

c_2 Invariants of Hourglass Chains via Quadratic Denominator Reduction

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Abstract. We introduce families of four-regular graphs consisting of chains of hourglasses which are attached to a finite kernel. We prove a formula for the c_2 invariant of these hourglass chains which only depends on the kernel. For different kernels these hourglass chains typically give rise to different c_2 invariants. An exhaustive search for the c_2 invariants of hourglass chains with kernels that have a maximum of ten vertices provides Calabi–Yau manifolds with point-counts which match the Fourier coefficients of modular forms whose weights and levels are [4,8], [4,16], [6,4], and [9,4]. Assuming the completion conjecture, we show that no modular form of weight 2 and level ≤ 1000 corresponds to the c_2 of such hourglass chains. This provides further evidence in favour of the conjecture that curves are absent in c_2 invariants of ϕ^4 quantum field theory.

Key words: c_2 invariant; denominator reduction; quadratic denominator reduction; Feynman period

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1 Introduction

Given a graph G , to each edge $e \in E(G)$ associate a variable α_e and define the *Kirchhoff polynomial* or *first Symanzik polynomial* to be

$$\Psi_G = \sum_T \prod_{e \notin T} \alpha_e,$$

where the sum is over all spanning trees T of G . When it converges define the *Feynman period* of G to be the projective integral

$$P_G = \int_{\alpha_e \geq 0} \frac{\Omega}{\Psi_G^2}, \quad (1.1)$$

where $\Omega = \sum_{i=1}^{|E(G)|} (-1)^i d\alpha_1 \cdots \widehat{d\alpha_i} \cdots d\alpha_{|E(G)|}$. The Feynman period P_G is the residue of its Feynman integral. It contributes to the β -function which controls the way the physical coupling

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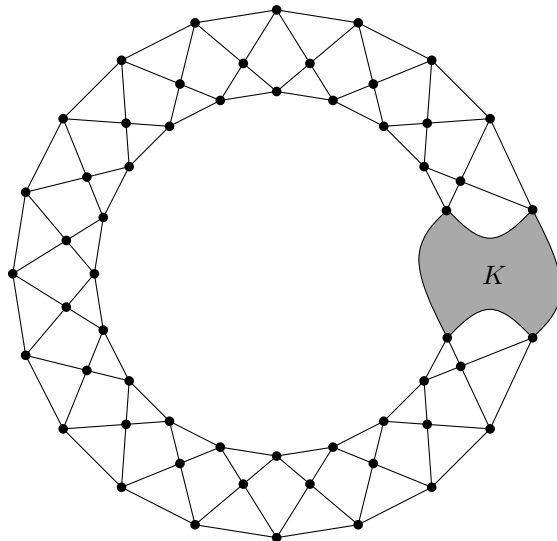


Figure 1. An illustration of the type of graph we consider. The shaded section is the kernel which is attached to the hourglass chain.

changes with momentum, see, e.g., [16]. The β -function also plays a prominent role for applications to phase transitions in statistical physics, see, e.g., [17, 33]. So, the Feynman period is of significance in various branches of physics. Furthermore, the Feynman period exhibits the arithmetic content of the Feynman integral, so it is very interesting for anyone studying the geometry and number theory underlying quantum field theory [1, 3, 21, 24].

In [4] Francis Brown introduced denominator reduction as a tool for studying Feynman periods algebraically. Denominator reduction describes how the denominators of the Feynman periods change through successive edge integrations. When these denominators fail to factor into linear pieces denominator reduction ends. Keeping track of the numerators in this process using multiple polylogarithms gives an algorithm for parametric Feynman integration which fails when denominator reduction stops [4, 5]. Improvements and variants of this integration approach have been implemented [20], but the constraint of the non-factoring denominators remains.

Denominator reduction itself is purely algebraic, but still carries substantial information about these integrals since the numerator polynomials in each integration step are typically of simpler geometry than the denominator. This simplicity of the numerators is inherited from the second integration step, where the polynomials in the numerator are mere coefficients of Ψ_G whereas the denominator is a resultant. The geometric domination of the denominator, however, is not a fully general feature. There exist (rare) examples where the Feynman integral P_G has a geometry which is missed by denominator reduction.

Denominator reduction tells us about weight drop [12] and can be used to compute the c_2 invariant, an arithmetic graph invariant that predicts properties of the Feynman periods [9, 25] (for a definition see Section 2.3).

In [27] one of us defined a generalized denominator reduction, called *quadratic denominator reduction*, that can always progress at least one more step than standard denominator reduction, and sometimes much further. At the cost of not working for even prime powers $\neq 2$, this quadratic denominator reduction can be used to compute c_2 invariants and so still tells us about the geometries underlying the Feynman periods.

The geometric idea behind quadratic denominator reduction is that in certain cases a denominator with a square root still has rational geometry as it defines a projective line.

As a demonstration of the power of quadratic denominator reduction, in this paper we will study infinite families of graphs built by attaching a chain of hourglasses to a finite kernel,

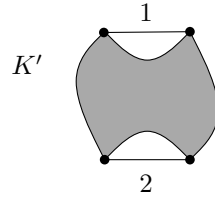


Figure 2. The graph K' is the kernel K with the two extra edges 1 and 2.

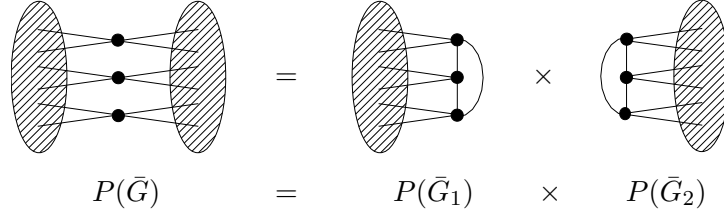


Figure 3. Vertex connectivity 3 leads to a product of periods.

see Figure 1. The c_2 invariant for any such graph will only depend on the kernel, giving arbitrarily many infinite families of graphs for which we know the c_2 invariant (for the prime 2 and odd prime powers).

With the notions of hourglass, kernel, and hourglass chain as illustrated in Figures 1 and 2 we can state our theorem.

Theorem 1.1. *Let K be a kernel and let $L \in \mathcal{G}_K$ be a graph of the type we consider with an hourglass chain of length at least 6, see Section 2.8 for definitions. Let v be a vertex of the hourglass chain that is shared by the second and third hourglasses from one end. Let K' be K with new edges joining the external vertices as in Figure 2. Index these new edges by 1 and 2 (with variables α_1 and α_2 , respectively). If q is an odd prime power then*

$$c_2^{(q)}(L - v) \equiv (\alpha_1 (\Psi_{K'}^{1,2}(\alpha))^2 \Psi_{K'}^{2,2}(\alpha) \Psi_{K',2}(\alpha))_q \pmod{q}, \tag{1.2}$$

where $(F)_q$ is the Legendre sum of the polynomial F , see Definition 2.8. For $q = 2$ the c_2 invariant of $L - v$ vanishes, $c_2^{(2)}(L - v) \equiv 0 \pmod{2}$.

The *Dodgson* polynomials on the right hand side of (1.2) are defined in Section 2. Good introductory references for the general role of Dodgson polynomial in physics are, e.g., [4, 14, 22].

Graphs G for which the Feynman period exists, sometimes have structures which lead to graphical reductions, see Section 2.7 and [24]. If G has a three vertex split the period factorizes, see Figure 3. Double triangles can be reduced without changing the c_2 [11], see Figure 4. Hourglass chains (for suitable kernels) have no such reductions. They establish families of the most complicated type, the *prime ancestors* [2, 24]. Hourglass chains are the first families of prime ancestors for which the c_2 invariant can be calculated. In a certain sense these hourglass chains can be considered as ‘telescopes’ that enable us to look into geometries of Feynman graphs at very high loop order (i.e., the number of independent cycles in $L - v$). This has never been achieved before: all previous techniques were either restricted to the analysis of small graphs, or they worked in a way which was fundamentally prime-by-prime [13, 30, 32] and hence did not lead to non-trivial graph families with the same underlying geometries.

The paper is organized as follows: in Section 2 we provide the necessary background information on denominator reduction and the c_2 invariant. Section 3 contains the proof of Theorem 1.1: the hourglass reductions.

Finally we use Theorem 1.1 in Section 4 for an exhaustive search for c_2 invariants in hourglass chains with kernels of at most six internal vertices (vertices which are not attached to the

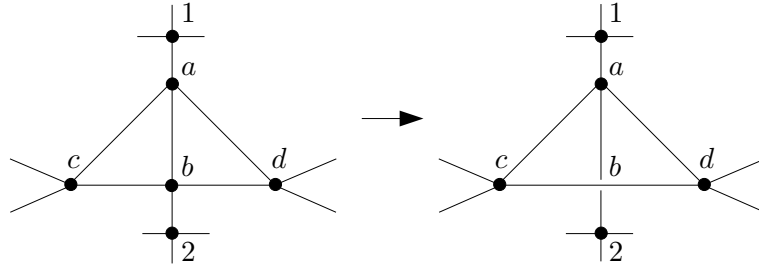


Figure 4. Double triangle reduction: Replace a joint vertex of two attached triangles by a crossing.

hourglasses, see Figure 19 and Table 2 for a maximum of five internal vertices). We find Legendre symbols (see Section 2) $(4/q)$ and $(-4/q)$ along with several modular forms. Explicitly the weight and level of the identified modular forms (given in the notation [weight,level], see [10, 27]) are [4,8], [4,16], [6,4], and [9,4]. The modular form [9,4] is a new addition to the table of modular forms in ϕ^4 theory as it was not found in ϕ^4 graphs of loop orders less or equal twelve which were studied in [27].

An important outcome of this article is providing further support for the conjecture that in ϕ^4 theory (corresponding to 4-regular graphs) the c_2 is free of curves (which correspond to weight 2 modular forms), [10, Conjecture 26], see also [27]. This puzzling conjecture is for the first time tested to any loop order for some non-trivial geometries. It seems to be connected to some deep algebraic structure in quantum field theories. Note that curves in c_2 invariants are ubiquitous if one lifts the (physical) restriction to 4-regular graphs.

For extra support of this “no-curves-puzzle” it might be worthwhile to study kernels in the future which lead out of ϕ^4 graphs. Which non- ϕ^4 kernels provide c_2 invariants that correspond to weight two modular forms? It is also possible to extend the c_2 -search to the rapidly increasing number of kernels with more than six internal vertices, see Table 1 and Question 3.1.

2 Background

2.1 Dodgsons

In order to define the denominator reductions we first need to give a determinantal expression for Ψ_G and define some related polynomials.

Assume for the rest of the paper that G is a connected graph.

Choose an order on the edges $E(G)$ and vertices $V(G)$ of G and choose a direction for each edge. Then the *signed incidence matrix* of G is a $|V(G)| \times |E(G)|$ matrix with entries $-1, 0, 1$, where the i, j th entry is -1 if edge j starts at vertex i , is 1 if edge j ends at vertex i , and is 0 otherwise. Let E be the signed incidence matrix with one row removed. Since G is connected the rank of the signed incidence matrix is $|V(G)| - 1$ and so E is full rank.

