ell^* = A_r^* S, \qquad SA_r^* = A_\ell^* S, \qquad SB_\ell^* = B_r^* S, \qquad SB_r^* = B_\ell^* S. \tag{7.4}$$

Proof. The equations (7.3) follow from Lemma 6.6. We illustrate with a detailed proof of $SA_\ell = A_r S$. For $v \in \mathbb{V}$,

$$SA_\ell(v) = S(A \star v) = S(v) \star S(A) = S(v) \star A = A_r S(v).$$

Therefore $SA_\ell = A_r S$. To obtain the equations (7.4), apply the adjoint map to each equation in (7.3). ∎

Lemma 7.7. *We have*

$$TA_\ell = B_\ell T, \qquad TA_r = B_r T, \qquad TB_\ell = A_\ell T, \qquad TB_r = A_r T, \tag{7.5}$$

$$TA_\ell^* = B_\ell^* T, \qquad TA_r^* = B_r^* T, \qquad TB_\ell^* = A_\ell^* T, \qquad TB_r^* = A_r^* T. \tag{7.6}$$

Proof. The equations (7.5) follow from Lemma 6.6. We illustrate with a detailed proof of $TA_\ell = B_\ell T$. For $v \in \mathbb{V}$,

$$TA_\ell(v) = T(A \star v) = T(A) \star T(v) = B \star T(v) = B_\ell T(v).$$

Therefore $TA_\ell = B_\ell T$. To obtain the equations (7.6), apply the adjoint map to each equation in (7.5). ∎

Lemma 7.8. *We have*

$$\theta A_L = A_\ell \theta, \qquad \theta A_R = A_r \theta, \qquad \theta B_L = B_\ell \theta, \qquad \theta B_R = B_r \theta, \tag{7.7}$$

$$\theta A_\ell^* = A_L^* \theta, \qquad \theta A_r^* = A_R^* \theta, \qquad \theta B_\ell^* = B_L^* \theta, \qquad \theta B_r^* = B_R^* \theta. \tag{7.8}$$

Proof. The equations (7.7) follow from the definition of θ below (6.6). We illustrate with a detailed proof of $\theta A_L = A_\ell \theta$. For $v \in \mathbb{V}$,

$$\theta A_L(v) = \theta(Av) = \theta(A) \star \theta(v) = A \star \theta(v) = A_\ell \theta(v).$$

Therefore $\theta A_L = A_\ell \theta$. To obtain the equations (7.8), apply the adjoint map to each equation in (7.7). ∎

8 Derivations

In this section we interpret the maps (4.1), (7.2) using the derivation concept from Section 2. We acknowledge that most if not all of the results in this section are well known to the experts; see for example [13, Sections 3 and 4].

Lemma 8.1. *For $u, v \in \mathbb{V}$,*

$$A_\ell^*(uv) = K(u)A_\ell^*(v) + A_\ell^*(u)v,$$
$$B_\ell^*(uv) = K^{-1}(u)B_\ell^*(v) + B_\ell^*(u)v,$$
$$A_r^*(uv) = uA_r^*(v) + A_r^*(u)K(v),$$
$$B_r^*(uv) = uB_r^*(v) + B_r^*(u)K^{-1}(v).$$

Proof. Without loss of generality, we may assume that u and v are words in $\mathbb{V}$. In this case the result is readily checked using Lemma 7.4. ∎

Corollary 8.2. *For the free algebra $\mathbb{V}$,*

(i) A_ℓ^* and $B_r^* K$ are (K, I)-derivations;

(ii) B_ℓ^* and $A_r^* K^{-1}$ are $\left(K^{-1}, I\right)$-derivations;

(iii) A_r^* and $B_\ell^* K$ are (I, K)-derivations;

(iv) B_r^* and $A_\ell^* K^{-1}$ are $\left(I, K^{-1}\right)$-derivations.

Proof. By Lemma 8.1 and the comments about derivations in Section 2. ∎

Lemma 8.3. *For $u, v \in \mathbb{V}$,*

$$A_L^*(u \star v) = K(u) \star A_L^*(v) + A_L^*(u) \star v,$$
$$B_L^*(u \star v) = K^{-1}(u) \star B_L^*(v) + B_L^*(u) \star v,$$
$$A_R^*(u \star v) = u \star A_R^*(v) + A_R^*(u) \star K(v),$$
$$B_R^*(u \star v) = u \star B_R^*(v) + B_R^*(u) \star K^{-1}(v).$$

Proof. Without loss, we may assume that u and v are words in $\mathbb{V}$. In this case the result is readily checked using (6.3), (6.4) and Lemma 4.3. ∎

Corollary 8.4. *For the q-shuffle algebra $\mathbb{V}$,*

(i) A_L^* and $B_R^* K$ are (K, I)-derivations;

(ii) B_L^* and $A_R^* K^{-1}$ are $\left(K^{-1}, I\right)$-derivations;

(iii) A_R^* and $B_L^* K$ are (I, K)-derivations;

(iv) B_R^* and $A_L^* K^{-1}$ are $\left(I, K^{-1}\right)$-derivations.

Proof. By Lemma 8.3 and the comments about derivations in Section 2. ∎

9 Some relations

Consider the maps $K^{\pm 1}$, (3.2), (4.1), (7.1), (7.2). In this section we describe how $K^{\pm 1}$, (4.1), (7.1) are related, and how $K^{\pm 1}$, (3.2), (7.2) are related. We also discuss the subspaces J and U. We acknowledge that most if not all of the relations in this section are well known to the experts, see for example [18, Sections 3.3 and 3.4].

Proposition 9.1. *The maps*

$$K, \quad K^{-1}, \quad A_L^*, \quad B_L^*, \quad A_R^*, \quad B_R^*, \quad A_\ell, \quad B_\ell, \quad A_r, \quad B_r \tag{9.1}$$

satisfy the following relations: $KK^{-1} = K^{-1}K = I$ *and*

$$
\begin{aligned}
&KA_L^* = q^{-2}A_L^*K, &&KB_L^* = q^2 B_L^*K, \\
&KA_R^* = q^{-2}A_R^*K, &&KB_R^* = q^2 B_R^*K, \\
&KA_\ell = q^2 A_\ell K, &&KB_\ell = q^{-2}B_\ell K, \\
&KA_r = q^2 A_r K, &&KB_r = q^{-2}B_r K, \\
&A_L^*A_R^* = A_R^*A_L^*, &&B_L^*B_R^* = B_R^*B_L^*, \\
&A_L^*B_R^* = B_R^*A_L^*, &&B_L^*A_R^* = A_R^*B_L^*, \\
&A_\ell A_r = A_r A_\ell, &&B_\ell B_r = B_r B_\ell, \\
&A_\ell B_r = B_r A_\ell, &&B_\ell A_r = A_r B_\ell, \\
&A_L^*B_r = B_r A_L^*, &&B_L^*A_r = A_r B_L^*, \\
&A_R^*B_\ell = B_\ell A_R^*, &&B_R^*A_\ell = A_\ell B_R^*, \\
&A_L^*B_\ell = q^{-2}B_\ell A_L^*, &&B_L^*A_\ell = q^{-2}A_\ell B_L^*, \\
&A_R^*B_r = q^{-2}B_r A_R^*, &&B_R^*A_r = q^{-2}A_r B_R^*, \\
&A_L^*A_\ell - q^2 A_\ell A_L^* = I, &&A_R^*A_r - q^2 A_r A_R^* = I, \\
&B_L^*B_\ell - q^2 B_\ell B_L^* = I, &&B_R^*B_r - q^2 B_r B_R^* = I, \\
&A_L^*A_r - A_r A_L^* = K, &&B_L^*B_r - B_r B_L^* = K^{-1}, \\
&A_R^*A_\ell - A_\ell A_R^* = K, &&B_R^*B_\ell - B_\ell B_R^* = K^{-1},
\end{aligned}
$$

$$A_\ell^3 B_\ell - [3]_q A_\ell^2 B_\ell A_\ell + [3]_q A_\ell B_\ell A_\ell^2 - B_\ell A_\ell^3 = 0,$$
$$B_\ell^3 A_\ell - [3]_q B_\ell^2 A_\ell B_\ell + [3]_q B_\ell A_\ell B_\ell^2 - A_\ell B_\ell^3 = 0,$$
$$A_r^3 B_r - [3]_q A_r^2 B_r A_r + [3]_q A_r B_r A_r^2 - B_r A_r^3 = 0,$$
$$B_r^3 A_r - [3]_q B_r^2 A_r B_r + [3]_q B_r A_r B_r^2 - A_r B_r^3 = 0.$$

Proof. The first 16 relations above are checked by applying each side to a word in $\mathbb{V}$. The next 16 relations are obtained by setting $u = A$ or $u = B$ or $v = A$ or $v = B$ in Lemma 8.3. The last four relations follow from (6.5), (6.6) and the fact that the q-shuffle product is associative. ∎

Proposition 9.2. *The maps*

$$K, \quad K^{-1}, \quad A_L, \quad B_L, \quad A_R, \quad B_R, \quad A_\ell^*, \quad B_\ell^*, \quad A_r^*, \quad B_r^* \tag{9.2}$$

satisfy the following relations: $KK^{-1} = K^{-1}K = I$ *and*

$$
\begin{aligned}
&KA_L = q^2 A_L K, &&KB_L = q^{-2}B_L K, \\
&KA_R = q^2 A_R K, &&KB_R = q^{-2}B_R K, \\
&KA_\ell^* = q^{-2}A_\ell^*K, &&KB_\ell^* = q^2 B_\ell^*K,
\end{aligned}
$$

$$KA_r^* = q^{-2}A_r^*K, \qquad KB_r^* = q^2B_r^*K,$$
$$A_LA_R = A_RA_L, \qquad B_LB_R = B_RB_L,$$
$$A_LB_R = B_RA_L, \qquad B_LA_R = A_RB_L,$$
$$A_\ell^*A_r^* = A_r^*A_\ell^*, \qquad B_\ell^*B_r^* = B_r^*B_\ell^*,$$
$$A_\ell^*B_r^* = B_r^*A_\ell^*, \qquad B_\ell^*A_r^* = A_r^*B_\ell^*,$$
$$A_LB_r^* = B_r^*A_L, \qquad B_LA_r^* = A_r^*B_L,$$
$$A_RB_\ell^* = B_\ell^*A_R, \qquad B_RA_\ell^* = A_\ell^*B_R,$$
$$A_LB_\ell^* = q^2B_\ell^*A_L, \qquad B_LA_\ell^* = q^2A_\ell^*B_L,$$
$$A_RB_r^* = q^2B_r^*A_R, \qquad B_RA_r^* = q^2A_r^*B_R,$$
$$A_\ell^*A_L - q^2A_LA_\ell^* = I, \qquad A_r^*A_R - q^2A_RA_r^* = I,$$
$$B_\ell^*B_L - q^2B_LB_\ell^* = I, \qquad B_r^*B_R - q^2B_RB_r^* = I,$$
$$A_r^*A_L - A_LA_r^* = K, \qquad B_r^*B_L - B_LB_r^* = K^{-1},$$
$$A_\ell^*A_R - A_RA_\ell^* = K, \qquad B_\ell^*B_R - B_RB_\ell^* = K^{-1},$$
$$(A_\ell^*)^3B_\ell^* - [3]_q(A_\ell^*)^2B_\ell^*A_\ell^* + [3]_qA_\ell^*B_\ell^*(A_\ell^*)^2 - B_\ell^*(A_\ell^*)^3 = 0,$$
$$(B_\ell^*)^3A_\ell^* - [3]_q(B_\ell^*)^2A_\ell^*B_\ell^* + [3]_qB_\ell^*A_\ell^*(B_\ell^*)^2 - A_\ell^*(B_\ell^*)^3 = 0,$$
$$(A_r^*)^3B_r^* - [3]_q(A_r^*)^2B_r^*A_r^* + [3]_qA_r^*B_r^*(A_r^*)^2 - B_r^*(A_r^*)^3 = 0,$$
$$(B_r^*)^3A_r^* - [3]_q(B_r^*)^2A_r^*B_r^* + [3]_qB_r^*A_r^*(B_r^*)^2 - A_r^*(B_r^*)^3 = 0.$$

