

Smooth Multisoliton Solutions of a 2-Component Peakon System with Cubic Nonlinearity

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Abstract. We present a reciprocal transformation which links the Geng–Xue equation to a particular reduction of the first negative flow of the Boussinesq hierarchy. We discuss two reductions of the reciprocal transformation for the Degasperis–Procesi and Novikov equations, respectively. With the aid of the Darboux transformation and the reciprocal transformation, we obtain a compact parametric representation for the smooth soliton solutions such as multi-kink solutions of the Geng–Xue equation.

Key words: soliton; Darboux transformation; Lax pair

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1 Introduction

The Degasperis–Procesi (DP) equation

$$m_t + 3u_x m + u m_x = 0, \quad m = u - u_{xx}$$

was derived by applying the method of asymptotic integrability to a many-parameter family of third order dispersive PDEs [9]. It may be viewed as an approximate model describing shallow water wave propagation in the small amplitude and long wavelength regime [7, 10, 16, 17]. The DP equation is a completely integrable equation. It admits a Lax pair, bi-Hamiltonian structure, and is reciprocally related to a negative flow in the Kaup–Kupershmidt hierarchy [8]. Moreover, the DP equation has been studied by the inverse scattering method [1, 6], and was shown to possess various periodic-wave solutions and travelling-wave solutions [19, 37]. Its smooth multi-soliton solutions were constructed by means of the τ function approach, Riemann–Hilbert method, dressing method and Darboux transformation (DT) [1, 5, 24, 30]. In particular, the DP equation is an equation of Camassa–Holm (CH) type, which has an unusual feature of admitting non-analytic solutions called peakons [26, 27]. There exists an interesting connection between the DP peakon lattice and the finite C-Toda lattice [3].

By using the method of perturbative symmetry to classifying integrable equations of generalised CH form, a new equation with cubic nonlinearity

$$m_t + u^2 m_x + 3u u_x m = 0, \quad m = u - u_{xx}$$

was discovered by Vladimir Novikov [33]. Soon afterwards, Hone and Wang [15] confirmed its integrability by presenting a Lax representation, infinitely many conserved quantities as

well as a bi-Hamiltonian structure. They also related this equation to a negative flow of the Sawada–Kotera hierarchy via a reciprocal transformation. Smooth multi-soliton solutions of the Novikov equation have been presented via several approaches such as the Hirota bilinear method, Riemann–Hilbert method and DT [2, 31, 39]. Furthermore, multipeakons of the Novikov equation may be computed by the inverse spectral method [14]. Dynamical system of the multipeakons of the Novikov equation is a Hamiltonian system, which is connected to the finite Toda lattice of BKP type [4, 15].

Subsequently, Geng and Xue [12] proposed a two-component generalization of the Novikov equation and the DP equation

$$\begin{aligned} m_t + 3u_x vm + uvm_x &= 0, \\ n_t + 3v_x un + un_x &= 0, \quad m = u - u_{xx}, \quad n = v - v_{xx}. \end{aligned} \quad (1.1)$$

Indeed, for $u = 1$ and $u = v$, the Geng–Xue equation (1.1) reduces to the DP equation and the Novikov equation, respectively. This equation is a completely integrable system with a Lax pair and bi-Hamiltonian structure [12, 22]. The homogeneous and local properties of the Hamiltonian functionals were discussed [20]. Also, the Geng–Xue equation is related to a negative flow in a modified Boussinesq hierarchy by a reciprocal transformation [23] and the behaviour of the bi-Hamiltonian structures under the transformation was studied [21]. Moreover, the Geng–Xue equation was shown to admit multi-peakon solutions [28, 29, 35] and its Cauchy problem was considered [13, 36]. However, to the best of our knowledge, smooth solutions such as multi-soliton solutions of the Geng–Xue equation have not been constructed.

The purpose of this paper is to propose a method for building soliton solutions of the Geng–Xue equation. To this end, we find it is convenient to relate the Geng–Xue equation to a particular reduction of the first negative flow in the Boussinesq hierarchy via a reciprocal transformation. Different from the works of [21, 23], this reciprocal transformation may be reduced to that of the DP equation and the Novikov equation. Furthermore, by combining the reciprocal transformation with DT of the negative flow in the Boussinesq hierarchy, we are able to obtain a parametric representation for the multi-kink solutions of the Geng–Xue equation.

The paper is arranged as follows. In Section 2, we introduce a reciprocal transformation and relate the Geng–Xue equation to a particular reduction of the first negative flow of the Boussinesq hierarchy. The reductions of the reciprocal transformation to the DP equation and the Novikov equation will also be discussed. In Section 3, with the aid of the reciprocal transformation and DT, we construct the smooth multisoliton or multi-kink solutions of the Geng–Xue equation. Interestingly the solutions will be represented in terms of Wronskians, and the simplest nontrivial cases will be given explicitly.

2 A reciprocal transformation and negative flow of the Boussinesq hierarchy

In this section, we present a proper reciprocal transformation and establish a link between the Geng–Xue equation and a special negative flow of the Boussinesq hierarchy. Also, we consider the possible reductions of the reciprocal transformation and show that the reciprocal transformations for both the DP equation and Novikov equation are recovered.

