Exact Formulas of the Transition Probabilities of the Multi-Species Asymmetric Simple Exclusion Process

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Abstract. We find the formulas of the transition probabilities of the N-particle multi-species asymmetric simple exclusion processes (ASEP), and show that the transition probabilities are written as a determinant when the order of particles in the final state is the same as the order of particles in the initial state.

Key words: ASEP; multi-species ASEP; integrable probability

2020 Mathematics Subject Classification: 82C22; 60J27

1 Introduction

The exact formulas of the transition probabilities of the *exactly solvable* models may be a good starting point to study interesting distributions and their asymptotic behaviours [2, 6, 7, 10, 14, 15, 18]. In the multi-species versions of these models, particles in the system may belong to different classes and there is a hierarchy of these classes. In this paper, we consider the multi-species version of the asymmetric simple exclusion processes (ASEP). (See [3, 5] for the multi-species version of the totally asymmetric zero range processes.) We consider N-particle systems which may consist of up to N species, labelled $1, \ldots, N$. The rules for the multi-species ASEP are as follows: a particle at $x \in \mathbb{Z}$ waits an exponential random time with rate 1 and then chooses $x+1$ with probability p or $x-1$ with probability $q = 1-p$ to jump. If the particle at x belongs to species l and the chosen site to jump is already occupied by another particle belonging to species $l' \geq l$, then the jump is prohibited, but if $l' < l$, the particle belonging to l jumps to the chosen site by interchanging sites with the particle belonging to l'. According to this rule, we may view an empty site as a "*particle*" labelled 0.

A state of the process is denoted by a pair (X, π) where $X = (x_1, \ldots, x_N) \in \mathbb{Z}^N$ with $x_1 < \cdots < x_N$ and $\pi = \pi_1 \pi_2 \cdots \pi_N$ is a permutation of a multi-set $\mathcal{M} = [i_1, \ldots, i_N]$ with elements taken from $\{1, \ldots, N\}$. Each x_i represents the position of the ith leftmost particle and π_i represents the species the ith leftmost particle belongs to.

The purposes of this paper are to provide the explicit formula of the probability $P_{(Y,\nu)}(X, \pi; t)$ that the system is in (X, π) at time t, given an initial state (Y, ν) for each $\mathcal{M}$, and to show that the formula of $P_{(Y,\nu)}(X, \pi; t)$ is in the form of a determinant in the case that all particles move only in one direction and $\nu = \pi$. If $\mathcal{M} = [i, i, \ldots, i]$, then the permutation of $[i, i, \ldots, i]$ is uniquely $ii \cdots i$, so the system is the ASEP. If $\mathcal{M} = [1, 2, \ldots, N]$, then all N particles in the system belong to all different species. The exact formulas of the transition probabilities of the ASEP, written $P_Y(X; t)$, were obtained by Tracy and Widom [15], generalizing Schütz's formulas [14] for the totally asymmetric simple exclusion process (TASEP). Tracy and Widom's

formula for the transition probabilities of the ASEP is an N-fold contour integral

$$P_Y(X;t) = \oint_c \cdots \oint_c \sum_{\sigma \in S_N} A_\sigma \prod_{i=1}^N \left(\xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\varepsilon(\xi_i)t}\right) d\xi_1 \cdots d\xi_N, \tag{1.1}$$

where the notation $\oint_c$ implies $\frac{1}{2\pi i} \int_c$ and the contour c is a circle centered at the origin with *sufficiently small* radius. The sum inside the integrals in (1.1) is over all permutations σ in the symmetric group S_N and

$$A_\sigma = \prod_{(\beta,\alpha)} S_{\beta\alpha}, \tag{1.2}$$

where

$$S_{\beta\alpha} = -\frac{p + q\xi_\alpha\xi_\beta - \xi_\beta}{p + q\xi_\alpha\xi_\beta - \xi_\alpha}$$

and

$$\varepsilon(\xi_i) = \frac{p}{\xi_i} + q\xi_i - 1.$$

The product in (1.2) is over all inversions (β, α) with $\beta > \alpha$ in σ, and if σ is the identity permutation, written Id, then we define $A_\sigma = 1$. Tracy and Widom have shown that there is a formula analogous to (1.1) for the transition probabilities of the multi-species ASEP, written as

$$P_{(Y,\nu)}(X, \pi; t) = \oint_c \cdots \oint_c \sum_{\sigma \in S_n} A_\sigma^{\nu,\pi} \prod_{i=1}^N \left(\xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\varepsilon(\xi_i)t}\right) d\xi_1 \cdots d\xi_N,$$

in our notation [17]. But, the formula of $A_\sigma^{\nu,\pi}$ was not given explicitly except a special case in [17, p. 458]. In this paper, we provide a method to find the formula of $A_\sigma^{\nu,\pi}$. This is elaborated in Section 2. In Section 3, we find a determinantal expression for transition probabilities.

