Exceptional Legendre Polynomials and Confluent Darboux Transformations

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Abstract. Exceptional orthogonal polynomials are families of orthogonal polynomials that arise as solutions of Sturm–Liouville eigenvalue problems. They generalize the classical families of Hermite, Laguerre, and Jacobi polynomials by allowing for polynomial sequences that miss a finite number of "exceptional" degrees. In this paper we introduce a new construction of multi-parameter exceptional Legendre polynomials by considering the isospectral deformation of the classical Legendre operator. Using confluent Darboux transformations and a technique from inverse scattering theory, we obtain a fully explicit description of the operators and polynomials in question. The main novelty of the paper is the novel construction that allows for exceptional polynomial families with an arbitrary number of real parameters.

Key words: exceptional orthogonal polynomials; Darboux transformations; isospectral deformations

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1 Introduction and main results

Exceptional orthogonal polynomials (XOPs) are complete families of orthogonal polynomials that arise as eigenfunctions of a Sturm–Liouville eigenvalue problem [11]. XOPs are more general than classical OPs, because the degree sequence of the polynomial family can have a finite number of missing, "exceptional" degrees. As in the classical theory, XOPs fall into three broad classes: Hermite, Laguerre and Jacobi, depending on whether the domain of orthogonality is the full line, the half-line or a finite interval [7]. Unlike the classical case, the corresponding exceptional second-order operator has rational rather than polynomial coefficients.

Exceptional polynomials appear in mathematical physics as bound states of exactly solvable rational extensions [10, 24, 26] and exact solutions to Dirac's equation [27]. They appear also in connection with super-integrable systems [21, 25] and finite-gap potentials [17]. From a mathematical point of view, the main results are concerned with the full classification of exceptional polynomials [7, 12], properties of their zeros [14, 18, 20], and recurrence relations [5, 13, 22, 23].

At the time of this writing, the most general construction of exceptional Jacobi polynomials [3, 6] involves a finite number of *discrete* parameters and is given in terms of a Wronskian-

like determinant of classical Jacobi polynomials, indexed by two partitions. The purpose of this note is to show that the class of exceptional orthogonal polynomials is much richer than previously thought. We do this by studying the class of exceptional Legendre polynomials, which cannot be obtained using the standard approach of multi-step Darboux transformations indexed by partitions. The main novelty of the new families is that they contain an arbitrary number of *continuous* deformation parameters. Another innovation is the use of integral rather than differential operators in the construction of the exceptional polynomials.

Definition 1.1. Let $\tau = \tau(z)$ be a polynomial. We say that the operator

$$\hat{T}(\tau) = \left(1 - z^2\right)\left(D_z^2 - 2\frac{\tau_z}{\tau}D_z + \frac{\tau_{zz}}{\tau}\right) - 2zD_z \tag{1.1}$$

is an *exceptional Legendre operator* if there exist polynomials $\left\{\hat{P}_i(z)\right\}_{i\in\mathbb{N}_0}$ and constants $\{\lambda_i\}_{i\in\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, \dots\}$, such that

$$\hat{T}(\tau)\hat{P}_i = \lambda_i\hat{P}_i$$

and such that the degree sequence $\left\{\deg\hat{P}_i\right\}_{i\in\mathbb{N}_0}$ is missing finitely many "exceptional" degrees [11].

Note that, in making this definition, we are *not* assuming that $\deg\hat{P}_i = i$.

Remark 1.2. As a direct consequence of this definition, if $\tau(z)$ has no zeros on $[-1, 1]$ and if the eigenvalues are distinct, then the resulting eigenpolynomials are orthogonal relative to the inner product

$$\int_{-1}^{1}\frac{\hat{P}_{i_1}(z)\hat{P}_{i_2}(z)}{\tau(z)^2}\mathrm{d}z = 0, \qquad i_1 \neq i_2. \tag{1.2}$$

In this case, the eigenpolynomials $\left\{\hat{P}_i(z)\right\}_{i\in\mathbb{N}_0}$ may define a complete orthogonal polynomial system, which motivates the following definition.

Definition 1.3. Let $\tau(z)$ be a polynomial that does not vanish in $[-1, 1]$. The set $\left\{\hat{P}_i(z)\right\}_{i\in\mathbb{N}_0}$ is a family of *exceptional Legendre polynomials* if

(i) $\left\{\hat{P}_i(z)\right\}_{i\in\mathbb{N}_0}$ are eigenfunctions of a Sturm–Liouville problem in $[-1, 1]$.

(ii) $\left\{\deg\hat{P}_i\right\}_{i\in\mathbb{N}_0}$ contains all but finitely many positive integers.

(iii) The polynomials $\left\{\hat{P}_i(z)\right\}_{i\in\mathbb{N}_0}$ satisfy the orthogonality relation (1.2).

(iv) The polynomials $\left\{\hat{P}_i(z)\right\}_{i\in\mathbb{N}_0}$ form a complete set in the Hilbert space $\mathrm{L}^2\left([-1, 1], \tau^{-2}\mathrm{d}z\right)$.

In other words, exceptional Legendre polynomials are just exceptional polynomials defined in $[-1, 1]$ with orthogonality weight $W(z) = \tau(z)^{-2}$, where $\tau(z)$ is a polynomial not vanishing in $[-1, 1]$. It should be noted, however, that the standard construction of exceptional Jacobi polynomials based on a multi-index determinant labelled by two partitions [6, equation (5.1)] does not allow parameters $\alpha = \beta = 0$ (see [3, equation (2.36)]). Thus, the construction of exceptional Legendre polynomials requires a different approach, which we present in this paper.

It is known [7, Theorem 1.2] that every exceptional operator can be related to a classical Bochner operator by a finite number of Darboux transformations. This is true in particular for an exceptional operator having the form (1.1), where the degree of $\tau(z)$ is equal to the number of exceptional degrees. The new exceptional polynomial families introduced in this paper do not invalidate the classification result [7, Theorem 1.2], but rather they highlight the fact that the

full class of Darboux transformations leading to exceptional polynomials is larger than previously thought.

As a matter of fact, we will consider in this paper a new class of exceptional operators that are obtained from the classical Legendre operator

$$T := \hat{T}(1) = (1 - z^2)D_z^2 - 2zD_z \tag{1.3}$$

by the application of a finite number of confluent Darboux transformations (CDTs) [16], also known as the "double commutator" method [9]. A CDT applied within a spectral gap of a second-order self-adjoint operator allows to add one eigenvalue to the spectrum. We will relate T to $\hat{T}(\tau)$ by a chain of CDTs, but the commutation procedure we consider is performed at an existing eigenvalue. The resulting spectral transformation for every confluent pair "deletes" an existing eigenvalue and then "adds" it back.[1] The overall effect is that of an isospectral transformation [19].

An important feature of confluent Darboux transformations is that every confluent pair of transformations naturally introduces an extra deformation parameter. Known instances of exceptional Jacobi polynomials are indexed by discrete parameters and cannot be continuously deformed into their classical counterparts. By contrast, after performing n CDTs on the classical Legendre operator (1.3) at distinct energy levels indexed by $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ we will arrive at an exceptional Legendre operator

$$T_{\boldsymbol{m}}(\boldsymbol{t_m}) = \hat{T}(\tau_{\boldsymbol{m}}(z; \boldsymbol{t_m}))$$

that depends on n real parameters $\boldsymbol{t_m} = (t_{m_1}, \ldots, t_{m_n}) \in \mathbb{R}^n$. The polynomial eigenfunctions of $T_{\boldsymbol{m}}(\boldsymbol{t_m})$ are exceptional Legendre polynomials $\{P_{\boldsymbol{m},i}(z; \boldsymbol{t_m})\}_{i \in \mathbb{N}_0}$, which depend on n real parameters $\boldsymbol{t_m} = (t_{m_1}, \ldots, t_{m_n})$, and can be continuously deformed to the classical Legendre polynomials by letting $\boldsymbol{t_m} \to 0$.

Adapting certain methodologies from the theory of inverse scattering [1, 4, 29], we are able to exhibit a determinantal representation of $\tau_{\boldsymbol{m}}(z; \boldsymbol{t_m})$ that is formally similar to the construction of KdV multi-solitons. The difference here is that, instead of dressing the zero potential, we isospectrally deform a particular instance of the Darboux–Poschl–Teller potential [15] by modifying the normalizations of a finite number of the corresponding bound states. Another feature of our approach is that, rather than working with a Schrödinger operator, we remain in a polynomial setting by utilizing the gauge and coordinate of the Legendre operator. The result is a constructive procedure that can be easily implemented using a computer algebra system.

1.1 Notation and definitions

The base case of the construction is the classical Legendre operator T, shown in (1.3), and the classical Legendre polynomials [30]

$$P_i(z) := \frac{2^{-i}}{i!} D_z^i (z^2 - 1)^i = 2^{-i} \sum_{k=0}^{i} \binom{i}{k}^2 (z-1)^{i-k}(z+1)^k, \qquad i \in \mathbb{N}_0. \tag{1.4}$$

These classical orthogonal polynomials do have $\deg P_i = i$, they satisfy the eigenvalue relation

$$TP_i = -i(i+1)P_i, \qquad i \in \mathbb{N}_0,$$

and they form an L^2-complete orthogonal family relative to the inner product

$$\int_{-1}^{1} P_{i_1}(z)P_{i_2}(z)\mathrm{d}z = \frac{2}{2i_1 + 1}\delta_{i_1 i_2}, \qquad i_1, i_2 \in \mathbb{N}_0.$$

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[1]This is true in a formal sense only, as the intermediate potential is singular.

Before we can state the main results of the paper, we would like to fix some notation conventions to be used throughout the paper. Bold symbols such as $\boldsymbol{m}$, $\boldsymbol{t}$ or $\boldsymbol{Q}$ will represent tuples of integers, real numbers or polynomials (one dimensional objects), while calligraphic symbols like $\mathcal{R}$ will denote matrices. To access the components of a vector or tensor we will employ square brackets, i.e., $[\mathcal{R}]_{k\ell}$ denotes the (k, ℓ) entry of $\mathcal{R}$. In addition, given an $n \times n$ matrix $\mathcal{R}$ and integers $1 \leq k < \ell \leq n$, we denote by $\mathcal{M}_{k,\ell}(\mathcal{R})$ the $(\ell - k + 1) \times (\ell - k + 1)$ square submatrix of $\mathcal{R}$ that includes the intersection of rows and columns from k to ℓ.