Proof. Apply the adjoint map to each relation in Proposition 9.1. ∎

Proposition 9.3. *The subspace U is invariant under each of the maps* (9.1). *On U,*

$$(A_L^*)^3B_L^* - [3]_q(A_L^*)^2B_L^*A_L^* + [3]_qA_L^*B_L^*(A_L^*)^2 - B_L^*(A_L^*)^3 = 0, \tag{9.3}$$
$$(B_L^*)^3A_L^* - [3]_q(B_L^*)^2A_L^*B_L^* + [3]_qB_L^*A_L^*(B_L^*)^2 - A_L^*(B_L^*)^3 = 0, \tag{9.4}$$
$$(A_R^*)^3B_R^* - [3]_q(A_R^*)^2B_R^*A_R^* + [3]_qA_R^*B_R^*(A_R^*)^2 - B_R^*(A_R^*)^3 = 0, \tag{9.5}$$
$$(B_R^*)^3A_R^* - [3]_q(B_R^*)^2A_R^*B_R^* + [3]_qB_R^*A_R^*(B_R^*)^2 - A_R^*(B_R^*)^3 = 0. \tag{9.6}$$

Proof. The subspace U is invariant under $K^{\pm 1}$ by Lemma 6.8. The subspace U is invariant under the last eight maps in (9.1), in view of Lemma 7.8. We illustrate with a detailed proof of $A_L^*(U) \subseteq U$. We have

$$A_L^*(U) = A_L^*\theta(\mathbb{V}) = \theta A_\ell^*(\mathbb{V}) \subseteq \theta(\mathbb{V}) = U.$$

Next we show that the relation (9.3) holds on U. Let X denote the map on the left in (9.3), and note that $X^*(v) = -J^+v$ for all $v \in \mathbb{V}$. We show that $XU = 0$. To do this, it suffices to show that $(XU, \mathbb{V}) = 0$. We have $(XU, \mathbb{V}) = (U, X^*\mathbb{V}) = (U, J^+\mathbb{V})$ and $J^+\mathbb{V} \subseteq J = U^\perp$. Therefore $(XU, \mathbb{V}) = 0$. We have shown that the relation (9.3) holds on U. One similarly shows that the relations (9.4)–(9.6) hold on U. ∎

Proposition 9.4. *The subspace J is invariant under each of the maps* (9.2). *On the quotient $\mathbb{V}/J$,*

$$A_L^3B_L - [3]_qA_L^2B_LA_L + [3]_qA_LB_LA_L^2 - B_LA_L^3 = 0, \tag{9.7}$$
$$B_L^3A_L - [3]_qB_L^2A_LB_L + [3]_qB_LA_LB_L^2 - A_LB_L^3 = 0, \tag{9.8}$$
$$A_R^3B_R - [3]_qA_R^2B_RA_R + [3]_qA_RB_RA_R^2 - B_RA_R^3 = 0, \tag{9.9}$$
$$B_R^3A_R - [3]_qB_R^2A_RB_R + [3]_qB_RA_RB_R^2 - A_RB_R^3 = 0. \tag{9.10}$$

Proof. The subspace J is invariant under $K^{\pm 1}$ by Lemma 5.6. The subspace J is invariant under A_L, B_L, A_R, B_R since J is a 2-sided ideal of the free algebra $\mathbb{V}$. The subspace J is invariant under A_ℓ^*, B_ℓ^*, A_r^*, B_r^* by (7.8) and $J = \ker(\theta)$. We verify that the relation (9.7) holds on $\mathbb{V}/J$. Let Y denote the map on the left in (9.7). To show that Y is zero on $\mathbb{V}/J$, it suffices to show that $Y\mathbb{V} \subseteq J$. This is the case, since $Y\mathbb{V} = J^+\mathbb{V} \subseteq J$. We have verified that (9.7) holds on $\mathbb{V}/J$. One similarly verifies that the relations (9.8)–(9.10) hold on $\mathbb{V}/J$. $\blacksquare$

10 The algebra $\square_q^\vee$

In this section we introduce an algebra $\square_q^\vee$ and describe how it is related to the free algebra $\mathbb{V}$. We also discuss the q-Serre relations. In the next section we will obtain sixteen $\square_q^\vee$-module structures on $\mathbb{V}$. Recall the cyclic group $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ of order 4.

Definition 10.1. Define the algebra $\square_q^\vee$ by generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

$$\frac{qx_i x_{i+1} - q^{-1}x_{i+1}x_i}{q - q^{-1}} = 1, \qquad i \in \mathbb{Z}_4. \tag{10.1}$$

Note 10.2. The algebra $\widetilde{\square}_q$ from [29, Definition 6.1] is related to $\square_q^\vee$ in the following way. There exists a surjective algebra homomorphism $\widetilde{\square}_q \to \square_q^\vee$ that sends $x_i \mapsto x_i$ and $c_i^{\pm 1} \mapsto 1$ for $i \in \mathbb{Z}_4$.

The algebra $\square_q^\vee$ is related to the free algebra $\mathbb{V}$ in the following way. Let $\left(\square_q^\vee\right)^{\text{even}}$ (resp. $\left(\square_q^\vee\right)^{\text{odd}}$) denote the subalgebra of $\square_q^\vee$ generated by x_0, x_2 (resp. x_1, x_3). Adapting the proof of [29, Proposition 6.17], we see that

(i) there exists an algebra isomorphism $\mathbb{V} \to \left(\square_q^\vee\right)^{\text{even}}$ that sends $A \mapsto x_0$ and $B \mapsto x_2$;

(ii) there exists an algebra isomorphism $\mathbb{V} \to \left(\square_q^\vee\right)^{\text{odd}}$ that sends $A \mapsto x_1$ and $B \mapsto x_3$;

(iii) the multiplication map $\left(\square_q^\vee\right)^{\text{even}} \otimes \left(\square_q^\vee\right)^{\text{odd}} \to \square_q^\vee$, $u \otimes v \mapsto uv$ is an isomorphism of vector spaces.

We need a fact about the q-Serre relations. We will take a moment to establish this fact, and then return to the main topic.

Lemma 10.3. Pick scalars $r, s \in \{q^2, q^{-2}\}$. Suppose we are given elements a, b, x, y, k, k^{-1} in any algebra such that $kk^{-1} = k^{-1}k = 1$ and

$$ax = xa, \qquad ay = ya, \qquad bx = xb, \qquad by = yb, \tag{10.2}$$

$$ka = rak, \qquad kb = r^{-1}bk, \qquad kx = sxk, \qquad ky = s^{-1}yk. \tag{10.3}$$

Then

$$
\begin{aligned}
\left(a - k^{-1}x\right)^3 &(b - ky) - [3]_q\left(a - k^{-1}x\right)^2(b - ky)\left(a - k^{-1}x\right) \\
&+ [3]_q\left(a - k^{-1}x\right)(b - ky)\left(a - k^{-1}x\right)^2 - (b - ky)\left(a - k^{-1}x\right)^3 \\
&= a^3b - [3]_q a^2ba + [3]_q aba^2 - ba^3 + \left(x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3\right)k^{-2}s^{-4}
\end{aligned}
$$

and

$$
\begin{aligned}
(b - ky)^3 &\left(a - k^{-1}x\right) - [3]_q(b - ky)^2\left(a - k^{-1}x\right)(b - ky) \\
&+ [3]_q(b - ky)\left(a - k^{-1}x\right)(b - ky)^2 - \left(a - k^{-1}x\right)(b - ky)^3 \\
&= b^3a - [3]_q b^2ab + [3]_q bab^2 - ab^3 + \left(y^3x - [3]_q y^2xy + [3]_q yxy^2 - xy^3\right)k^2s^{-4}.
\end{aligned}
$$

Proof. To verify each equation, expand the left-hand side and evaluate the result using (10.2), (10.3). ∎

Corollary 10.4. *With the notation and assumptions of Lemma* 10.3*, if a, b and x, y satisfy the q-Serre relations, then so do any of the following pairs:* (i) $a - k^{-1}x$, $b - ky$; (ii) $a - xk^{-1}$, $b - yk$; (iii) $a - kx$, $b - k^{-1}y$; (iv) $a - xk$, $b - yk^{-1}$.

Proof. (i) By Lemma 10.3.
   (ii) Apply (i) above with a, b replaced by $s^{-1}a$, $s^{-1}b$.
   (iii) Apply (i) above with k, r, s replaced by k^{-1}, r^{-1}, s^{-1}.
   (iv) Apply (ii) above with k, r, s replaced by k^{-1}, r^{-1}, s^{-1}. ∎

11   Sixteen $\square_q^\vee$-module structures on $\mathbb{V}$

In Definition 10.1 we defined the algebra $\square_q^\vee$. In this section we describe sixteen $\square_q^\vee$-module structures on $\mathbb{V}$. The first eight involve the maps from (9.2), and the rest involve the maps from (9.1). For notational convenience define $Q = 1 - q^2$.

Proposition 11.1. *For each row in the tables below, the vector space $\mathbb{V}$ becomes a $\square_q^\vee$-module on which the generators $\{x_i\}_{i \in \mathbb{Z}_4}$ act as indicated.*

module label	x_0	x_1	x_2	x_3
I	A_L	$Q(A_\ell^* - B_r^*K)$	B_L	$Q\big(B_\ell^* - A_r^*K^{-1}\big)$
IS	A_R	$Q(A_r^* - B_\ell^*K)$	B_R	$Q\big(B_r^* - A_\ell^*K^{-1}\big)$
IT	B_L	$Q\big(B_\ell^* - A_r^*K^{-1}\big)$	A_L	$Q(A_\ell^* - B_r^*K)$
IST	B_R	$Q\big(B_r^* - A_\ell^*K^{-1}\big)$	A_R	$Q(A_r^* - B_\ell^*K)$

module label	x_0	x_1	x_2	x_3
II	$Q(A_L - KB_R)$	A_ℓ^*	$Q\big(B_L - K^{-1}A_R\big)$	B_ℓ^*
IIS	$Q(A_R - KB_L)$	A_r^*	$Q\big(B_R - K^{-1}A_L\big)$	B_r^*
IIT	$Q\big(B_L - K^{-1}A_R\big)$	B_ℓ^*	$Q(A_L - KB_R)$	A_ℓ^*
IIST	$Q\big(B_R - K^{-1}A_L\big)$	B_r^*	$Q(A_R - KB_L)$	A_r^*

On each $\square_q^\vee$-module in the tables, the actions of x_1, x_3 satisfy the q-Serre relations.