2 Transition probabilities

We first review the basic concepts used in [1, 8, 9] to extend the results for the TASEP with second class particles there to the multi-species ASEP. The state space of the multi-species ASEP with N particles is countable, so we may view $P_{(Y,\nu)}(X, \pi; t)$ as a matrix element of an infinite matrix, denoted $\mathbf{P}(t)$, which is a member of a probability semigroup $\{\mathbf{P}(t) \colon t \geq 0\}$. We assume that the rows are labelled by (X, π) and the columns are labelled by (Y, ν), following a rule that if $\pi \prec \pi'$ where $\prec$ is the lexicographical order, then the row (X, π') is below the row (X, π) for a given X, and similarly, the column (Y, ν') is to the right of the column (Y, ν) if $\nu \prec \nu'$ for a given Y. For fixed X and Y, let $\mathbf{P}_Y(X; t)$ be a sub-matrix of $\mathbf{P}(t)$, consisting of the rows labelled $(X, \cdot)$ and the columns labelled $(Y, \cdot)$ of $\mathbf{P}(t)$. Then, $\mathbf{P}_Y(X; t)$ is the $N^N \times N^N$ matrix in the form of

$$\mathbf{P}_Y(X;t) = \begin{bmatrix} P_{(Y,1\cdots1)}(X, 1\cdots11; t) & P_{(Y,1\cdots12)}(X, 1\cdots11; t) & \cdots & P_{(Y,N\cdots N)}(X, 1\cdots11; t) \\ P_{(Y,1\cdots1)}(X, 1\cdots12; t) & P_{(Y,1\cdots12)}(X, 1\cdots12; t) & \cdots & P_{(Y,N\cdots N)}(X, 1\cdots12; t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{(Y,1\cdots1)}(X, N\cdots N; t) & P_{(Y,1\cdots12)}(X, N\cdots N; t) & & P_{(Y,N\cdots N)}(X, N\cdots N; t) \end{bmatrix}.$$

The labels of the rows and the columns of this matrix are the permutations of the multi-sets of cardinality N with elements taken from $\{1, \ldots, N\}$ in the lexicographical order from the top to the bottom and from the left to the right, respectively.

The semigroup $\{\mathbf{P}(t) : t \geq 0\}$ is uniform, and $\mathbf{P}(t)$ is the unique solution to the forward equation which is an element-wise matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{P}(t) = \mathbf{G} \mathbf{P}(t)$$

subject to the initial condition $\mathbf{P}(0) = \mathbf{I}_\infty$ where $\mathbf{I}_\infty$ is the infinite identity matrix and $\mathbf{G}$ is the generator. (We use the notation $\mathbf{I}_n$ for $n \times n$ identity matrix and $\mathbf{0}_n$ for $n \times n$ zero matrix.) The initial condition $\mathbf{P}(0) = \mathbf{I}_\infty$ implies that the sub-matrices $\mathbf{P}_Y(X; t)$ satisfy

$$\mathbf{P}_Y(X; 0) = \begin{cases} \mathbf{I}_{N^N} & \text{if } X = Y, \\ \mathbf{0}_{N^N} & \text{otherwise.} \end{cases}$$

By the simple extension of Section 2.1 in [9] for the TASEP with second class particles to the ASEP with second class particles, we can obtain a matrix which generalizes (2.11) in [9],

$$\widetilde{\mathbf{R}}_{\beta\alpha} = \begin{bmatrix} S_{\beta\alpha} & 0 & 0 & 0 \\ 0 & P_{\beta\alpha} & pT_{\beta\alpha} & 0 \\ 0 & qT_{\beta\alpha} & Q_{\beta\alpha} & 0 \\ 0 & 0 & 0 & S_{\beta\alpha} \end{bmatrix},$$

where

$$S_{\beta\alpha} = -\frac{p + q\xi_\alpha\xi_\beta - \xi_\beta}{p + q\xi_\alpha\xi_\beta - \xi_\alpha}, \qquad P_{\beta\alpha} = \frac{(p - q\xi_\alpha)(\xi_\beta - 1)}{p + q\xi_\alpha\xi_\beta - \xi_\alpha},$$

$$T_{\beta\alpha} = \frac{\xi_\beta - \xi_\alpha}{p + q\xi_\alpha\xi_\beta - \xi_\alpha}, \qquad Q_{\beta\alpha} = \frac{(p - q\xi_\beta)(\xi_\alpha - 1)}{p + q\xi_\alpha\xi_\beta - \xi_\alpha}.$$