If $\boldsymbol{m} = (m_1, \ldots, m_n)$ is an n-tuple and $k \in \{1, \ldots, n\}$, we will denote by $\boldsymbol{m}^{\langle k \rangle}$ the $(n-1)$-tuple where the element $[\boldsymbol{m}]_k$ is removed, i.e., $\boldsymbol{m}^{\langle k \rangle} = (m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n)$, and by $\boldsymbol{m}_k$ the k-tuple formed by the first k elements of $\boldsymbol{m}$, i.e., $\boldsymbol{m}_k = (m_1, \ldots, m_k)$. In particular, we may write explicitly $\boldsymbol{m}_n$ instead of $\boldsymbol{m}$ whenever the context requires to emphasize the length of the tuple, mostly in the proofs by induction or recurrence relations.

Associated to an n-tuple of integers $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$, we will define the n-tuple of real parameters $\boldsymbol{t_m} = (t_{m_1}, \ldots, t_{m_n}) \in \mathbb{R}^n$. Semicolons will be used to separate objects of different nature. Commas will be used for tuple concatenation, e.g., if $i_1, \ldots, i_k \in \mathbb{N}_0$, $(\boldsymbol{m}, i_1, \ldots, i_k)$ denotes the $(n+k)$-tuple $(m_1, \ldots, m_n, i_1, \ldots, i_k,)$. Similarly, we have $(\boldsymbol{t_m}, t_{i_1}, \ldots, t_{i_k}) = \boldsymbol{t}_{(\boldsymbol{m}, i_1, \ldots, i_k)}$. Often, we will omit the parentheses when denoting 1-tuples, e.g., m_1 instead of (m_1). Finally, we will use, depending on the context, the following notation for derivatives of a function f with respect to z: $D_z f$, f' and f_z.

With this notation in mind, we proceed to define the main objects of this paper.

Definition 1.4. Given an n-tuple $\boldsymbol{m} \in \mathbb{N}_0^n$ and the associated $\boldsymbol{t_m} \in \mathbb{R}^n$, we define $\mathcal{R}_{\boldsymbol{m}}(z; \boldsymbol{t_m})$ as the $n \times n$ matrix with polynomial entries given by

$$[\mathcal{R}_{\boldsymbol{m}}(z; \boldsymbol{t_m})]_{k\ell} = \delta_{k\ell} + t_{m_\ell} R_{m_k m_\ell}(z), \qquad k, \ell \in \{1, \ldots, n\},$$

where

$$R_{m_k m_\ell}(z) := \int_{-1}^{z} P_{m_k}(u) P_{m_\ell}(u) \mathrm{d}u, \tag{1.5}$$

and $P_i(z)$ denote the classical Legendre polynomials (1.4). We denote its determinant by

$$\tau_{\boldsymbol{m}}(z; \boldsymbol{t_m}) := \det \mathcal{R}_{\boldsymbol{m}}(z; \boldsymbol{t_m}). \tag{1.6}$$

We define the n-tuple of polynomials

$$\boldsymbol{Q}_{\boldsymbol{m}}^{\mathrm{T}}(z; \boldsymbol{t_m}) := \tau_{\boldsymbol{m}}(z; \boldsymbol{t_m}) \mathcal{R}_{\boldsymbol{m}}(z; \boldsymbol{t_m})^{-1} \big(P_{m_1}(z), \ldots, P_{m_n}(z) \big)^{\mathrm{T}}, \tag{1.7}$$

Finally, for $i \in \mathbb{N}_0$, we define the polynomials

$$P_{\boldsymbol{m}; i}(z; \boldsymbol{t_m}) := \big[\boldsymbol{Q}_{(\boldsymbol{m}, i)}(z; \boldsymbol{t}_{(\boldsymbol{m}, i)}) \big]_{n+1}. \tag{1.8}$$

Note that, by construction, $\tau_{\boldsymbol{m}}(z; \boldsymbol{t_m})$ is symmetric in $\boldsymbol{m}$ and $\boldsymbol{Q}_{\boldsymbol{m}}(z; \boldsymbol{t_m})$ is equivariant with respect to permutations of $\boldsymbol{m}$. In addition, $P_{\boldsymbol{m}; i}(z; \boldsymbol{t_m})$ is symmetric in $\boldsymbol{m}$ and does not depend on t_i since $\tau_{(\boldsymbol{m}, i)}(z; \boldsymbol{t}_{(\boldsymbol{m}, i)}) \big[\mathcal{R}_{(\boldsymbol{m}, i)}(z; \boldsymbol{t}_{(\boldsymbol{m}, i)})^{-1} \big]_{n+1, j}$ correspond to the minors of the last column of $\mathcal{R}_{(\boldsymbol{m}, i)}(z; \boldsymbol{t}_{(\boldsymbol{m}; i)})$, the only column where t_i appears.

For example, for $m_1, m_2 \in \mathbb{N}_0$ we have

$$\tau_{m_1}(z; t_{m_1}) = 1 + t_{m_1} R_{m_1 m_1}(z),$$

$$\mathcal{R}_{(m_1, m_2)}(z; \boldsymbol{t}_{(m_1, m_2)}) = \begin{pmatrix} 1 + t_{m_1} R_{m_1 m_1}(z) & t_{m_2} R_{m_1 m_2}(z) \\ t_{m_1} R_{m_2 m_1}(z) & 1 + t_{m_2} R_{m_2 m_2}(z) \end{pmatrix},$$

$$\tau_{(m_1, m_2)}(z; \boldsymbol{t}_{(m_1, m_2)}) = 1 + t_{m_1} R_{m_1 m_1}(z) + t_{m_2} R_{m_2 m_2}(z)$$
$$+ t_{m_1} t_{m_2} \big(R_{m_1 m_1}(z) R_{m_2 m_2}(z) - R_{m_1 m_2}^2(z) \big),$$

$$\boldsymbol{Q}_{(m_1, m_2)}(z; \boldsymbol{t}_{(m_1, m_2)}) = \begin{pmatrix} \big(1 + t_{m_2} R_{m_2 m_2}(z)\big) P_{m_1}(z) - t_{m_2} R_{m_1 m_2}(z) P_{m_2}(z) \\ \big(1 + t_{m_1} R_{m_1 m_1}(z)\big) P_{m_2}(z) - t_{m_1} R_{m_2 m_1}(z) P_{m_1}(z) \end{pmatrix},$$

$$P_{m_1; i}(z; t_{m_1}) = \big(1 + t_{m_1} R_{m_1 m_1}(z)\big) P_i(z) - t_{m_1} R_{i m_1}(z) P_{m_1}(z), \qquad i \in \mathbb{N}_0.$$

After defining these objects, we are now ready to state the results.

1.2 Main results

The main result of this paper states that the polynomials $\{P_{\boldsymbol{m}; i}(z; \boldsymbol{t}_{\boldsymbol{m}})\}_{i \in \mathbb{N}_0}$ defined by (1.5)–(1.8) are exceptional Legendre polynomials, provided the real parameters $\boldsymbol{t}_{\boldsymbol{m}}$ satisfy certain constraints to ensure that $\tau_{\boldsymbol{m}}(z; t_{\boldsymbol{m}})$ has constant sign on $z \in [-1, 1]$.

Theorem 1.5. *For $\boldsymbol{m} \in \mathbb{N}_0^n$, consider the operator*

$$T_{\boldsymbol{m}}(\boldsymbol{t}_{\boldsymbol{m}}) := \hat{T}(\tau_{\boldsymbol{m}}(z; \boldsymbol{t}_{\boldsymbol{m}})),$$

given by (1.1) and (1.6). Then $T_{\boldsymbol{m}}(\boldsymbol{t}_{\boldsymbol{m}})$ is an exceptional Legendre operator that satisfies

$$T_{\boldsymbol{m}}(\boldsymbol{t}_{\boldsymbol{m}}) P_{\boldsymbol{m}; i}(z; \boldsymbol{t}_{\boldsymbol{m}}) = -i(i+1) P_{\boldsymbol{m}; i}(z; \boldsymbol{t}_{\boldsymbol{m}}), \qquad i \in \mathbb{N}_0, \tag{1.9}$$

with $P_{\boldsymbol{m}; i}(z; \boldsymbol{t}_{\boldsymbol{m}})$ as in (1.8).

In light of (1.9), we may refer to $P_{\boldsymbol{m}; i}(z; \boldsymbol{t}_{\boldsymbol{m}})$, where $i \in \mathbb{N}_0$ varies and $\boldsymbol{m}$ and $\boldsymbol{t}_{\boldsymbol{m}}$ are fixed, as exceptional Legendre polynomials. This requires according to Remark 1.2 and Definition 1.3 that $\tau_{\boldsymbol{m}}(z; \boldsymbol{t}_{\boldsymbol{m}})$ does not vanish on $[-1, 1]$. The following theorem gives necessary and sufficient conditions for this to be true. In that case, like their classical counterparts, the polynomials $\{P_{\boldsymbol{m}; i}(z; \boldsymbol{t}_{\boldsymbol{m}})\}_{i \in \mathbb{N}_0}$ are orthogonal and complete.

Theorem 1.6. *For $\boldsymbol{m} \in \mathbb{N}_0^n$ with $m_1, \ldots, m_n$ distinct, the polynomial $\tau_{\boldsymbol{m}}(z; \boldsymbol{t}_{\boldsymbol{m}})$ in (1.6) has no zeros on $[-1, 1]$ if and only if*

$$t_{m_j} > -m_j - \frac{1}{2}, \qquad j \in \{1, \ldots, n\}. \tag{1.10}$$

If the above conditions hold, then $\{P_{\boldsymbol{m}; i}(z; \boldsymbol{t}_{\boldsymbol{m}})\}_{i \in \mathbb{N}_0}$ are exceptional Legendre polynomials with

$$\int_{-1}^{1} \frac{P_{\boldsymbol{m}; i_1}(u; \boldsymbol{t}_{\boldsymbol{m}}) P_{\boldsymbol{m}; i_2}(u; \boldsymbol{t}_{\boldsymbol{m}})}{\tau_{\boldsymbol{m}}(z; \boldsymbol{t}_{\boldsymbol{m}})^2} \mathrm{d}u = \frac{2}{1 + 2i_1 + 2(\delta_{i_1 m_1} + \cdots + \delta_{i_1 m_n}) t_{i_1}} \delta_{i_1 i_2}, \qquad i_1, i_2 \in \mathbb{N}_0.$$

and L^2-completeness in $[-1, 1]$ relative to the measure $\tau_{\boldsymbol{m}}(z; \boldsymbol{t}_{\boldsymbol{m}})^{-2} \mathrm{d}z$.