2.1 A reciprocal transformation of the Geng–Xue equation

The Geng–Xue equation (1.1) has a Lax representation [12], namely it is the compatibility condition of

$$\Phi_x = M\Phi, \quad \Phi_t = N\Phi, \quad (2.1)$$

where $\Phi = (\varphi_1, \varphi_2, \varphi_3)^T$ and

$$M = \begin{pmatrix} 0 & m\lambda & 1 \\ 0 & 0 & n\lambda \\ 1 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -u_x v & \frac{u_x}{\lambda} - uv m \lambda & u_x v_x \\ \frac{v}{\lambda} & u_x v - uv_x - \frac{1}{\lambda^2} & -uv n \lambda - \frac{v_x}{\lambda} \\ -uv & \frac{u}{\lambda} & uv_x \end{pmatrix}.$$

Use of the above linear spectral problem (2.1) and a standard algorithm lead to infinitely many conservation laws for the Geng–Xue equation. One of them is

$$h_t = -(uvh)_x, \quad h = (mn)^{\frac{1}{3}},$$

which allows us to introduce new independent variables y and τ via the following reciprocal transformation

$$dy = h dx - uvh dt, \quad d\tau = dt. \quad (2.2)$$

Setting $w = (\frac{m}{n})^{\frac{1}{3}}$ and eliminating φ_1, φ_2 , we find that the linear system (2.1) may be rewritten in terms of $\varphi \equiv \varphi_3$ as

$$\begin{aligned} [(wh)^{-\frac{3}{2}}(h(h\varphi_y)_y - \varphi)]_y - \lambda^2 h^{\frac{1}{2}} w^{-\frac{3}{2}} \varphi &= 0, \\ \varphi_\tau - \lambda^{-2} u (wh)^{-\frac{3}{2}} [h(h\varphi_y)_y - \varphi] - uv_y h \varphi &= 0. \end{aligned} \quad (2.3)$$

To bring (2.3) into a familiar form, we introduce a gauge transformation

$$\varphi = h^{-\frac{1}{2}} w^{\frac{1}{2}} \phi, \quad (2.4)$$

and have

$$\phi_{yyy} + \xi \phi_y + \eta \phi = \lambda^2 \phi, \quad \phi_\tau - \frac{p}{\lambda^2} [\phi_{yy} + q \phi_y + (\xi - q_y + q^2) \phi] = 0, \quad (2.5)$$

where

$$p = uh^{\frac{1}{2}} w^{-\frac{3}{2}}, \quad q = \frac{w_y}{w} \quad (2.6)$$

and

$$\xi = \frac{3w_{yy}}{2w} - \frac{9w_y^2}{4w^2} - \frac{h_{yy}}{2h} + \frac{h_y^2 - 4}{4h^2}, \quad \eta = \xi_y - \frac{w_{yyy} + \xi w_y}{w} + \frac{6w_y w_{yy}}{w^2} - \frac{6w_y^3}{w^3}.$$

It is noted that the first equation of (2.5) is the linear spectral problem of the Boussinesq hierarchy. Now, the compatibility condition of the two equations of (2.5) yields the associated Geng–Xue equation, which reads

$$\xi_\tau = -3p_y, \quad s_{1y} = 0, \quad \eta_\tau = -3p_{yy} - 3(pq)_y, \quad s_2 = 0, \quad (2.7)$$

where

$$s_1 = p_{yy} + 3p_y q + 3pq^2 + \xi p + 1, \quad s_2 = \eta - \xi_y + q_{yy} - 3qq_y + \xi q + q^3.$$

It is mentioned that the Geng–Xue equation under the transformation (2.2) implies the associated Geng–Xue equation (2.7). As system (2.7) possesses a Lax pair, one may construct infinitely many conserved quantities for it. Furthermore, it passes the standard Painlevé test of Weiss, Tabor and Canevale (WTC) [38].

As mentioned, the spatial part of the spectral problem (2.5) is the one for the Boussinesq hierarchy. The first equation in this hierarchy is the Boussinesq equation [34, 41]

$$\xi_t = -\xi_{xx} + 2\eta_x, \quad \eta_t = \eta_{xx} - \frac{2}{3}(\xi_{xxx} + \xi\xi_x).$$

Thus the associated Geng–Xue equation (2.7) should have connection with a particular flow of this hierarchy. To see it, let us consider a more general spectral problem

$$\begin{aligned} \phi_{yyy} + \xi\phi_y + \eta\phi &= \lambda^2\phi, \\ \phi_\tau - \frac{1}{\lambda^2} \left[a\phi_{yy} + b\phi_y + \frac{1}{3}(2a\xi - 3b_y - a_{yy})\phi \right] &= 0. \end{aligned}$$

Its compatibility condition yields

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}_\tau = J_1 \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv J_2 \begin{pmatrix} a \\ b \end{pmatrix} = 0, \quad (2.8)$$

where

$$\begin{aligned} J_1 &= \begin{pmatrix} -3\partial & 0 \\ -3\partial^2 & -3\partial \end{pmatrix}, \\ J_2 &= \begin{pmatrix} \partial^4 + \xi\partial^2 - \partial\xi_y + 3\eta\partial + 2\eta_y & 2\partial^3 + \xi\partial + \partial\xi \\ \frac{1}{3}(\partial^5 + \xi\partial^3 - 2\partial^3\xi - 2\xi\partial\xi) + 3\partial\eta\partial + \eta_{yy} & \partial^4 + \xi\partial^2 + 3\eta\partial + \eta_y \end{pmatrix}. \end{aligned}$$

It is not difficult to check that $J_2J_1^{-1}$ is the well-known recursion operator of the Boussinesq hierarchy, and hence the system (2.8) is just the first negative flow in the Boussinesq hierarchy. Now, setting $a = p$, $b = pq$, we have the following relation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \partial - q & 2p\partial + 3p_y \\ \frac{1}{3}\partial^2 + (q_y - \frac{2}{3}\xi - q^2) & p\partial^2 + (3p_y + pq)\partial + 3p_{yy} + 3(pq)_y \end{pmatrix} \begin{pmatrix} s_1y \\ s_2 \end{pmatrix}.$$

Thus, the associated Geng–Xue equation (2.7) indeed is a particular reduction of the negative Boussinesq equation (2.8).

