On Gorenstein Fano Threefolds
with an Action of a Two-Dimensional Torus

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Abstract. We classify the non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one that admit an effective action of a two-dimensional algebraic torus.

Key words: Fano threefolds; torus action

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1 Introduction

This article contributes to the classification of singular Fano threefolds. We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. By a Fano variety we mean a normal projective variety X over $\mathbb{K}$ admitting an ample anticanonical divisor $-\mathcal{K}_X$. In the smooth case, the classifications by Iskovskikh [23, 24] and Mori–Mukai [26] provide us with a far developed picture in dimension three. In the singular case, the situation is less explored. As a landmark, we have in dimension two the classifications by Alexeev/Nikulin [1] and Nakayama [27] of the log terminal del Pezzo surfaces X of Gorenstein index $\imath_X \leq 2$. Here, log terminal means discrepancies greater than -1 and $\imath_X$ is the smallest positive integer with $\imath_X\mathcal{K}_X$ Cartier; so, $\imath_X = 1$ merely means that X is Gorenstein. In dimension three, the classification problem for singular Fano varieties is widely open. The Mori–Fano threefolds, that means the terminal $\mathbb{Q}$-factorial Fano threefolds of Picard number one, are intensely studied; see in particular Prokhorov's classifications for higher index and degree cases [28, 29, 30, 31].

Once we restrict to Fano varieties with many symmetries, the singular case is more accessible. A sample case are toric Fano varieties, where we mention Kasprzyk's classification of the canonical toric Fano threefolds [25], comprising in particular the toric Mori–Fano threefolds. Going one step beyond the toric case, one considers threefolds coming with an effective action of a two-dimensional torus. In this setting, the Mori–Fano threefolds have been classified in [8], using the so-called anticanonical complex; see also [21] for generalizations. Moreover, in [22], a classification algorithm for Gorenstein canonical Fano varieties with a torus action of complexity one has been proposed using the approach via polyhedral divisors [2]. However, as soon as we leave the surface case, feasability becomes a serious question.

In the present article we classify the non-toric $\mathbb{Q}$-factorial Gorenstein log terminal Fano threefolds X of Picard number one that come with an effective action of a two-dimensional torus. We use the Cox ring based approach to rational varieties with a torus action of complexity one developed in [16, 19]; see also Section 2 for a brief reminder. The Cox ring of a normal projective variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ is defined as

$$\mathcal{R}(X) = \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

where we refer to [4] for the details. For our Fano threefolds X of Picard number one acted on by a two-dimensional torus, the divisor class group $\mathrm{Cl}(X)$ is of the form $\mathbb{Z} \oplus \Gamma$ with a finite abelian torsion part Γ and the Cox ring $\mathcal{R}(X)$ is a finitely generated complete intersection ring with a very specific system of trinomial relations. Moreover, the variety X can be reconstructed from the list of generator degrees in $\mathrm{Cl}(X)$ and the defining relations of the Cox ring $\mathcal{R}(X)$ which allows us to encode X via these Cox ring data in a compact manner. Let us briefly describe the main result of the article; the detailed classification lists, containing in particular the Cox ring data, are provided in Section 4.

Classification 1.1. We obtain 538 families of non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one acted on effectively by a two-dimensional torus. Listed according to the possible divisor class groups, we have:

Divisor class group	Sporadic varieties	True families
$\mathbb{Z}$	242	3 one-dimensional
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	163	4 one-dimensional
$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$	46	5 one-dimensional, 1 two-dimensional
$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^3$	6	1 one-dimensional
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	4	1 one-dimensional
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	1	0
$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	26	1 one-dimensional
$\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	1	0
$\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	18	1 one-dimensional
$\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	4	0
$\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	8	0
$\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	2	0

Moreover, every non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold of Picard number one with an effective action of a two-dimensional torus is isomorphic to precisely one member of these 538 families.

Note that being Gorenstein and log terminal, all varieties from Classification 1.1 are canonical. The overlap with the classification of non-toric Mori–Fano threefolds coming with an action of a two-dimensisonal torus given in [8] consists precisely of the smooth quadric in $\mathbb{P}_4$. The defining data of each of our 538 families are stored in the file [7]. Moreover, we store in this file geometric invariants such as genus, codimension, anticanonical self intersection, Hilbert series, etc., which allows to extract varieties with given properties.