Remark 1.7. Note that we could reformulate the above result without the assumption that $m_1, \ldots, m_n$ are distinct. However, there is no extra benefit in doing this, as demonstrated by Proposition 3.6 below. Assuming that the indices $m_1, \ldots, m_n$ are all distinct does not entail any loss of generality.

The rest of the paper is organized as follows: in Section 2 we study exceptional Legendre operators connected by a single step rational confluent Darboux transformation, which involves in fact two Darboux transformations at the same factorization energy. We will iterate these results in Section 3 to consider any number of confluent Darboux transformations and we will relate this construction with the objects in Definition 1.4, thus yielding the proofs of the main theorems. Finally, in Section 4 we give some explicit examples of the new exceptional Legendre families.

2   One step confluent Darboux transformations

In this section we collect a number of relevant Propositions for the proofs of the Theorems 1.5 and 1.6. We introduce the concept of a rational confluent Darboux transformation and we show that this transformation preserves the class of exceptional Legendre operators.

Before introducing rational confluent Darboux transformation, we recall 2-step ordinary Darboux transformations between two operators T_1 and T_2 with rational coefficients. If A_1, A_2, B_1, B_2 are first-order differential operators with rational coefficients, and λ_1, λ_2 two constants, consider the following 2-step rational Darboux transformation:

$$T_1 = B_1 A_1 + \lambda_1,$$
$$\tilde{T} = A_1 B_1 + \lambda_1 = A_2 B_2 + \lambda_2,$$
$$T_2 = B_2 A_2 + \lambda_2.$$

The first transformation at energy level λ_1 maps T_1 to $\tilde{T}$ and is state-deleting, while the second transformation at energy level λ_2 maps $\tilde{T}$ to T_2 and is state-adding (or equivalently, the inverse transformation from T_2 to $\tilde{T}$ is state-deleting). The confluent version arises when $\lambda_1 = \lambda_2$, i.e., we use seed functions at each of the two steps which are (at least formally) eigenfunctions of the corresponding (formal) operator with the same eigenvalue. A full discussion of confluent Darboux transformations from this point of view can be seen, for instance, in [28]. We can make this notion more precise in the following definition.

Definition 2.1. Let T_1, T_2 be second-order operators with rational coefficients. We will say that T_1 and T_2 are related by a *rational confluent Darboux transformation* if there exist first-order operators A_1, A_2, B_1, B_2, all with rational coefficients, and a constant λ such that

$$A_1 B_1 = A_2 B_2, \qquad T_1 = B_1 A_1 + \lambda, \qquad T_2 = B_2 A_2 + \lambda.$$

Given polynomials $\tau(z)$, $\phi(z)$, we define the rational operators

$$A(\tau, \phi) := \tau^{-1} \left(\phi D_z - \phi_z \right),$$
$$B(\phi, \tau) := A(\phi, \tau) \circ \left(1 - z^2 \right) = \phi^{-1} \left(\left(1 - z^2 \right) \left(\tau D_z - \tau_z \right) - 2z\tau \right). \tag{2.1}$$

The form of these operators coincides with the general form of the first-order operators appearing in factorization of operators given in [7, Proposition 3.5], with a particular choice that ensures that operator $\hat{T}(\tau) = B(\phi, \tau) A(\tau, \phi)$ is in the natural gauge [7, Definition 5.1]. In the proofs, we will use the fact that, for a given function f, we have

$$A(\tau, \phi) f = \tau^{-1} \operatorname{Wr}(\phi, f),$$

where Wr denotes the Wronskian determinant.

Throughout this section, we consider an exceptional Legendre operator $\hat{T}(\tau)$ with polynomials $\{\pi_i(z)\}_{i \in \mathbb{N}_0}$ that satisfy

$$\hat{T}(\tau) \pi_i = \lambda_i \pi_i, \qquad \lambda_{i_1} \neq \lambda_{i_2} \quad \text{if} \quad i_1 \neq i_2, \quad i, i_1, i_2 \in \mathbb{N}_0. \tag{2.2}$$

Our goal is to apply a rational CDT on this operator. To this end, for $m \in \mathbb{N}_0$ and $t \in \mathbb{R}$, let us define the following objects:

$$\rho_{i_1 i_2}(z) := \int_{-1}^{z} \frac{\pi_{i_1}(u)\pi_{i_2}(u)}{\tau(u)^2} du, \qquad i_1, i_2 \in \mathbb{N}_0, \tag{2.3}$$

$$\tau_m(z;t) := \tau(z)\left(1 + t\rho_{mm}(z)\right), \tag{2.4}$$

$$\pi_{m;i}(z;t) := (1 + t\rho_{mm}(z))\pi_i(z) - t\rho_{im}(z)\pi_m(z), \qquad i \in \mathbb{N}_0. \tag{2.5}$$

For a lighter notation, we may omit the t dependence and write τ_m instead of $\tau_m(z;t)$. Note that τ might already depend on a number of real parameters, so τ_m will depend on the same parameters as τ, plus an extra parameter t.

Remark 2.2. In the rest of this Section, i.e., for the following four Propositions, we shall assume that $\rho_{i_1 i_2}(z)$ defined by (2.3) is a rational function that vanishes at $z = -1$ and $\tau_m(z)$ and $\pi_{m;i}(z,t)$ defined by (2.4)–(2.5) are polynomials in z.

If we start from an exceptional Legendre operator (1.1) for a given τ polynomial with eigenpolynomials π_i, these assumptions are far from obvious by looking at (2.3)–(2.5). In the next section we will see that the assumptions hold whenever (2.2) does, i.e., that the rational CDT between exceptional Legendre families is well defined.

Proposition 2.3. *For $m \in \mathbb{N}_0$, let $\rho_{mm}(z)$ and $\tau_m(z)$ be defined by (2.3)–(2.4) and satisfy the assumptions of Remark 2.2. Then, $\hat{T}(\tau)$ and $\hat{T}(\tau_m)$ are related by a rational confluent Darboux transformation with*

$$A(\tau, \pi_m)B(\pi_m, \tau) = A(\tau_m, \pi_m)B(\pi_m, \tau_m),$$

$$\hat{T}(\tau) = B(\pi_m, \tau)A(\tau, \pi_m) + \lambda_m,$$

$$\hat{T}(\tau_m) = B(\pi_m, \tau_m)A(\tau_m, \pi_m) + \lambda_m.$$

Proof. The results follow from direct calculation with the previous definitions. ∎

The following lemma examines the behaviour at the endpoint $z = -1$ of a combination of these objects, and it will be necessary to prove some of the following propositions.

Lemma 2.4. *Let $\{\pi_i\}_{i \in \mathbb{N}_0}$ be polynomials that satisfy the eigenvalue equation (2.2) with (1.1) and let $\rho_{i_1,i_2}(z)$ be the rational functions defined by (2.3) that vanish at $z = -1$. Then,*

$$\left(1 - z^2\right)\frac{\mathrm{Wr}(\pi_i, \pi_m)}{\tau^2}\bigg|_{z=-1} = 0.$$

Proof. Since $\rho_{im}(z)$ is rational and vanishes at $z = -1$, we can write for given $\alpha, \beta \in \mathbb{N}_0$

$$\rho_{im}(z) = (1+z)^{1+\alpha}q(z), \qquad q(-1) = C_q \neq 0, \qquad \alpha > 0,$$

$$\tau(z) = (1+z)^{\beta}p(z), \qquad p(-1) = C_p \neq 0,$$

where q is a rational function and p is a polynomial. Thus

$$\rho'_{im}(-1) = \frac{\pi_i\pi_m}{\tau^2}\bigg|_{z=-1} = \begin{cases} C_q & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha \geq 2. \end{cases}$$

Using the eigenvalue equation we have

$$(1-z^2)(\pi_i''\pi_m - \pi_i\pi_m'') - 2\left((1-z^2)\frac{\tau'}{\tau} + z\right)(\pi_i'\pi_m - \pi_i\pi_m') = (\lambda_i - \lambda_m)\pi_i\pi_m.$$

Evaluating both sides at $z = -1$, we have

$$\left.\frac{\mathrm{Wr}(\pi_i, \pi_m)}{\tau^2}\right|_{z=-1} = \left.\frac{(\lambda_i - \lambda_m)}{2(1 - 2\beta)} \frac{\pi_i \pi_m}{\tau^2}\right|_{z=-1} < \infty.$$

Since the previous expression is bounded, the desired result is proved. ∎

The next proposition shows how to build the eigenpolynomials of the transformed operator, by the use of the second order intertwining relations for CDTs

$$(B_2 A_1)T_1 = T_2(B_2 A_1), \qquad T_1(B_1 A_2) = (B_1 A_2)T_2.$$

Proposition 2.5. *For $m \in \mathbb{N}_0$, let $\rho_{im}(z)$, $\tau_m(z)$ and $\pi_{m;i}(z, t)$ be defined by (2.3)–(2.5) and satisfy the assumptions of Remark 2.2.*

Then, for $i \in \mathbb{N}_0$, we have

$$(\lambda_m - \lambda_i)\rho_{im} = (1 - z^2)\tau^{-1}A(\tau, \pi_m)\pi_i, \tag{2.6}$$

$$(\lambda_m - \lambda_i)\pi_{m;i} = B(\pi_m, \tau_m)A(\tau, \pi_m)\pi_i, \tag{2.7}$$

$$\hat{T}(\tau_m)\pi_{m;i} = \lambda_i \pi_{m;i}. \tag{2.8}$$

Proof. We start by noticing that

$$(\lambda_m - \lambda_i)\rho'_{im} = (\lambda_m - \lambda_i)\frac{\pi_i \pi_m}{\tau^2} = \left((1 - z^2)\tau^{-1}A(\tau, \pi_m)\pi_i\right)',$$

where for the last equality we use the eigenvalue equation (2.2), or equivalently the Sturm–Liouville equation

$$\left((1 - z^2)\tau^{-2}\pi'_i\right)' + (1 - z^2)\tau_{zz}\tau^{-3}\pi_i = \lambda_i \tau^{-2}\pi_i.$$

The first result follows by integration since Lemma 2.4 ensures that

$$\left.(1 - z^2)\tau^{-1}A(\tau, \pi_m)\pi_i\right|_{z=-1} = \left.(1 - z^2)\frac{\mathrm{Wr}(\pi_m, \pi_i)}{\tau^2}\right|_{z=-1} = 0.$$

The second identity follows by direct calculation using the definitions and previous identities. Indeed, using (2.1) and (2.6) we have

$$\begin{aligned}
B(\pi_m, \tau_m)A(\tau, \pi_m)\pi_i &= A(\pi_m, \tau_m) \circ (1 - z^2)\left(\frac{(\lambda_m - \lambda_i)\rho_{im}}{(1 - z^2)\tau^{-1}}\right) \\
&= (\lambda_m - \lambda_i)\pi_m^{-1}\mathrm{Wr}(\tau_m, \tau\rho_{im}) \\
&= (\lambda_m - \lambda_i)\pi_m^{-1}\tau^2\mathrm{Wr}(1 + t\rho_{mm}, \rho_{im})
\end{aligned}$$

and deriving ρ_{im} in (2.3) and using (2.5) leads to the desired result (2.7).