2.2 Reductions of the reciprocal transformation

Recall that the Geng–Xue equation may be reduced to the DP equation and the Novikov equation by setting $u = 1$ and $u = v$ respectively. So it is interesting to consider the reciprocal transformation (2.2) under these reductions.

Case 1: $u = 1$. In this case, the reciprocal transformation (2.2) becomes

$$dy = h dx - v h dt, \quad d\tau = dt,$$

where $h = n^{\frac{1}{3}}$. A direct calculation shows that the associated equation (2.7) and the spectral problem (2.5) reduce to

$$\xi_\tau = -6hh_y, \quad 2h_{yyy} + h\xi_y + 2\xi h_y = 0,$$

and

$$\phi_{yyy} + \xi\phi_y + \frac{1}{2}\xi_y\phi = \lambda^2\phi, \quad \phi_\tau = \frac{1}{\lambda^2} [h^2\phi_{yy} - hh_y\phi_y + (h_y^2 - hh_{yy} - 1)\phi],$$

respectively. These are just the associated DP equation and its spectral problem [8, 24].

Case 2: $u = v$. Now, the reciprocal transformation (2.2) turns into

$$dy = h dx - u^2 h dt, \quad d\tau = dt,$$

where $h = m^{\frac{2}{3}}$. In this case, the spectral problem (2.5) becomes

$$\phi_{yyy} + \xi\phi_y + \xi_y\phi = \lambda^2\phi, \quad \phi_\tau = \frac{p}{\lambda^2}(\phi_{yy} + \xi\phi), \quad (2.9)$$

where

$$\xi = -\frac{h_{yy}}{2h} + \frac{h_y^2}{4h^2} - \frac{1}{h^2}, \quad p = uh^{\frac{1}{2}}.$$

Meanwhile, (2.7) yields

$$\xi_\tau = -3p_y, \quad (p_{yy} + p\xi)_y = 0. \quad (2.10)$$

Setting $\phi = \psi_y$, from (2.9) we have

$$\psi_{yyy} + \xi\psi_y = \lambda^2\psi, \quad \psi_\tau = \frac{1}{\lambda^2}(p\psi_{yy} - p_y\psi_y + p_{yy}\psi + p\xi\psi). \quad (2.11)$$

It is easy to see that by integrating the second equation of (2.10) and substituting it into (2.11), we reach the results appeared in [15, 39].

3 Multisoliton solutions of the Geng–Xue equation

In the previous section, we have related the Geng–Xue equation (1.1) to the associated equation (2.7), which was shown to be a reduction of the first negative flow in the Boussinesq hierarchy. In what follows, we will take a similar approach as done for the CH, modified CH and DP equations [24, 25, 40] and explain that this connection, together with the DT for the associated Geng–Xue equation, allows us to propose an algorithm to build multisoliton solutions of the Geng–Xue equation.

As a first step, we have the following

Proposition 3.1. *The spectral problem (2.5) is covariant with respect to the following N -DT*

$$\begin{aligned} \phi[N] &= \frac{W(f_1, f_2, \dots, f_N, \phi)}{W_N}, \\ \xi[N] &= \xi + 3(\ln W_N)_{yy}, \\ \eta[N] &= \eta + \left(N\xi + \frac{3}{2}[(\ln W_N)_y]^2 + 3(\ln W_N)_{yy} - \frac{3F_N}{W_N} \right)_y, \\ p[N] &= p - (\ln W_N)_{y\tau}, \\ q[N] &= \frac{1}{p[N]} \left(pq + Np_y - (\ln W_N)_y(\ln W_N)_{y\tau} + \left(\frac{F_N}{W_N} \right)_\tau \right), \end{aligned} \quad (3.1)$$

where f_1, \dots, f_N are solutions of the spectral problem (2.5) at $\lambda = \lambda_1, \dots, \lambda_N$, respectively. W denotes the Wronskian,

$$W_N = W(f_1, f_2, \dots, f_N) = \begin{vmatrix} f_1 & \cdots & f_1^{(N-1)} \\ \vdots & & \vdots \\ f_N & \cdots & f_N^{(N-1)} \end{vmatrix} = |f, \dots, f^{(N-1)}| = |\widehat{N-1}|,$$

and

$$F_N = \begin{cases} 0, & N = 1, \\ W(f_{1y}, f_{2y}), & N = 2, \\ |\widehat{N-3, N-1, N}|, & N \geq 3. \end{cases}$$

Here and in the sequel all determinant notations are adopted according to [11].

Proof. For the Lax operator $L = \partial_y^3 + \xi \partial_y + \eta$, it is well known that the operator of the 1-DT is given by $T_1 = \partial_y - \frac{f_{1y}}{f_1}$ (see [18, 32]). It is straightforward to check that the transformed variables solve

$$\begin{aligned} (\phi[1])_{yyy} + \xi[1](\phi[1])_y + \eta[1]\phi[1] &= \lambda^2 \phi[1], \\ (\phi[1])_\tau - \frac{p[1]}{\lambda^2} ((\phi[1])_{yy} + q[1](\phi[1])_y + (\xi[1] - (q[1])_y + q[1]^2)\phi[1]) &= 0, \end{aligned}$$

namely the proposition is valid for $N = 1$ case. For the general case, we may assume that the operator of the N -DT takes the form $T = \partial_y^N + a_1 \partial_y^{N-1} + \dots + a_N$, where coefficients a_1, \dots, a_N are functions of f_1, \dots, f_N and their derivatives with respect to y . Hence, for this iterated DT, we have

$$\hat{L}T = TL, \quad \hat{L} = \partial_y^3 + \xi[N]\partial_y + \eta[N]. \quad (3.2)$$