Define the *expanded Laplacian* to be the matrix

$$L_G = \begin{bmatrix} \Lambda & E^t \\ E & 0 \end{bmatrix},$$

where Λ is the diagonal matrix with diagonal entries α_e , $e \in E(G)$, in the edge order chosen above. This matrix is called the expanded Laplacian because it behaves very much like the Laplacian (with a matching row and column removed), but the pieces of it have been expanded out into a larger block matrix.

Proposition 2.1.

$$\Psi_G = (-1)^{|V(G)|-1} \det L_G.$$

This proposition is at its core the matrix-tree theorem [4, Section 2.2]. More specifically, the form of the matrix tree theorem that is most useful here is the form that says given a $|V(G) - 1| \times |V(G) - 1|$ submatrix of E , this matrix has determinant ± 1 if the edges corresponding to the columns of the submatrix form a spanning tree of G and has determinant 0 otherwise.

In the following we also need the extension to minors of L_G . Polynomials from minors of L_G are called *Dodgson polynomials* in [4]. In general, these polynomials have signs which depend on the sequence of edges [27]. While in classical denominator reduction signs are often insignificant, they play an important role in quadratic denominator reduction. By the special structure of (1.2) it is sufficient for our purpose to define the sign of Dodgson polynomials in a trivial case.

Definition 2.2. For any subsets I, J, K of the edges of a connected graph G with $|I| = |J|$ we define the Dodgson polynomials $\Psi_K^{I,J}$ as

$$\Psi_{G,K}^{I,J} = \pm \det L_G^{I,J} \Big|_{\alpha_k=0, k \in K},$$

where $L_G^{I,J}$ is L_G with rows in I and columns in J deleted. In the case $I = J$ the sign is $(-1)^{|V(G)|-1}$.

The contraction-deletion formula (see [4], [27, Lemma 11])

$$\Psi_{G,K}^{I,J} = \alpha_e \Psi_{G,K}^{Ie,Je} + \Psi_{G,Ke}^{I,J} = \alpha_e \Psi_{G \setminus e, K}^{I,J} + \Psi_{G/e, K}^{I,J} \tag{2.1}$$

relates Dodgson polynomials to minors. Note that in the context of Dodgson polynomials graphs may have multiple edges and self-loops (which contract to zero, i.e., every Dodgson polynomial of a graph with a contracted self loop vanishes).

Example 2.3. A tree has the graph polynomial 1. The Dodgson polynomial of a circle C is the sum of its edge-variables. In this case $\Psi_C^{e,f} = \pm 1$ for any two edges $e, f \in E(G)$.

We get the following vanishing cases (see [4], [11, Section 2.2 (4)], [27, Lemma 13]):

$$\begin{aligned} \Psi_{G,K}^{I,J} &= 0 && \text{if } I \text{ or } J \text{ cut } G, \\ \Psi_{G,K}^{I,J} &= 0 && \text{if } (I \cup K) \setminus J \text{ or } (J \cup K) \setminus I \text{ contain a cycle.} \end{aligned} \tag{2.2}$$

An important Dodgson identity is (see [4] and [27, Lemma 18])

$$\Psi_{G,2}^{1,1} \Psi_{G,1}^{2,2} - \Psi_G^{12,12} \Psi_{G,12} = (\Psi_G^{1,2})^2 \tag{2.3}$$

for any two edges 1, 2 in G . Many more identities for Dodgson polynomials can be found in [4, 27].

2.2 Spanning forest polynomials

Dodgson polynomials can also be written in terms of spanning forests. To that end, given a partition P of a subset of the vertices of G , define the *spanning forest polynomial*

$$\Phi_G^P = \sum_F \prod_{e \notin F} \alpha_e,$$

where the sum is over spanning forests with the property that there is a bijection between the parts of P and the trees of the forest such that every vertex in a part of the partition is in the corresponding tree.

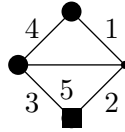


Figure 5. An example of a graph with a partition of some of the vertices marked by the shape of the large vertices.

For example, given the graph illustrated in Figure 5, with the partition P indicated by the shape of the large vertices, the corresponding spanning forest polynomial is $\alpha_3(\alpha_4\alpha_2 + \alpha_1\alpha_2 + \alpha_1\alpha_5 + \alpha_2\alpha_5)$. This technique of marking the partition by the shape of the large vertices will be used without further comment in the main hourglass chain reduction argument.

We have the following proposition

Proposition 2.4.

$$\Psi_{G,K}^{I,J} = \sum f_k \Phi_{G \setminus (I \cup J \cup K)}^{P_k},$$

where the P_k run over partitions of the ends of I , J , and K , and $f_k \in \{-1, 0, 1\}$. Furthermore, those f_k which are nonzero are exactly those where each forest in the polynomial becomes a tree in $G \setminus I / (J \cup K)$ and in $G \setminus J / (I \cup K)$; if any forest in the polynomial has this property then all of them do.

This proposition is a particular interpretation of the all minors matrix tree theorem, for a proof see [12, Propositions 8 and 12]. To see the connection to the matrix tree theorem briefly, consider a term in the determinant giving $\Psi_{G,K}^{I,J}$. The variables indicate edges of G , where the corresponding columns are removed in E and corresponding rows are removed in E^t . The rows indexed by I are removed in E^t but not in E and the columns indexed by J are removed in E but not in E^t . By the matrix tree theorem as summarized after Proposition 2.1, both these sets of edges must simultaneously be spanning trees. Furthermore the variables indexed by K are set to 0. So they must not be in the monomial and hence must be in the trees. An edge which must be in a tree can be contracted, and splitting apart vertices which were identified via a contraction splits the tree into a forest with constraints on which vertices belong in which tree of the forest. Working out the details gives the result, see [12] for details.

The signs in the spanning forest polynomial expansion of a Dodgson polynomial can be tricky, however, the only case we will need for the argument below is given in the following lemma.

Lemma 2.5. *With notation as in Proposition 2.4, if P_1 and P_2 both have nonzero coefficients, and P_1 and P_2 differ by swapping two vertices which are in the same component of J viewed as a subgraph of G , then $f_1 = -f_2$.*

This is a special case of [12, Corollary 17].

Another useful observation, see [12, Proposition 21], is that if G is formed as the 2-sum of G_1 and G_2 , that is G_1 and G_2 each have a distinguished edge, e_1 and e_2 respectively, and G is the result of identifying e_1 and e_2 and then removing this new identified edge while leaving the induced identifications on the incident vertices, then

$$\Psi_G = \Psi_{G_1 \setminus e_1} \Psi_{G_2 / e_2} + \Psi_{G_1 / e_1} \Psi_{G_2 \setminus e_2} = \Psi_{G_1 \setminus e_1} \Phi_{G_2 \setminus e_2}^{\{v_1\}, \{v_2\}} + \Phi_{G_1 \setminus e_1}^{\{v_1\}, \{v_2\}} \Psi_{G_2 \setminus e_2}, \quad (2.4)$$

where v_1 and v_2 are the ends of e_1 and e_2 .

Note that all Dodgson polynomials and all spanning forest polynomials are explicitly linear in all their variables.

2.3 The c_2 invariant

A general aim in the mathematical theory of Feynman periods is to understand what kind of numbers can appear [3, 24, 26]. The interest in this topic was recently intensified by the (conjectural) discovery of a Galois coaction structure on these numbers [6, 7, 21]. In general, the Feynman period (1.1) is hard to analyze. Even the zero locus of the graph hypersurface $\Psi_G = 0$ has a complicated geometric structure.

The number theoretic content of the Feynman period is intimately related to the motivic structure of its integral. In general, the motivic setup is a deep superstructure to the cohomology theory of integrals in algebraic geometry. Going back to ideas of A. Grothendieck it lifts Galois theory to higher dimensions (for first reading we recommend [23]). This motivic structure unifies all fields and thus finite fields \mathbb{F}_q capture information on the geometry of the Feynman period. The motivic information can be extracted from the number of elements on the singular locus of the integrand. This is, e.g., visible in the action of the Frobenius homomorphism in the classical Lefschetz fixed point theorem [18]. While the knowledge of this point-count for a single q is still not very informative, its value for all (or many) q carries important number theoretical information on the (*framing* of the) Feynman period [8].

We define

$$[F]_q = |\{F = 0 \text{ in } \mathbb{F}_q\}|$$

as the point-count of the zero locus of the polynomial F . In the context of Feynman periods the important information of the point-count is hidden in the first non-trivial reduction modulo q . For any connected graph G with at least three edges we define [25]

$$c_2^{(q)}(G) \equiv \frac{[\Psi_G]_q}{q^2} \pmod{q} \tag{2.5}$$

as the c_2 invariant of the Feynman graph G . The above definition implies that the point-count of the graph hypersurface is always divisible by q^2 (the index 2 in c_2 refers to this square). For a given graph G one should think of the c_2 as the infinite sequence $(c_2^{(q)})_{q=2,3,4,5,7,8,9,11,\dots}$ of remainders modulo q .

The benefit of the reduction modulo q is that the point-count is combinatorially quite accessible. In practice, non-trivial prime powers are still harder to come by, so that the c_2 invariant is often studied for pure primes only. It is conjectured [27, Conjecture 2], that the knowledge of the c_2 for all primes determines the c_2 for all prime powers.

The c_2 has been studied quite deeply in the context of ϕ^4 quantum field theory (Section 2.7). The focus of these studies can either be the general mathematical structure of the c_2 [11, 15, 31] or the zoology of the geometries identified by c_2 s [9, 10, 13, 27, 30, 32]. The nature of this article is more in the latter direction, particularly when we analyze the c_2 s of small kernels in Section 4. We would like to emphasize that although identifying c_2 s can have an experimental flavour it might be of high importance to understanding the algebraic structure of quantum field theories.

We can get rid of the division by q^2 in (2.5) by using Dodgsons instead of the graph polynomial (see [9, Corollary 28 and Theorem 29])

$$c_2^{(q)}(G) \equiv -[\Psi_G^{13,23} \Psi_{G,3}^{1,2}]_q \pmod{q}$$

for every connected graph G with a degree 3 vertex v . In this version the c_2 can be further simplified by denominator reductions and quadratic denominator reductions as will be outlined in the next sections.

2.4 Denominator reduction

Successive integration of the Feynman period (1.1) leads to denominators which are linear in the next integration variable. After three initial steps the new denominator will be the resultant of the old denominator with respect to the integration variable. Eventually the denominator may cease to factor into linear pieces and then one typically enters very complicated territory. This successive taking of resultants has a point-count version which says that under certain conditions

$$[(A\alpha + B)(C\alpha + D)]_q \equiv -[AD - BC]_q \pmod{q}.$$

If the resultant $AD - BC$ factors in some new variable then the reduction can be repeated. The size of the polynomial that has to be counted reduces rapidly and when no more reduction is possible then one can still resort to brute force counting at the last step. More precisely, we obtain the following result.

Definition 2.6 (denominator reduction [4, Definition 120 and Proposition 126]). Given a connected graph G with at least three edges and a sequence of edges $1, 2, \dots, |E(G)|$ we define

$${}^3\Psi_G(1, 2, 3) = \pm\Psi_G^{13,23}\Psi_G^{1,2}.$$

Suppose ${}^n\Psi_G$ for $n \geq 3$ factors as

$${}^n\Psi_G(1, \dots, n) = (A\alpha_{n+1} + B)(C\alpha_{n+1} + D)$$

then we define

$${}^{n+1}\Psi_G(1, \dots, n+1) = \pm(AD - BC).$$

Otherwise denominator reduction terminates at step n . If it exists we call ${}^n\Psi_G$ an n -invariant of G .