The third identity (2.8) follows trivially from (2.7) and the intertwining relation

$$\hat{T}(\tau_m)B(\pi_m, \tau_m)A(\tau, \pi_m) = B(\pi_m, \tau_m)A(\tau, \pi_m)\hat{T}(\tau). \qquad \blacksquare$$

The next result derives the transformation rule for $\rho_{i_1 i_2}$ under a CDT, and it is key to obtain the norming constants of the transformed polynomials in Proposition 2.7.

Proposition 2.6. *For $m, i_1, i_2 \in \mathbb{N}_0$, let $\rho_{i_1 i_2}(z)$, $\tau_m(z)$ and $\pi_{m;i}(z, t)$ be defined by (2.3)–(2.5) and satisfy the assumptions of Remark 2.2. Then,*

$$\int_{-1}^{z} \frac{\pi_{m;i_1}(u; t)\pi_{m;i_2}(u; t)}{\tau_m(u)^2}\mathrm{d}u = \rho_{i_1 i_2}(z) - \frac{t\rho_{i_1 m}(z)\rho_{i_2 m}(z)}{1 + t\rho_{mm}(z)}, \qquad i_1, i_2 \in \mathbb{N}_0. \tag{2.9}$$

Proof. The identity between the derivatives of both sides can be easily proved by direct computation using the definitions. The desired result follows then by integration since $\rho_{i_1 i_2}(-1) = 0$. ∎

The last proposition of this section shows that the CDT of an exceptional Legendre family falls into the same class under suitable bound on the introduced parameter t.

Proposition 2.7. *Assume that τ does not vanish in $[-1, 1]$ and that $\{\pi_i\}_{i \in \mathbb{N}_0}$ are exceptional Legendre polynomials with*

$$\int_{-1}^1 \frac{\pi_{i_1}(u)\pi_{i_2}(u)}{\tau(u)^2}\mathrm{d}u = \nu_{i_1}\delta_{i_1 i_2},$$

for constants $\nu_i > 0$, $i \in \mathbb{N}_0$ and completeness in $\mathrm{L}^2\big([-1, 1], \tau^{-2}\mathrm{d}z\big)$. Let $m \in \mathbb{N}_0$ and set

$$\nu_{m;i} := \begin{cases} \nu_i & \text{if } i \neq m, \\ (t + \nu_m^{-1})^{-1} & \text{if } i = m. \end{cases}$$

Then, $\tau_m(z) > 0$ on $[-1, 1]$ if and only if $\nu_{m;m} > 0$. In that case, the set $\{\pi_{m;i}(z; t)\}_{i \in \mathbb{N}_0}$ is a family of exceptional Legendre polynomials with

$$\int_{-1}^1 \frac{\pi_{m;i_1}(u)\pi_{m;i_2}(u)}{\tau_m(u)^2}\mathrm{d}u = \nu_{m;i_1}\delta_{i_1 i_2}, \qquad i_1, i_2, m \in \mathbb{N}_0, \tag{2.10}$$

and completeness in $\mathrm{L}^2\big([-1, 1], \tau_m^{-2}\mathrm{d}z\big)$.

Proof. First, note that (2.10) is true in a formal sense. By (2.9), the rational function

$$\rho_{m;i_1 i_2}(z; t) := \rho_{i_1 i_2}(z) - \frac{t\rho_{i_1 m}(z)\rho_{i_2 m}(z)}{1 + t\rho_{mm}(z)}$$

is defined by the integral on the l.h.s. of (2.9). Furthermore, since we are assuming that

$$\rho_{i_1 i_2}(1) = \delta_{i_1 i_2}\nu_{i_1},$$

we have

$$\rho_{m;i_1 i_2}(1; t) = \delta_{i_1 i_2}\left(\nu_{i_1} - \delta_{i_1 m}\frac{t\nu_m^2}{1 + t\nu_m}\right) = \delta_{i_1 i_2}\nu_{m;i_1}.$$

By (2.4), $\tau_m(z)$ is positive on $z \in [-1, 1]$ if and only if the same is true for $1 + t\rho_{mm}(z)$. Since $\rho_{mm}(z)$ is an increasing function, the latter is true if and only if $1 + t\nu_m > 0$. Observe that

$$\nu_{m;m}^{-1} = t + \nu_m^{-1} = \nu_m^{-1}(1 + t\nu_m).$$

Hence $\tau_m(z)$ is positive on $z \in [-1, 1]$ if and only if $\nu_{m;m} > 0$.

Finally, we prove completeness. We assume that the eigenpolynomials $\{\pi_i(z)\}_{i \in \mathbb{N}_0}$ are L^2-complete in $[-1, 1]$ relative to $\tau(z)^{-2}\mathrm{d}z$. Following an argument adapted from the appendix of [1], we re-express the completeness assumption as

$$\sum_{i \in \mathbb{N}_0} \nu_i^{-1} \frac{\pi_i(z)}{\tau(z)} \frac{\pi_i(w)}{\tau(w)} = \delta(z - w),$$

where the equality is understood in distributional sense on $[-1, 1] \times [-1, 1]$. Rewriting (2.3) as

$$\rho_{i_1 i_2}(z) = \int_{-1}^1 \theta(z - u)\frac{\pi_{i_1}(u)\pi_{i_2}(u)}{\tau(u)^2}\mathrm{d}u, \qquad i_1, i_2 \in \mathbb{N}_0,$$

where $\theta(z)$ denotes the Heaviside step function, it follows that for $j \in \mathbb{N}_0$

$$\sum_{i \in \mathbb{N}_0} \nu_i^{-1} \frac{\pi_i(z)}{\tau(z)} \rho_{ij}(w) = \theta(w - z) \frac{\pi_j(z)}{\tau(z)},$$

$$\sum_{i \in \mathbb{N}_0} \nu_i^{-1} \rho_{ij}(z) \rho_{ij}(w) = \theta(w - z) \rho_{jj}(z) + \theta(z - w) \rho_{jj}(w).$$

By (2.4) and (2.5), we have

$$\frac{\pi_{m;i}(z)}{\tau_m(z)} = \frac{\pi_i(z)}{\tau(z)} - t \rho_{im}(z) \frac{\pi_m(z)}{\tau_m(z)}, \qquad i \in \mathbb{N}_0,$$

$$\frac{t \rho_{mm}(z)}{\tau_m(z)} = \frac{1}{\tau(z)} - \frac{1}{\tau_m(z)}.$$

Therefore, making use of the previous identities,

$$\sum_{i \in \mathbb{N}_0} \nu_{m;i}^{-1} \frac{\pi_{m;i}(z)}{\tau_m(z)} \frac{\pi_{m;i}(w)}{\tau_m(w)} == t \frac{\pi_{m;m}(z)}{\tau_m(z)} \frac{\pi_{m;m}(w)}{\tau_m(w)} + \sum_{i \in \mathbb{N}_0} \nu_i^{-1} \frac{\pi_{m;i}(z)}{\tau_m(z)} \frac{\pi_{m;i}(w)}{\tau_m(w)}$$

$$= t \frac{\pi_m(z)}{\tau_m(z)} \frac{\pi_m(w)}{\tau_m(w)} + \sum_{i \in \mathbb{N}_0} \nu_i^{-1} \left(\frac{\pi_i(z)}{\tau(z)} - t \rho_{im}(z) \frac{\pi_m(z)}{\tau_m(z)} \right) \left(\frac{\pi_i(w)}{\tau(w)} - t \rho_{im}(w) \frac{\pi_m(w)}{\tau_m(w)} \right)$$

$$= t \frac{\pi_m(z)}{\tau_m(z)} \frac{\pi_m(w)}{\tau_m(w)} + \delta(z - w) - t \, \theta(w - z) \frac{\pi_m(z)}{\tau(z)} \frac{\pi_m(w)}{\tau_m(w)} - t \, \theta(z - w) \frac{\pi_m(z)}{\tau_m(z)} \frac{\pi_m(w)}{\tau(w)}$$

$$+ t^2 \big(\theta(w - z) \rho_{mm}(z) + \theta(z - w) \rho_{mm}(w) \big) \frac{\pi_m(z)}{\tau_m(z)} \frac{\pi_m(w)}{\tau_m(w)} = \delta(z - w). \qquad \blacksquare$$

3 Recursive construction and proof of theorems

The strategy to prove the main theorems is the following. First, we will define some polynomials and rational functions recursively, starting the recursion at the objects corresponding to the classical Legendre Sturm–Liouville problem. The recursion formulas coincide with (1.6), (2.5) and (2.9). Next, we show in Proposition 3.2 that these recursively defined objects coincide with those defined in Definition 1.4, and thus they satisfy the rationality and polynomiality conditions of Remark 2.2. Propositions 2.3–2.7 then ensure that at each step of the recursion we have an exceptional Legendre Sturm–Liouville problem, provided the parameters are chosen in the right range.