Plugging the expressions of T , L into the first equation of (3.2) and considering the coefficients of ∂^{N+1} and ∂^N , we obtain

$$\xi[N] = \xi - 3a_{1y}, \quad \eta[N] = \eta + N\xi_y + 3a_1 a_{1y} - 3a_{1yy} - 3a_{2y}. \quad (3.3)$$

The coefficients of the operator T may be determined from the conditions $Tf_i = 0$, $1 \leq i \leq N$. In other words, we have

$$a_N f_i + a_{N-1} f_i^{(1)} + \dots + a_1 f_i^{(N-1)} = -f_i^{(N)}, \quad i = 1, \dots, N. \quad (3.4)$$

Using Cramer's rule, we obtain from the system (3.4) that

$$a_1 = -\frac{|\widehat{N-2, N}|}{|\widehat{N-1}|}, \quad a_2 = \frac{|\widehat{N-3, N-1, N}|}{|\widehat{N-1}|}.$$

Substituting them into (3.3), we have

$$\xi[N] = \xi + 3(\ln W_N)_{yy}, \quad \eta[N] = \eta + \left(N\xi + \frac{3}{2}[(\ln W_N)_y]^2 + 3(\ln W_N)_{yy} - \frac{3F_N}{W_N} \right)_y.$$

In addition, the iterated formulae for p , q may be acquired by substituting $\xi[N]$, $\eta[N]$ into the associated Geng–Xue equation (2.7). This completes the proof. \blacksquare

Next, let us start with the trivial solution $u = u_0$, $v = v_0$ of the Geng–Xue equation, where u_0, v_0 are positive constants. Then the corresponding seed solution in the above DT reads $\xi = -k^{-2}$, $p = k^2$, $\eta = q = 0$, where $k = (u_0 v_0)^{\frac{1}{3}}$. To calculate the solutions f_j of the system (2.5) at λ_j , $j = 1, \dots, N$, it is convenient to assume $\alpha_j, \beta_j, -\alpha_j - \beta_j$ being three distinct roots of the cubic equation $\omega^3 - k^{-2}\omega = \lambda_j^2$. Now, without loss of generality, from (2.5) we have

$$f_j = e^{\vartheta_j} + \delta_j e^{\sigma_j}, \quad 1 \leq j \leq N, \quad (3.5)$$

where

$$\vartheta_j = \alpha_j y + \frac{k^2}{\alpha_j} \tau + c_{1j}, \quad \sigma_j = \beta_j y + \frac{k^2}{\beta_j} \tau + c_{2j}.$$

Herein c_{1j} , c_{2j} , δ_j are arbitrary constants. To build real solutions of the Geng–Xue equation, let us assume $\alpha_j - \beta_j = p_j > 0$ and rewrite (3.5) as

$$f_j = e^{\frac{\vartheta_j + \sigma_j}{2}} (e^{\theta_j} + \delta_j e^{-\theta_j}), \quad \theta_j = \frac{1}{2} \left(p_j y + \frac{3k^2 p_j}{p_j^2 - k^{-2}} \tau + c_{1j} - c_{2j} \right).$$

It is noted that 1 , $e^{y/k}$ and $e^{-y/k}$ constitute a fundamental set of solutions of the first equation of (2.5) at $\lambda = 0$ with the seed $\xi = -k^{-2}$, $p = k^2$, $\eta = q = 0$. With the help of the f_j 's given by (3.5), it follows immediately from Proposition 3.1 that for the N -th iterated spectral problem

$$(\phi[N])_{yyy} + \xi[N](\Phi[N])_y + \eta[N]\Phi[N] = 0$$

at $\lambda = 0$ we may calculate its solutions, and in particular we have

$$\begin{aligned} \phi[N] &= \nu_1 \phi^1[N] + \nu_2 \phi^2[N] + \nu_3 \phi^3[N], \\ p[N] &= k_1^2 - (\ln W_N)_{y\tau}, \\ q[N] &= \frac{1}{p[N]} \left(\left(\frac{F_N}{W_N} \right)_\tau - (\ln W_N)_y (\ln W_N)_{y\tau} \right), \end{aligned} \quad (3.6)$$

where ν_i , $i = 1, 2, 3$, are arbitrary constants and

$$\begin{aligned} \phi^1[N] &= \frac{W(f_1, \dots, f_N, 1)}{W_N}, & \phi^2[N] &= \frac{W(f_1, \dots, f_N, e^{\frac{1}{k}y})}{W_N}, \\ \phi^3[N] &= \frac{W(f_1, \dots, f_N, e^{-\frac{1}{k}y})}{W_N}. \end{aligned} \quad (3.7)$$

Direct calculations show that the asymptotic behaviours of the wave functions $\phi^1[N]$, $\phi^2[N]$, $\phi^3[N]$ are given by

$$\begin{aligned} \phi^1[N] &\sim \begin{cases} (-1)^N \prod_{i=1}^N \alpha_i, & y \rightarrow +\infty, \\ (-1)^N \prod_{i=1}^N \beta_i, & y \rightarrow -\infty, \end{cases} & \phi^2[N] &\sim \begin{cases} e^{\frac{1}{k}y} \prod_{i=1}^N \left(\frac{1}{k} - \alpha_i \right), & y \rightarrow +\infty, \\ e^{\frac{1}{k}y} \prod_{i=1}^N \left(\frac{1}{k} - \beta_i \right), & y \rightarrow -\infty, \end{cases} \\ \phi^3[N] &\sim \begin{cases} e^{-\frac{1}{k}y} \prod_{i=1}^N \left(-\frac{1}{k} - \alpha_i \right), & y \rightarrow +\infty, \\ e^{-\frac{1}{k}y} \prod_{i=1}^N \left(-\frac{1}{k} - \beta_i \right), & y \rightarrow -\infty. \end{cases} \end{aligned} \quad (3.8)$$