Proof. By the relations in Proposition 9.2 along with Corollary 10.4. ∎

Proposition 11.2. *For each row in the tables below, the vector space $\mathbb{V}$ becomes a $\square_q^\vee$-module on which the generators $\{x_i\}_{i \in \mathbb{Z}_4}$ act as indicated.*

module label	x_0	x_1	x_2	x_3
III	A_ℓ	$Q(A_L^* - B_R^*K)$	B_ℓ	$Q\big(B_L^* - A_R^*K^{-1}\big)$
IIIS	A_r	$Q(A_R^* - B_L^*K)$	B_r	$Q\big(B_R^* - A_L^*K^{-1}\big)$
IIIT	B_ℓ	$Q\big(B_L^* - A_R^*K^{-1}\big)$	A_ℓ	$Q(A_L^* - B_R^*K)$
IIIST	B_r	$Q\big(B_R^* - A_L^*K^{-1}\big)$	A_r	$Q(A_R^* - B_L^*K)$

module label	x_0	x_1	x_2	x_3
IV	$Q(A_\ell - KB_r)$	A_L^*	$Q\big(B_\ell - K^{-1}A_r\big)$	B_L^*
IVS	$Q(A_r - KB_\ell)$	A_R^*	$Q\big(B_r - K^{-1}A_\ell\big)$	B_R^*
IVT	$Q\big(B_\ell - K^{-1}A_r\big)$	B_L^*	$Q(A_\ell - KB_r)$	A_L^*
IVST	$Q\big(B_r - K^{-1}A_\ell\big)$	B_R^*	$Q(A_r - KB_\ell)$	A_R^*

On each $\square_q^\vee$-module in the tables, the actions of x_0, x_2 satisfy the q-Serre relations.

Proof. By the relations in Proposition 9.1, along with Corollary 10.4. ∎

Note 11.3. Going forward, the $\square_q^\vee$-module $\mathbb{V}$ with label I will be denoted $\mathbb{V}_\mathrm{I}$, and so on.

We now describe the sixteen $\square_q^\vee$-modules in more detail.

Lemma 11.4. *For each $\square_q^\vee$-module $\mathbb{V}$ in Propositions 11.1, 11.2 we have*

$$x_0 \mathbb{V}_n \subseteq \mathbb{V}_{n+1}, \qquad x_1 \mathbb{V}_n \subseteq \mathbb{V}_{n-1}, \qquad x_2 \mathbb{V}_n \subseteq \mathbb{V}_{n+1}, \qquad x_3 \mathbb{V}_n \subseteq \mathbb{V}_{n-1}$$

for $n \in \mathbb{N}$.

Proof. The result for x_0 and x_2 comes from Lemmas 3.3, 7.2. The result for x_1 and x_3 comes from Lemmas 4.4, 7.5. ∎

Lemma 11.5. *For each $\square_q^\vee$-module in Proposition 11.1, the elements x_1 and x_3 act on the free algebra $\mathbb{V}$ as a derivation of the following sort:*

module label	x_1	x_3
I, II	(K, I)-derivation	(K^{-1}, I)-derivation
IS, IIS	(I, K)-derivation	(I, K^{-1})-derivation
IT, IIT	(K^{-1}, I)-derivation	(K, I)-derivation
IST, IIST	(I, K^{-1})-derivation	(I, K)-derivation

Proof. By Corollary 8.2. ∎

Lemma 11.6. *For each $\square_q^\vee$-module in Proposition 11.2, the elements x_1 and x_3 act on the q-shuffle algebra $\mathbb{V}$ as a derivation of the following sort:*

module label	x_1	x_3
III, IV	(K, I)-derivation	(K^{-1}, I)-derivation
IIIS, IVS	(I, K)-derivation	(I, K^{-1})-derivation
IIIT, IVT	(K^{-1}, I)-derivation	(K, I)-derivation
IIIST, IVST	(I, K^{-1})-derivation	(I, K)-derivation

Proof. By Corollary 8.4. ∎

12 Some homomorphisms between the sixteen $\square_q^\vee$-modules

In the previous section we gave sixteen $\square_q^\vee$-module structures on $\mathbb{V}$. In this section we describe some homomorphisms between them.

Lemma 12.1. *The map $S \in \mathrm{End}(\mathbb{V})$ is an isomorphism of $\square_q^\vee$-modules from*

$$\mathbb{V}_{\mathrm{I}} \leftrightarrow \mathbb{V}_{\mathrm{IS}}, \qquad \mathbb{V}_{\mathrm{IT}} \leftrightarrow \mathbb{V}_{\mathrm{IST}}, \qquad \mathbb{V}_{\mathrm{II}} \leftrightarrow \mathbb{V}_{\mathrm{IIS}}, \qquad \mathbb{V}_{\mathrm{IIT}} \leftrightarrow \mathbb{V}_{\mathrm{IIST}},$$
$$\mathbb{V}_{\mathrm{III}} \leftrightarrow \mathbb{V}_{\mathrm{IIIS}}, \qquad \mathbb{V}_{\mathrm{IIIT}} \leftrightarrow \mathbb{V}_{\mathrm{IIIST}}, \qquad \mathbb{V}_{\mathrm{IV}} \leftrightarrow \mathbb{V}_{\mathrm{IVS}}, \qquad \mathbb{V}_{\mathrm{IVT}} \leftrightarrow \mathbb{V}_{\mathrm{IVST}}.$$

Proof. By Lemmas 5.3, 7.6 and since S is a bijection. ∎

Lemma 12.2. *The map $T \in \mathrm{End}(\mathbb{V})$ is an isomorphism of $\square_q^\vee$-modules from*

$$\mathbb{V}_{\mathrm{I}} \leftrightarrow \mathbb{V}_{\mathrm{IT}}, \qquad \mathbb{V}_{\mathrm{IS}} \leftrightarrow \mathbb{V}_{\mathrm{IST}}, \qquad \mathbb{V}_{\mathrm{II}} \leftrightarrow \mathbb{V}_{\mathrm{IIT}}, \qquad \mathbb{V}_{\mathrm{IIS}} \leftrightarrow \mathbb{V}_{\mathrm{IIST}},$$
$$\mathbb{V}_{\mathrm{III}} \leftrightarrow \mathbb{V}_{\mathrm{IIIT}}, \qquad \mathbb{V}_{\mathrm{IIIS}} \leftrightarrow \mathbb{V}_{\mathrm{IIIST}}, \qquad \mathbb{V}_{\mathrm{IV}} \leftrightarrow \mathbb{V}_{\mathrm{IVT}}, \qquad \mathbb{V}_{\mathrm{IVS}} \leftrightarrow \mathbb{V}_{\mathrm{IVST}}.$$

Proof. By Lemmas 5.5, 7.7 and since T is a bijection. ∎

Lemma 12.3. *The map $\theta \in \mathrm{End}(\mathbb{V})$ is a homomorphism of $\square_q^\vee$-modules from*

$$\mathbb{V}_{\mathrm{I}} \to \mathbb{V}_{\mathrm{III}}, \qquad \mathbb{V}_{\mathrm{IS}} \to \mathbb{V}_{\mathrm{IIIS}}, \qquad \mathbb{V}_{\mathrm{IT}} \to \mathbb{V}_{\mathrm{IIIT}}, \qquad \mathbb{V}_{\mathrm{IST}} \to \mathbb{V}_{\mathrm{IIIST}},$$
$$\mathbb{V}_{\mathrm{II}} \to \mathbb{V}_{\mathrm{IV}}, \qquad \mathbb{V}_{\mathrm{IIS}} \to \mathbb{V}_{\mathrm{IVS}}, \qquad \mathbb{V}_{\mathrm{IIT}} \to \mathbb{V}_{\mathrm{IVT}}, \qquad \mathbb{V}_{\mathrm{IIST}} \to \mathbb{V}_{\mathrm{IVST}}.$$

Proof. By Lemma 7.8 and the first equation in Lemma 6.7. ∎

Our next goal is to display a map $\varphi \in \mathrm{End}(\mathbb{V})$ such that φ is a $\square_q^\vee$-module isomorphism from $\mathbb{V}_{\mathrm{I}} \to \mathbb{V}_{\mathrm{II}}$ and $\mathbb{V}_{\mathrm{IT}} \to \mathbb{V}_{\mathrm{IIT}}$, and φ^* is a $\square_q^\vee$-module isomorphism from $\mathbb{V}_{\mathrm{III}} \to \mathbb{V}_{\mathrm{IV}}$ and $\mathbb{V}_{\mathrm{IIIT}} \to \mathbb{V}_{\mathrm{IVT}}$.

Definition 12.4. Define a map $\varphi \in \mathrm{End}(\mathbb{V})$ as follows. For a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$ we have

$$\varphi(v) = \widehat{v}_1 \widehat{v}_2 \cdots \widehat{v}_n(1), \tag{12.1}$$

where

$$\widehat{A} = Q(A_L - K B_R), \qquad \widehat{B} = Q(B_L - K^{-1} A_R). \tag{12.2}$$

In particular $\varphi(1) = 1$.

Lemma 12.5. We have $\varphi \mathbb{V}_n \subseteq \mathbb{V}_n$ for $n \in \mathbb{N}$.

Proof. By Lemma 3.3 and Definition 12.4. ∎

Shortly we will describe φ from an another point of view. In this description we use the following notation. Let $v = v_1 v_2 \cdots v_n$ denote a word in $\mathbb{V}$. For a subset $\Omega \subseteq \{1, 2, \ldots, n\}$ we define a word v_Ω as follows. Write $\Omega = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Then $v_\Omega = v_{i_1} v_{i_2} \cdots v_{i_k}$. Let $\overline{\Omega}$ denote the complement of Ω in $\{1, 2, \ldots, n\}$. The word v_Ω is obtained from $v_1 v_2 \cdots v_n$ by deleting v_j for each $j \in \overline{\Omega}$. Note that $v_\varnothing = 1$.

Lemma 12.6. For a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$\varphi(v) = Q^n \sum_\Omega v_{\overline{\Omega}} S T(v_\Omega) (-1)^{|\Omega|} q^{-2|\Omega|} \left(\prod_{\substack{i,j \in \Omega \\ i < j}} q^{-\langle v_i, v_j \rangle} \right) \left(\prod_{\substack{i \in \Omega,\, j \in \overline{\Omega} \\ i < j}} q^{\langle v_i, v_j \rangle} \right),$$

where the sum is over all subsets Ω of $\{1, 2, \ldots, n\}$. The maps S and T are from Definitions 5.2 and 5.4, respectively.

Proof. Expand the right-hand side of (12.1) using (12.2). ∎

Lemma 12.7. We have $T\varphi = \varphi T$ and $T\varphi^* = \varphi^* T$.