Notice that the denominators of the nonzero elements in $\widetilde{\mathbf{R}}_{\beta\alpha}$ have the same form as the denominator of $S_{\beta\alpha}$ in [15, 16]. This fact makes it easy to prove *the initial condition* later. The matrix $\widetilde{\mathbf{R}}_{\beta\alpha}$ is a building block for N-particle system. In N-particle ASEP with multi-species, a matrix corresponding to $\mathbf{B}$ in (2.6) of [9] which is for a two-particle TASEP is an $N^2 \times N^2$ matrix $\mathbf{B}$ with

$$\left[\mathbf{B}\right]_{ij,kl} = \begin{cases} 1 & \text{if } ij = kl \text{ with } i = j, \\ p & \text{if either } ij = kl \text{ or } ij = lk \text{ with } i < j, \\ q & \text{if either } ij = kl \text{ or } ij = lk \text{ with } i > j, \\ 0 & \text{for all other cases,} \end{cases}$$

where ij and kl are labels for rows and columns, represented by $11, 12, \ldots, NN$. In the same way as $\widetilde{\mathbf{R}}_{\beta\alpha}$ is obtained, we define an $N^2 \times N^2$ matrix $\mathbf{R}_{\beta\alpha}$ by

$$\mathbf{R}_{\beta\alpha} = -\left[(p + q\xi_\alpha\xi_\beta)\mathbf{I}_{N^2} - \xi_\alpha\mathbf{B}\right]^{-1}\left[(p + q\xi_\alpha\xi_\beta)\mathbf{I}_{N^2} - \xi_\beta\mathbf{B}\right],$$

where $\alpha, \beta = 1, \ldots, N$ and $\alpha \neq \beta$. Here, $\mathbf{I}_{N^2}$ is the $N^2 \times N^2$ identity matrix. The entries of $\mathbf{R}_{\beta\alpha}$ are given by

$$\left[\mathbf{R}_{\beta\alpha}\right]_{ij,kl} = \begin{cases} S_{\beta\alpha} & \text{if } ij = kl \text{ with } i = j, \\ P_{\beta\alpha} & \text{if } ij = kl \text{ with } i < j, \\ Q_{\beta\alpha} & \text{if } ij = kl \text{ with } i > j, \\ pT_{\beta\alpha} & \text{if } ij = lk \text{ with } i < j, \\ qT_{\beta\alpha} & \text{if } ij = lk \text{ with } i > j, \\ 0 & \text{for all other cases.} \end{cases}$$

If $\mathbf{U}(X;t) = \mathbf{U}(x_1,\ldots,x_N;t)$ be an $N^N \times N^N$ matrix whose entries are functions on $\mathbb{Z}^N \times [0,\infty)$ which satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}(X;t) = \sum_{i=1}^{N} \big[p\mathbf{U}(x_1,\ldots,x_{i-1},x_i-1,x_{i+1},\ldots,x_N;t)$$
$$+ q\mathbf{U}(x_1,\ldots,x_{i-1},x_i+1,x_{i+1},\ldots,x_N;t)\big] - N\mathbf{U}(x_1,\ldots,x_N;t), \quad (2.1)$$

and the initial condition

$$\mathbf{U}(x_1,\ldots,x_N;0) = \begin{cases} \mathbf{I}_{N^N} & \text{if } (x_1,\ldots,x_N) = (y_1,\ldots,y_N) \text{ and } x_1 < \cdots < x_N, \\ \mathbf{0}_{N^N} & \text{if } (x_1,\ldots,x_N) \neq (y_1,\ldots,y_N) \text{ and } x_1 < \cdots < x_N \end{cases} \quad (2.2)$$

for a given initial positions of particles $(y_1,\ldots,y_N)$ with $y_1 < \cdots < y_N$, and the *boundary condition*