Let us spend a few words on the methods used in the classifications of non-toric Fano threefolds X of Picard number one coming with an action of a two-dimensional torus. As mentioned before, the main tool of [8], which settles the terminal case, is the anticanonical complex $\mathcal{A}_X^c$ associated with X, a polyhedral complex extending directly the features of the Fano polytope from toric geometry: X is terminal if and only if $\mathcal{A}_X^c$ has only the origin as an interior lattice point. This allows to bound the possible Cox ring data via the volumes of suitable lattice polytopes constructed out of the complex. In the log terminal Gorenstein case, one could think of proceeding analogously by using canonicity, which, however, appears to end in overflowing

computations, even when building on the classification of canonical threefold singularities with action of a two-dimensional torus provided in [9]. Instead we can benefit in a completely different way and much more directly from the Gorenstein property: it gives rise to unit fraction identities involving the Cox ring data that admit only a finite number of integral solutions; see Proposition 3.3 for a sample. Moreover, the computation of these integral solutions turns out to be easily feasible, which at the end makes the classification possible.

Remark 1.2. The following figure shows how the 538 families from the Classification 1.1 are distributed over the genus-codimension landscape of Fano threefolds presented in [10, Figure 1]:

Codim \ Genus	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	21	5																									
2	15	54	31																								
3				22																							
4			37		17																						
5						62																					
6				71																							
7								51																			
8									41																		
9										10																	
10											1																
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26																											3

Here the genus of a Fano threefold X is $h^0(X, -\mathcal{K}_X) - 2$ and the codimension is taken with respect to embedding into a weighted projective space by means of a minimal system of homogeneous generators of the anticanonical ring

$$A_X = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(X, -n\mathcal{K}_X).$$

The article is organized as follows. Section 2 serves to provide the necessary background on the approach to rational projective varieties X with a torus action of complexity one via the Cox ring based on [16, 19]. In Picard number one, this approach represents any family of $\mathbb{Q}$-factorial varieties X in terms of an integral matrix P. Very first constraints arise from log terminality: Proposition 2.24, originally due to [8], shows that log terminality leaves us with eight types of matrices P to consider. In Section 3, we exemplarily discuss one of these eight cases, showing basically all the necessary arguments. In particular, we see how to establish in several refining steps appropriate bounds on the entries of the defining matrix P making a computational treatment feasible. The full elaboration of all cases will be presented elsewhere. Section 4

presents the classification tables and in Section 5, we compute the Hilbert–Poincaré series of our varieties (also accessible via clicking on the items in the genus-codimension landscape).

2   Torus actions of complexity one

We recall the necessary background on rational varieties with a torus action of complexity one and fix our notation. The reader is assumed to be familiar with the very basics of toric geometry, in particular the correspondence between fans and toric varieties; see [12, 13, 14]. We restrict ourselves to spending just a few words on Cox's quotient presentation [11] of a toric variety arising from a fan.

Construction 2.1. Let Z be the toric variety defined by a fan Σ in a lattice N such that the primitive generators $v_1, \ldots, v_r$ of the rays of Σ span the rational vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. We have a linear map

$$P \colon \quad \mathbb{Z}^r \to N, \qquad e_i \mapsto v_i.$$

In case $N = \mathbb{Z}^n$, we also speak of the *generator matrix* $P = [v_1, \ldots, v_r]$ of Σ. The divisor class group and the Cox ring of Z are

$$\mathrm{Cl}(Z) = K := \mathbb{Z}^r / \mathrm{im}(P^*), \qquad \mathcal{R}(Z) = \mathbb{K}[T_1, \ldots, T_r], \qquad \deg(T_i) = Q(e_i),$$

where P^* denotes the dual map of P and $Q \colon \mathbb{Z}^r \to K$ the projection. Now, one defines a fan $\hat{\Sigma}$ in $\mathbb{Z}^r$ consisting of faces of the positive orthant of $\mathbb{Q}^r$ by

$$\hat{\Sigma} := \{\delta_0 \preceq \mathbb{Q}^r_{\geq 0}; \ P(\delta_0) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

The toric variety $\hat{Z}$ associated with $\hat{\Sigma}$ is an open toric subset in $\bar{Z} := \mathbb{K}^r$. As P is a map of the fans $\hat{\Sigma}$ and Σ, it defines a toric morphism $p \colon \hat{Z} \to Z$. The quasitorus

$$H = \mathrm{Spec}\, \mathbb{K}[K] = \ker(p) \subseteq \mathbb{T}^r = (\mathbb{K}^*)^r$$

acts as a subgroup of the torus $\mathbb{T}^r$ on $\hat{Z}$ and the morphism $p \colon \hat{Z} \to Z$ turns out to be a good quotient with respect to the H-action.