Proof. We first show that $T\varphi = \varphi T$. By Definition 5.4, $T(A) = B$ and $T(B) = A$. By Lemma 5.5 and (12.2), $T\widehat{A} = \widehat{B}T$ and $T\widehat{B} = \widehat{A}T$. By these comments $T\widehat{A} = \widehat{T(A)}T$ and $T\widehat{B} = \widehat{T(B)}T$. We show that $T\varphi(v) = \varphi T(v)$ for all $v \in \mathbb{V}$. Without loss of generality, we may assume that v is a word in $\mathbb{V}$. Write $v = v_1 v_2 \cdots v_n$. By Definition 12.4 and the above comments,

$$T\varphi(v) = T\varphi(v_1 v_2 \cdots v_n) = T\widehat{v}_1 \widehat{v}_2 \cdots \widehat{v}_n(1) = \widehat{T(v_1)} \widehat{T(v_2)} \cdots \widehat{T(v_n)} T(1)$$
$$= \widehat{T(v_1)} \widehat{T(v_2)} \cdots \widehat{T(v_n)}(1) = \varphi\big(T(v_1) T(v_2) \cdots T(v_n)\big) = \varphi T(v_1 v_2 \cdots v_n) = \varphi T(v).$$

We have shown that $T\varphi = \varphi T$. In this equation, apply the adjoint map to each side and use $T^* = T$ to obtain $T\varphi^* = \varphi^* T$. ∎

Lemma 12.8. On $\mathbb{V}$,

$$\varphi A_L = Q(A_L - K B_R)\varphi, \qquad \varphi B_L = Q(B_L - K^{-1} A_R)\varphi, \tag{12.3}$$
$$\varphi(A_\ell^* - B_r^* K)Q = A_\ell^* \varphi, \qquad \varphi(B_\ell^* - A_r^* K^{-1})Q = B_\ell^* \varphi. \tag{12.4}$$

Proof. We first obtain (12.3). For $x \in \{A, B\}$ we show that $\varphi x_L = \widehat{x}\varphi$, where $\widehat{x}$ is from (12.2). It suffices to show that $\varphi x_L(v) = \widehat{x}\varphi(v)$ for all words v in $\mathbb{V}$. Let the word v be given, and write $v = v_1 v_2 \cdots v_n$. Using (12.1),

$$\varphi x_L(v) = \varphi x_L(v_1 v_2 \cdots v_n) = \varphi(x v_1 v_2 \cdots v_n) = \widehat{x}\widehat{v_1}\widehat{v_2}\cdots\widehat{v_n}(1) = \widehat{x}\varphi(v).$$

We have obtained (12.3). Next we obtain the equation on the left in (12.4). For that equation let Δ denote the left-hand side minus the right-hand side. We show that $\Delta = 0$. For notational convenience define

$$x = A_L, \qquad\qquad y = Q(A_\ell^* - B_r^* K), \qquad z = B_L,$$
$$X = Q(A_L - K B_R), \qquad Y = A_\ell^*, \qquad\qquad\qquad Z = Q(B_L - K^{-1} A_R).$$

Note that $\Delta = \varphi y - Y\varphi$. Referring to Proposition 11.1, from the $\mathbb{V}_\mathrm{I}$ data

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \qquad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1,$$

and from the $\mathbb{V}_\mathrm{II}$ data

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = 1, \qquad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = 1.$$

By (12.3),

$$\varphi x = X\varphi, \qquad \varphi z = Z\varphi.$$

We will show that

$$X\Delta = q^{-2}\Delta x, \qquad Z\Delta = q^2 \Delta z. \tag{12.5}$$

We have

$$X\Delta = X\varphi y - XY\varphi = \varphi xy - XY\varphi = q^{-2}(\varphi yx - YX\varphi) = q^{-2}(\varphi yx - Y\varphi x) = q^{-2}\Delta x.$$

Similarly

$$Z\Delta = Z\varphi y - ZY\varphi = \varphi zy - ZY\varphi = q^2(\varphi yz - YZ\varphi) = q^2(\varphi yz - Y\varphi z) = q^2\Delta z.$$

We have shown (12.5). We can now easily show that $\Delta = 0$. We define $W = \{v \in \mathbb{V} \,|\, \Delta v = 0\}$ and show that $W = \mathbb{V}$. By (12.5), W is invariant under $x = A_L$ and $z = B_L$. Note that $\Delta(1) = 0$, since $y(1) = 0$, $Y(1) = 0$, $\varphi(1) = 1$. Therefore $1 \in W$. By these comments and Lemma 3.4 we obtain $W = \mathbb{V}$, so $\Delta = 0$. We have obtained the equation on the left in (12.4). In this equation, multiply each side on the left by T and on the right by T^{-1}. Simplify the result using the first equation in Lemma 12.7, together with the equations

$$TKT^{-1} = K^{-1}, \qquad TA_\ell^* T^{-1} = B_\ell^*, \qquad TB_r^* T^{-1} = A_r^*$$

from Lemmas 5.5, 7.7. This yields the equation on the right in (12.4). ∎

Corollary 12.9. *The map φ from Definition 12.4 is a homomorphism of $\Box_q^\vee$-modules from $\mathbb{V}_\mathrm{I} \to \mathbb{V}_\mathrm{II}$ and $\mathbb{V}_\mathrm{IT} \to \mathbb{V}_\mathrm{IIT}$.*

Proof. The $\Box_q^\vee$-modules in the corollary statement are described in Proposition 11.1. Using these descriptions and Lemma 12.8, we find that for $i \in \mathbb{Z}_4$ the relation $\varphi x_i = x_i\varphi$ holds on $\mathbb{V}_\mathrm{I}$ and $\mathbb{V}_\mathrm{IT}$. The result follows. ∎

Next we consider the adjoint φ^*.

Lemma 12.10. *We have $\varphi^* \mathbb{V}_n \subseteq \mathbb{V}_n$ for $n \in \mathbb{N}$. Moreover $\varphi^*(1) = 1$.*

Proof. The first assertion follows from Lemma 12.5 and the fact that the summands in (3.1) are mutually orthogonal. To obtain the last assertion, use $\varphi(1) = 1$ and the fact that 1 is a basis for $\mathbb{V}_0$. ∎

Lemma 12.11. *On $\mathbb{V}$,*

$$A_L^* \varphi^* = \varphi^*(A_L^* - B_R^* K)Q, \qquad B_L^* \varphi^* = \varphi^*\big(B_L^* - A_R^* K^{-1}\big)Q,$$
$$Q(A_\ell - KB_r)\varphi^* = \varphi^* A_\ell, \qquad Q\big(B_\ell - K^{-1}A_r\big)\varphi^* = \varphi^* B_\ell.$$

Proof. For each equation in Lemma 12.8, apply the adjoint map to each side. ∎

Corollary 12.12. *The map φ^* is a homomorphism of $\Box_q^\vee$-modules from $\mathbb{V}_{\mathrm{III}} \to \mathbb{V}_{\mathrm{IV}}$ and $\mathbb{V}_{\mathrm{IIIT}} \to \mathbb{V}_{\mathrm{IVT}}$.*

Proof. The $\Box_q^\vee$-modules in the corollary statement are described in Proposition 11.2. Using these descriptions and Lemma 12.11, we find that for $i \in \mathbb{Z}_4$ the relation $\varphi^* x_i = x_i \varphi^*$ holds on $\mathbb{V}_{\mathrm{III}}$ and $\mathbb{V}_{\mathrm{IIIT}}$. The result follows. ∎

Lemma 12.13. *We have $\varphi^* \theta = \theta \varphi$.*

Proof. We define $\Delta = \varphi^* \theta - \theta \varphi$ and show that $\Delta = 0$. By Lemma 12.3 and Corollaries 12.9, 12.12 we find that each of $\varphi^* \theta$ and $\theta \varphi$ is a $\Box_q^\vee$-module homomorphism $\mathbb{V}_{\mathrm{I}} \to \mathbb{V}_{\mathrm{IV}}$. So Δ is a $\Box_q^\vee$-module homomorphism $\mathbb{V}_{\mathrm{I}} \to \mathbb{V}_{\mathrm{IV}}$. By Proposition 11.1, x_0 and x_2 act on $\mathbb{V}_{\mathrm{I}}$ as A_L and B_L, respectively. By Proposition 11.2, x_0 and x_2 act on $\mathbb{V}_{\mathrm{IV}}$ as $Q(A_\ell - KB_r)$ and $Q\big(B_\ell - K^{-1}A_r\big)$, respectively. By these comments

$$\Delta A_L = Q(A_\ell - KB_r)\Delta, \qquad \Delta B_L = Q\big(B_\ell - K^{-1}A_r\big)\Delta. \tag{12.6}$$

Let W denote the kernel of Δ on $\mathbb{V}$. By (12.6), W is invariant under A_L and B_L. We have $\Delta(1) = 0$ since $\theta(1) = 1$, $\varphi(1) = 1$, $\varphi^*(1) = 1$. Therefore $1 \in W$. By these comments and Lemma 3.4 we obtain $W = \mathbb{V}$, so $\Delta = 0$. ∎

Next we show that φ and φ^* are bijections.

Definition 12.14. Define $\psi \in \mathrm{End}(\mathbb{V})$ by

$$\psi = KB_R A_L^* + K^{-1} A_R B_L^*. \tag{12.7}$$

Lemma 12.15. *We have $\psi \mathbb{V}_0 = 0$, and $\psi \mathbb{V}_n \subseteq \mathbb{V}_n$ for $n \geq 1$.*

Proof. By Lemmas 3.3, 4.4. ∎

Lemma 12.16. *The map ψ acts on the standard basis for $\mathbb{V}$ as follows. We have $\psi(1) = 0$. For a nontrivial word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,*

$$\psi(v) = v_2 v_3 \cdots v_n T(v_1) q^{\langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle + \cdots + \langle v_1, v_n \rangle - 2}.$$

Proof. Use Definition 5.1 and the comments below it, along with Lemma 4.3 and Definition 12.14. ∎

Lemma 12.17. *The following* (i)–(iv) *hold for $n \geq 1$.*

  (i) *The equation $\psi^n = q^{-2n} T$ holds on $\mathbb{V}_n$.*

(ii) *The equation $\psi^{2n} = q^{-4n}I$ holds on $\mathbb{V}_n$.*

(iii) *The map $I - \psi$ is invertible on $\mathbb{V}_n$.*

(iv) *On $\mathbb{V}_n$,*

$$I - \psi = (A_L - KB_R)A_L^* + (B_L - K^{-1}A_R)B_L^*.$$

Proof. (i) Pick a word $v = v_1v_2\cdots v_n$ in $\mathbb{V}$. Repeatedly applying Lemma 12.16 we obtain

$$\psi^n(v) = T(v_1)T(v_2)\cdots T(v_n)q^{-2n} = T(v)q^{-2n}.$$

The result follows.

(ii) By (i) above and since $T^2 = I$.

(iii) Pick a vector $v \in \mathbb{V}_n$ such that $(I - \psi)v = 0$. We show that $v = 0$. By construction $\psi v = v$. By this and (ii) we obtain $v = \psi^{2n}v = q^{-4n}v$. So $(1 - q^{-4n})v = 0$. In this equation the coefficient of v is nonzero, so $v = 0$.