$$p\mathbf{U}(x_1,\ldots,x_{i-1},x_i,x_i,x_{i+2},\ldots,x_N;t) + q\mathbf{U}(x_1,\ldots,x_{i-1},x_i+1,x_i+1,x_{i+2},\ldots,x_N;t)$$
$$= \big(\mathbf{I}_N^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{\otimes(N-i-1)}\big)\mathbf{U}(x_1,\ldots,x_{i-1},x_i,x_i+1,x_{i+2},\ldots,x_N;t) \quad (2.3)$$

for all $i = 1,\ldots,N-1$, then we may assert that the restriction of $\mathbf{U}(x_1,\ldots,x_N;t)$ on $\{(x_1,\ldots,x_N) \in \mathbb{Z}^N : x_1 < \cdots < x_N\}$ is $\mathbf{P}_Y(X;t)$. The solution of (2.1) by the Bethe ansatz is given

$$\sum_{\sigma \in S_N} \mathbf{A}_\sigma \prod_{i=1}^{N} \big(\xi_{\sigma(i)}^{x_i-y_{\sigma(i)}-1} \mathrm{e}^{\varepsilon(\xi_i)t}\big), \quad (2.4)$$

where

$$\varepsilon(\xi_i) = \frac{p}{\xi_i} + q\xi_i - 1$$

for $\xi_i \in \mathbb{C} \setminus \{0\}$, where $\mathbf{A}_\sigma$ is an $N^N \times N^N$ matrix whose entries are independent of $x_1,\ldots,x_N$ and t. Next, we will construct the matrix $\mathbf{A}_\sigma$ so that (2.4) satisfies the boundary condition (2.3).

Step 1. Let T_i be a simple transposition which interchanges the ith entry and the $(i+1)st$ entry and leaves the other entries fixed, that is,

$$T_i\sigma = \sigma',$$

where $\sigma = \sigma(1)\cdots\sigma(i)\sigma(i+1)\cdots\sigma(N)$ and $\sigma' = \sigma(1)\cdots\sigma(i-1)\sigma(i+1)\sigma(i)\sigma(i+2)\cdots\sigma(N)$. Since $T_1,\ldots,T_{N-1}$ generate S_N, there is a finite sequence $a_1,\ldots,a_n$ where each a_i belongs to $\{1,2,\ldots,N-1\}$ such that

$$\sigma = T_{a_n}\cdots T_{a_1}$$

for any given $\sigma \in S_N$.

Step 2. For each $1 \le l \le N-1$, we define

$$\mathbf{T}_l(\beta,\alpha) := \mathbf{I}_N^{\otimes(l-1)} \otimes \mathbf{R}_{\beta\alpha} \otimes \mathbf{I}_N^{\otimes(N-l-1)}. \quad (2.5)$$

Step 3. Let us denote $T_{a_k}\cdots T_{a_1}$ by $\sigma^{(k)}$ and let $\sigma^{(0)}$ be the identity permutation. Define

$$\mathbf{A}_\sigma := \mathbf{T}_{a_n}\big(\sigma^{(n-1)}(a_n+1),\sigma^{(n-1)}(a_n)\big)\cdots\mathbf{T}_{a_1}\big(\sigma^{(0)}(a_1+1),\sigma^{(0)}(a_1)\big). \quad (2.6)$$

Lemma 2.1. *If $\mathbf{A}_\sigma$ is given as in (2.6), then (2.4) satisfies (2.3) for each i.*

Proof. Substituting (2.4) into (2.3) for i,

$$\sum_{\sigma \in S_N} \big(p\mathbf{A}_\sigma \xi_{\sigma(1)}^{x_1} \cdots \xi_{\sigma(i-1)}^{x_{i-1}} \xi_{\sigma(i)}^{x_i} \xi_{\sigma(i+1)}^{x_i} \xi_{\sigma(i+2)}^{x_{i+2}} \cdots \xi_{\sigma(N)}^{x_N}$$

$$+ q\mathbf{A}_\sigma \xi_{\sigma(1)}^{x_1} \cdots \xi_{\sigma(i-1)}^{x_{i-1}} \xi_{\sigma(i)}^{x_i+1} \xi_{\sigma(i+1)}^{x_i+1} \xi_{\sigma(i+2)}^{x_{i+2}} \cdots \xi_{\sigma(N)}^{x_N} \tag{2.7}$$

$$- \big(\mathbf{I}_N^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{\otimes(N-i-1)}\big) \mathbf{A}_\sigma \xi_{\sigma(1)}^{x_1} \cdots \xi_{\sigma(i-1)}^{x_{i-1}} \xi_{\sigma(i)}^{x_i} \xi_{\sigma(i+1)}^{x_i+1} \xi_{\sigma(i+2)}^{x_{i+2}} \cdots \xi_{\sigma(N)}^{x_N} \big) = \mathbf{0}_{N^N}.$$