Note that the n -invariants are only defined up to sign. The 4-invariant always factorizes [4, Lemma 82]

$${}^4\Psi_G = \pm\Psi_G^{14,23}\Psi_G^{13,24}. \tag{2.6}$$

Therefore the 5-invariant always exists. In Lemma 87 of [4] it is proved that for $n \geq 5$ the n -invariants become independent of the sequence of the reduced edges (they only depend on the set of reduced variables). Denominator reduction is compatible with the c_2 invariant in the following sense.

Theorem 2.7 ([9, Theorem 29]). *Let G be a connected graph with at least three edges and $h_1(G) \leq |E(G)|/2$ independent cycles. Then*

$$c_2^{(q)}(G) \equiv (-1)^n [{}^n\Psi_G]_q \pmod{q}$$

whenever ${}^n\Psi_G$ exists for $n < |E(G)|$.

The theorem was proved for ≥ 5 edges in [9]. The proof trivially extends to the case of three or four edges. If ${}^n\Psi_G = 0$ for some sequence of edges and some n (and hence for all subsequent n) then G has *weight drop*. In this case the c_2 invariant vanishes [12].

2.5 Quadratic denominator reduction

Only in a few particularly simple cases does denominator reduction go through to the very end where all variables are reduced. Brute force point-counting after the last step of denominator reduction can be very time consuming and does not lend itself to more theoretical understanding. Therefore it is desirable to continue the reduction as far as possible. The integration of a denominator that does not factor produces a square root. The existence of further reduction steps is suggested by the fact that even in the presence of squares and square roots integrals may stay rational in a geometrical sense. Let us exemplify this by the following toy integrals [27],

$$\int_0^\infty \frac{d\alpha}{A\alpha^2 + B\alpha + C} = \frac{\log(X)}{\sqrt{B^2 - 4AC}},$$

for some algebraic expression X in A, B, C and

$$\int_0^\infty \frac{d\alpha}{\sqrt{D\alpha^2 + E\alpha + F(H\alpha + J)}} = \frac{\log(Y)}{\sqrt{DJ^2 - EHJ + FH^2}}$$

for some algebraic expression Y in D, E, F, H, J .

For the formal implementation of this idea we pass from point-counts to Legendre sums.

Definition 2.8 ([27, Definition 29]). Let q be an odd prime power. For any $a \in \mathbb{F}_q$ the Legendre symbol $(a/q) \in \{-1, 0, 1\}$ is defined by

$$\left(\frac{a}{q}\right) = |\{x \in \mathbb{F}_q : x^2 = a\}| - 1.$$

For any polynomial $F \in \mathbb{Z}[\alpha_1, \dots, \alpha_N]$ we define

$$(F)_q = \sum_{\alpha \in \mathbb{F}_q^N} \left(\frac{F(\alpha)}{q}\right),$$

where the sum is in \mathbb{Z} .

The Legendre symbol is multiplicative, $(ab/q) = (a/q)(b/q)$ for $a, b \in \mathbb{F}_q$ and trivial for squares $(a^2/q) = 1 - \delta_{a,0}$, where the delta is the characteristic function on \mathbb{F}_q . This leads to $(F^2)_q = q^N - [F]_q$ for any polynomial F in N variables. If $N \geq 1$ we get

$$[F]_q \equiv -(F^2)_q \pmod{q}.$$

With the above equation we can translate point-counts to Legendre sums. Quadratic denominator reduction knows two cases,

$$\begin{aligned} ((A\alpha^2 + B\alpha + C)^2)_q &\equiv -(B^2 - 4AC)_q \pmod{q}, \\ ((D\alpha^2 + E\alpha + F)(H\alpha + J)^2)_q &\equiv -(DJ^2 - EHJ + FH^2)_q \pmod{q} \end{aligned}$$

if the total degree of the polynomials on the left hand sides does not exceed twice the number of their variables. The proof of these identities is in [27, Section 7]. It uses a Chevalley–Warning–Ax theorem for double covers of affine space which is proved by F. Knop in [27, Appendix]. As examples, $(W\alpha + X)^3(Y\alpha + Z)$ reduces by the second case to zero whereas $(U\alpha^2 + V\alpha + W)(X\alpha^2 + Y\alpha + Z)$ or $(U\alpha + V)(W\alpha + X)(Y\alpha + Z)$ do not reduce in general.

Note that in the case that both quadratic reductions are applicable one is back to the case of standard denominator reduction. Then both reductions lead to the same result.

We define quadratic n -invariants ${}^n\Psi_G^2$ in analogy to Definition 2.6.

Definition 2.9 (quadratic denominator reduction [27, Definition 34]). Given a connected graph G with at least three edges and a sequence of edges $1, 2, \dots, |E(G)|$ we define

$${}^3\Psi_G^2(1, 2, 3) = ({}^3\Psi_G(1, 2, 3))^2. \quad (2.7)$$

Suppose ${}^n\Psi_G^2$ for $n \geq 3$ is of the form

$${}^n\Psi_G^2(1, \dots, n) = (A\alpha_{n+1}^2 + B\alpha_{n+1} + C)^2 \quad (2.8)$$

then we define

$${}^{n+1}\Psi_G^2(1, \dots, n+1) = B^2 - 4AC.$$

Suppose ${}^n\Psi_G^2$ is of the form

$${}^n\Psi_G^2(1, \dots, n) = (D\alpha_{n+1}^2 + E\alpha_{n+1} + F)(H\alpha_{n+1} + J)^2 \quad (2.9)$$

then we define

$${}^{n+1}\Psi_G^2(1, \dots, n+1) = DJ^2 - EHJ + FH^2.$$

Otherwise quadratic denominator reduction terminates at step n . If it exists we call ${}^n\Psi_G^2$ a quadratic n -invariant of G . If ${}^n\Psi_G^2 = 0$ for some sequence of edges and some n then we say that G has weight drop.

Note that quadratic n -invariants have no sign ambiguity. The connection to the c_2 invariant is similar to the standard case, with a restriction to $q = 2$ or odd prime powers q .

Theorem 2.10 ([27, Theorem 36 and Remark 37]). *Let q be an odd prime power and G be a connected graph with at least three edges and $h_1(G) \leq |E(G)|/2$ independent cycles. Then*

$$c_2^{(q)}(G) \equiv (-1)^{n-1} ({}^n\Psi_G^2)_q \pmod{q}$$

whenever ${}^n\Psi_G^2$ exists. If ${}^n\Psi_G^2 \equiv 0 \pmod{2}$ then $c_2^{(2)}(G) \equiv 0 \pmod{2}$.

If the n -invariant ${}^n\Psi_G$ exists, we get ${}^n\Psi_G^2 = [{}^n\Psi_G]^2$, generalizing (2.7). In many cases quadratic denominator reduction goes significantly beyond standard denominator reduction.

2.6 Scaling

Even when quadratic denominator reduction stops it is often possible to simplify further by scaling some variables (see Section 3.4). In the context of Legendre sums this technique is based on the elimination of square factors from the Legendre symbol. For classical point-counts scaling was already used in [25] and later in [9], with some observations on some combinatorial conditions which allow it in [29]. In this article we have a case where the result after scaling can be further reduced by additional steps of quadratic denominator reduction. This makes the scaling technique particularly powerful.

2.7 ϕ^4 theory

The most interesting graphs for us are the primitive 4-point ϕ^4 graphs. Rephrased in a purely graph theoretic language, this means we are most interested in graphs which can be obtained by taking a 4-regular graph and removing one vertex. The 4-regular graph needs to be internally 6-edge connected, that is the only 4-edge cuts of the 4-regular graph are those which separate

one vertex from the rest of the graph. For graphs obtained from an internally 6-edge connected 4-regular graph in this way the Feynman period is convergent [24]. For such graphs we have (quadratic) denominator reduction, Theorems 2.7 and 2.10 as well as the Chevalley–Warning theorem as extra tool (see [30, Lemma 2.6] for an exposition).

Furthermore, the Feynman period of a graph obtained by removing a vertex of an internally 6-edge connected 4-regular graph does not depend on the choice of vertex removed. This is the *completion invariance* of the Feynman period. The analogous invariance for the c_2 invariant is conjectural [9], but an approach based on counting edge partitions has enabled a proof when $q = 2$ and the 4-regular graph has an odd number of vertices [31]. Upcoming work of one of us with Simone Hu will complete the $q = 2$ proof. In our hourglass chain graphs, we will be removing the most convenient vertex; if we assume the conjecture then this is equivalent to removing any other vertex.

With the completion conjecture we can also ignore 4-regular graphs with three vertex splits (*reducible* graphs in [24], see Figure 3): By deleting one of the three split vertices the decompleted graph inherits a two-vertex split which renders the c_2 trivial [11]. The Feynman period of a graph with a 3-vertex split factorizes [24]. Another reduction is obtained by ignoring graphs with double triangles (a pair of triangles with a common edge, see Figure 4). It was shown in [11] that double triangles can be reduced to single triangles (one of the common vertices becomes a crossing) without changing the c_2 invariant. With all reductions (internally 6-connected, 3-vertex connected, double-triangle-free) we are lead to considering prime ancestors [2, 24]. Note that for suitable kernels K the hourglass chains in this paper provide infinite families of prime ancestors.

2.8 Hourglasses

By an *hourglass* we mean two triangles sharing one common vertex. Taking two of the degree two vertices of an hourglass which are not in the same triangle and joining to two such vertices in another hourglass, we obtain a *bihourglass*, see Figure 6. Continuing by joining a third hourglass in the same way to the remaining degree 2 vertices of the second hourglass, and so on, we obtain longer *hourglass chains*, where the hourglass and bihourglass are the hourglass chains of length 1 and 2 respectively.

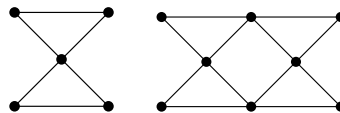


Figure 6. An hourglass and a bihourglass.

Hourglass chains of any length have four degree 2 vertices and all remaining vertices of degree 4. Take another fixed graph K which has four degree 2 vertices and all remaining vertices of degree 4. Additionally, fix a bipartition of the degree 2 vertices of K into two parts of size 2. We will call K (with this choice of bipartition) the *kernel*, see Figure 19. Let \mathcal{G}_K be the family of graphs obtained by taking an hourglass chain of any length, joining the two degree 2 vertices at one end of the chain to the two degree 2 vertices in one part of the bipartition given with K , and joining the 2 vertices at the other end of the chain to the two degree 2 vertices in the other part of the bipartition. See Figure 1 for an illustration. Note there are two ways to join on any hourglass chain compatible with the bipartition, differing by a half (Möbius) twist. In the end, this half twist will not affect the c_2 invariant, and so we include both in \mathcal{G}_K .