(iv) By (12.7) and since $I = A_L A_L^* + B_L B_L^*$ on $\mathbb{V}_n$. ∎

Lemma 12.18. *For $n \geq 1$ we have*

$$\mathbb{V}_n = (A_L - KB_R)\mathbb{V}_{n-1} + (B_L - K^{-1}A_R)\mathbb{V}_{n-1}.$$

Proof. The inclusion $\supseteq$ follows from Lemma 3.3. Concerning the inclusion $\subseteq$, use parts (iii), (iv) of Lemma 12.17 along with Lemma 4.4 to obtain

$$\begin{aligned}
\mathbb{V}_n = (I - \psi)\mathbb{V}_n &= \left((A_L - KB_R)A_L^* + (B_L - K^{-1}A_R)B_L^*\right)\mathbb{V}_n \\
&\subseteq (A_L - KB_R)A_L^*\mathbb{V}_n + (B_L - K^{-1}A_R)B_L^*\mathbb{V}_n \\
&\subseteq (A_L - KB_R)\mathbb{V}_{n-1} + (B_L - K^{-1}A_R)\mathbb{V}_{n-1}.
\end{aligned}$$
∎

Lemma 12.19. *For $n \in \mathbb{N}$ we have*

(i) $\varphi\mathbb{V}_n = \mathbb{V}_n$;

(ii) $\varphi^*\mathbb{V}_n = \mathbb{V}_n$.

Proof. (i) By induction on n. For $n = 0$ the result holds, because 1 is a basis for $\mathbb{V}_0$ and $\varphi(1) = 1$. Next assume that $n \geq 1$. The sum $\mathbb{V}_n = A\mathbb{V}_{n-1} + B\mathbb{V}_{n-1}$ is direct. In this equation apply φ to each side. By Lemma 12.8 and induction,

$$\begin{aligned}
\varphi(A\mathbb{V}_{n-1}) &= \varphi A_L \mathbb{V}_{n-1} = (A_L - KB_R)\varphi\mathbb{V}_{n-1} = (A_L - KB_R)\mathbb{V}_{n-1}, \\
\varphi(B\mathbb{V}_{n-1}) &= \varphi B_L \mathbb{V}_{n-1} = (B_L - K^{-1}A_R)\varphi\mathbb{V}_{n-1} = (B_L - K^{-1}A_R)\mathbb{V}_{n-1}.
\end{aligned}$$

By these comments and Lemma 12.18,

$$\varphi\mathbb{V}_n = \varphi(A\mathbb{V}_{n-1}) + \varphi(B\mathbb{V}_{n-1}) = (A_L - KB_R)\mathbb{V}_{n-1} + (B_L - K^{-1}A_R)\mathbb{V}_{n-1} = \mathbb{V}_n.$$

(ii) By Lemma 12.10 and (i) above, along with the fact that the dimension of $\mathbb{V}_n$ is finite. ∎

Lemma 12.20. *Each of φ, φ^* is a bijection.*

Proof. By (3.1) and Lemma 12.19. ∎

Proposition 12.21. *The map φ is an isomorphism of $\square_q^\vee$-modules from $\mathbb{V}_I \to \mathbb{V}_{II}$ and $\mathbb{V}_{IT} \to \mathbb{V}_{IIT}$. Moreover φ^* is an isomorphism of $\square_q^\vee$-modules from $\mathbb{V}_{III} \to \mathbb{V}_{IV}$ and $\mathbb{V}_{IIIT} \to \mathbb{V}_{IVT}$.*

Proof. By Corollaries 12.9, 12.12 and Lemma 12.20. ∎

Lemma 12.22. *We have $\varphi(J) = J$.*

Proof. We invoke Lemmas 12.13, 12.20 and $J = \ker(\theta)$. For $v \in \mathbb{V}$,

$$v \in J \iff \theta(v) = 0 \iff \varphi^* \theta(v) = 0 \iff \theta\varphi(v) = 0 \iff \varphi(v) \in J.$$

The result follows.   ∎

Lemma 12.23. *We have $\varphi^*(U) = U$.*

Proof. By Lemmas 12.13, 12.20 and $\theta(\mathbb{V}) = U$,

$$\varphi^*(U) = \varphi^* \theta(\mathbb{V}) = \theta\varphi(\mathbb{V}) = \theta(\mathbb{V}) = U.$$

    ∎

13   Sixteen $\square_q$-module structures on $\mathbb{V}/J$ or U

In Section 1 we informally discussed the algebra $\square_q$. In this section we formally bring in $\square_q$, and review its basic properties. We then display sixteen $\square_q$-module structures on $\mathbb{V}/J$ or U. In order to motivate things, we mention a result about the $\square_q^\vee$-modules from Propositions 11.1 and 11.2.

Lemma 13.1. *The following* (i), (ii) *hold.*

(i) *For each $\square_q^\vee$-module $\mathbb{V}$ in Proposition 11.1, the subspace J is a $\square_q^\vee$-submodule. On the quotient $\square_q^\vee$-module $\mathbb{V}/J$ the actions of x_0, x_2 satisfy the q-Serre relations.*

(ii) *For each $\square_q^\vee$-module $\mathbb{V}$ from Proposition 11.2, the subspace U is a $\square_q^\vee$-submodule on which the actions of x_1, x_3 satisfy the q-Serre relations.*

Proof. (i) Use Propositions 9.4, 11.1 and Corollary 10.4.
(ii) Use Propositions 9.3, 11.2 and Corollary 10.4.   ∎

Definition 13.2 (see [29, Definition 5.1])**.** Define the algebra $\square_q$ by generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

$$\frac{q x_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = 1,$$
$$x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0.$$

Lemma 13.3. *There exists an algebra homomorphism $\square_q^\vee \to \square_q$ that sends $x_i \mapsto x_i$ for $i \in \mathbb{Z}_4$. This homomorphism is surjective.*

Proof. Compare Definitions 10.1, 13.2.   ∎

Recall the algebra $U_q^+ = \mathbb{V}/J$ from Definition 3.6. This algebra is related to $\square_q$ in the following way. Let $\square_q^{\text{even}}$ (resp. $\square_q^{\text{odd}}$) denote the subalgebra of $\square_q$ generated by x_0, x_2 (resp. x_1, x_3). Then by [29, Proposition 5.5],

(i) there exists an algebra isomorphism $U_q^+ \to \square_q^{\text{even}}$ that sends $A \mapsto x_0$ and $B \mapsto x_2$;

(ii) there exists an algebra isomorphism $U_q^+ \to \square_q^{\text{odd}}$ that sends $A \mapsto x_1$ and $B \mapsto x_3$;

(iii) the multiplication map $\square_q^{\text{even}} \otimes \square_q^{\text{odd}} \to \square_q$, $u \otimes v \mapsto uv$ is an isomorphism of vector spaces.

Theorem 13.4. *For each row in the tables below, the vector space $\mathbb{V}/J$ becomes a $\square_q$-module on which the generators $\{x_i\}_{i\in\mathbb{Z}_4}$ act as indicated.*

module label	x_0	x_1	x_2	x_3
I	A_L	$Q(A_\ell^* - B_r^* K)$	B_L	$Q\big(B_\ell^* - A_r^* K^{-1}\big)$
IS	A_R	$Q(A_r^* - B_\ell^* K)$	B_R	$Q\big(B_r^* - A_\ell^* K^{-1}\big)$
IT	B_L	$Q\big(B_\ell^* - A_r^* K^{-1}\big)$	A_L	$Q(A_\ell^* - B_r^* K)$
IST	B_R	$Q\big(B_r^* - A_\ell^* K^{-1}\big)$	A_R	$Q(A_r^* - B_\ell^* K)$

module label	x_0	x_1	x_2	x_3
II	$Q(A_L - K B_R)$	A_ℓ^*	$Q\big(B_L - K^{-1} A_R\big)$	B_ℓ^*
IIS	$Q(A_R - K B_L)$	A_r^*	$Q\big(B_R - K^{-1} A_L\big)$	B_r^*
IIT	$Q\big(B_L - K^{-1} A_R\big)$	B_ℓ^*	$Q(A_L - K B_R)$	A_ℓ^*
IIST	$Q\big(B_R - K^{-1} A_L\big)$	B_r^*	$Q(A_R - K B_L)$	A_r^*

Proof. By Proposition 11.1 and Lemma 13.1(i). ∎

Theorem 13.5. *For each row in the tables below, the vector space U becomes a $\square_q$-module on which the generators $\{x_i\}_{i\in\mathbb{Z}_4}$ act as indicated.*

module label	x_0	x_1	x_2	x_3
III	A_ℓ	$Q(A_L^* - B_R^* K)$	B_ℓ	$Q\big(B_L^* - A_R^* K^{-1}\big)$
IIIS	A_r	$Q(A_R^* - B_L^* K)$	B_r	$Q\big(B_R^* - A_L^* K^{-1}\big)$
IIIT	B_ℓ	$Q\big(B_L^* - A_R^* K^{-1}\big)$	A_ℓ	$Q(A_L^* - B_R^* K)$
IIIST	B_r	$Q\big(B_R^* - A_L^* K^{-1}\big)$	A_r	$Q(A_R^* - B_L^* K)$

module label	x_0	x_1	x_2	x_3
IV	$Q(A_\ell - K B_r)$	A_L^*	$Q\big(B_\ell - K^{-1} A_r\big)$	B_L^*
IVS	$Q(A_r - K B_\ell)$	A_R^*	$Q\big(B_r - K^{-1} A_\ell\big)$	B_R^*
IVT	$Q\big(B_\ell - K^{-1} A_r\big)$	B_L^*	$Q(A_\ell - K B_r)$	A_L^*
IVST	$Q\big(B_r - K^{-1} A_\ell\big)$	B_R^*	$Q(A_r - K B_\ell)$	A_R^*

Proof. By Proposition 11.2 and Lemma 13.1(ii). ∎

Note 13.6. Going forward, the $\square_q$-module $\mathbb{V}/J$ with label I will be denoted $(\mathbb{V}/J)_{\mathrm{I}}$, and so on.

Proposition 13.7. *For each $\square_q$-module in Theorem 13.4 the elements x_1 and x_3 act on the algebra $\mathbb{V}/J$ as a derivation of the following sort:*

module label	x_1	x_3
I, II	(K, I)-derivation	$\big(K^{-1}, I\big)$-derivation
IS, IIS	(I, K)-derivation	$\big(I, K^{-1}\big)$-derivation
IT, IIT	$\big(K^{-1}, I\big)$-derivation	(K, I)-derivation
IST, IIST	$\big(I, K^{-1}\big)$-derivation	(I, K)-derivation

Proof. By Lemma 11.5 and the comments about derivations in Section 2. ∎

Proposition 13.8. *For each $\square_q$-module in Theorem 13.5 the elements x_1 and x_3 act on the algebra U as a derivation of the following sort:*

module label	x_1	x_3
III, IV	(K, I)-derivation	$\big(K^{-1}, I\big)$-derivation
IIIS, IVS	(I, K)-derivation	$\big(I, K^{-1}\big)$-derivation
IIIT, IVT	$\big(K^{-1}, I\big)$-derivation	(K, I)-derivation
IIIST, IVST	$\big(I, K^{-1}\big)$-derivation	(I, K)-derivation

Proof. By Lemma 11.6 and the comments about derivations in Section 2. ∎

14   The sixteen $\square_q$-modules are mutually isomorphic and irreducible

In the previous section we gave sixteen $\square_q$-module structures on $\mathbb{V}/J$ or U. In this section we show that these $\square_q$-modules are mutually isomorphic and irreducible.