Definition 3.1. Let $i, i_1, i_2 \in \mathbb{N}_0$, $\boldsymbol{m} = \boldsymbol{m}_n = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ and $\boldsymbol{m}_j = (m_1, \dots, m_j)$, and define recursively functions $R_{\boldsymbol{m};i_1 i_2}(z; \boldsymbol{t_m})$, $\tilde{\tau}_{\boldsymbol{m}}(z; \boldsymbol{t_m})$ and $\tilde{P}_{\boldsymbol{m};i}(z; \boldsymbol{t_m})$. For $j = 0$, we start the recursion at $\boldsymbol{m}_0 = \varnothing$ with

$$R_{\boldsymbol{m}_0;i_1 i_2}(z; \boldsymbol{t_{m_0}}) = R_{i_1 i_2}(z),$$

$$\tilde{\tau}_{\boldsymbol{m}_0}(z; \boldsymbol{t_{m_0}}) = 1,$$

$$\tilde{P}_{\boldsymbol{m}_0;i}(z; \boldsymbol{t_{m_0}}) = P_i(z),$$

where $R_{i_1 i_2}(z)$ are given by (1.5) and $P_i(z)$ are the classical Legendre operators. For $j = 1, \dots, n$ we define recursively

$$R_{\boldsymbol{m}_j;i_1 i_2}(z; \boldsymbol{t_{m_j}}) = R_{\boldsymbol{m}_{j-1};i_1 i_2}(z; \boldsymbol{t_{m_{j-1}}})$$
$$- \frac{t_{m_j} R_{\boldsymbol{m}_{j-1};i_1 m_j}(z; \boldsymbol{t_{m_{j-1}}}) R_{\boldsymbol{m}_{j-1};i_2 m_j}(z; \boldsymbol{t_{m_{j-1}}})}{1 + t_{m_j} R_{\boldsymbol{m}_{j-1};m_j m_j}(z; \boldsymbol{t_{m_{j-1}}})}, \tag{3.1}$$

$$\tilde{\tau}_{\boldsymbol{m}_j}(z;\boldsymbol{t}_{\boldsymbol{m}_j}) = \left(1 + t_{m_j} R_{\boldsymbol{m}_{j-1};m_j m_j}(z;\boldsymbol{t}_{\boldsymbol{m}_{j-1}})\right) \tilde{\tau}_{\boldsymbol{m}_{j-1}}(z;\boldsymbol{t}_{\boldsymbol{m}_{j-1}}), \tag{3.2}$$

$$\tilde{P}_{\boldsymbol{m}_j;i}(z;\boldsymbol{t}_{\boldsymbol{m}_j}) = (1 + t_{m_j} R_{\boldsymbol{m}_{j-1};m_j m_j}(z;\boldsymbol{t}_{\boldsymbol{m}_{j-1}}))\tilde{P}_{\boldsymbol{m}_{j-1};i}(z;\boldsymbol{t}_{\boldsymbol{m}_{j-1}})$$
$$- t_{m_j} R_{\boldsymbol{m}_{j-1};i m_j}(z;\boldsymbol{t}_{\boldsymbol{m}_{j-1}})\tilde{P}_{\boldsymbol{m}_{j-1};m_j}(z;\boldsymbol{t}_{\boldsymbol{m}_{j-1}}). \tag{3.3}$$

The next proposition states that these recursively defined functions coincide with the polynomials and rational functions introduced in Definition 1.4.

Proposition 3.2. *For $i,i_1,i_2 \in \mathbb{N}_0$ and $\boldsymbol{m} = (m_1,\dots,m_n) \in \mathbb{N}_0^n$, let $R_{\boldsymbol{m};i_1 i_2}(z;\boldsymbol{t}_{\boldsymbol{m}})$, $\tilde{\tau}_{\boldsymbol{m}}(z;\boldsymbol{t}_{\boldsymbol{m}})$ and $\tilde{P}_{\boldsymbol{m};i}(z;\boldsymbol{t}_{\boldsymbol{m}})$ be the functions defined in Definition 3.1. Let $\tau_{\boldsymbol{m}}(z;\boldsymbol{t}_{\boldsymbol{m}})$ and $P_{\boldsymbol{m};i}(z;\boldsymbol{t}_{\boldsymbol{m}})$ be the polynomials defined in Definition 1.4. Then,*

$$\tilde{\tau}_{\boldsymbol{m}}(z;\boldsymbol{t}_{\boldsymbol{m}}) = \tau_{\boldsymbol{m}}(z;\boldsymbol{t}_{\boldsymbol{m}}), \tag{3.4}$$

$$\tilde{P}_{\boldsymbol{m};i}(z;\boldsymbol{t}_{\boldsymbol{m}}) = P_{\boldsymbol{m};i}(z;\boldsymbol{t}_{\boldsymbol{m}}), \tag{3.5}$$

and

$$R_{\boldsymbol{m};i_1 i_2}(z;\boldsymbol{t}_{\boldsymbol{m}}) = \int_{-1}^z \frac{P_{\boldsymbol{m};i_1}(u;\boldsymbol{t}_{\boldsymbol{m}})P_{\boldsymbol{m};i_2}(u;\boldsymbol{t}_{\boldsymbol{m}})}{\tau_{\boldsymbol{m}}(u;\boldsymbol{t}_{\boldsymbol{m}})^2}\mathrm{d}u, \tag{3.6}$$

where, again, the integral denotes an anti-derivative that vanishes at $z = -1$.

The consequence of this proposition is to ensure that the rational CDTs applied iteratively on the classical Legendre operator are always well defined, i.e., the conditions specified in Remark 2.2 will hold at each step of the chain. Before we can address the proof of Proposition 3.2, we need to establish the following technical lemma.

Lemma 3.3. *Let $\boldsymbol{m} \in \mathbb{N}_0^n$ with $n \geq 3$. Then*

$$\begin{pmatrix} 1 + t_{m_{n-1}} R_{\boldsymbol{m}_{n-2};m_{n-1}m_{n-1}}(z;\boldsymbol{t}_{\boldsymbol{m}_{n-2}}) & t_{m_n} R_{\boldsymbol{m}_{n-2};m_{n-1}m_n}(z;\boldsymbol{t}_{\boldsymbol{m}_{n-2}}) \\ t_{m_{n-1}} R_{\boldsymbol{m}_{n-2};m_{n-1}m_n}(z;\boldsymbol{t}_{\boldsymbol{m}_{n-2}}) & 1 + t_{m_n} R_{\boldsymbol{m}_{n-2};m_n m_n;}(z;\boldsymbol{t}_{\boldsymbol{m}_{n-2}}) \end{pmatrix}^{-1}$$
$$= \mathcal{M}_{n-1,n}\left(\mathcal{R}_{\boldsymbol{m}}(z;t_{\boldsymbol{m}})^{-1}\right), \tag{3.7}$$

where $\mathcal{M}_{n-1,n}$ denotes the bottom right 2×2 submatrix of the indicated matrix and $\mathcal{R}_{\boldsymbol{m}}(z;t_{\boldsymbol{m}})$ is given in Definition 1.4.

Proof. The result follows by iteration and the Sylvester determinant identity [2]. Let us start by showing the argument for $n = 3$. If we write

$$\mathcal{R}_{\boldsymbol{m}}(z;\boldsymbol{t}_{\boldsymbol{m}}) = \left(\begin{array}{c|c} a & \boldsymbol{b}_1 \\ \hline \boldsymbol{b}_2^{\mathrm{T}} & \mathcal{A} \end{array}\right),$$

where $\boldsymbol{b}_j$ are 2-tuples for $j = 1,2$ and $\mathcal{A}$ is the $\mathcal{M}_{2,3}$ submatrix, by the Sylvester determinant identity,

$$\mathcal{M}_{2,3}\left(\mathcal{R}_{\boldsymbol{m}}(z;\boldsymbol{t}_{\boldsymbol{m}})^{-1}\right) = \left(\mathcal{A} - \boldsymbol{b}_2^{\mathrm{T}} a^{-1} \boldsymbol{b}_1\right)^{-1}.$$

Identifying the elements in the decomposition of $\mathcal{R}_{\boldsymbol{m}}(z;\boldsymbol{t}_{\boldsymbol{m}})$ according to its definition, by direct calculation and the recursive formulation (3.1) we obtain

$$\mathcal{A} - \boldsymbol{b}_1 a^{-1} \boldsymbol{b}_2^{\mathrm{T}} = \begin{pmatrix} 1 + t_{m_2} R_{m_1;m_2 m_2}(z;t_{m_1}) & t_{m_3} R_{m_1;m_2 m_3}(z;t_{m_1}) \\ t_{m_2} R_{m_1;m_2 m_3}(z;t_{m_1}) & 1 + t_{m_3} R_{m_1;m_3 m_3}(z;t_{m_1}) \end{pmatrix}$$

as desired.

For $n > 3$, let $\mathcal{R}_{\boldsymbol{m}}^{[j]}(z; \boldsymbol{t_m})$ be the $(n-j) \times (n-j)$ matrix for any $j \in \{1, n-2\}$ with entries

$$\left[\mathcal{R}_{\boldsymbol{m}}^{[j]}(z; \boldsymbol{t_m})\right]_{k\ell} := \delta_{k\ell} + t_{m_{j+\ell}} R_{\boldsymbol{m}j; m_{j+k}m_{j+\ell}}(z; \boldsymbol{t_{m_j}}), \qquad k, \ell \in \{1, \ldots, n-j\}. \tag{3.8}$$

We aim to show that $\mathcal{R}_{\boldsymbol{m}}^{[n-2]}(z; \boldsymbol{t_m})^{-1} = \mathcal{M}_{n-1,n}\big(\mathcal{R}_{\boldsymbol{m}}(z; \boldsymbol{t_m})^{-1}\big)$. Arguing as above, we have $\mathcal{R}_{\boldsymbol{m}}^{[n-2]}(z; \boldsymbol{t_m})^{-1} = \mathcal{M}_{2,3}\big(\mathcal{R}_{\boldsymbol{m}}^{[n-3]}(z; \boldsymbol{t_m})^{-1}\big)$. Applying analogously the Sylvester determinant identity, we also obtain

$$\mathcal{R}_{\boldsymbol{m}}^{[j+1]}(z; \boldsymbol{t_m})^{-1} = \mathcal{M}_{2,n-j}\big(\mathcal{R}_{\boldsymbol{m}}^{[j]}(z; \boldsymbol{t_m})^{-1}\big)$$

for $j \in \{0, \ldots, n-3\}$. The result then follows by iteration. ∎

Proof of Proposition 3.2. Throughout this proof, we are going to omit the explicit dependence on z and $\boldsymbol{t_m}$ of the objects, which must be understood from the dependence on $\boldsymbol{m}$, i.e., we will write $\mathcal{R}_{\boldsymbol{m}}$ instead of $\mathcal{R}_{\boldsymbol{m}}(z; \boldsymbol{t_m})$. By definition, we have

$$\mathcal{R}_{\boldsymbol{m}} = \left(\begin{array}{c|c} \mathcal{R}_{\boldsymbol{m}_{n-1}} & \vdots \\ \hline \cdots & 1 + t_{m_n} R_{m_n m_n} \end{array} \right).$$

Applying the expression for the inverse with the adjoint of the cofactors matrix, we obtain that

$$\left[\mathcal{R}_{\boldsymbol{m}}^{-1}\right]_{nn} = \frac{\det \mathcal{R}_{\boldsymbol{m}_{n-1}}}{\det \mathcal{R}_{\boldsymbol{m}_n}} = \frac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_n}}.$$

Applying Lemma 3.3 after computing the inverse of the matrix in the left hand side of (3.7) and using the recursion (3.1) allows to prove that the $\tau_{\boldsymbol{m}}$ and $\tau_{\boldsymbol{m}_{n-1}}$ satisfy the recursion relation (3.2). Since $\tilde{\tau}_{\boldsymbol{m}_1} = \tau_{\boldsymbol{m}_1}$, we see that (3.4) holds.