Next, we work out the coordinate transformation between the independent variables x , t and y , τ . To this end, we consider the spectral problem in (2.1) at $\lambda = 0$ which yields

$$\varphi_{xx} - \varphi = 0, \quad (3.9)$$

for $\varphi = \varphi_3$. As e^x and e^{-x} form a fundamental set of solutions of (3.9), in view of the gauge transformation (2.4), they may be represented as

$$\begin{aligned} e^x &= \left(\frac{w}{h}\right)^{1/2} (c_1\phi^1[N] + c_2\phi^2[N] + c_3\phi^3[N]), \\ e^{-x} &= \left(\frac{w}{h}\right)^{1/2} (d_1\phi^1[N] + d_2\phi^2[N] + d_3\phi^3[N]), \end{aligned}$$

where the coefficients $c_k, d_k, k = 1, 2, 3$, are independent of y . Taking account of the asymptotic behaviours (3.8), we find

$$e^x = c_2 \left(\frac{w}{h}\right)^{1/2} \phi^2[N], \quad e^{-x} = d_3 \left(\frac{w}{h}\right)^{1/2} \phi^3[N],$$

which further imply

$$x = c + \frac{1}{2} \ln \left(\frac{\phi^2[N]}{\phi^3[N]} \right),$$

where $c = c(\tau)$. Differentiating above equation with respect to τ and using the reciprocal transformation (2.2), we have

$$x_\tau = c_\tau + \frac{1}{2} \left(\ln \frac{\phi^2[N]}{\phi^3[N]} \right)_\tau = uv,$$

then by taking the limit $y \rightarrow \infty$, we find $c_\tau = u_0 v_0 = k^3$, which leads to $c = k^3\tau + d$ with d as an integration constant. Thus we obtain

$$x = k^3\tau + \frac{1}{2} \ln \left(\frac{\phi^2[N]}{\phi^3[N]} \right) + d. \quad (3.10)$$

For temporal variables, from (2.2) we have

$$t = \tau. \quad (3.11)$$

Finally, we need to work out the transformation formulae for the field variables u, v , which can be done by means of (2.6) and the reciprocal transformation (2.2). In particular, we may deduce w from $q[N] = \frac{w_y}{w}$. It is interesting to observe that for $p[N]$ and $q[N]$, we have

$$p[N] = W(g_1, \dots, g_N) W(f'_1, \dots, f'_N) \left(\frac{k}{W_N} \right)^2, \quad (3.12)$$

$$q[N] = - \left(\ln \frac{W(g_1, g_2, \dots, g_N)}{W_N} \right)_y, \quad (3.13)$$

where $g_i = \lambda_i^{-2} (f_{iyy} - k^{-2} f_i)$, $i \geq 1$. The proof of (3.12) and (3.13) is presented in the appendix. Summarizing above discussions, we have

Proposition 3.2. *The Geng–Xue equation admits the parametric representation of the multi-kink solutions*

$$u = k^2 \ell W(f'_1, \dots, f'_N) \left(\frac{x_y}{W_N W(g_1, g_2, \dots, g_N)} \right)^{\frac{1}{2}}, \quad v = \frac{x_\tau}{u},$$

where x, t are defined by (3.10) and (3.11), ℓ is an integration constant, and $\phi^2[N], \phi^3[N]$ are given by (3.7), respectively.

It may be checked that for the solution presented above both potentials u and v have different asymptotic behaviors as $y \rightarrow \infty$ and $y \rightarrow -\infty$. This is an indication that the solution is of kink-type.

In the rest part of this section, we consider the simplest cases and present two examples.

Example 3.3 (1-kink solution). For $N = 1$, let us take

$$f_1 = e^{\vartheta_1} + \delta_1 e^{\sigma_1}, \quad g_1 = \lambda_1^{-2}(f_{1yy} - k^{-2}f_1),$$

where δ_1 may be chosen as ± 1 . Direct computations show that the DT (3.1) yields

$$\begin{aligned} \phi^2[1] &= \left(\frac{1}{k} - \frac{f_{1y}}{f_1}\right) e^{\frac{1}{k}y}, & \phi^3[1] &= -\left(\frac{1}{k} + \frac{f_{1y}}{f_1}\right) e^{-\frac{1}{k}y}, \\ p[1] &= k^2 - k^2 \left(\frac{g_1}{f_1}\right)_y, & q[1] &= -\frac{f_1}{g_1} \left(\frac{g_1}{f_1}\right)_y. \end{aligned}$$

Then, application of Proposition 3.2 allows us to have the following solution of the Geng–Xue equation

$$\begin{aligned} x &= \frac{1}{k}y + k^3\tau + \frac{1}{2} \ln \left(\frac{\bar{k}_1 f_{1y} - \frac{1}{k}f_1}{f_{1y} + \frac{1}{k}f_1} \right), & t &= \tau, \\ u &= k^2 \frac{f_{1y}}{f_1} \left(\bar{k}_2 \frac{f_1}{g_1} x_y \right)^{\frac{1}{2}}, & v &= \frac{k^3 f_{1y} (f_{1y} - \frac{1}{k^2}g_1)}{(f_{1y} - \frac{1}{k}f_1)(f_{1y} + \frac{1}{k}f_1)u}. \end{aligned} \quad (3.14)$$