If σ' is an even permutation, then $T_i\sigma'$ is an odd permutation, and vice-versa. So, if we express (2.7) as a sum over the alternating group A_N,

$$\sum_{\sigma' \in A_N} \Big(\big[\mathbf{I}_{N^N} \big(p + q\xi_{\sigma'(i)}\xi_{\sigma'(i+1)} \big) - \big(\mathbf{I}_N^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{\otimes(N-i-1)} \big) \xi_{\sigma'(i+1)} \big] \mathbf{A}_{\sigma'}$$

$$+ \big[\mathbf{I}_{N^N} \big(p + q\xi_{\sigma'(i+1)}\xi_{\sigma'(i)} \big) - \big(\mathbf{I}_N^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{\otimes(N-i-1)} \big) \xi_{\sigma'(i)} \big] \mathbf{A}_{T_i\sigma'} \big)$$

$$\times \xi_{\sigma'(1)}^{x_1} \cdots \xi_{\sigma'(i-1)}^{x_{i-1}} \xi_{\sigma'(i)}^{x_i} \xi_{\sigma'(i+1)}^{x_i} \xi_{\sigma'(i+2)}^{x_{i+2}} \cdots \xi_{\sigma'(N)}^{x_N} = \mathbf{0}_{N^N}. \tag{2.8}$$

A sufficient condition for (2.8) is that for each $\sigma \in S_N$,

$$\mathbf{A}_{T_i\sigma} = -\big[\mathbf{I}_{N^N} \big(p + q\xi_{\sigma(i+1)}\xi_{\sigma(i)} \big) - \big(\mathbf{I}_N^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{\otimes(N-i-1)} \big) \xi_{\sigma(i)} \big]^{-1}$$

$$\times \big[\mathbf{I}_{N^N} \big(p + q\xi_{\sigma(i)}\xi_{\sigma(i+1)} \big) - \big(\mathbf{I}_N^{\otimes(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{\otimes(N-i-1)} \big) \xi_{\sigma(i+1)} \big] \mathbf{A}_\sigma$$

$$= -\big(\mathbf{I}_{N^{(i-1)}} \otimes \big[\mathbf{I}_{N^2} \big(p + q\xi_{\sigma(i+1)}\xi_{\sigma(i)} \big) - \mathbf{B}\xi_{\sigma(i)} \big]^{-1} \otimes \mathbf{I}_{N^{(N-i-1)}} \big)$$

$$\times \big(\mathbf{I}_{N^{(i-1)}} \otimes \big[\mathbf{I}_{N^2} \big(p + q\xi_{\sigma(i+1)}\xi_{\sigma(i)} \big) - \mathbf{B}\xi_{\sigma(i+1)} \big] \otimes \mathbf{I}_{N^{(N-i-1)}} \big) \mathbf{A}_\sigma$$

$$= \big(\mathbf{I}_{N^{(i-1)}} \otimes \mathbf{R}_{\sigma(i+1)\sigma(i)} \otimes \mathbf{I}_{N^{(N-i-1)}} \big) \mathbf{A}_\sigma. \tag{2.9}$$

Now, it remains to show that if $\mathbf{A}_\sigma$ is given by (2.6), then (2.9) is satisfied. This is immediately obtained because

$$\mathbf{A}_{T_i\sigma} = \mathbf{T}_i \mathbf{T}_{a_n} \cdots \mathbf{T}_{a_1} = \mathbf{T}_i\big(\sigma(i+1), \sigma(i)\big)\mathbf{A}_\sigma$$

$$= \big(\mathbf{I}_{N^{(i-1)}} \otimes \mathbf{R}_{\sigma(i+1)\sigma(i)} \otimes \mathbf{I}_{N^{(N-i-1)}} \big) \mathbf{A}_\sigma$$

by (2.5). $\blacksquare$

Remark 2.2. The consistency relations

$$\mathbf{T}_i(\beta, \alpha)\mathbf{T}_j(\delta, \gamma) = \mathbf{T}_j(\delta, \gamma)\mathbf{T}_i(\beta, \alpha) \qquad \text{if } |i - j| \geq 2,$$

$$\mathbf{T}_i(\gamma, \beta)\mathbf{T}_j(\gamma, \alpha)\mathbf{T}_i(\beta, \alpha) = \mathbf{T}_j(\beta, \alpha)\mathbf{T}_i(\gamma, \alpha)\mathbf{T}_j(\gamma, \beta) \qquad \text{if } |i - j| = 1,$$

$$\mathbf{T}_i(\beta, \alpha)\mathbf{T}_i(\alpha, \beta) = \mathbf{I}_{N^N} \tag{2.10}$$

can be directly verified by the matrix multiplication. In particular, (2.10) with $p = 1$ was verified in [1].