3 Hourglass reductions

The goal of this section is to prove Theorem 1.1. Note that after using the theorem one can continue to reduce any variable of the kernel K . If K is the kernel of a 4-regular hourglass chain then K' has four vertices of degree three at the ends of the extra edges 1 and 2. The topology of degree three vertices simplifies the structure of related Dodgsons [4, 9], [27, Lemma 19]. Particularly simple is the case of edge 2 whose variable is absent in (1.2) (note that the choice of labels 1 and 2 is arbitrary). Experiments suggest that it might always be possible to reduce both edges $\neq 2$ of any vertex adjacent to edge 2.

Question 3.1. Let v be a degree three vertex attached to edge 2 in K' . Let α_3 and α_4 be the edges of v which are in K . Is it always possible to quadratically denominator reduce the right hand side of (1.2) with respect to α_3 and α_4 ? If yes, what expression does one get after the quadratic reduction of α_3 and α_4 ?

In general it is quite helpful for analyzing larger kernels to have closed expressions for the c_2 with as many reductions as possible.

Figure 7 gives an overview of how the reductions will proceed. Following the specified order in an explicit example using a computer for the reductions can also be a helpful way to follow through the general argument.

3.1 Initial reductions

The first step of the proof is to begin a conventional denominator reduction on $L - v$, see Section 2.4.

Consider the two hourglasses around v and label the edges as in Figure 8. Then beginning our denominator reduction at the 4-invariant with (see (2.6))

$$\Psi_{L-v}^{14,23} \Psi_{L-v}^{13,24},$$

the reductions of 5 and 6 are forced to avoid contracting the triangles 145 and 236 in the first factor, and then the reductions of 7 and 8 are forced to avoid disconnecting the degree three vertices 457 and 268 in the first factor, see (2.2). This yields

$$\Psi_{L-v,78}^{1456,2356} \Psi_{L-v,56}^{1378,2478}.$$

At this point it is more convenient to consider the situation in terms of spanning forest polynomials via Proposition 2.4 and Lemma 2.5,

$$\Psi_{L-v,78}^{1456,2356} \Psi_{L-v,56}^{1378,2478} = \pm \left(\Phi_{L'}^{\{u_1, u_4\}, \{u_2, u_3\}} - \Phi_{L'}^{\{u_1, u_3\}, \{u_2, u_4\}} \right) \Phi_{L'}^{\{u_2\}, \{u_3\}},$$

where L' is L without v , the edges 1 through 8, and without the three vertices isolated by those removals. Label the triangle of L' containing u_1 as in Figure 9. Reduce edge 9 by the general deletion and contraction reduction formula (2.1). Notice that in both terms where 9 was deleted, 10 is an isthmus and u_1 , the vertex at the isolated end of 10, is not in a part by itself. Thus 10 cannot be cut in these terms, forcing 10 to be contracted in the other factors. This gives

$$\pm \left(\Phi_{L'-u_1}^{\{u_5, u_4\}, \{u_2, u_3\}} - \Phi_{L'-u_1}^{\{u_5, u_3\}, \{u_2, u_4\}} - \Phi_{L'-u_1}^{\{u_6, u_4\}, \{u_2, u_3\}} + \Phi_{L'-u_1}^{\{u_6, u_3\}, \{u_2, u_4\}} \right) \Phi_{L'-u_1}^{\{u_2\}, \{u_3\}},$$

where the vertices are as labelled in Figure 9.

Consider which trees the vertices u_6 and u_7 can belong to in the first factor of the previous expression. This factor is illustrated in Figure 10. Both trees of the forest appear in the portion of the graph including vertices u_2, u_5, u_6 and u_7 and so must exit from this portion to the rest

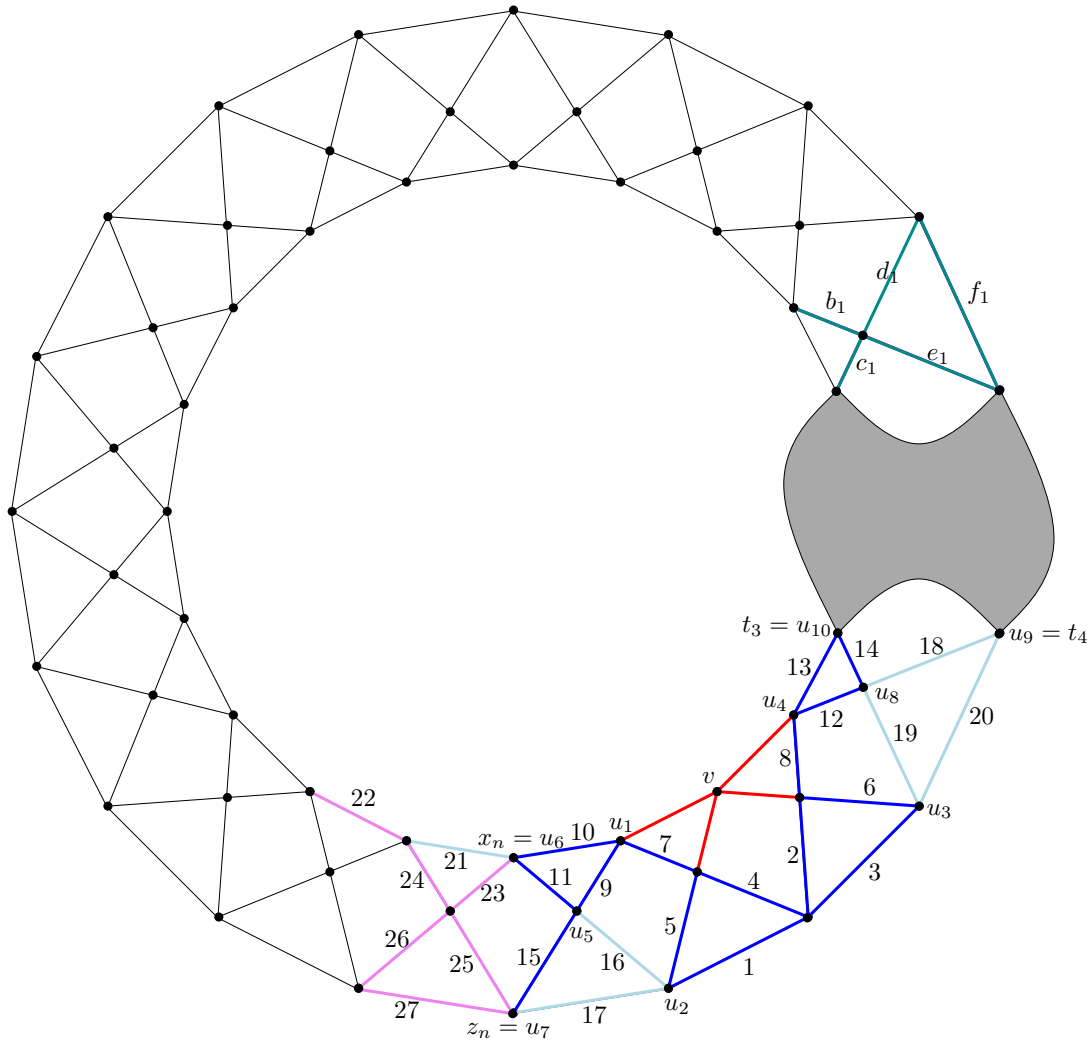


Figure 7. The order in which the edges should be reduced. First decomplate at v , thus removing the red edges. Then reduce the dark blue edges with conventional denominator reduction. Next reduce the light blue edges with quadratic denominator reduction. Continue with quadratic denominator reduction to reduce the six pink edges. The same form of denominator reappears, and so inductively we can reduce the analogous six edges in each subsequent hourglass until we reach the last hourglass. Finally reduce the remains of the last hourglass, the five dark cyan edges, according to Section 3.4.

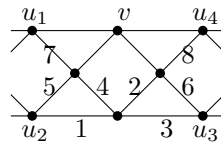


Figure 8. Edge labelling around v .

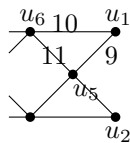


Figure 9. Edge labelling for the triangle containing u_1 .

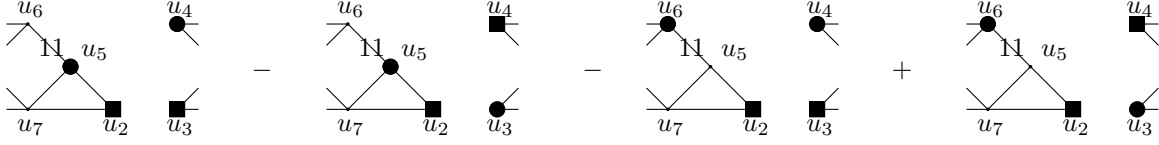


Figure 10. The first factor in the expression after reducing edge 10.

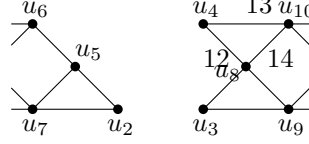


Figure 11. The vertex labellings for the hourglass containing u_3 and u_4 .

of the graph via the only possible vertices, u_6 and u_7 . Since there are two trees and two vertices, one must use u_6 and the other must use u_7 . In view of the shape of the graph, the only way this can happen is that the tree corresponding to the square vertices exits via u_7 and the tree corresponding to the circle vertices exits via u_6 . Then the only thing undetermined in how the trees go through the illustrated part of the graph, is the tree to which u_5 belongs in the third and fourth terms. Summing over both possibilities we see that one of the possibilities cancels with the first two terms, and so what remains of the entire expression is

$$\pm \left(\Phi_{L'-u_1}^{\{u_2, u_5, u_7, u_3\}\{u_6, u_4\}} - \Phi_{L'-u_1}^{\{u_2, u_5, u_7, u_4\}\{u_6, u_3\}} \right) \Phi_{L'-u_1}^{\{u_2\}, \{u_3\}}.$$

In the first factor, edge 11 must always be deleted since its two ends are in different parts and so we obtain

$$\pm \left(\Phi_{(L'-u_1)\setminus 11}^{\{u_2, u_5, u_7, u_3\}\{u_6, u_4\}} - \Phi_{(L'-u_1)\setminus 11}^{\{u_2, u_5, u_7, u_4\}\{u_6, u_3\}} \right) \Phi_{(L'-u_1)/11}^{\{u_2\}, \{u_3\}}.$$

The only way that the hourglass with u_3 and u_4 affected this computation was to guarantee that both trees had to leave the part of the graph illustrated on the left. In other words, all we needed to know is that both trees appeared in the part of the graph illustrated on the right. Swapping left and right this remains true and so we can use the same argument as above on the triangle involving u_4 . Labelling the vertices of the hourglass including u_3 and u_4 as in Figure 11, this calculation gives

$$\pm \left(\Phi_{L_1}^{\{u_2, u_5, u_7, u_3, u_8, u_9\}, \{u_6, u_{10}\}} - \Phi_{L_1}^{\{u_2, u_5, u_7, u_{10}\}, \{u_3, u_8, u_9, u_6\}} \right) \Phi_{L_2}^{\{u_2\}, \{u_3\}},$$

where $L_1 = (L' - \{u_1, u_4\}) \setminus \{11, 14\}$ and $L_2 = (L' - \{u_1, u_4\}) / \{11, 14\}$. Now comes the key observation that the first factor can be factored. First, the triangles u_2, u_5, u_7 and u_3, u_8, u_9 factor off since they are only joined at a vertex and the tree to which that vertex belongs is known. Additionally, similarly to the observations used above, both trees need to propagate through each hourglass remaining in the chain since both trees appear on both sides. However, the trees cannot cross within an hourglass, so one tree must run down one side of the hourglasses and the other tree down the other side; only the middle vertex could be in either tree.