In order to prove (3.5), we first observe that (3.3) can be rewritten as

$$\tilde{P}_{\boldsymbol{m};i} = \frac{\tau_{(\boldsymbol{m};i)}}{\tau_{\boldsymbol{m}_{n-1}}} \left[\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-1]}\right)^{-1} \left(\tilde{P}_{\boldsymbol{m}_{n-1}; m_n}, \tilde{P}_{\boldsymbol{m}_{n-1}; i}\right)^{\mathrm{T}} \right]_1,$$

where $\mathcal{R}_{(\boldsymbol{m},i)}^{[n-1]} = \mathcal{R}_{(\boldsymbol{m},i)}^{[n-1]}(z; \boldsymbol{t}_{(\boldsymbol{m},i)})$ is given by (3.8). The proof follows by induction. It is clear that (3.5) holds for $\boldsymbol{m}_1$ by definition. We assume that it also holds for $\boldsymbol{m}_j$ with $j = 1, \ldots, n-1$, and we must prove it also hods for $\boldsymbol{m}_n$. We start by proving that,

$$\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-1]}\right)^{-1} \begin{pmatrix} P_{\boldsymbol{m}_{n-1}; m_n} \\ P_{\boldsymbol{m}_{n-1}; i} \end{pmatrix} = \frac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_{n-2}}} \left(\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1} \begin{pmatrix} P_{\boldsymbol{m}_{n-2}; m_{n-1}} \\ P_{\boldsymbol{m}_{n-2}; m_n} \\ P_{\boldsymbol{m}_{n-2}; i} \end{pmatrix} \right)^{\langle 1 \rangle}, \tag{3.9}$$

where in the right hand side we have the last two components of a vector of three entries. This may be verified as follows. First note that, by assumption, we have $\tilde{P}_{\boldsymbol{m}_j; i} = P_{\boldsymbol{m}_j; i}$ for all $i \in \mathbb{N}_0$ and $j = 1, \ldots, n-1$. By Lemma 3.3 and (3.3), the left hand side is equal to

$$\mathcal{M}_{2,3}\left(\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1} \right) \begin{pmatrix} -t_{m_{n-1}} R_{\boldsymbol{m}_{n-2}; m_{n-1}, m_n} & \dfrac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_{n-2}}} & 0 \\ -t_{m_{n-1}} R_{\boldsymbol{m}_{n-2}; m_{n-1}, i)} & 0 & \dfrac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_{n-2}}} \end{pmatrix} \begin{pmatrix} P_{\boldsymbol{m}_{n-2}; m_{n-1}} \\ P_{\boldsymbol{m}_{n-2}; m_n} \\ P_{\boldsymbol{m}_{n-2}; i} \end{pmatrix},$$

where we have used by (3.2)

$$1 + t_{m_{n-1}} R_{\boldsymbol{m}_{n-2}; m_{n-1}, m_{n-1}} = \frac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_{n-2}}}.$$

Now we can conclude that

$$\mathcal{M}_{2,3}\left(\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}\right)\begin{pmatrix} -t_{m_{n-1}}R_{\boldsymbol{m}_{n-2};m_{n-1},m_n} & \dfrac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_{n-2}}} & 0 \\ -t_{m_{n-1}}R_{\boldsymbol{m}_{n-2};m_{n-1},i} & 0 & \dfrac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_{n-2}}} \end{pmatrix} \tag{3.10}$$

corresponds to the last two rows of $\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}$ multiplied by $\frac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_{n-2}}}$. For the second block, corresponding to $\mathcal{M}_{2,3}\left(\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}\right)$, the correspondence is clear. In order to verify the result for the elements $(2,1)$ and $(3,1)$ of $\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}$, we identify the components of the second matrix in (3.10) as elements of $\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}$ and we rely on the fact that we are multiplying elements of the matrix $\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}$ and its inverse. In fact, for instance the element $(2,1)$ corresponds to

$$-\left[\left(\mathcal{R}_{(\boldsymbol{m},i)}^{n-2}\right)^{-1}\right]_{2,2}\left[\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right]_{2,1} - \left[\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}\right]_{2,3}\left[\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right]_{3,1}$$

$$= -\left[\mathcal{I}\right]_{2,1} + \left[\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}\right]_{2,1}\left[\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right]_{1,1}$$

$$= \left(1 + t_{m_{n-1}}R_{\boldsymbol{m}_{n-2};m_{n-1}m_{n-1}}\right)\left[\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}\right]_{2,1} = \frac{\tau_{\boldsymbol{m}_{n-1}}}{\tau_{\boldsymbol{m}_{n-2}}}\left[\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}\right]_{2,1},$$

where $\mathcal{I}$ is the identity matrix. Similarly to (3.9), we can show the following identity for $j = 1, \dots, n-2$:

$$\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[j]}\right)^{-1}\begin{pmatrix} P_{\boldsymbol{m}_j;m_{j+1}} \\ \vdots \\ P_{\boldsymbol{m}_j;m_n} \\ P_{\boldsymbol{m}_j;i} \end{pmatrix} = \frac{\tau_{\boldsymbol{m}_j}}{\tau_{\boldsymbol{m}_{j-1}}}\left(\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[j-1]}\right)^{-1}\begin{pmatrix} P_{\boldsymbol{m}_{j-1};m_j} \\ \vdots \\ P_{\boldsymbol{m}_{j-1};m_n} \\ P_{\boldsymbol{m}_{j-1};i} \end{pmatrix}\right)^{\langle 1 \rangle}.$$

Combining the previous identities yields (3.5):

$$\tilde{P}_{\boldsymbol{m};i} = \frac{\tau_{(\boldsymbol{m};i)}}{\tau_{\boldsymbol{m}_{n-1}}}\left[\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-1]}\right)^{-1}\left(\tilde{P}_{\boldsymbol{m}_{n-1};m_n},\tilde{P}_{\boldsymbol{m}_{n-1};i}\right)^{\mathrm{T}}\right]_1,$$

$$= \frac{\tau_{(\boldsymbol{m};i)}}{\tau_{\boldsymbol{m}_{n-2}}}\left[\left(\mathcal{R}_{(\boldsymbol{m},i)}^{[n-2]}\right)^{-1}\left(P_{\boldsymbol{m}_{n-2};m_{n-1}},P_{\boldsymbol{m}_{n-2};m_n},P_{\boldsymbol{m}_{n-2};i}\right)^{\mathrm{T}}\right]_1 = \cdots$$

$$= \tau_{(\boldsymbol{m};i)}\left[\left(\mathcal{R}_{(\boldsymbol{m},i)}\right)^{-1}\left(P_{m_1},\dots P_{m_n},P_i\right)^{\mathrm{T}}\right]_1 = P_{\boldsymbol{m};i}.$$

Finally, relation (3.6) follows by Proposition 2.6 and by induction on j. ∎

Proposition 3.2 together with the results in Section 2 allow now to prove the main theorems.

Proof of Theorems 1.5 and 1.6. The key to the proof is to observe that starting form the classical Legendre operator, the application of a rational confluent Darboux transformation indexed by an integer m_i introduces an extra real parameter t_{m_i} and leads to a well defined Sturm–Liouville problem defining a family of exceptional Legendre polynomials. Indeed, the equivalence of the objects defined by the CDT recursion (2.3)–(2.5) and those defined by matrix multiplication in Definition 1.4 show that for any $\boldsymbol{m} \in \mathbb{N}_0^n$, $\tau_{\boldsymbol{m}}(z;\boldsymbol{t_m})$ and $P_{\boldsymbol{m};i}(z;\boldsymbol{t_m})$ are polynomials and $\mathcal{R}_{\boldsymbol{m}}(z;\boldsymbol{t_m})$ is a matrix of rational functions that satisfy the premises of Propositions 2.3–2.7 (see Remark 2.2). Theorem 1.5 follows then from Proposition 2.5 and Theorem 1.6 follows by induction from Proposition 2.7, which establishes the bounds on the parameters (1.10) that ensure the regularity of $T_{\boldsymbol{m}}(\boldsymbol{t_m})$ and the positivity of the measure in $[-1,1]$ ∎

As mentioned above, the degree of the i-th exceptional Legendre polynomial $P_{\boldsymbol{m};i}$ indexed by $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ is not i. The next proposition provides this result.