Here and in the sequel, all \bar{k}_i , $i \in \mathbb{Z}$, are assumed to be arbitrary constants. We take $\alpha_1, \beta_1, \lambda_1^2$ as real constants so that we have the real-valued solutions. Furthermore, assuming that $\delta_1 = 1$, $0 < kp_1 < 1$, our solutions will be non singular. Under these assumptions and substituting the expressions of f_1, g_1 into the solution (3.14), we obtain a parameter representation of 1-kink solution of the Geng–Xue equation

$$\begin{aligned} x &= \bar{k}_3 + \frac{1}{k}y + k^3\tau + \frac{1}{2} \ln \frac{\left(\frac{2}{k} - r_1\right) \cosh \theta_1 - p_1 \sinh \theta_1}{\left(\frac{2}{k} + r_1\right) \cosh \theta_1 + p_1 \sinh \theta_1}, & t &= \tau, \\ u &= \frac{\bar{k}_4 (p_1 \sinh \theta_1 + r_1 \cosh \theta_1)}{\sqrt{r_1^2 \cosh(2\theta_1) - p_1 r_1 \sinh(2\theta_1) + p_1^2 + r_1^2}}, \\ v &= \frac{k^3 r_1 [3k^2 p_1 r_1 \sinh \theta_1 - (k^2 p_1^2 + 2) \cosh \theta_1]}{\bar{k}_4 (k^2 p_1^2 - 1) \sqrt{r_1^2 \cosh(2\theta_1) - p_1 r_1 \sinh(2\theta_1) + p_1^2 + r_1^2}}, \end{aligned}$$

where $\bar{k}_3 = \ln(-\bar{k})$ and $\theta_1 = \frac{1}{2}(p_1 y + \frac{3k^2 p_1}{p_1^2 - k^{-2}}\tau + \theta_{10})$, $r_1 = \alpha_1 + \beta_1 = \pm \sqrt{\frac{4}{3k^2} - \frac{1}{3}p_1^2}$ with θ_{10} any constant.

We now consider the first a few conserved quantities for the 1-kink solution. The Geng–Xue equation is known to possess infinitely many conserved quantities [12, 20, 22, 23], and among them the first three are

$$I_1 = \int_{-\infty}^{\infty} h \, dx, \quad I_2 = \int_{-\infty}^{\infty} un \, dx, \quad I_3 = \int_{-\infty}^{\infty} u_x n \, dx.$$

Since 1-kink solution of the Geng–Xue equation is not explicit and written in parametric form, we consider I_i under (2.2), namely,

$$I_1 = \int_{-\infty}^{\infty} 1 \, dy, \quad I_2 = \int_{-\infty}^{\infty} \frac{un}{h} \, dy, \quad I_3 = \int_{-\infty}^{\infty} u_y n \, dy.$$

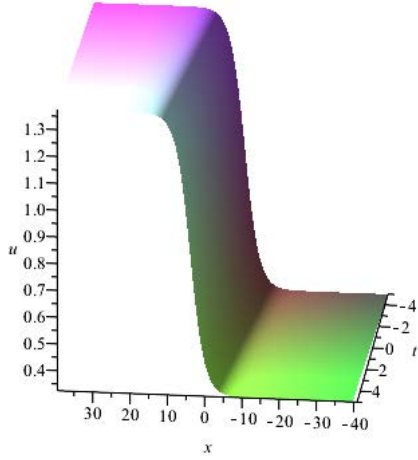


Figure 1. One-kink for u at $\bar{k}_3 = \theta_{10} = 0$, $p_1 = 1$, $k_1 = \frac{1}{2}$, $\bar{k}_4 = 1$, $r_1 = \sqrt{5}$.

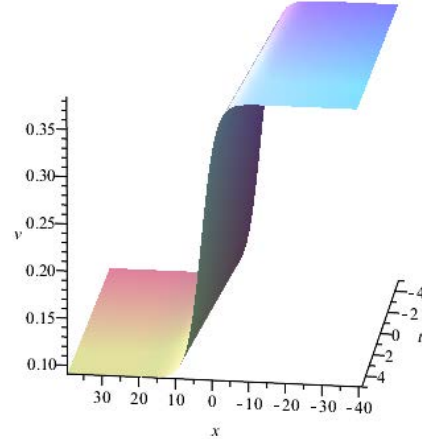


Figure 2. One-antikink for v at $\bar{k}_3 = \theta_{10} = 0$, $p_1 = 1$, $k_1 = \frac{1}{2}$, $\bar{k}_4 = 1$, $r_1 = \sqrt{5}$.

Thus, for the 1-kink solution we have

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} p \, dy = k^2 \int_{-\infty}^{\infty} \left[1 - \left(\frac{g_1}{f_1} \right)_y \right] dy = k^2 I_1 - k^2 \int_{-\infty}^{\infty} \left(\frac{g_1}{f_1} \right)_y dy \\ &= k^2 I_1 - \frac{k^2}{\lambda_1^2} (\alpha_1^2 - \beta_1^2) \equiv -\frac{k^2}{\lambda_1^2} p_1 r_1 \pmod{I_1}. \end{aligned}$$

For I_3 , using definition of p together with

$$x_y = \frac{1}{k} + \frac{1}{2} \left(\frac{f_{1yy} - \frac{1}{k} f_{1y}}{f_{1y} - \frac{1}{k} f_1} - \frac{f_{1yy} + \frac{1}{k} f_{1y}}{f_{1y} + \frac{1}{k} f_1} \right) = \frac{\lambda_1^2 f_1 g_1}{k(f_{1y}^2 - \frac{1}{k^2} f_1^2)},$$

we arrive at

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} \frac{u_y}{u} p h \, dy = \int_{-\infty}^{\infty} \frac{k^2 - k^2 \left(\frac{g_1}{f_1} \right)_y}{x_y} \left[\frac{f_{1yy}}{f_{1y}} - \frac{f_{1y}}{2f_1} + \frac{x_{yy}}{2x_y} - \frac{g_{1y}}{2g_1} \right] dy \\ &= \int_{-\infty}^{\infty} \frac{k^2 g_1 f_{1y}}{x_y f_1^2} \left[\frac{f_{1yy}}{f_{1y}} - \frac{(f_{1y}^2 - \frac{1}{k^2} f_1^2)_y}{2(f_{1y}^2 - \frac{1}{k^2} f_1^2)} \right] dy = -\frac{k}{\lambda_1^2} \int_{-\infty}^{\infty} \left(\frac{f_{1y}}{f_1} \right)_y dy = -\frac{k}{\lambda_1^2} p_1. \end{aligned}$$