To write this down nicely we need some more systematic notation. The edges and vertices of each hourglass will be labelled as in Figure 12, where the hourglasses X_i are indexed by i . When it is useful to talk about an hourglass generically we will leave out the subscripts. Additionally, let t_1, t_2, t_3 and t_4 be the degree 2 vertices of K with the bipartition being $\{t_1, t_2\}, \{t_3, t_4\}$. With

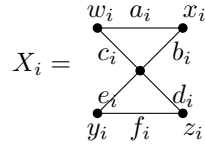


Figure 12. Labels for hourglasses.

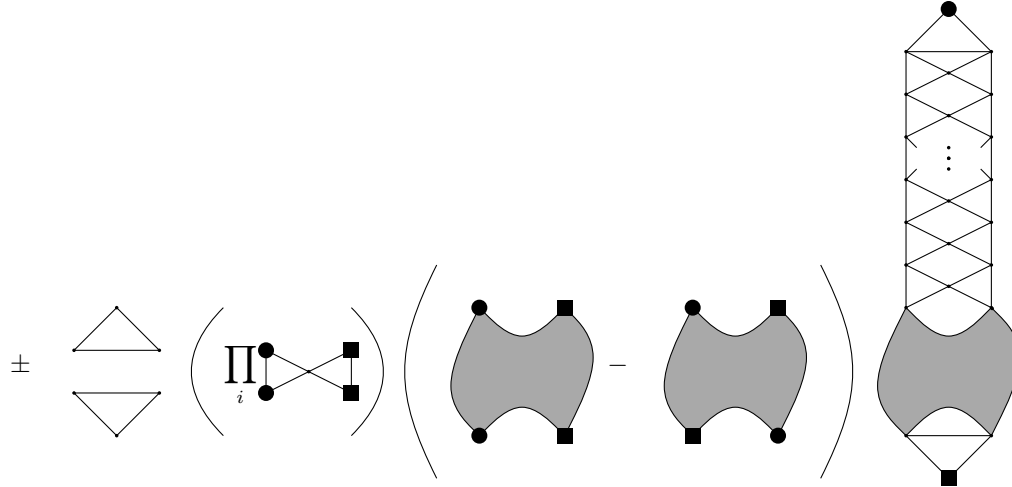


Figure 13. Diagrammatic representation of (3.1).

this notation, the factorization observation allows us to rewrite the denominator expression so far as

$$\begin{aligned} & \pm \Psi_{L[u_2, u_5, u_7]} \Psi_{L[u_3, u_8, u_9]} \left(\prod_i \Phi_{X_i}^{\{w_i, x_i\}, \{y_i, z_i\}} \right) \left(\Phi_K^{\{t_1, t_3\}, \{t_2, t_4\}} - \Phi_K^{\{t_1, t_4\}, \{t_2, t_3\}} \right) \\ & \times \Phi_{L_2}^{\{u_2\}, \{u_3\}}, \end{aligned} \tag{3.1}$$

where $L[S]$ for a set of vertices S indicates the induced subgraph of L given by the vertices of S , that is the subgraph with the vertices of S and all edges in L that have both ends in S . In this case the two induced subgraphs are both triangles. Diagrammatically, this expression can be represented as in Figure 13. The reader is encouraged to draw all the steps diagrammatically, as this gives the most insight into the calculation.

Note that due to the choice of v , one of the triangles u_2, u_5, u_7 and u_3, u_8, u_9 shares two vertices with K , while the other does not. Without loss of generality say that u_2, u_5, u_7 is the one that is adjacent to other hourglasses, if there are any. This is the one drawn pointing upwards in Figure 13, and u_2 is the top vertex.

3.2 The two triangles

The plan of attack now is to reduce the edges in the two triangles u_2, u_5, u_7 and u_3, u_8, u_9 . After the first such edge, we will need to pass to quadratic denominator reduction. Note that the only factors in (3.1) containing edge variables from the triangle u_2, u_5, u_7 are $\Psi_{L[u_2, u_5, u_7]}$ and $\Phi_{L_2}^{\{u_2\}, \{u_3\}}$. Let the edge between u_5 and u_7 be 15 and let 16 and 17 be the other two edges of the triangle u_2, u_5, u_7 . Reduce 15 in the usual way to obtain

$$(\alpha_{16} + \alpha_{17}) \Phi_{L_2 \setminus 15}^{\{u_2\}, \{u_3\}} - \Phi_{L_2/15}^{\{u_2\}, \{u_3\}}$$

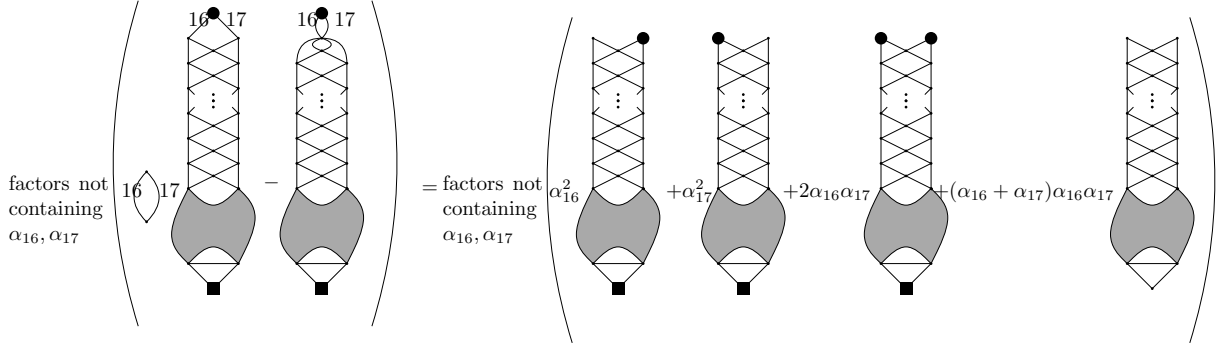


Figure 14. Diagrammatic rendition of the result of reducing edge 15.

times the factors not involving the triangle u_2, u_5, u_7 . Expanding out α_{16} and α_{17} we get

$$\begin{aligned}
 & (\alpha_{16} + \alpha_{17})\Phi_{L_2 \setminus 15}^{\{u_2\}, \{u_3\}} - \Phi_{L_2/15}^{\{u_2\}, \{u_3\}} \\
 &= (\alpha_{16} + \alpha_{17}) \left(\alpha_{16} \Phi_{L_3}^{\{z_n\}, \{u_3\}} + \alpha_{17} \Phi_{L_3}^{\{x_n\}, \{u_3\}} + \alpha_{16} \alpha_{17} \Psi_{L_3} \right) - \alpha_{16} \alpha_{17} \Psi_{L_4} \\
 &= \alpha_{16}^2 \Phi_{L_3}^{\{z_n\}, \{u_3\}} + \alpha_{17}^2 \Phi_{L_3}^{\{x_n\}, \{u_3\}} + 2\alpha_{16} \alpha_{17} \Phi_{L_3}^{\{x_n, z_n\}, \{u_3\}} + (\alpha_{16} + \alpha_{17}) \alpha_{16} \alpha_{17} \Psi_{L_3}
 \end{aligned}$$

times the factors not involving the triangle u_3, u_5, u_7 , where

- n is the index of the hourglass adjacent to the triangle u_2, u_5, u_7 with the vertices x_n and z_n being the same vertices as u_5 (which has also been contracted with the original u_6) and u_7 respectively,
- 16 the edge from $x_n = u_5$ to u_2 and 17 the edge from $z_n = u_7$ to u_2 ,
- $L_3 = (L_2 - u_2) \setminus 15$ and $L_4 = (L_2 - u_2) / 15$.

This is illustrated diagrammatically in Figure 14. Note that in these calculations we used that merging two vertices is equivalent to not having them merged but having them be in different parts of the vertex partition defining the spanning forest polynomial, and when there was originally only a single tree, there is no need to consider how the parts interact with any existing parts. We also used that if a vertex is not in a part of a partition for a spanning forest polynomial then we can sum over all possibilities for putting that vertex into a part.

Now we need to move to quadratic denominator reduction, see Section 2.5. Reducing α_{16} according to quadratic denominator reduction (2.8) we obtain

$$\left(2\alpha_{17} \Phi_{L_3}^{\{x_n, z_n\}, \{u_3\}} + \alpha_{17}^2 \Psi_{L_3} \right)^2 - 4 \left(\Phi_{L_3}^{\{z_n\}, \{u_3\}} + \alpha_{17} \Psi_{L_3} \right) \alpha_{17}^2 \Phi_{L_3}^{\{x_n\}, \{u_3\}}$$

times the square of the factors not involving the triangle u_3, u_5, u_7 . Note that α_{17}^2 factors out of this expression and so following (2.9) quadratic denominator reduction of α_{17} gives

$$4 \left(\left(\Phi_{L_3}^{\{x_n, z_n\}, \{u_3\}} \right)^2 - \Phi_{L_3}^{\{z_n\}, \{u_3\}} \Phi_{L_3}^{\{x_n\}, \{u_3\}} \right)$$

times the square of the factors not involving the triangle u_3, u_5, u_7 .

Next consider the u_3, u_8, u_9 triangle. Unfortunately we cannot simply use the same argument as above since we are not starting in conventional denominator reduction this time. This part of the argument is rather gruesome, but fortunately, it is the last bit of messy work before we get to the systematic part.

Let 18 be the edge between u_8 and u_9 , and let 19 and 20 be the edges from u_3 to t_3 and t_4 respectively. For the purposes of these edges, the factors we need to consider are the one given explicitly above and the factor for the u_3, t_3, t_4 triangle itself. Namely we have

$$(\alpha_{18} + \alpha_{19} + \alpha_{20})^2 \left((\Phi_{L_3}^{\{x_n, z_n\}, \{u_3\}})^2 - \Phi_{L_3}^{\{z_n\}, \{u_3\}} \Phi_{L_3}^{\{x_n\}, \{u_3\}} \right)$$

times the square of the factors not involving the triangles u_3, u_5, u_7 and u_3, t_3, t_4 . We have included the constant 4 among the suppressed factors as it also is simply carried along through the calculations that follow.