Proposition 14.1. *The map $\mathbb{V}/J \to \mathbb{V}/J$, $x + J \mapsto S(x) + J$ is an isomorphism of $\square_q$-modules from*

$$(\mathbb{V}/J)_{\mathrm{I}} \leftrightarrow (\mathbb{V}/J)_{\mathrm{IS}}, \qquad (\mathbb{V}/J)_{\mathrm{IT}} \leftrightarrow (\mathbb{V}/J)_{\mathrm{IST}},$$
$$(\mathbb{V}/J)_{\mathrm{II}} \leftrightarrow (\mathbb{V}/J)_{\mathrm{IIS}}, \qquad (\mathbb{V}/J)_{\mathrm{IIT}} \leftrightarrow (\mathbb{V}/J)_{\mathrm{IIST}}.$$

The map $U \to U$, $x \mapsto S(x)$ is an isomorphism of $\square_q$-modules from

$$U_{\mathrm{III}} \leftrightarrow U_{\mathrm{IIIS}}, \qquad U_{\mathrm{IIIT}} \leftrightarrow U_{\mathrm{IIIST}}, \qquad U_{\mathrm{IV}} \leftrightarrow U_{\mathrm{IVS}}, \qquad U_{\mathrm{IVT}} \leftrightarrow U_{\mathrm{IVST}}.$$

Proof. By Lemma 12.1, together with the fact that $S(J) = J$ by (5.6) and $S(U) = U$ by Lemma 6.8.   ∎

Proposition 14.2. *The map $\mathbb{V}/J \to \mathbb{V}/J$, $x + J \mapsto T(x) + J$ is an isomorphism of $\square_q$-modules from*

$$(\mathbb{V}/J)_{\mathrm{I}} \leftrightarrow (\mathbb{V}/J)_{\mathrm{IT}}, \qquad (\mathbb{V}/J)_{\mathrm{IS}} \leftrightarrow (\mathbb{V}/J)_{\mathrm{IST}},$$
$$(\mathbb{V}/J)_{\mathrm{II}} \leftrightarrow (\mathbb{V}/J)_{\mathrm{IIT}}, \qquad (\mathbb{V}/J)_{\mathrm{IIS}} \leftrightarrow (\mathbb{V}/J)_{\mathrm{IIST}}.$$

The map $U \to U$, $x \mapsto T(x)$ is an isomorphism of $\square_q$-modules from

$$U_{\mathrm{III}} \leftrightarrow U_{\mathrm{IIIT}}, \qquad U_{\mathrm{IIIS}} \leftrightarrow U_{\mathrm{IIIST}}, \qquad U_{\mathrm{IV}} \leftrightarrow U_{\mathrm{IVT}}, \qquad U_{\mathrm{IVS}} \leftrightarrow U_{\mathrm{IVST}}.$$

Proof. By Lemma 12.2, together with the fact that $T(J) = J$ by (5.6) and $T(U) = U$ by Lemma 6.8.   ∎

Proposition 14.3. *The map $\mathbb{V}/J \to U$, $x + J \mapsto \theta(x)$ is an isomorphism of $\square_q$-modules from*

$$(\mathbb{V}/J)_{\mathrm{I}} \to U_{\mathrm{III}}, \qquad (\mathbb{V}/J)_{\mathrm{IS}} \to U_{\mathrm{IIIS}}, \qquad (\mathbb{V}/J)_{\mathrm{IT}} \to U_{\mathrm{IIIT}}, \qquad (\mathbb{V}/J)_{\mathrm{IST}} \to U_{\mathrm{IIIST}},$$
$$(\mathbb{V}/J)_{\mathrm{II}} \to U_{\mathrm{IV}}, \qquad (\mathbb{V}/J)_{\mathrm{IIS}} \to U_{\mathrm{IVS}}, \qquad (\mathbb{V}/J)_{\mathrm{IIT}} \to U_{\mathrm{IVT}}, \qquad (\mathbb{V}/J)_{\mathrm{IIST}} \to U_{\mathrm{IVST}}.$$

Proof. By Lemma 12.3 and the fact that J is the kernel of θ.   ∎

Proposition 14.4. *The map $\mathbb{V}/J \to \mathbb{V}/J$, $x + J \mapsto \varphi(x) + J$ is an isomorphsim of $\square_q$-modules from*

$$(\mathbb{V}/J)_{\mathrm{I}} \to (\mathbb{V}/J)_{\mathrm{II}}, \qquad (\mathbb{V}/J)_{\mathrm{IT}} \to (\mathbb{V}/J)_{\mathrm{IIT}}.$$

The map $U \to U$, $x \mapsto \varphi^*(x)$ is an isomorphism of $\square_q$-modules from

$$U_{\mathrm{III}} \to U_{\mathrm{IV}}, \qquad U_{\mathrm{IIIT}} \to U_{\mathrm{IVT}}.$$

Proof. By Proposition 12.21 and Lemmas 12.22, 12.23.   ∎

Theorem 14.5. *The following $\square_q$-modules are mutually isomorphic:*

$$(\mathbb{V}/J)_{\mathrm{I}}, \qquad (\mathbb{V}/J)_{\mathrm{IS}}, \qquad (\mathbb{V}/J)_{\mathrm{IT}}, \qquad (\mathbb{V}/J)_{\mathrm{IST}},$$
$$(\mathbb{V}/J)_{\mathrm{II}}, \qquad (\mathbb{V}/J)_{\mathrm{IIS}}, \qquad (\mathbb{V}/J)_{\mathrm{IIT}}, \qquad (\mathbb{V}/J)_{\mathrm{IIST}},$$
$$U_{\mathrm{III}}, \qquad U_{\mathrm{IIIS}}, \qquad U_{\mathrm{IIIT}}, \qquad U_{\mathrm{IIIST}},$$
$$U_{\mathrm{IV}}, \qquad U_{\mathrm{IVS}}, \qquad U_{\mathrm{IVT}}, \qquad U_{\mathrm{IVST}}.$$

Proof. By Propositions 14.1–14.4. ∎

Our next goal is to show that the $\square_q$-modules in Theorem 14.5 are irreducible.

Lemma 14.6. *Let W denote a nonzero $\square_q$-submodule of U_{IV}. Then $1 \in W$.*

Proof. By Proposition 11.2 the generators x_1 and x_3 act on U_{IV} as A_L^* and B_L^*, respectively. Now $1 \in W$ by Lemma 4.6. ∎

Lemma 14.7. *Let W denote a proper $\square_q$-submodule of U_{III}. Then $1 \notin W$.*

Proof. We assume $1 \in W$ and get a contradiction. By Proposition 11.2 the generators x_0 and x_2 act on U_{III} as A_ℓ and B_ℓ, respectively. By Lemma 7.3, W is not proper. This contradicts our assumptions. ∎

Theorem 14.8. *The $\square_q$-modules in Theorem 14.5 are irreducible.*

Proof. By Theorem 14.5 along with Lemmas 14.6, 14.7 we find that none of the listed $\square_q$-modules contain a nonzero proper $\square_q$-submodule. The result follows. ∎

15   The NIL modules for $\square_q$

In this section we characterize the $\square_q$-modules in Theorem 14.5, using the notion of a NIL $\square_q$-module.

We start with some comments about $\square_q^\vee$. Reformulating the relations (10.1),

$$x_1 x_0 = q^2 x_0 x_1 + 1 - q^2, \qquad x_1 x_2 = q^{-2} x_2 x_1 + 1 - q^{-2},$$
$$x_3 x_2 = q^2 x_2 x_3 + 1 - q^2, \qquad x_3 x_0 = q^{-2} x_0 x_3 + 1 - q^{-2}.$$

Next we express these relations in a uniform way.

Lemma 15.1. *For $u \in \{x_0, x_2\}$ and $v \in \{x_1, x_3\}$ the following holds in $\square_q^\vee$:*

$$vu = uvq^{\langle u,v \rangle} + 1 - q^{\langle u,v \rangle},$$

where

$\langle\,,\,\rangle$	x_1	x_3
x_0	2	-2
x_2	-2	2

The following formula will be useful.

Lemma 15.2. *Pick $n \in \mathbb{N}$. Referring to the algebra $\square_q^\vee$, pick $u_i \in \{x_0, x_2\}$ for $1 \leq i \leq n$, and also $v \in \{x_1, x_3\}$. Then*

$$vu_1 u_2 \cdots u_n = u_1 u_2 \cdots u_n v q^{\langle u_1,v \rangle + \langle u_2,v \rangle + \cdots + \langle u_n,v \rangle}$$
$$+ \sum_{i=1}^{n} u_1 \cdots u_{i-1} u_{i+1} \cdots u_n q^{\langle u_1,v \rangle + \cdots + \langle u_{i-1},v \rangle} \left(1 - q^{\langle u_i,v \rangle}\right),$$

where $\langle\,,\,\rangle$ is from Lemma 15.1.

Proof. By Lemma 15.1 and induction on n. ∎

Corollary 15.3. *Let V denote a $\Box_q^\vee$-module. Pick $\xi \in V$ such that $x_1\xi = 0$ and $x_3\xi = 0$. Then for $n \in \mathbb{N}$ and $u_1, u_2, \ldots, u_n \in \{x_0, x_2\}$ and $v \in \{x_1, x_3\}$,*

$$vu_1 u_2 \cdots u_n \xi = \sum_{i=1}^n u_1 \cdots u_{i-1} u_{i+1} \cdots u_n \xi q^{\langle u_1, v \rangle + \cdots + \langle u_{i-1}, v \rangle} \left(1 - q^{\langle u_i, v \rangle}\right),$$

where $\langle \, , \, \rangle$ is from Lemma 15.1.

Proof. Referring to the equation displayed in Lemma 15.2, apply each side to ξ and note that $v\xi = 0$. ∎

Recall that $\left(\Box_q^\vee\right)^{\text{even}}$ is the subalgebra of $\Box_q^\vee$ generated by x_0, x_2. Recall the free algebra $\mathbb{V}$ with generators A, B. By our comments below Note 10.2, there exists an algebra isomorphism $\mathbb{V} \to \left(\Box_q^\vee\right)^{\text{even}}$ that sends $A \mapsto x_0$ and $B \mapsto x_2$. Denote this isomorphism by κ. Recall the $\Box_q^\vee$-module $\mathbb{V}_{\mathrm{I}}$ from Proposition 11.1.