Example 3.4 (2-kink solution). For $N = 2$, we may take

$$f_i = e^{\vartheta_i} + \delta_i e^{\sigma_i}, \quad g_i = \frac{1}{\lambda_i^2} (f_{iyy} - k^{-2} f_i), \quad i = 1, 2,$$

where δ_1, δ_2 are allowed to be ± 1 . From (3.6), it follows that

$$\begin{aligned} \phi^2[2] &= \frac{W(f_1, f_2, e^{\frac{1}{k}y})}{W(f_1, f_2)}, & \phi^3[2] &= \frac{W(f_1, f_2, e^{-\frac{1}{k}y})}{W(f_1, f_2)}, \\ p[2] &= k^2 - [\ln W(f_1, f_2)]_{y\tau}, & q[2] &= \frac{1}{p[2]} \left(\frac{W(f_{1y}, f_{2y})}{W(f_1, f_2)} - \frac{1}{2} \left(\frac{W_y(f_1, f_2)}{W(f_1, f_2)} \right)^2 \right)_\tau. \end{aligned}$$

Notice that $g_{iy} = f_i$, $f_{i\tau} = k^2 g_i$, $i = 1, 2$. Then, after some direct calculations, we find

$$\frac{\phi^2[2]}{\phi^3[2]} = e^{\frac{2}{k}y} \frac{G_1}{G_2}, \quad p[2] = k^2 \frac{W(g_1, g_2) W(f_{1y}, f_{2y})}{W^2(f_1, f_2)}, \quad q[2] = \left(\ln \frac{W(f_1, f_2)}{W(g_1, g_2)} \right)_y,$$

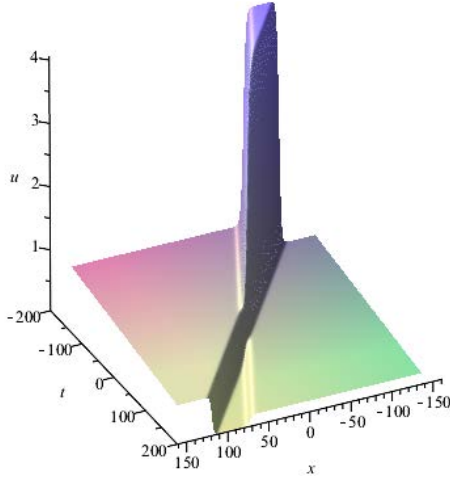


Figure 3. Two-kink for u .

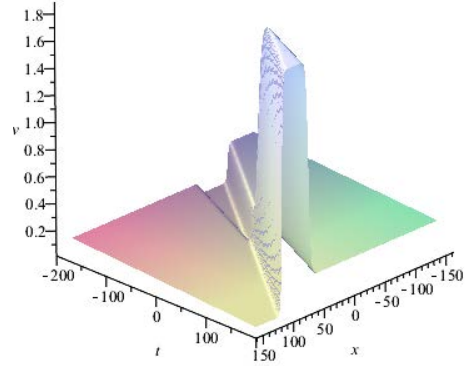


Figure 4. Two-kink for v .

where

$$G_1 = W\left(f_{1y} - \frac{1}{k}f_1, f_{2y} - \frac{1}{k}f_2\right), \quad G_2 = G_1|_{k \rightarrow -k}.$$

Following Proposition 3.2 and after tedious calculations, we may get a parameter representation of exact solution

$$x = \bar{k}_5 + \frac{1}{k}y + k^3\tau + \frac{1}{2} \ln \frac{G_1}{G_2}, \quad t = \tau,$$

$$u = \bar{k}_6 \frac{W(f_{1y}, f_{2y})}{\sqrt{G_1 G_2}}, \quad v = \frac{k^3}{\bar{k}_6 \sqrt{G_1 G_2}} W\left(f_{1y} - \frac{1}{k^2}g_1, f_{2y} - \frac{1}{k^2}g_2\right).$$

Hereafter, to obtain real solution without singularity, let us assume that $\alpha_i, \beta_i, \lambda_i^2$ are real and $\delta_1 = \delta_2 = 1$. Then we may establish the parameter representation of the 2-kink solution of the Geng–Xue equation. A profile of 2-kink solution is plotted in Figures 3 and 4 for the parameters $p_1 = 2, p_2 = \frac{3}{2}, r_1 = \alpha_1 + \beta_1 = \sqrt{7}, r_2 = \alpha_2 + \beta_2 = -\sqrt{\frac{91}{12}}, \bar{k}_5 = \theta_{10} = \theta_{20} = 0, k = \frac{2}{5}, \bar{k}_6 = 1$.