Now, we choose $\mathbf{A}_{1\cdots N} = \mathbf{I}_{N^N}$ and take a multiple-fold contour integral of each element of the matrix (2.6) with respect to $\xi_1, \ldots, \xi_N$ over sufficiently small circles centered at the origin, denoting the matrix by

$$\oint_c \cdots \oint_c \sum_{\sigma \in S_N} \mathbf{A}_\sigma \prod_{i=1}^N \big(\xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\varepsilon(\xi_i)t} \big) d\xi_1 \cdots d\xi_N. \tag{2.11}$$

Finally, it should be proved that (2.11) satisfies the initial condition (2.2), but its proof is essentially the same as the proof for the ASEP in [16, 17] because the poles arising in the integrands are the same as the poles in the ASEP formulas. So, we omit the proof and conclude that

$$\mathbf{P}_Y(X; t) = \oint_c \cdots \oint_c \sum_{\sigma \in S_N} \mathbf{A}_\sigma \prod_{i=1}^N \big(\xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\varepsilon(\xi_i)t} \big) d\xi_1 \cdots d\xi_N. \tag{2.12}$$

3   Determinantal formulas

In this section we focus on the diagonal of the matrix (2.12) for the totally asymmetric case $p = 1$, that is, the transition probabilities $P_{(Y,\pi)}(X, \pi; t)$. In the case of the totally asymmetric model ($p = 1$), if the order of particles at $t = 0$ is represented by a permutation π of a given multi-set and the order of particles at time $t = t_0 > 0$ is also given by π, then the order of particles at each time $0 \le t \le t_0$ must be π. If all particles belong to the same species, then $P_{(Y,i\cdots i)}(X, i \cdots i; t)$ is simply the TASEP's transition probability. Schütz found its determinantal formula [14]. For the TASEP with second class particles (two-species TASEP), Chatterjee and Schütz found the determinantal formulas for the transition probabilities when there is no change in order of particles [1]. We will extend Chatterjee and Schütz's result to the multi-species TASEP. Let

$$F_n(x; t) = \int_c \xi^{x-1}(1 - \xi)^{-n} e^{(1/\xi - 1)t} d\xi.$$

For a given permutation $\pi = \pi_1 \cdots \pi_N$ of a multi-set, let $\#_{ij}$ be the number of pairs of places $(i, i+1)$ between the ith place and jth place such that $\pi_i > \pi_{i+1}$. For example, if $\pi = 214311$, then $\#_{12} = 1$, $\#_{14} = 2$, and $\#_{15} = \#_{16} = 3$. Obviously, $\#_{ij} = \#_{ji}$. Let

$$n(i, j) = \text{sgn}(i - j) \times (|i - j| - \#_{ij}).$$

If we denote the diagonal elements of $\mathbf{U}(x_1, \ldots, x_N; t)$ by $U_{\pi,\pi}(x_1, \ldots, x_N; t)$ where $\pi = \pi_1 \cdots \pi_N$, then the equations for the diagonal of (2.1), (2.2) and (2.3) are

$$\frac{d}{dt} U_{\pi,\pi}(x_1, \ldots, x_N; t) = \sum_{i=1}^{N} U_{\pi,\pi}(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_N; t)$$
$$- N U_{\pi,\pi}(x_1, \ldots, x_N; t), \tag{3.1}$$

$$U_{\pi,\pi}(x_1, \ldots, x_N; 0) = \prod_{i=1}^{N} \delta_{x_i y_i} \qquad \text{when} \quad x_1 < \cdots < x_N \ \text{ for given } y_1 < \cdots < y_N \tag{3.2}$$

and

$$U_{\pi,\pi}(x_1, \ldots, x_i, x_i, x_{i+2}, \ldots, x_N; t)$$
$$= \begin{cases} U_{\pi,\pi}(x_1, \ldots, x_i, x_i + 1, x_{i+2}, \ldots, x_N; t), & \text{if } \pi_i \le \pi_{i+1}, \\ 0 & \text{if } \pi_i > \pi_{i+1}. \end{cases} \tag{3.3}$$

The following theorem gives the solution to (3.1), (3.2), and (3.3).