Reducing edge 18 we obtain

$$\begin{aligned} & (\alpha_{19} + \alpha_{20})^2 \left((\Phi_{L_3 \setminus 18}^{\{x_n, z_n\}, \{u_3\}})^2 - \Phi_{L_3 \setminus 18}^{\{z_n\}, \{u_3\}} \Phi_{L_3 \setminus 18}^{\{x_n\}, \{u_3\}} \right) \\ & - (\alpha_{19} + \alpha_{20}) \left(2\Phi_{L_3 \setminus 18}^{\{x_n, z_n\}, \{u_3\}} \Phi_{L_3/18}^{\{x_n, z_n\}, \{u_3\}} - \Phi_{L_3 \setminus 18}^{\{z_n\}, \{u_3\}} \Phi_{L_3/18}^{\{x_n\}, \{u_3\}} - \Phi_{L_3 \setminus 18}^{\{x_n\}, \{u_3\}} \Phi_{L_3/18}^{\{z_n\}, \{u_3\}} \right) \\ & + \left((\Phi_{L_3/18}^{\{x_n, z_n\}, \{u_3\}})^2 - \Phi_{L_3/18}^{\{z_n\}, \{u_3\}} \Phi_{L_3/18}^{\{x_n\}, \{u_3\}} \right) \end{aligned}$$

times the square of the factors not involving either triangle. Expanding out all the α_{19} and α_{20} explicitly and freely using the observations on spanning forest polynomials used before, there are many nice cancellations and we obtain

$$\begin{aligned} & (\alpha_{19} + \alpha_{20})^2 \left(\alpha_{19}^2 \left((\Phi_{M_n}^{\{x_n, z_n\}, \{t_4\}})^2 - \Phi_{M_n}^{\{z_n\}, \{t_4\}} \Phi_{M_n}^{\{x_n\}, \{t_4\}} \right) \right. \\ & + \alpha_{20}^2 \left((\Phi_{M_n}^{\{x_n, z_n\}, \{t_3\}})^2 - \Phi_{M_n}^{\{z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n\}, \{t_3\}} \right) \\ & + \alpha_{19}\alpha_{20} \left(2\Phi_{M_n}^{\{x_n, z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n, z_n\}, \{t_4\}} - \Phi_{M_n}^{\{z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n\}, \{t_4\}} - \Phi_{M_n}^{\{z_n\}, \{t_4\}} \Phi_{M_n}^{\{x_n\}, \{t_3\}} \right. \\ & \left. + \Phi_{M_n}^{\{x_n\}, \{z_n\}} \Phi_{M_n}^{\{t_3\}, \{t_4\}} \right) - \alpha_{19}\alpha_{20}(\alpha_{19} + \alpha_{20})\Psi_{M_n} \Phi_{M_n}^{\{x_n\}, \{z_n\}} \left. \right) \end{aligned}$$

times the square of the factors not involving either triangle and where $M_n = (L_3 - u_3) \setminus 18$ (the n refers to the index of the last remaining hourglass). This calculation uses the fact that $\Phi_{M_n}^{\{z_n\}, \{t_4\}} + \Phi_{M_n}^{\{x_n\}, \{t_4\}} - 2\Phi_{M_n}^{\{x_n, z_n\}, \{t_4\}} = \Phi_{M_n}^{\{z_n\}, \{t_3\}} + \Phi_{M_n}^{\{x_n\}, \{t_3\}} - 2\Phi_{M_n}^{\{x_n, z_n\}, \{t_3\}} = \Phi_{M_n}^{\{x_n\}, \{z_n\}}$ which can be seen to be true by expanding all the spanning forest polynomials so that each of x_n, z_n, t_3, t_4 is in each partition.

Because of the factor of $(\alpha_{19} + \alpha_{20})^2$ we can proceed to reduce edge 19 to obtain

$$\begin{aligned} & \alpha_{20}^2 \left((\Phi_{M_n}^{\{x_n, z_n\}, \{t_4\}})^2 - \Phi_{M_n}^{\{z_n\}, \{t_4\}} \Phi_{M_n}^{\{x_n\}, \{t_4\}} - \alpha_{20} \Psi_{M_n} \Phi_{M_n}^{\{x_n\}, \{z_n\}} \right) \\ & - \alpha_{20} \left(\alpha_{20} \left(2\Phi_{M_n}^{\{x_n, z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n, z_n\}, \{t_4\}} - \Phi_{M_n}^{\{z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n\}, \{t_4\}} - \Phi_{M_n}^{\{z_n\}, \{t_4\}} \Phi_{M_n}^{\{x_n\}, \{t_3\}} \right) \right. \\ & \left. + \Phi_{M_n}^{\{x_n\}, \{z_n\}} \Phi_{M_n}^{\{t_3\}, \{t_4\}} \right) - \alpha_{20}^2 \Psi_{M_n} \Phi_{M_n}^{\{x_n\}, \{z_n\}} \left. \right) \\ & + \alpha_{20}^2 \left((\Phi_{M_n}^{\{x_n, z_n\}, \{t_3\}})^2 - \Phi_{M_n}^{\{z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n\}, \{t_3\}} \right) \end{aligned}$$

times the square of the factors not involving either triangle. There is a factor of α_{20}^2 in this expression, so we can reduce α_{20} to obtain

$$\begin{aligned} & (\Phi_{M_n}^{\{x_n, z_n\}, \{t_4\}})^2 - \Phi_{M_n}^{\{z_n\}, \{t_4\}} \Phi_{M_n}^{\{x_n\}, \{t_4\}} - 2\Phi_{M_n}^{\{x_n, z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n, z_n\}, \{t_4\}} + \Phi_{M_n}^{\{z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n\}, \{t_4\}} \\ & + \Phi_{M_n}^{\{z_n\}, \{t_4\}} \Phi_{M_n}^{\{x_n\}, \{t_3\}} - \Phi_{M_n}^{\{x_n\}, \{z_n\}} \Phi_{M_n}^{\{t_3\}, \{t_4\}} + (\Phi_{M_n}^{\{x_n, z_n\}, \{t_3\}})^2 - \Phi_{M_n}^{\{z_n\}, \{t_3\}} \Phi_{M_n}^{\{x_n\}, \{t_3\}} \end{aligned}$$

times the square of the factors not involving either triangle. Expanding over all possibilities for assigning whichever of x_n, z_n, t_3, t_4 are not in the partition in each term, cancelling and then recollecting terms we can simplify the expression above to

$$\left(\Phi_{M_n}^{\{x_n, t_4\}, \{z_n, t_3\}} - \Phi_{M_n}^{\{x_n, t_3\}, \{z_n, t_4\}} \right)^2 - \Phi_{M_n}^{\{x_n\}, \{z_n\}} \Phi_{M_n}^{\{t_3\}, \{t_4\}} \quad (3.2)$$

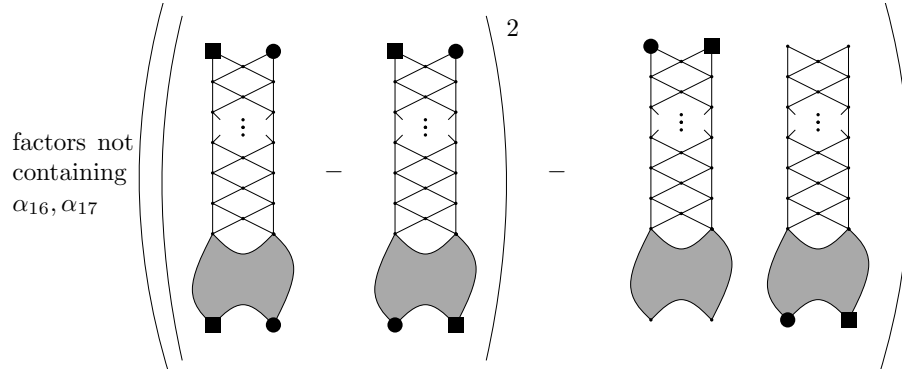


Figure 15. Diagrammatic representation of (3.2).

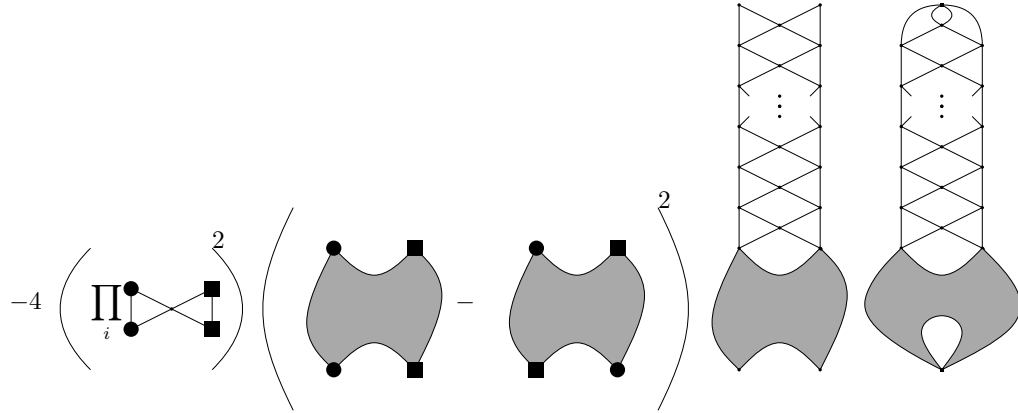


Figure 16. Diagrammatic representation of (3.3).

times the square of the factors not involving either triangle. This expression is more symmetric than the notation makes it seem. Figure 15 shows the symmetry better. Note that other than to choose the labelling of the vertices, we have not used the fact that all the remaining hourglasses are on one side of K . If we followed the same calculation beginning with v in the middle of the hourglasses, then at this point we would have hourglasses on each side of K making the expression nicely symmetric.

To proceed, we can apply a Dodgson identity to the last factor. Let M'_n be the graph obtained by adding an edge labelled 1 joining z_n and x_n and an edge labelled 2 joining t_3 and t_4 and let M''_n . Then (3.2) can be written as $(\Psi_{M'_n}^{1,2})^2 - \Psi_{2,M'_n}^{1,1} \Psi_{1,M'_n}^{2,2}$, so by the Dodgson identity (2.3) we find that (3.2) is also $-\Psi_{M'_n,12} \Psi_{M''_n}^{12,12}$.

Putting back in the factors we have been ignoring we get

$$-4 \left(\prod_i \Phi_{X_i}^{\{w,x\},\{y,z\}} \right)^2 \left(\Phi_K^{\{t_1,t_3\},\{t_2,t_4\}} - \Phi_K^{\{t_1,t_4\},\{t_2,t_3\}} \right)^2 \Psi_{M'_n,12} \Psi_{M''_n}^{12,12}. \quad (3.3)$$

For an illustration of this equation see Figure 16.

3.3 Systematic hourglass reduction

Notice that the two right hand factors of (3.3) are the same except that x_n and z_n , the top two vertices as illustrated, are identified or not and likewise for t_3 and t_4 , the bottom two vertices.

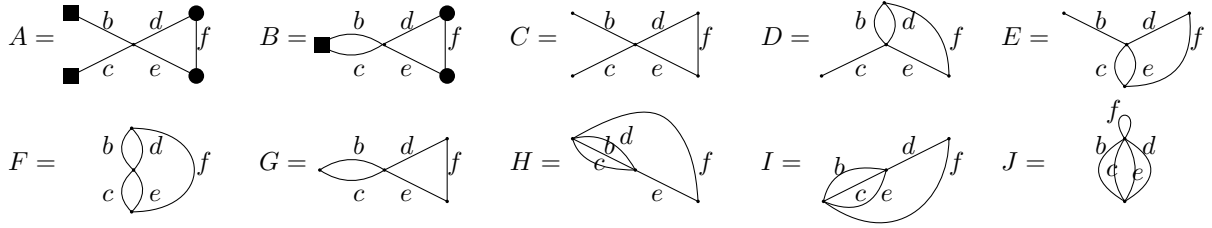


Figure 17. The hourglass pieces that occur.