A Proof of (3.12) and (3.13)

For $N = 1, 2$, the validity of (3.12), (3.13) has been shown in Examples 3.3 and 3.4. In general case, we first introduce the following two matrices M, \bar{M} and N -dimensional column vectors $\mathbf{g}, \mathbf{f}, \mathbf{a}, \mathbf{b}, \mathbf{d}$ as

$$M = \begin{pmatrix} f'_1 & \cdots & f_1^{(N-2)} \\ \vdots & \vdots & \vdots \\ f'_N & \cdots & f_N^{(N-2)} \end{pmatrix}, \quad \bar{M} = \begin{pmatrix} f'_1 & \cdots & f_1^{(N-3)} & f_1^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ f'_N & \cdots & f_N^{(N-3)} & f_N^{(N-1)} \end{pmatrix},$$

$$(\mathbf{g}, \mathbf{f}, \mathbf{a}, \mathbf{b}, \mathbf{d}) = \begin{pmatrix} g_1 & f_1 & f_1^{(N-2)} & f_1^{(N-1)} & f_1^{(N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_N & f_N & f_N^{(N-2)} & f_N^{(N-1)} & f_N^{(N)} \end{pmatrix}.$$

Now we assume $N \geq 3$ and note $f_{j\tau} = k^2 g_j$, then we deduce

$$p[N] = k^2 - k^2 \left(\frac{|\mathbf{g}, M, \mathbf{b}|}{W_N} \right)_y$$

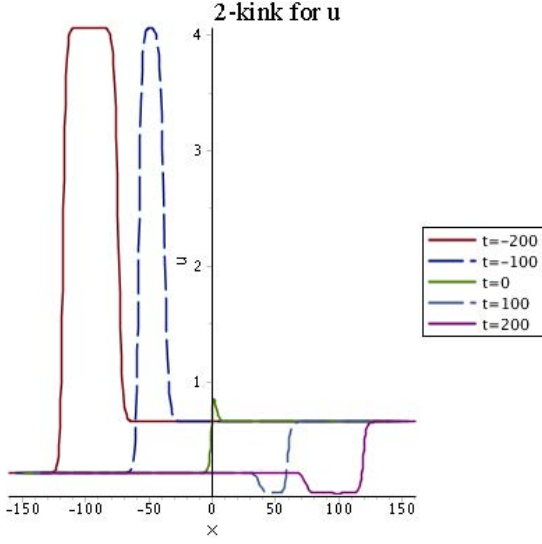


Figure 5. Two-kink solution for u at $\tau = -200, -100, 0, 100, 200$.

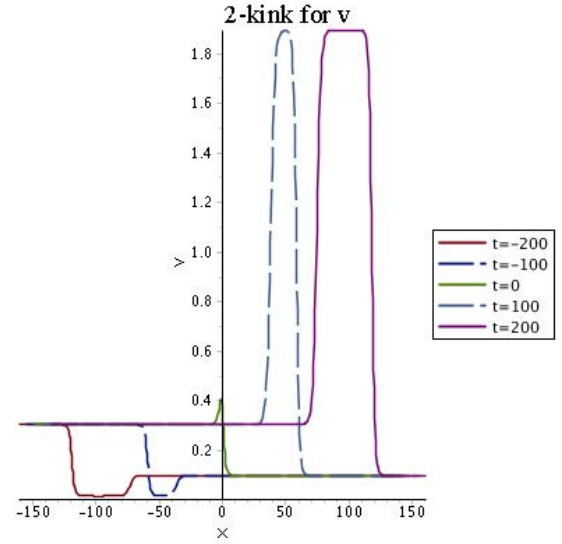


Figure 6. Two-kink solution for v at $\tau = -200, -100, 0, 100, 200$.

$$\begin{aligned}
&= \frac{k^2}{W_N^2} (|\mathbf{g}, M, \mathbf{b}|W_{Ny} - |\mathbf{g}, M, \mathbf{d}|W_N) \\
&= \frac{k^2}{W_N^2} (|M, \mathbf{g}, \mathbf{b}||M, \mathbf{f}, \mathbf{d}| - |M, \mathbf{f}, \mathbf{b}||M, \mathbf{g}, \mathbf{d}|) \\
&= \frac{k^2}{W_N^2} |M, \mathbf{g}, \mathbf{f}||M, \mathbf{b}, \mathbf{d}| \\
&= \frac{k^2}{W_N^2} W(g_1, \dots, g_N)W(f'_1, \dots, f'_N),
\end{aligned}$$

where the Plücker identity

$$|M, \mathbf{g}, \mathbf{f}||M, \mathbf{b}, \mathbf{d}| - |M, \mathbf{g}, \mathbf{b}||M, \mathbf{f}, \mathbf{d}| + |M, \mathbf{g}, \mathbf{d}||M, \mathbf{f}, \mathbf{b}| = 0 \quad (\text{A.1})$$

is used.

Next we consider (3.13). Indeed, we have

$$\begin{aligned}
q[N] &= \frac{1}{p[N]} ((F_N/W_N)_\tau - (\ln W_N)_y (\ln W_N)_{y\tau}) \\
&= \frac{1}{p[N]} \left(\frac{W_{Ny}(p[N] - k^2)}{W_N} + \frac{F_{N\tau}W_N - F_NW_{N\tau}}{W_N^2} \right) \\
&= \frac{W_{Ny}}{W_N} + \frac{F_{N\tau}W_N - F_NW_{N\tau} - k^2W_NW_{Ny}}{k^2\hat{W}_N} \\
&= \frac{W_{Ny}}{W_N} + \frac{|\bar{M}, \mathbf{g}, \mathbf{d}||\bar{M}, \mathbf{a}, \mathbf{f}| + |\bar{M}, \mathbf{g}, \mathbf{a}||\bar{M}, \mathbf{f}, \mathbf{d}|}{\hat{W}_N} \\
&= \frac{W_{Ny}}{W_N} + \frac{|\bar{M}, \mathbf{a}, \mathbf{d}||\bar{M}, \mathbf{g}, \mathbf{f}|}{\hat{W}_N} \\
&= \frac{W_{Ny}}{W_N} - \frac{W_y(g_1, \dots, g_N)}{W(g_1, \dots, g_N)},
\end{aligned}$$

where $\hat{W}_N = W(g_1, \dots, g_N)W(f'_1, \dots, f'_N)$ and an identity similar to (A.1) is used again.

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