Theorem 3.1. *For given $Y = (y_1, \ldots, y_N)$ with $y_1 < \cdots < y_N$ and π, define*

$$G(x_1, \ldots, x_N; t) = \begin{bmatrix} F_{n(1,1)}(x_1 - y_1; t) & F_{n(1,2)}(x_1 - y_2; t) & \cdots & F_{n(1,N)}(x_1 - y_N; t) \\ F_{n(2,1)}(x_2 - y_1; t) & F_{n(2,2)}(x_2 - y_2; t) & \cdots & F_{n(2,N)}(x_2 - y_N; t) \\ \vdots & \vdots & \ddots & \vdots \\ F_{n(N,1)}(x_N - y_1; t) & F_{n(N,2)}(x_N - y_2; t) & \cdots & F_{(N,N)}(x_N - y_N; t) \end{bmatrix}.$$

Then,

$$P_{(Y,\pi)}(x_1, \ldots, x_N, \pi; t) = \det G(x_1, \ldots, x_N; t).$$

Proof. (i) To show that $\det G(x_1, \ldots, x_N; t)$ satisfies (3.1). Since

$$\frac{d}{dt} F_{n(i,j)}(x_i - y_j; t) = -F_{n(i,j)}(x_i - y_j; t) + F_{n(i,j)}(x_i - 1 - y_j; t),$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \det G(x_1, \ldots, x_N; t)$$

$$= \begin{vmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}F_{n(1,1)}(x_1 - y_1; t) & \dfrac{\mathrm{d}}{\mathrm{d}t}F_{n(1,2)}(x_1 - y_2; t) & \cdots & \dfrac{\mathrm{d}}{\mathrm{d}t}F_{n(1,N)}(x_1 - y_N; t) \\ F_{n(2,1)}(x_2 - y_1; t) & F_{n(2,2)}(x_2 - y_2; t) & \cdots & F_{n(2,N)}(x_2 - y_N; t) \\ \vdots & \vdots & \ddots & \vdots \\ F_{n(N,1)}(x_N - y_1; t) & F_{n(N,2)}(x_N - y_2; t) & \cdots & F_{(N,N)}(x_N - y_N; t) \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} F_{n(1,1)}(x_1 - y_1; t) & F_{n(1,2)}(x_1 - y_2; t) & \cdots & F_{n(1,N)}(x_1 - y_N; t) \\ F_{n(2,1)}(x_2 - y_1; t) & F_{n(2,2)}(x_2 - y_2; t) & \cdots & F_{n(2,N)}(x_2 - y_N; t) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\mathrm{d}}{\mathrm{d}t}F_{n(N,1)}(x_N - y_1; t) & \dfrac{\mathrm{d}}{\mathrm{d}t}F_{n(N,2)}(x_N - y_2; t) & \cdots & \dfrac{\mathrm{d}}{\mathrm{d}t}F_{(N,N)}(x_N - y_N; t) \end{vmatrix}$$

$$= -N \det G(x_1, \ldots, x_N; t) + \sum_{i=1}^{N} \det G(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_N; t).$$

(ii) To show that $\det G(x_1, \ldots, x_N; t)$ satisfies (3.3): Suppose that $\pi_i > \pi_{i+1}$. First, we will show that $n(i, j) = n(i + 1, j)$. If $j > i + 1$, then

$$n(i, j) = (-1) \times (j - i - \#_{ij}) = (-1) \times \big[j - (i + 1) - \underbrace{(\#_{ij} - 1)}_{\#_{(i+1)j}} \big] = n(i + 1, j).$$

If $j < i - 1$, then

$$n(i, j) = 1 \times (i - j - \#_{ij}) = 1 \times \big[i + 1 - j - \underbrace{(\#_{ij} + 1)}_{\#_{(i+1)j}} \big] = n(i + 1, j).$$

If $i = j$ or $j = i \pm 1$, then

$$n(i, j) = 0 = n(i + 1, j).$$

Hence, the ith row and the $(i + 1)$th row of $G(x_1, \ldots, x_i, x_i, x_{i+2}, \ldots, x_N; t)$ are the same and

$$\det G(x_1, \ldots, x_i, x_i, x_{i+2}, \ldots, x_N; t) = 0.$$

Next, suppose that $\pi_i \leq \pi_{i+1}$, so $\#_{ij} = \#_{(i+1)j}$. We will show that $n(i, j) = n(i + 1, j) - 1$. If $j > i + 1$, then

$$n(i + 1, j) = (-1) \times (j - i - 1 - \#_{(i+1)j}) = (-1) \times (j - i - \#_{ij}) + 1 = n(i, j) + 1.$$