Proposition 3.4. *Let* $\boldsymbol{m} = (m_1, \ldots, m_n)$ *and suppose that* $m_1, \ldots, m_n$ *are distinct. Let* $\tau_{\boldsymbol{m}}$, $P_{\boldsymbol{m};i}$ *be as defined in* (1.6) *and* (1.8). *Then,*

$$\deg \tau_{\boldsymbol{m}}(z; \boldsymbol{t_m}) = 2(m_1 + \ldots + m_n) + n, \tag{3.11}$$

$$\deg P_{\boldsymbol{m};i}(z; \boldsymbol{t_m}) = 2(m_1 + \ldots + m_n) + n + i$$
$$- (\delta_{i,m_1} + \cdots + \delta_{i,m_n})(2i+1), \qquad i \in \mathbb{N}_0. \tag{3.12}$$

Moreover,

$$P_{\boldsymbol{m};m_k}(z; \boldsymbol{t_m}) = P_{\boldsymbol{m}^{\langle k \rangle};m_k}(z; \boldsymbol{t_{m^{\langle k \rangle}}}), \qquad k = 1, \ldots, n, \tag{3.13}$$

where $\boldsymbol{m}^{\langle k \rangle} \in \mathbb{N}_0^{n-1}$ *denotes the tuple obtained by removing the* k^{th} *entry of* $\boldsymbol{m}$.

Notice that (3.13) accounts for the above Kronecker delta terms in (3.12). It is also worth noting that, as opposed to the "traditional" exceptional families, the degree sequence for exceptional Legendre polynomials is not an increasing sequence, which is further evidence of the different construction.

Proof. The identity (3.13) follows directly from applying (3.3) to this specific choice and the symmetry with respect to permutations in $\boldsymbol{m}$.

Relations (3.11) and (3.12) can be proved by induction. It is clear that they hold for $\boldsymbol{m}_1 = m_1$, since $\deg R_{i_1 i_2}(z) = i_1 + i_2 + 1$. Notice that in $P_{m_1;i}(z; t_{m_1})$, the coefficients of $R_{m_1 m_1}(z) P_i(z)$ and $R_{m_1 i}(z) P_{m_1}(z)$ do not coincide if $i \neq m_1$, so $\deg P_{m_1;i}(z; t_{m_1}) = 2m_1 + i + 1$. If $i = m_1$, by (3.13) we obtain $\deg P_{m_1;m_1}(z; t_{m_1}) = \deg P_{m_1}(z) = m_1$

Now we assume (3.11) and (3.12) hold for $\boldsymbol{m}_{j-1}$. By (3.6), the degree of $R_{\boldsymbol{m}_{j-1};im_j}(z; \boldsymbol{t_{m_{j-1}}})$, understood as the difference between the degree of the polynomial numerator and of the polynomial denominator, is $m_j + i + 1$ if $i \neq m_1, \ldots, m_{j-1}$. Then,

$$\deg \tau_{\boldsymbol{m}_j}(z; \boldsymbol{t_{m_j}}) = \deg \tau_{\boldsymbol{m}_{j-1}}(z; \boldsymbol{t_{m_{j-1}}}) + 2m_j + 1 = 2(m_1 + \cdots + m_j) + j.$$

Arguing as above to verify that there is no cancellation between the highest order contributions, we obtain

$$\deg P_{\boldsymbol{m}_j;i}(z; \boldsymbol{t_{m_j}}) = 2(m_1 + \cdots + m_{j-1}) + (j-1) + 2m_j + i + 1$$

if $i \neq m_1, \ldots, m_j$. In order to prove the result for i equal to some component of $\boldsymbol{m}_j$, we employ relation (3.13). ∎

Remark 3.5. From Proposition 3.4 we see that the codimension (number of missing degrees) of the exceptional Legendre family indexed by $\boldsymbol{m} = (m_1, \ldots, n)$ is $2(m_1 + \cdots + m_n) + n$. This coincides with the degree of $\tau_{\boldsymbol{m}}$, as it happens for all exceptional polynomials [7].

So far we have considered the case when $\boldsymbol{m} = (m_1, \ldots, m_n)$ contains distinct indices. As announced in Remark (1.7), let us show that this choice entails no loss of generality.

Proposition 3.6. *Let* $\boldsymbol{m} \in \mathbb{N}_0^n$ *and let* $\tau_{\boldsymbol{m}}(z; \boldsymbol{t_m})$ *and* $P_{\boldsymbol{m};i}(z; \boldsymbol{t_m})$ *be as defined in* (1.6) *and* (1.8). *Then, for any* $j \in \mathbb{N}_0$ *we have*

$$\tau_{(\boldsymbol{m},j,j)}\big(z; (\boldsymbol{t_m}, t_j, t_j)\big) = \tau_{(\boldsymbol{m},j)}\big(z; (\boldsymbol{t_m}, 2t_j)\big),$$
$$P_{(\boldsymbol{m},j,j);i}\big(z; (\boldsymbol{t_m}, t_j, t_j)\big) = P_{(\boldsymbol{m},j);i}\big(z; (\boldsymbol{t_m}, 2t_j)\big).$$

We see thus that the repeated application of a 2-step confluent Darboux transformation at the same eigenvalue only serves to modify the deformation parameter. In general, if the two parameters at the repeated j are different, we would have similarly

$$\tau_{(\boldsymbol{m},j,j)}(z;(\boldsymbol{t_m},t_j,t_j')) = \tau_{(\boldsymbol{m},j)}(z;(\boldsymbol{t_m},t_j+t_j'))),$$

$$P_{(\boldsymbol{m},j,j);i}(z;(\boldsymbol{t_m},t_j,t_j')) = P_{(\boldsymbol{m},j);i}(z;(\boldsymbol{t_m},t_j+t_j')).$$

Proof. We omit again explicit dependence on z and $t_{\boldsymbol{m}},t_j$ if no confusion arises. We apply Proposition 3.2 and Definition 3.1 twice to obtain

$$\tau_{(\boldsymbol{m},j,j)} = (1+t_j R_{(\boldsymbol{m},j);jj})\tau_{(\boldsymbol{m},j)} = \left(1+t_j R_{\boldsymbol{m};jj} - \frac{t_j^2 R_{\boldsymbol{m};jj}^2}{1+t_j R_{\boldsymbol{m};jj}}\right)(1+t_j R_{\boldsymbol{m};jj})\tau_{\boldsymbol{m}}$$

$$= (1+2t_j R_{\boldsymbol{m};jj})\tau_{\boldsymbol{m}} = \tau_{(\boldsymbol{m},j)}\big((z;(\boldsymbol{t_m},2t_j)\big),$$

$$P_{(\boldsymbol{m},j,j);i} = (1+t_j R_{(\boldsymbol{m},j);jj})P_{(\boldsymbol{m},j);i} - t_j R_{(\boldsymbol{m},j);ij}P_{(\boldsymbol{m},j);j}$$

$$= (1+t_j R_{(\boldsymbol{m},j);jj})\big((1+t_j R_{\boldsymbol{m};jj})P_{\boldsymbol{m};i} - t_j R_{\boldsymbol{m};ij}P_{\boldsymbol{m};i}\big)$$

$$- t_j \left(R_{\boldsymbol{m};ij} - \frac{t_j R_{\boldsymbol{m};jj}R_{\boldsymbol{m};ij}}{1+t_j R_{\boldsymbol{m};jj}}\right)P_{\boldsymbol{m};j}$$

$$= (1+2t_j R_{\boldsymbol{m};jj})P_{\boldsymbol{m};i} - 2t_j R_{\boldsymbol{m};ij}P_{\boldsymbol{m};j} = P_{(\boldsymbol{m},j);i}(z;(\boldsymbol{t_m},2t_j)). \qquad \blacksquare$$

4   Examples

To conclude, we present some examples of exceptional Legendre polynomials and orthogonality relations for the cases of $n=1$ and $n=2$.

4.1   The 1-parameter exceptional Legendre family

In the 1-parameter case, we have $\boldsymbol{m} = (m_1) \in \mathbb{N}_0$, and

$$\tau_{m_1}(z;t_{m_1}) = 1 + t_{m_1}R_{m_1 m_1}(z),$$

$$P_{m_1;i}(z;t_{m_1}) = (1+t_{m_1}R_{m_1 m_1}(z))P_i(z) - t_{m_1}R_{im_1}(z)P_{m_1}(z), \qquad i \in \mathbb{N}_0.$$

Note that $P_{m_1;m_1}(z) = P_{m_1}(z)$. The degrees of the other polynomials are

$$\deg P_{m_1;i}(z) = i + 2m_1 + 1, \qquad i \neq m_1.$$

The corresponding exceptional operator is $T_{m_1}(t_{m_1}) = \hat{T}(\tau_{m_1}(z;t_{m_1}))$ with the latter as per (1.1).

The polynomial $\tau_{m_1}(z;t_{m_1})$ does not vanish in $[-1,1]$ provided that

$$t_{m_1} > -\frac{1}{R_{m_1 m_1}(1)} = -m_1 - \frac{1}{2}.$$

In this case, $\{P_{m_1;i}(z;t_{m_1})\}_{i\in\mathbb{N}_0}$ is a family of exceptional Legendre polynomials, with orthogonality weight

$$W_{m_1}(z;t_{m_1}) = \frac{1}{\tau_{m_1}(z;t_{m_1})^2}.$$

The above set is a complete orthogonal polynomial basis of the space $\mathrm{L}^2([-1,1], W_{m_1}(z;t_{m_1})\mathrm{d}z)$. The orthogonality relations are

$$\int_{-1}^{1} \frac{P_{m_1;i_1}(u;t_{m_1})P_{m_1;i_2}(u;t_{m_1})}{\tau_{m_1}(u;t_{m_1})^2}\mathrm{d}u = \frac{2}{1+2i_1}\delta_{i_1 i_2}, \qquad i_1,i_2 \in \mathbb{N}_0\backslash\{m_1\},$$

$$\int_{-1}^{1} \frac{P_{m_1;m_1}(u;t_{m_1})^2}{\tau_{m_1}(u;t_{m_1})^2}\mathrm{d}u = \frac{2}{1+2m_1+2t_{m_1}}.$$
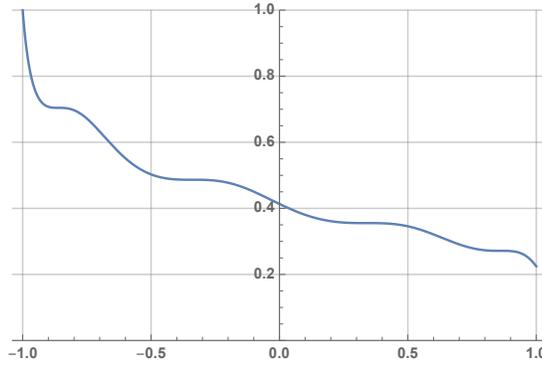
Figure 1. Exceptional Legendre weight $\tau_{m_1}^{-2}(z, t_{m_1})$ for $m_1 = 4$ and $t_{m_1} = 5.2$.