Lemma 15.4. *Let V denote a $\Box_q^\vee$-module. Pick $\xi \in V$ such that $x_1\xi = 0$ and $x_3\xi = 0$. Then the map $\mathbb{V}_I \to V$, $v \mapsto \kappa(v)\xi$ is a $\Box_q^\vee$-module homomorphism.*

Proof. We show that for $i \in \mathbb{Z}_4$ the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{V}_{\mathrm{I}} & \xrightarrow{v \mapsto \kappa(v)\xi} & V \\
{\scriptstyle x_i}\Big\downarrow & & \Big\downarrow{\scriptstyle x_i} \\
\mathbb{V}_{\mathrm{I}} & \xrightarrow{v \mapsto \kappa(v)\xi} & V.
\end{array}
$$

Referring to the above diagram, the action $x_i \colon \mathbb{V}_{\mathrm{I}} \to \mathbb{V}_{\mathrm{I}}$ is described in Proposition 11.1 along with Definition 5.1 and Lemma 7.4. The action $x_i \colon V \to V$ is clear for i even, and described in Corollary 15.3 for i odd. Using these descriptions we chase each word in $\mathbb{V}$ around the diagram, and confirm that the diagram commutes. ∎

Let V denote a $\Box_q$-module. We view V as a $\Box_q^\vee$-module on which

$$x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0, \qquad i \in \mathbb{Z}_4.$$

Lemma 15.5. *Let V denote a $\Box_q$-module that contains a nonzero vector ξ such that $x_1\xi = 0$ and $x_3\xi = 0$. Then the map in Lemma 15.4 has kernel J. Moreover the map $(\mathbb{V}/J)_{\mathrm{I}} \to V$, $v + J \mapsto \kappa(v)\xi$ is an injective $\Box_q$-module homomorphism.*

Proof. Let L denote the kernel of the map in Lemma 15.4. This map sends $1 \mapsto \xi$, and ξ is nonzero, so $1 \notin L$. Observe that L is a left ideal of the free algebra $\mathbb{V}$. The set $H = \{v \in \mathbb{V} | \kappa(v)V = 0\}$ is a 2-sided ideal of the free algebra $\mathbb{V}$. By construction $H \subseteq L$. Recall the elements $J^\pm$ from (3.3), (3.4). We have

$$\kappa(J^+) = x_0^3 x_2 - [3]_q x_0^2 x_2 x_0 + [3]_q x_0 x_2 x_0^2 - x_2 x_0^3, \tag{15.1}$$

$$\kappa(J^-) = x_2^3 x_0 - [3]_q x_2^2 x_0 x_2 + [3]_q x_2 x_0 x_2^2 - x_0 x_2^3. \tag{15.2}$$

Since V is a $\Box_q$-module, the elements (15.1), (15.2) are zero on V. Therefore $J^\pm \in H$. Recall that J is the 2-sided ideal of the free algebra $\mathbb{V}$ generated by $J^\pm$. Consequently $J \subseteq H$. We mentioned earlier that $H \subseteq L$, so $J \subseteq L$. By this and Lemma 15.4, the map $(\mathbb{V}/J)_{\mathrm{I}} \to V$, $v + J \mapsto \kappa(v)\xi$ is a $\Box_q$-module homomorphism that has kernel L/J. This kernel L/J is a $\Box_q$-submodule of the $\Box_q$-module $(\mathbb{V}/J)_{\mathrm{I}}$, and the $\Box_q$-module $(\mathbb{V}/J)_{\mathrm{I}}$ is irreducible by Theorem 14.8, so $L/J = 0$ or $L/J = \mathbb{V}/J$. Thus $L = J$ or $L = \mathbb{V}$. We have $L \neq \mathbb{V}$ since $1 \notin L$, so $L = J$. Consequently the map $(\mathbb{V}/J)_{\mathrm{I}} \to V$, $v + J \mapsto \kappa(v)\xi$ is injective. The result follows. ∎

Let V denote a nonzero $\Box_q$-module. For $\xi \in V$, we say that V is *generated by* ξ whenever V does not have a proper $\Box_q$-submodule that contains ξ.

Proposition 15.6. *Let V denote a $\Box_q$-module that is generated by a nonzero vector ξ such that $x_1\xi = 0$ and $x_3\xi = 0$. Then the map $(\mathbb{V}/J)_{\mathrm{I}} \to V$, $v + J \mapsto \kappa(v)\xi$ is an isomorphism of $\Box_q$-modules.*

Proof. By Lemma 15.5, the map $(\mathbb{V}/J)_{\mathrm{I}} \to V$, $v + J \mapsto \kappa(v)\xi$ is an injective $\Box_q$-module homomorphism. For this map the image is a $\Box_q$-submodule of V that contains ξ, so this image is equal to V. By these comments the map $(\mathbb{V}/J)_{\mathrm{I}} \to V$, $v + J \mapsto \kappa(v)\xi$ is an isomorphism of $\Box_q$-modules. ∎

Theorem 15.7. *For a $\Box_q$-module V the following are equivalent:*

(i) *V is isomorphic to the $\Box_q$-modules in Theorem 14.5;*

(ii) *v is generated by a nonzero vector ξ such that $x_1\xi = 0$ and $x_3\xi = 0$.*

Proof. (i) $\Rightarrow$ (ii): Without loss of generality, we may identify the $\Box_q$-module V with the $\Box_q$-module U_{III} listed in Theorem 14.5. The vector $\xi = 1$ has the desired properties.

(ii) $\Rightarrow$ (i): By Theorem 14.5 and Proposition 15.6. ∎

Definition 15.8. Let V denote a $\Box_q$-module. A vector $\xi \in V$ is called NIL whenever $x_1\xi = 0$ and $x_3\xi = 0$ and $\xi \neq 0$. The $\Box_q$-module V is called *NIL* whenever it is generated by a NIL vector.

By Theorem 15.7, up to isomorphism there exists a unique NIL $\Box_q$-module, which we denote by $\mathbf{U}$. Also by Theorem 15.7, the $\Box_q$-module $\mathbf{U}$ is isomorphic to each of the $\Box_q$-modules from Theorem 14.5. By Theorem 14.8 the $\Box_q$-module $\mathbf{U}$ is irreducible. The $\Box_q$-module $\mathbf{U}$ is infinite-dimensional; indeed it is isomorphic to U_q^+ as a vector space, as we now clarify. Recall the algebra isomorphism $U_q^+ \to \Box_q^{\mathrm{even}}$ from below Lemma 13.3.

Lemma 15.9. *Identify the algebra U_q^+ with $\Box_q^{\mathrm{even}}$ via the algebra isomorphism from below Lemma 13.3. Let ξ denote a NIL vector in $\mathbf{U}$. Then the map $U_q^+ \to \mathbf{U}$, $u \mapsto u\xi$ is an isomorphism of vector spaces.*

Proof. By Proposition 15.6. ∎

Theorem 15.10. *The $\Box_q$-module $\mathbf{U}$ has a unique sequence of subspaces $\{\mathbf{U}_n\}_{n \in \mathbb{N}}$ such that:*

(i) *$\mathbf{U}_0 \neq 0$;*

(ii) *the sum $\mathbf{U} = \sum_{n \in \mathbb{N}} \mathbf{U}_n$ is direct;*

(iii) *for $n \in \mathbb{N}$,*

$$x_0\mathbf{U}_n \subseteq \mathbf{U}_{n+1}, \qquad x_1\mathbf{U}_n \subseteq \mathbf{U}_{n-1}, \qquad x_2\mathbf{U}_n \subseteq \mathbf{U}_{n+1}, \qquad x_3\mathbf{U}_n \subseteq \mathbf{U}_{n-1},$$

where $\mathbf{U}_{-1} = 0$.

The sequence $\{\mathbf{U}_n\}_{n \in \mathbb{N}}$ is described as follows. The subspace $\mathbf{U}_0$ has dimension 1. The nonzero vectors in $\mathbf{U}_0$ are precisely the NIL vectors in $\mathbf{U}$, and each of these vectors generates $\mathbf{U}$. Let ξ denote a NIL vector in $\mathbf{U}$. Then for $n \in \mathbb{N}$, $\mathbf{U}_n$ is spanned by the vectors

$$u_1 u_2 \cdots u_n \xi, \qquad u_i \in \{x_0, x_2\}, \qquad 1 \leq i \leq n. \tag{15.3}$$

Proof. Concerning existence, without loss of generality we may identify the $\square_q$-module $\mathbf{U}$ with the $\square_q$-module U_{III} listed in Theorem 14.5. Recall that U is the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by A, B. For $n \in \mathbb{N}$ define $\mathbf{U}_n = U \cap \mathbb{V}_n$. One checks that the sequence $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ satisfies the above conditions (i)–(iii). We have established existence. Going forward let $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ denote any sequence of subspaces that satisfies the above conditions (i)–(iii). Let ξ denote a NIL vector in $\mathbf{U}$. We claim that $\xi \in \mathbf{U}_0$. To prove the claim, note by condition (ii) that there exists $n \in \mathbb{N}$ and $\xi_i \in \mathbf{U}_i$ $(0 \le i \le n)$ such that $\xi_n \ne 0$ and $\xi = \sum_{i=0}^{n} \xi_i$. Since ξ is NIL, $x_1\xi = 0$ and $x_3\xi = 0$. In the sum $\xi = \sum_{i=0}^{n} \xi_i$, apply x_1 to each term and use condition (iii) to find $x_1\xi_n = 0$. Similarly $x_3\xi_n = 0$, so ξ_n is NIL. Since the $\square_q$-module $\mathbf{U}$ is irreducible, it is generated by any nonzero vector in $\mathbf{U}$. In particular the $\square_q$-module $\mathbf{U}$ is generated by ξ_n. By Proposition 15.6 the map $(\mathbb{V}/J)_{\text{I}} \to \mathbf{U}$, $v + J \mapsto \kappa(v)\xi_n$ is an isomorphism of $\square_q$-modules. Consider the image of this map. On one hand, the image is equal to $\mathbf{U}$. On the other hand, by (iii) and the definition of κ above Lemma 15.4, the image is contained in $\sum_{i\in\mathbb{N}} \mathbf{U}_{n+i}$. By these comments and (i), (ii) we obtain $n = 0$, so $\xi = \xi_0 \in \mathbf{U}_0$. We have proven the claim. For $n \in \mathbb{N}$ let $\mathbf{U}'_n$ denote the subspace of $\mathbf{U}$ spanned by the vectors (15.3). We claim that $\mathbf{U}'_n = \mathbf{U}_n$ for $n \in \mathbb{N}$. By (iii) we have $\mathbf{U}'_n \subseteq \mathbf{U}_n$ for $n \in \mathbb{N}$. Earlier we mentioned an isomorphism $(\mathbb{V}/J)_{\text{I}} \to \mathbf{U}$; its existence shows that the vector space $\mathbf{U}$ is spanned by the vectors

$$u_1 u_2 \cdots u_n \xi, \qquad n \in \mathbb{N}, \qquad u_i \in \{x_0, x_2\}, \qquad 1 \le i \le n.$$

So $\mathbf{U} = \sum_{n\in\mathbb{N}} \mathbf{U}'_n$. By these comments and (ii) we obtain $\mathbf{U}'_n = \mathbf{U}_n$ for $n \in \mathbb{N}$. The claim is proven. By the claims, the sequence $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ satisfying conditions (i)–(iii) is unique, and it fits the description given in the last paragraph of the theorem statement. ∎

A   Data on the q-shuffle product

Recall the q-shuffle algebra $\mathbb{V}$ from Section 6. In the following tables we express some q-shuffle products in terms of the standard basis for $\mathbb{V}$.