If $j < i - 1$, then

$$n(i + 1, j) = 1 \times (i + 1 - j - \#_{(i+1)j}) = 1 \times (i - j - \#_{ij}) + 1 = n(i, j) + 1.$$

If $i = j$, then $n(i + 1, j) = 1 = n(i, i) + 1$. If $j = i + 1$, then $n(i + 1, j) = 0 = n(i, j) + 1$. If $j = i - 1$, then

$$n(i + 1, j) = (+1) \times (2 - \#_{(i-1)(i+1)}) = (+1) \times (2 - \#_{(i-1)i})$$
$$= (+1) \times (1 - \#_{(i-1)i}) + 1 = n(i, i - 1) + 1 = n(i, j) + 1. \tag{3.4}$$

Also, using the identity

$$F_n(x+1;t) = F_n(x;t) - F_{n-1}(x;t),$$

we obtain

$$
\begin{vmatrix}
\vdots & & \vdots & & \vdots \\
F_{n(i,1)}(x_i - y_1;t) & \cdots & & F_{n(i,N)}(x_i - y_N;t) \\
F_{n(i+1,1)}(x_i + 1 - y_1;t) & \cdots & & F_{n(i+1,N)}(x_i + 1 - y_N;t) \\
\vdots & & \vdots & & \vdots
\end{vmatrix}
$$

$$
=
\begin{vmatrix}
\vdots & & \vdots & & \vdots \\
F_{n(i,1)}(x_i - y_1;t) & \cdots & & F_{n(i,N)}(x_i - y_N;t) \\
F_{n(i+1,1)}(x_i - y_1;t) & \cdots & & F_{n(i+1,N)}(x_i - y_N;t) \\
\vdots & & \vdots & & \vdots
\end{vmatrix}
$$

$$
-
\begin{vmatrix}
\vdots & & \vdots & & \vdots \\
F_{n(i,1)}(x_i - y_1;t) & \cdots & & F_{n(i,N)}(x_i - y_N;t) \\
F_{n(i+1,1)-1}(x_i - y_1;t) & \cdots & & F_{n(i+1,N)-1}(x_i - y_N;t) \\
\vdots & & \vdots & & \vdots
\end{vmatrix},
\tag{3.5}
$$

where the second term on the right-hand side of (3.5) is zero by (3.4). Hence,

$$\det G(x_1, \ldots, x_i, x_i + 1, x_{i+2}, \ldots, x_N;t) = \det G(x_1, \ldots, x_i, x_i, x_{i+2}, \ldots, x_N;t).$$

(iii) To show that $\det G(x_1, \ldots, x_N)$ satisfies (3.2): Since $F_{n(i,i)}(x_i - y_i;0) = \delta_{x_i y_i}$ and $F_{n(i,j)}(x_i - y_j;0) = 0$ for $i > j$ because $x_i - y_j \geq 1$ when $x_1 < \cdots < x_N$ for given $y_1 < \cdots < y_N$,

$$\det G(x_1, \ldots, x_N;0) = \prod_{i=1}^{N} \delta_{x_i y_i}$$

when $x_1 < \cdots < x_N$ for given $y_1 < \cdots < y_N$.   ■

When $\pi = (1\ 1\ \cdots\ 1)$, that is, for the TASEP with single species, $F_{n(i,j)}(x_i - y_j;t) = F_{i-j}(x_i - y_j;t)$ has some interesting properties, for example, [11, equation (4)], which are needed to find one-point distributions and multi-point distributions [11, 12, 13]. At this moment, it is not clear if $F_{n(i,j)}(x_i - y_j;t)$ has similar properties for general π. It would be interesting to see if there is a π other than $(1\ 1\ \cdots\ 1)$ whose $F_{n(i,j)}(x_i - y_j;t)$ has nice properties to enable us to find the one-point and the multi-point distributions. Also, it is notable that Kuan recently obtained a determinantal expression for a multi-point distribution in a different context (that is, of *the maximum species number among all particles to the left of a point* (see [4] for detailed definition) in the inhomogeneous multi-species TASEP [4]. But, the determinantal formulas in this paper are for the transition probabilities ($\pi \to \pi$) for all possible cases that particles belong to some species.

Acknowledgements

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