The above example illustrates perfectly the isospectral nature of the CDT that relates $T(1) \to T_{m_1}(t_{m_1})$. The eigenvalues of the two operators are the same. As for the eigenfunctions, if $t_{m_1} \neq 0$, then for $i \neq m_1$ they are transformed, but their norms stay the same. On the other hand, for $i = m_1$ the opposite happens: the eigenfunction does not change but its norm does.

Since $\tau'_{m_1}(z) = t_{m_1} P_{m_1}^2(z)$, it follows that the weight $W_{m_1}(z)$ is a decreasing (resp. increasing) function for $t_{m_1} > 0$ (resp. $t_{m_1} < 0$), which has m_1 saddle points in $[-1, 1]$ at the zeros of P_{m_1}, but no local minima or maxima.

For instance, for $m_1 = 4$, we have

$$\tau_4(z, t_4) = 1 + \frac{1}{576} t_4 \left(64 + 81z - 540z^3 + 1998z^5 - 2700z^7 + 1225z^9 \right)$$

and the weight is shown in Fig. 1. The first few polynomials for this choice are

$$P_{4;0}(z, t_4) = 1 + \frac{t_4}{144} \left(16 + 135z^3 - 459z^5 + 585z^7 - 245z^9 \right),$$

$$P_{4;1}(z, t_4) = z + \frac{t_4}{1152} \left(-9 + 128z + 171z^2 + 30z^4 - 1314z^6 + 2475z^8 - 1225z^{10} \right),$$

$$P_{4;2}(z, t_4) = P_2(z) - \frac{t_4}{576} \left(32 - 96z^2 + 189z^3 - 756z^5 + 1278z^7 - 1300z^9 + 525z^{11} \right),$$

$$P_{4;3}(z, t_4) = P_3(z) - \frac{t_4}{9216} \left(243 - 1536z - 3402z^2 + 2560z^3 + 3645z^4 + 7668z^6 - 17955z^8 \right.$$
$$\left. + 16950z^{10} - 6125z^{12} \right),$$

$$P_{4;4}(z, t_4) = P_4(z),$$

$$P_{4;5}(z, t_4) = P_5(z) + \frac{t_4}{9216} \left(243 + 1920z - 1215z^2 - 8960z^3 - 3645z^4 + 8064z^5 + 17145z^6 \right.$$
$$\left. - 42255z^8 + 66171z^{10} - 50855z^{12} + 15435z^{14} \right).$$

We display the above polynomials for $t_4 = 0$ (classical Legendre) and for $t_4 = 5.2$ in Fig. 2. Observe that all polynomials undergo a continuous deformation as t_4 changes, except for $P_{4;4} = P_4$ that stays the same.

4.2   The 2-parameter exceptional Legendre family

In the 2-parameter case, we have $\boldsymbol{m} = (m_1, m_2) \in \mathbb{N}_0^2$, $\boldsymbol{t_m} = (t_{m_1}, t_{m_2})$ and

$$\tau_{\boldsymbol{m}}(z; \boldsymbol{t_m}) = \tau_{m_1}(z; t_{m_1}) \tau_{m_2}(z; t_{m_2}) - t_{m_1} t_{m_2} R_{m_1 m_2}(z)^2,$$
$$P_{\boldsymbol{m};i}(z; \boldsymbol{t_m}) = P_i(z) \tau_{\boldsymbol{m}}(z; \boldsymbol{t_m}) - P_{m_1}(z) t_{m_1} \tau_{m_2}(z; t_{m_2}) R_{m_1; m_2, i}(z; t_{m_2})$$
$$- P_{m_2}(z) t_{m_2} \tau_{m_1}(z; t_{m_1}) R_{m_1; m_2, i}(z; t_{m_1}), \qquad i \in \mathbb{N}_0,$$
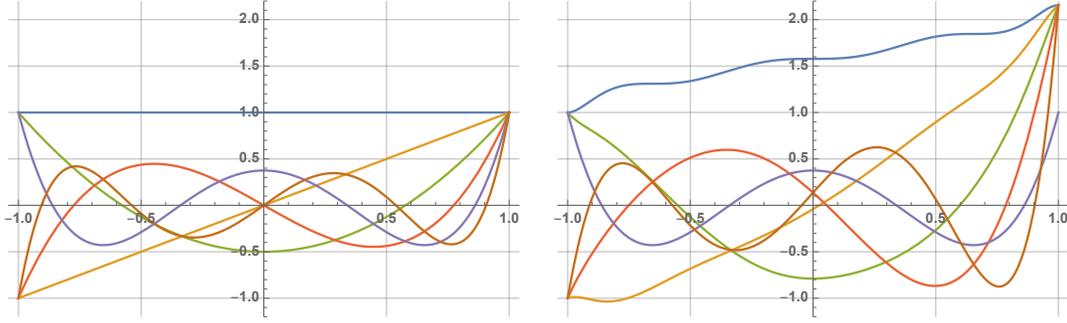
Figure 2. First few exceptional Legendre polynomials $P_{m_1;i}(z;t_{m_1})$ for $m_1 = 4$, with $t_{m_1} = 0$ (left) and $t_{m_1} = 5.2$ (right).

where

$$
\begin{aligned}
R_{m_1;i_1i_2}(z;t_m) &= \int_{-1}^{z} \frac{P_{m_1;i_1}(u;t_{m_1})P_{m_1;i_2}(u;t_{m_1})}{\tau_{m_1}(u;t_{m_1})^2}\mathrm{d}u \\
&= R_{i_1i_2}(z) - \frac{t_{m_1}R_{i_1m_1}(z)R_{i_2m_1}(z)}{1 + t_{m_1}R_{m_1m_1}}, \qquad i_1, i_2 \in \mathbb{N}_0,
\end{aligned}
$$

Supposing that $m_1 \neq m_2$ we have

$$
\begin{aligned}
\deg P_{m;i}(z;t_m) &= i + 2m_1 + 2m_2 + 2, \qquad i \in \mathbb{N}_0 \backslash \{m_1, m_2\}, \\
\deg P_{m,m_1}(z;t_m) &= m_1 + 2m_2 + 1, \\
\deg P_{m;m_2}(z;t_m) &= m_2 + 2m_1 + 1.
\end{aligned}
$$

The polynomial $\tau_m(z;t_{m_1,t_{m_2}})$ does not vanish in $[-1,1]$ provided that

$$
t_{m_1} > -m_1 - \frac{1}{2}, \qquad t_{m_2} > -m_2 - \frac{1}{2}.
$$

In this case, $\{P_{m;i}(z;t_m)\}_{i\in\mathbb{N}_0}$ is a family of exceptional Legendre polynomials, with orthogonality weight

$$
W_m(z;t_m) = \frac{1}{\tau_m(z;t_m)^2}.
$$

The above set is a complete orthogonal polynomial basis of the space $\mathrm{L}^2([-1,1], W_m(z;t_m)\mathrm{d}z)$.

The orthogonality relations are

$$
\begin{aligned}
\int_{-1}^{1} \frac{P_{m;i_2}(u;t_m)P_{m;i_2}(u;t_m)}{\tau_m(u;t_m)^2}\mathrm{d}u &= \frac{2}{1+2i_1}\delta_{i_1i_2}, \qquad i_1, i_2 \in \mathbb{N}_0\backslash\{m_1, m_2\}, \\
\int_{-1}^{1} \frac{P_{m;m_1}(u;t_m)^2}{\tau_m(u;t_m)^2}\mathrm{d}u &= \frac{2}{1+2m_1+2t_{m_1}}, \\
\int_{-1}^{1} \frac{P_{m;m_2}(u;t_m)^2}{\tau_m(u;t_m)^2}\mathrm{d}u &= \frac{2}{1+2m_2+2t_{m_2}}.
\end{aligned}
$$

For instance, for $m = (m_1, m_1) = (1,2)$, we have

$$
\begin{aligned}
\tau_{(1,2)}\big(z;(t_1,t_2)\big) &= 1 + \frac{1}{3}t_1\big(1+z^3\big) + \frac{1}{20}t_2\big(4+5z-10z^3+9z^5\big) \\
&\quad + \frac{1}{960}t_1t_2(1+z)^4\big(49-116z+110z^2-36z^3+9z^4\big).
\end{aligned}
$$

For certain values of $t_m = (t_{m_1}, t_{m_2})$ the weight is displayed in Fig. 3. To the best of our knowledge, this is the first example of an exceptional orthogonal polynomial system whose weight is not monotonic or unimodal.
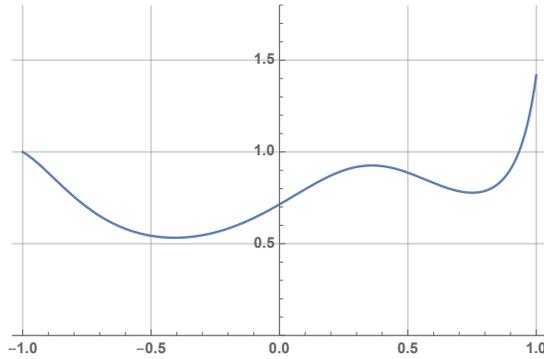
Figure 3. Exceptional Legendre weight $\tau_{\boldsymbol{m}}^{-2}(z, \boldsymbol{t_m})$ for $\boldsymbol{m} = (m_1, m_2) = (1, 2)$ and $(t_{m_1}, t_{m_2}) = (2, -1.6)$.

5 Summary

In this paper we show that the class of exceptional orthogonal polynomials is much larger than previously thought. A new construction based on confluent Darboux transformations leads to new exceptional families with some similarities and differences with respect to the other exceptional families. In common, they are also Sturm–Liouville problems with rational coefficients, each family is indexed by a set of integers, and it defines a complete basis of polynomial eigenfunctions, whose degree sequence has missing degrees. But as opposed to the other exceptional families, there are no gaps in the spectrum and the new families contain an arbitrary number of real deformation parameters, so the construction can be seen as an isospectral deformation of the classical operators. We illustrate the new construction by describing the full class of exceptional Legendre polynomials, which cannot be derived through the standard construction. The same method can be applied with minor modifications to the Jacobi operator. A more exhaustive description of these matters will be provided in a forthcoming publication, together with a discussion of the implications for the classification of exceptional polynomials [8].

Acknowledgements

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