This will be important, so it will be useful to have a compact notation for this; write

$$\begin{matrix} T \\ \\ \\ A \end{matrix} \quad \text{for} \quad \Psi_{M'_n, 2}^{1,1},$$

that is the top vertices together (T) and the bottom two vertices apart (A), and similarly for

$$\begin{matrix} T \\ \\ \\ A \end{matrix} = \Psi_{M'_n, 12}, \quad \begin{matrix} A \\ \\ \\ T \end{matrix} = \Psi_{M'_n}^{12,12}, \quad \begin{matrix} A \\ \\ \\ T \end{matrix} = \Psi_{M'_n, 2}^{1,1}.$$

The next order of business is to start reducing the top hourglass X_n . Recall the hourglass notation as in Figure 12.

First we reduce a_n . We are going to need to keep track to all the polynomials that come about from the remains of X_n as they appear in each term after reducing a_n . Writing generically for every hourglass for the moment, define

$$\begin{aligned} A &= de, & B &= de(b+c) + bc(d+e+f), \\ C &= d+e+f, & D &= bd + (b+d)(e+f), \\ E &= ce + (c+e)(d+f), & F &= bd(c+e) + ce(b+d) + f(c+e)(b+d), \\ G &= (b+c)(d+e+f), & H &= bcd + (bc+bd+cd)(e+f), \\ I &= bce + (bc+be+ce)(d+f), & J &= f(bcd + bce + bde + cde). \end{aligned}$$

These are the Kirchhoff polynomials and spanning forest polynomials corresponding to the graphs shown in Figure 17.

Using (2.4) along with the notation above, reducing a_n gives

$$\begin{aligned} & -4 \left(\prod_{i < n} \Phi_{X_i}^{\{w,x\}, \{y,z\}} \right)^2 \left(\Phi_K^{\{t_1, t_3\}, \{t_2, t_4\}} - \Phi_K^{\{t_1, t_4\}, \{t_2, t_3\}} \right)^2 \\ & \times \left(B_n^2 \left(\begin{matrix} T & T \\ C_n D_n & E_n D_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} T & T \\ C_n F_n & E_n F_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} A & T \\ I_n D_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} T & A \\ G_n D_n & G_n F_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} A & A \\ I_n D_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} \right) \right. \\ & - A_n B_n \left(\begin{matrix} T & T \\ G_n D_n & I_n D_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} A & T \\ I_n D_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} T & A \\ G_n F_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} A & A \\ I_n D_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} \right) \\ & - A_n B_n \left(\begin{matrix} T & T \\ C_n H_n & E_n H_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} A & T \\ I_n H_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} T & A \\ C_n J_n & E_n J_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} A & A \\ I_n H_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} \right) \\ & \left. + A_n^2 \leq \left(\begin{matrix} T & T \\ G_n H_n & I_n H_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} A & T \\ I_n H_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} T & A \\ G_n J_n & E_n J_n \\ A & T \end{matrix}^{n-1n-1} + \begin{matrix} A & A \\ I_n H_n & I_n F_n \\ A & T \end{matrix}^{n-1n-1} \right) \right). \end{aligned}$$

This expression factors giving

$$\begin{aligned}
& -4 \left(\prod_{i < n} \Phi_{X_i}^{\{w,x\},\{y,z\}} \right)^2 \left(\Phi_K^{\{t_1,t_3\},\{t_2,t_4\}} - \Phi_K^{\{t_1,t_4\},\{t_2,t_3\}} \right)^2 \\
& \quad \times \left(\frac{T}{A} (B_n C_n - A_n G_n) + \frac{A}{A} (B_n E_n - A_n I_n) \right) \\
& \quad \times \left(\frac{T}{T} (B_n D_n - A_n H_n) + \frac{A}{T} (B_n F_n - A_n J_n) \right).
\end{aligned}$$

Plugging in the expressions for the polynomials this simplifies further to

$$\begin{aligned}
& -4 \left(\prod_{i < n} \Phi_{X_i}^{\{w,x\},\{y,z\}} \right)^2 \left(\Phi_K^{\{t_1,t_3\},\{t_2,t_4\}} - \Phi_K^{\{t_1,t_4\},\{t_2,t_3\}} \right)^2 (d_n + e_n + f_n) c_n Z_n \\
& \quad \times \left(\frac{T}{A} b_n (d_n + e_n + f_n) + \frac{A}{A} Y_n \right) \left(\frac{T}{T} b_n (d_n + e_n + f_n) + \frac{A}{T} Y_n \right), \tag{3.4}
\end{aligned}$$

where

$$\begin{aligned}
Y &= bcd + bce + bde + cde + bcf + bef, \\
Z &= bcd + bce + bde + cde + bcf + cdf.
\end{aligned}$$

Next we reduce a_{n-1} , yielding

$$\begin{aligned}
& -4 \left(\prod_{i < n-1} \Phi_{X_i}^{\{w,x\},\{y,z\}} \right)^2 \left(\Phi_K^{\{t_1,t_3\},\{t_2,t_4\}} - \Phi_K^{\{t_1,t_4\},\{t_2,t_3\}} \right)^2 (d_n + e_n + f_n) c_n Z_n \\
& \quad \times \left(\frac{T}{A} ((d_n + e_n + f_n) b_n (B_{n-1} D_{n-1} - A_{n-1} H_{n-1}) + Y_n (B_{n-1} C_{n-1} - A_{n-1} G_{n-1})) \right. \\
& \quad \quad \left. + \frac{A}{A} ((d_n + e_n + f_n) b_n (B_{n-1} F_{n-1} - A_{n-1} J_{n-1}) + Y_n (B_{n-1} E_{n-1} - A_{n-1} I_{n-1})) \right) \\
& \quad \times \left(\frac{T}{T} ((d_n + e_n + f_n) b_n (B_{n-1} D_{n-1} - A_{n-1} H_{n-1}) + Y_n (B_{n-1} C_{n-1} - A_{n-1} G_{n-1})) \right. \\
& \quad \quad \left. + \frac{A}{T} ((d_n + e_n + f_n) b_n (B_{n-1} F_{n-1} - A_{n-1} J_{n-1}) + Y_n (B_{n-1} E_{n-1} - A_{n-1} I_{n-1})) \right).
\end{aligned}$$

Substituting in the expressions for the polynomials something special happens; a square factor appears:

$$\begin{aligned}
& -4 \left(\prod_{i < n-1} \Phi_{X_i}^{\{w,x\},\{y,z\}} \right)^2 \left(\Phi_K^{\{t_1,t_3\},\{t_2,t_4\}} - \Phi_K^{\{t_1,t_4\},\{t_2,t_3\}} \right)^2 (d_n + e_n + f_n) c_n Z_n \\
& \quad \times ((d_n + e_n + f_n) b_n Z_{n-1} + Y_n c_{n-1} (d_{n-1} + e_{n-1} + f_{n-1}))^2 \\
& \quad \times \left(\frac{T}{A} (d_{n-1} + e_{n-1} + f_{n-1}) b_{n-1} + \frac{A}{A} Y_{n-1} \right) \left(\frac{T}{T} (d_{n-1} + e_{n-1} + f_{n-1}) b_{n-1} + \frac{A}{T} Y_{n-1} \right).
\end{aligned}$$

We have a square term $T_{n-1,n}^2$, where $T_{n-1,n} = (d_n + e_n + f_n) b_n Z_{n-1} + Y_n c_{n-1} (d_{n-1} + e_{n-1} + f_{n-1})$. The factor $T_{n-1,n}$ involves edge variables from both of the top two hourglasses. It is the

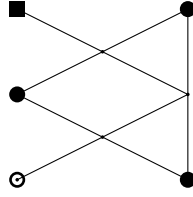


Figure 18. Spanning forest polynomial giving the factor $T_{n-1,n}$.

spanning forest polynomial illustrated in Figure 18. We use this factor to reduce the variables of the hourglass X_n (in any order). This part of the calculation is routine and can be done by hand or rigorously on a computer (reductions are, e.g., implemented in [28]), obtaining

$$(d_n + e_n + f_n)c_n Z_n T_{n-1,n}^2 \rightarrow c_{n-1}(d_{n-1} + e_{n-1} + f_{n-1})Z_{n-1}$$

which exactly leads to (3.4) with $n - 1$ in place of n .

Inductively, we can reduce until only pieces from one hourglass X_1 remain. At this step the quadratic denominator reduction will give

$$\begin{aligned} & -4 \left(\Phi_K^{\{t_1, t_3\}, \{t_2, t_4\}} - \Phi_K^{\{t_1, t_4\}, \{t_2, t_3\}} \right)^2 c_1 (d_1 + e_1 + f_1) Z_1 \\ & \times \begin{pmatrix} T \\ 0 \\ A \end{pmatrix} (d_1 + e_1 + f_1) b_1 + \begin{pmatrix} A \\ 0 \\ A \end{pmatrix} Y_1 \begin{pmatrix} T \\ 0 \\ T \end{pmatrix} (d_1 + e_1 + f_1) b_1 + \begin{pmatrix} A \\ 0 \\ T \end{pmatrix} Y_1. \end{aligned}$$

Next we want to rewrite the parts involving the kernel in terms of Dodgson polynomials rather than spanning forest polynomials or the apart-together notation (which no longer has any hourglasses in it). Using K' as in the statement of the theorem this gives (see Figure 2)

$$\begin{aligned} & -4c_1 (d_1 + e_1 + f_1) Z_1 (\Psi_{K'}^{1,2})^2 (\Psi_{K',1}^{2,2} (d_1 + e_1 + f_1) b_1 + \Psi_{K'}^{12,12} Y_1) \\ & \times (\Psi_{K',12} (d_1 + e_1 + f_1) b_1 + \Psi_{K',2}^{1,1} Y_1). \end{aligned} \tag{3.5}$$

3.4 The endgame

Because the expression (3.5) vanishes modulo 2 we get the result for $q = 2$ from the last statement in Theorem 2.10.

We now restrict ourselves to odd prime powers and show that the last hourglass can be eliminated. With (3.5) we achieved the following situation: we have two sets of variables, the five variables b_1, c_1, d_1, e_1, f_1 from the hourglass X_1 and some variables α_i in the Dodgson polynomials with the kernel K' . The expression (3.5) does not depend on the two variables α_1 and α_2 which are associated to the extra edges 1 and 2 in K' . Let d be the degree of $\Psi_{K'}^{12,12}$. Then $\Psi_{K',1}^{2,2}$, $\Psi_{K'}^{1,2}$, and $\Psi_{K',2}^{1,1}$ have degree $d + 1$, while $\Psi_{K',12}$ has degree $d + 2$ (see, e.g., [27]). The total degree of (3.5) is $4d + 14$ which equals twice the total number of its variables.

Quadratic denominator reduction stops at (3.5). To obtain further reductions we use a scaling technique which was first used in [25] and later adopted in [9] to exhibit a K3 structure in ϕ^4 theory at loop order eight. Considering (3.5) as a denominator of an integrand it is clear that the variables separate under a scaling transformation of all α_i by $S = Y_1/[b_1 c_1 (d_1 + e_1 + f_1)]$. Because S has total degree zero, homogeneity of (3.5) is preserved.