	$A \star A \star B$	$A \star B \star A$	$B \star A \star A$
AAB	$q[2]_q$	$q^{-1}[2]_q$	$q^{-3}[2]_q$
ABA	$q^{-1}[2]_q$	$q^{-1}[2]_q$	$q^{-1}[2]_q$
BAA	$q^{-3}[2]_q$	$q^{-1}[2]_q$	$q[2]_q$

	$B \star B \star A$	$B \star A \star B$	$A \star B \star B$
BBA	$q[2]_q$	$q^{-1}[2]_q$	$q^{-3}[2]_q$
BAB	$q^{-1}[2]_q$	$q^{-1}[2]_q$	$q^{-1}[2]_q$
ABB	$q^{-3}[2]_q$	$q^{-1}[2]_q$	$q[2]_q$

	$A \star A \star A \star B$	$A \star A \star B \star A$	$A \star B \star A \star A$	$B \star A \star A \star A$
$AAAB$	$q^3[3]_q[2]_q$	$q[3]_q[2]_q$	$q^{-1}[3]_q[2]_q$	$q^{-3}[3]_q[2]_q$
$AABA$	$q[3]_q[2]_q$	$(2q + q^{-1})[2]_q$	$(q + 2q^{-1})[2]_q$	$q^{-1}[3]_q[2]_q$
$ABAA$	$q^{-1}[3]_q[2]_q$	$(q + 2q^{-1})[2]_q$	$(2q + q^{-1})[2]_q$	$q[3]_q[2]_q$
$BAAA$	$q^{-3}[3]_q[2]_q$	$q^{-1}[3]_q[2]_q$	$q[3]_q[2]_q$	$q^3[3]_q[2]_q$

	$B \star B \star B \star A$	$B \star B \star A \star B$	$B \star A \star B \star B$	$A \star B \star B \star B$
$BBBA$	$q^3[3]_q[2]_q$	$q[3]_q[2]_q$	$q^{-1}[3]_q[2]_q$	$q^{-3}[3]_q[2]_q$
$BBAB$	$q[3]_q[2]_q$	$(2q + q^{-1})[2]_q$	$(q + 2q^{-1})[2]_q$	$q^{-1}[3]_q[2]_q$
$BABB$	$q^{-1}[3]_q[2]_q$	$(q + 2q^{-1})[2]_q$	$(2q + q^{-1})[2]_q$	$q[3]_q[2]_q$
$ABBB$	$q^{-3}[3]_q[2]_q$	$q^{-1}[3]_q[2]_q$	$q[3]_q[2]_q$	$q^3[3]_q[2]_q$

B   Some matrix representations

Consider the free algebra $\mathbb{V}$ generated by A, B. Earlier in the paper we described many maps in $\text{End}(\mathbb{V})$. For such a map X, consider the matrix that represents X with respect to the standard basis for $\mathbb{V}$. The rows and columns are indexed by the words in $\mathbb{V}$. For words u, v the (u, v)-entry is equal to (u, Xv). We will display some of these entries shortly. For the above matrix and $r, s \in \mathbb{N}$ the submatrix $X(r, s)$ has rows and columns indexed by the words of length r and s, respectively. The matrix $X(r, s)$ has dimensions $2^r \times 2^s$. We are going to display the nonzero $X(r, s)$ such that $0 \leq r, s \leq 3$. For this display we use the following word order:

word length	word order
0	1
1	A, B
2	AA, AB, BA, BB
3	AAA, AAB, ABA, BAA, ABB, BAB, BBA, BBB

For each X we now display the nonzero $X(r, s)$ such that $0 \leq r, s \leq 3$. From the dimensions of $X(r, s)$ it is clear what is r and s, so we do not state this explicitly. We have

$$
A_L: \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad
\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
A_L^*: \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},
$$

$$
B_L: \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad
\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
B_L^*: \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad
\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
A_R: \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$A_R^*:\ (1\ \ 0),\qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_R:\ \begin{pmatrix} 0 \\ 1 \end{pmatrix},\qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},\qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_R^*:\ (0\ \ 1),\qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_\ell:\ \begin{pmatrix} 1 \\ 0 \end{pmatrix},\qquad \begin{pmatrix} q[2]_q & 0 \\ 0 & 1 \\ 0 & q^{-2} \\ 0 & 0 \end{pmatrix},\qquad \begin{pmatrix} q^2[3]_q & 0 & 0 & 0 \\ 0 & q[2]_q & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & q^{-1}[2]_q & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-2} \\ 0 & 0 & 0 & q^{-4} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_\ell^*:\ (1\ \ 0),\qquad \begin{pmatrix} q[2]_q & 0 & 0 & 0 \\ 0 & 1 & q^{-2} & 0 \end{pmatrix},\qquad \begin{pmatrix} q^2[3]_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q[2]_q & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & q^{-1}[2]_q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & q^{-2} & q^{-4} & 0 \end{pmatrix},$$

$$B_\ell:\ \begin{pmatrix} 0 \\ 1 \end{pmatrix},\qquad \begin{pmatrix} 0 & 0 \\ q^{-2} & 0 \\ 1 & 0 \\ 0 & q[2]_q \end{pmatrix},\qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-4} & 0 & 0 & 0 \\ q^{-2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & q^{-1}[2]_q & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & q[2]_q & 0 \\ 0 & 0 & 0 & q^2[3]_q \end{pmatrix},$$

$$B_\ell^*:\ (0\ \ 1),\qquad \begin{pmatrix} 0 & q^{-2} & 1 & 0 \\ 0 & 0 & 0 & q[2]_q \end{pmatrix},\qquad \begin{pmatrix} 0 & q^{-4} & q^{-2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-1}[2]_q & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & q[2]_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2[3]_q \end{pmatrix},$$

$$A_r:\ \begin{pmatrix} 1 \\ 0 \end{pmatrix},\qquad \begin{pmatrix} q[2]_q & 0 \\ 0 & q^{-2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix},\qquad \begin{pmatrix} q^2[3]_q & 0 & 0 & 0 \\ 0 & q^{-1}[2]_q & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & q[2]_q & 0 \\ 0 & 0 & 0 & q^{-4} \\ 0 & 0 & 0 & q^{-2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_r^*\colon \ (1 \ \ 0), \qquad \begin{pmatrix} q[2]_q & 0 & 0 & 0 \\ 0 & q^{-2} & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} q^2[3]_q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1}[2]_q & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & q[2]_q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-4} & q^{-2} & 1 & 0 \end{pmatrix},$$

$$B_r\colon \ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ q^{-2} & 0 \\ 0 & q[2]_q \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ q^{-2} & 0 & 0 & 0 \\ q^{-4} & 0 & 0 & 0 \\ 0 & q[2]_q & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & q^{-1}[2]_q & 0 \\ 0 & 0 & 0 & q^2[3]_q \end{pmatrix},$$

$$B_r^*\colon \ (0 \ \ 1), \qquad \begin{pmatrix} 0 & 1 & q^{-2} & 0 \\ 0 & 0 & 0 & q[2]_q \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & q^{-2} & q^{-4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q[2]_q & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & q^{-1}[2]_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2[3]_q \end{pmatrix},$$

$$K\colon \ (1), \qquad \begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix}, \qquad \begin{pmatrix} q^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-4} \end{pmatrix},$$

$$\begin{pmatrix} q^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-6} \end{pmatrix},$$

$$S\colon \ (1), \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T\colon \ (1), \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\theta\colon \ (1), \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} q[2]_q & 0 & 0 & 0 \\ 0 & 1 & q^{-2} & 0 \\ 0 & q^{-2} & 1 & 0 \\ 0 & 0 & 0 & q[2]_q \end{pmatrix},$$

$$[2]_q \begin{pmatrix} q^3[3]_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & q^{-1} & q^{-3} & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & q^{-1} & q^{-1} & 0 & 0 & 0 & 0 \\ 0 & q^{-3} & q^{-1} & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & q^{-1} & q^{-3} & 0 \\ 0 & 0 & 0 & 0 & q^{-1} & q^{-1} & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & q^{-3} & q^{-1} & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^3[3]_q \end{pmatrix},$$

$$\varphi: \quad \begin{pmatrix} 1 \end{pmatrix}, \qquad Q\begin{pmatrix} 1 & -q^{-2} \\ -q^{-2} & 1 \end{pmatrix}, \qquad Q^2\begin{pmatrix} 1 & -q^{-2} & -q^{-4} & q^{-6} \\ -q^{-1}[2]_q & q^{-1}[2]_q & 0 & 0 \\ 0 & 0 & q^{-1}[2]_q & -q^{-1}[2]_q \\ q^{-6} & -q^{-4} & -q^{-2} & 1 \end{pmatrix},$$

$$Q^3\begin{pmatrix} 1 & -q^{-2} & -q^{-4} & -q^{-6} & q^{-6} & q^{-8} & q^{-10} & -q^{-12} \\ -[3]_q & 2+q^{-2} & q^{-2} & 0 & -q^{-4} & 0 & 0 & 0 \\ 0 & 0 & q^{-1}[2]_q & q^{-3}[2]_q & -q^{-1}[2]_q & -q^{-3}[2]_q & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -q^{-2} & -q^{-2}-2q^{-4} & q^{-4}[3]_q \\ q^{-4}[3]_q & -q^{-2}-2q^{-4} & -q^{-2} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -q^{-3}[2]_q & -q^{-1}[2]_q & q^{-3}[2]_q & q^{-1}[2]_q & 0 & 0 \\ 0 & 0 & 0 & -q^{-4} & 0 & q^{-2} & 2+q^{-2} & -[3]_q \\ -q^{-12} & q^{-10} & q^{-8} & q^{-6} & -q^{-6} & -q^{-4} & -q^{-2} & 1 \end{pmatrix},$$

$$\varphi^*: \quad \begin{pmatrix} 1 \end{pmatrix}, \qquad Q\begin{pmatrix} 1 & -q^{-2} \\ -q^{-2} & 1 \end{pmatrix}, \qquad Q^2\begin{pmatrix} 1 & -q^{-1}[2]_q & 0 & q^{-6} \\ -q^{-2} & q^{-1}[2]_q & 0 & -q^{-4} \\ -q^{-4} & 0 & q^{-1}[2]_q & -q^{-2} \\ q^{-6} & 0 & -q^{-1}[2]_q & 1 \end{pmatrix},$$

$$Q^3\begin{pmatrix} 1 & -[3]_q & 0 & 0 & q^{-4}[3]_q & 0 & 0 & -q^{-12} \\ -q^{-2} & 2+q^{-2} & 0 & 0 & -q^{-2}-2q^{-4} & 0 & 0 & q^{-10} \\ -q^{-4} & q^{-2} & q^{-1}[2]_q & 0 & -q^{-2} & -q^{-3}[2]_q & 0 & q^{-8} \\ -q^{-6} & 0 & q^{-3}[2]_q & 1 & 0 & -q^{-1}[2]_q & -q^{-4} & q^{-6} \\ q^{-6} & -q^{-4} & -q^{-1}[2]_q & 0 & 1 & q^{-3}[2]_q & 0 & -q^{-6} \\ q^{-8} & 0 & -q^{-3}[2]_q & -q^{-2} & 0 & q^{-1}[2]_q & q^{-2} & -q^{-4} \\ q^{-10} & 0 & 0 & -q^{-2}-2q^{-4} & 0 & 0 & 2+q^{-2} & -q^{-2} \\ -q^{-12} & 0 & 0 & q^{-4}[3]_q & 0 & 0 & -[3]_q & 1 \end{pmatrix}.$$

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