For Legendre sums over finite fields we need the following argument. If $S \in \mathbb{F}_q^\times$ the α_i -variables in the Legendre sum can be multiplied by S yielding

$$-b_1 (d_1 + e_1 + f_1)^2 Y_1 Z_1 (\Psi_{K'}^{1,2})^2 (\Psi_{K',1}^{2,2} + c_1 \Psi_{K'}^{12,12}) (\Psi_{K',12} + c_1 \Psi_{K',2}^{1,1}) (2S^{2d+2})^2.$$

The last factor is a non-zero square which can be dropped from the Legendre sum. This suggests that

$$\begin{aligned} & (\text{expression (3.5)})_q \\ &= \left(-b_1(d_1 + e_1 + f_1)^2 Y_1 Z_1 (\Psi_{K'}^{1,2})^2 (\Psi_{K',1}^{2,2} + c_1 \Psi_{K'}^{12,12}) (\Psi_{K',12} + c_1 \Psi_{K',2}^{1,1}) \right)_q. \end{aligned} \quad (3.6)$$

In \mathbb{F}_q , we cannot ignore the singular locus of the scaling transformation. We need the following lemma.

Lemma 3.2. *Let P be a homogeneous polynomial of odd degree and let q be an odd prime power. Then $(P)_q = 0$.*

Proof. Because q is odd there exists an $x \in \mathbb{F}_q^\times$ which is not a square (half the elements in \mathbb{F}_q^\times are non-squares). Scaling all variables by x gives

$$(P)_q = (Px^D)_q = (P)_q \left(\frac{x}{q} \right)^D = -(P)_q,$$

where we used that the degree D of P is odd. ■

We observe that in any situation where S is singular (i.e., some of the $Y_1, b_1, c_1, d_1 + e_1 + f_1$ are zero) both expressions – (3.5) and the polynomial on the right hand side of (3.6) – are either zero or the product of two factors in separate variables which are homogeneous of odd degree. The validity of (3.6) follows from the Lemma with an inclusion-exclusion argument.

The term on the right hand side of (3.6) is homogeneous of degree $4d + 14$ which equals twice the number of variables. We may use quadratic denominator reduction in the variables f_1, e_1, d_1, b_1 (in this sequence) yielding (use, e.g., [28])

$$(\text{expression (3.5)})_q \equiv \left(c_1 (\Psi_{K'}^{1,2})^2 (\Psi_{K',1}^{2,2} + c_1 \Psi_{K'}^{12,12}) (\Psi_{K',12} + c_1 \Psi_{K',2}^{1,1}) \right)_q \pmod{q}.$$

By contraction-deletion (2.1) the polynomial on the right hand side is $\alpha_1 (\Psi_{K'}^{1,2})^2 \Psi_{K'}^{2,2} \Psi_{K',2}$, where we renamed c_1 to α_1 .

Theorem 1.1 follows from Theorem 2.10 because the number of reduced variables is odd.

4 Kernels

In this section we study kernels which lead to 4-regular hourglass chains. This implies that the kernel K is internally 4-regular while every external vertex has two incident edges (see Figure 19).

4.1 Trivial kernels

Periods which admit a 3-vertex split are products (see Figure 3). Assuming the completion conjecture, their c_2 invariants vanish. We hence skip kernels with a 3-vertex split. If a kernel has a double triangle (see Figure 4), the c_2 invariant is equal to the c_2 of a smaller graph, where the double triangle is reduced [11, 24]. We also exclude these cases in K because their c_2 s are found in smaller kernels. Moreover, we ignore kernels which have an external hourglass in such a way that it adds to the chain (with the exception that K is an hourglass).

Assume a kernel K with at least two internal vertices has two or more edges between external vertices. Then, every hourglass chain $L \in \mathcal{G}_K$ splits if one cuts the four or less edges of K which have exactly one external vertex. The chain L has a subdivergence, its period diverges and the c_2 vanishes [11]. The same holds true if K has a non-trivial internal four edge cut.

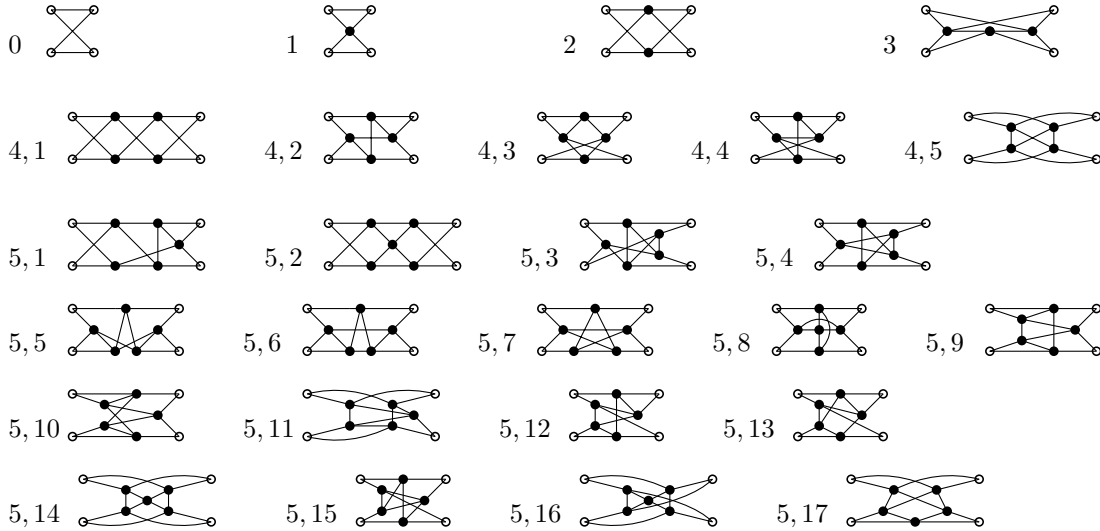


Figure 19. The kernels with up to five internal vertices. Here, edges 1 and 2 in K' would be vertical edges on either sides of the depicted kernels K .

Table 1. The number of effectively different kernels in ϕ^4 theory increases rapidly with the number of internal vertices.

# internal vertices	0	1	2	3	4	5	6	7	8	9	10
# different kernels	1	1	1	1	5	17	78	497	3882	33587	316860

So, for kernels K with ≥ 2 internal vertices we restrict ourselves to the case that K has at most one edge between external vertices. If such an edge e shares its vertices with edge 1 or 2 in K' , then any $L \in \mathcal{G}_K$ has a double-triangle. After two double-triangle reductions we are left with a graph which has a three vertex split and the c_2 (conjecturally) vanishes. We are effectively left with the case that e joins the vertices of the hourglasses from opposite ends of the chain. By Theorem 1.1 Möbius twists can be ignored as they lead to equal c_2 s. The case of one edge between external vertices of K thus reduces to a single setup (which becomes relevant for kernels with ≥ 6 internal vertices).

If K has no edge between external vertices, there exist three potentially distinct cases how to glue the kernel K into the hourglass chain (corresponding to the 2,2 set partitions of the external vertices).

4.2 Small kernels

We generated all effectively different kernels with up to ten internal vertices. We use the fact that every kernel K can be made 4-regular by adding a square to the external edges. Non-trivial kernels with at least one internal vertex can be found in 4-regular graphs which are internally six-connected and do not have a three vertex cut. Such 4-regular graphs are called irreducible primitive in [24]. From opening these graphs along all their squares we obtain the number of effectively different kernels given in Table 1 (graphs were generated with nauty [19]). See Figure 19 for the cases with at most five internal vertices.

The connection between c_2 invariants and geometries is described in [10, 27]. Here, we investigated all kernels with at most six internal vertices. For kernels with up to five internal vertices the c_2 invariants are listed in Table 2. Note that most results are not proven but

Table 2. The c_2 invariants for kernels with up to five internal vertices. The first column in the table refers to the number of internal vertices and the label in Figure 19. Unidentified sequences in the second column are specified by their prime prefix $(-c_2^{(p)} \bmod p)_{p=2,3,5,7,11,\dots}$ (note that modularity, e.g., applies to the negative c_2).

kernel	c_2 invariant
0	Legendre symbol $(-4/q)$
1	Legendre symbol $(4/q)$
2	Legendre symbol $(4/q)$
3	modular form of weight 4 and level 8
4,1	Legendre symbol $(-4/q)$
4,2	unidentified sequence 0, 2, 3, 2, 3, 8, 15, 9, 6, 27, 11, 32 . . .
4,3	Legendre symbol $(4/q)$
4,4	unidentified sequence 0, 0, 0, 0, 0, 7, 16, 0, 0, 22, 0, 19, . . .
4,5	unidentified sequence 0, 1, 3, 5, 8, 8, 15, 10, 17, 27, 20, 32 . . .
5,1	modular form of weight 4 and level 16
5,2	Legendre symbol $(4/q)$
5,3	modular form of weight 9 and level 4
5,4	unidentified sequence 0, 1, 1, 1, 2, 4, . . .
5,5	unidentified sequence 0, 1, 1, 0, 2, 0, 3, . . .
5,6	unidentified sequence 0, 1, 4, 2, 0, 3, 1, . . .
5,7	unidentified sequence 0, 2, 4, 2, 8, 4, . . .
5,8	unidentified sequence 0, 2, 0, 6, 10, 9, 10, . . .
5,9	unidentified sequence 0, 2, 4, 5, 0, 3, 1, 2, . . .
5,10	unidentified sequence 0, 1, 1, 1, 1, 7, 6, 17, 2, . . .
5,11	unidentified sequence 0, 1, 1, 6, 1, 11, 2, . . .
5,12	unidentified sequence 0, 1, 4, 5, 4, 4, 11, . . .
5,13	unidentified sequence 0, 0, 1, 5, 1, 3, 16, . . .
5,14	modular form of weight 6 and level 4
5,15	unidentified sequence 0, 0, 0, 4, 5, 10, 3, . . .
5,16	unidentified sequence 0, 0, 3, 0, 0, 3, 1, 0, 0, 25, . . .
5,17	modular form of weight 6 and level 4

obtained by identifying finite c_2 prefixes (indexed by primes). In general it is convenient to restrict prefix calculations to primes because (1) computations are simpler and faster, (2) it is assumed that prime prefixes determine the geometry, see [27, Conjecture 2], and (3) modularity only uses primes, see [10, Definition 21]. All identified modular c_2 s are confirmed up to prime 29 using the Maple package `HyperlogProcedures` by the first author [28]. Computations for the primes 31 and 37 are ongoing.

Full reductions were possible for the Legendre symbol $(-4/q)$ in kernel 4,1 and for the Legendre symbol $(4/q)$ in kernel 5,2 and in a kernel with six internal vertices. For these hourglass families of ϕ^4 ancestors the c_2 is proved (except for non-trivial even prime powers).

The modular form [9,4] in kernel 5,3 was not found in the c_2 invariants of ϕ^4 graphs of loop order ≤ 12 (see [27]). It is the first form of weight 9 that has been found in ϕ^4 theory. Note that 4 is the lowest level of all forms of weight 9 which fits into the picture that forms in ϕ^4 have very low level. No new modular forms were found in hourglass chains of kernels with six internal vertices.

Assuming the completion conjecture we could show that no weight 2 modular form of level ≤ 1000 (corresponding to point-counts of curves) exists in hourglass chains of kernels with at most six internal vertices. This result provides substantial extra support for the no-curve conjecture in [10, 27].

The analysis of kernels with seven (or more) internal vertices requires significantly more computing power. We did not pursue this here.

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