The quotient presentation of toric varieties is a central piece in the Cox ring based approach to rational varieties with a torus action of complexity one provided by [16, 19]; see also [4, Section 3.4]. The first step, however, is the following purely algebraic construction of a certain class of graded algebras; see [4, Construction 3.4.2.1] and more generally [17, Constructions 3.5 and 6.3].

Construction 2.2. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \cdots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The input data are matrices

$$A = [a_0, \ldots, a_r] \in \mathrm{Mat}(2, r+1; \mathbb{K}), \qquad P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix} \in \mathrm{Mat}(r+s, n+m; \mathbb{Z}),$$

where A has pairwise linearly independent columns and P is built from an $(s \times n)$-block d, an $(s \times m)$-block d' and an $(r \times n)$-block L of the shape

$$L = \begin{bmatrix} -l_0 & l_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \ldots & l_r \end{bmatrix}, \qquad l_i = (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geq 1}$$

such that the columns v_{ij}, v_k of P are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a cone. Consider the polynomial algebra

$$\mathbb{K}[T_{ij}, S_k] := \mathbb{K}[T_{ij}, S_k; \; 0 \le i \le r, 1 \le j \le n_i, 1 \le k \le m].$$

Denote by $\mathfrak{J}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \le i_1 < i_2 < i_3 \le r$ and define for any $I \in \mathfrak{J}$ a trinomial

$$g_I := g_{i_1, i_2, i_3} := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}, \qquad T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Consider the factor group $K := \mathbb{Z}^{n+m}/\operatorname{im}(P^*)$ and the projection $Q \colon \mathbb{Z}^{n+m} \to K$. We define a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) := \omega_{ij} := Q(e_{ij}), \qquad \deg(S_k) := \omega_k := Q(e_k).$$

Then the trinomials g_I just introduced are K-homogeneous and they all share the same K-degree. In particular, we obtain a K-graded factor algebra

$$R(A, P) := \mathbb{K}[T_{ij}, S_k]/\langle g_I; \; I \in \mathfrak{J} \rangle.$$

Example 2.3. We choose $r = 2$, moreover $n_0 = 2$, $n_1 = n_2 = 1$ and $m = 1$ and, finally $s = 2$. In this setting, consider the defining matrices

$$A := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad P := \begin{bmatrix} -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -2 & 4 & 0 & 0 \end{bmatrix}.$$

The algebra $R(A, P)$ arising from these matrices comes due to $r = 2$ with a single trinomial relation and is explicitly given by

$$R(A, P) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, S_1]/\langle T_{01}T_{02} + T_{11}^4 + T_{21}^2 \rangle.$$

We have $K = \mathbb{Z}^5/\operatorname{im}(P^*) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and the degrees of the T_{ij} and S_1 are the columns of the *degree matrix*

$$Q = \begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}.$$

Theorem 2.4 (see [4, Theorem 3.4.2.3], also [17, Theorems 3.10 and 6.5]). *The ring $R(A, P)$ produced by Construction 2.2 is a normal complete intersection ring and its ideal of relations is generated by the trinomials $g_i = g_{i,i+1,i+2}$, where $i = 0, \dots, r-2$.*

Remark 2.5. We call a defining matrix P *irredundant* if we have $l_{i1}n_i \ge 2$ for $i = 0, \dots, r$. Each $R(A, P)$ is isomorphic as a graded algebra to some $R(A', P')$ with P' irredundant. Note that for $r \ge 2$ and an irredundant P, the ring $R(A, P)$ is not a polynomial ring.

Remark 2.6. Consider a defining matrix P as in Construction 2.2. By an *admissible operation* on the matrix P we mean one of the following:

  (i) adding a multiple of one of the upper r rows to one of the lower s rows,

  (ii) applying a unimodular matrix from the left to the (d, d') block,

(iii) swapping two columns v_{ij_1} and v_{ij_2} inside a leaf $v_{i1}, \ldots, v_{in_i}$,

(iv) swapping two leafs $v_{i1}, \ldots, v_{in_i}$ and $v_{j1}, \ldots, v_{jn_j}$ and rearranging the L-block by elementary row operations into its required shape,

(v) swapping two columns v_{k_1} and v_{k_2} of the d'-block.

If P' arises from P via admissible operations, then with a suitable A', the graded rings $R(A, P)$ and $R(A', P')$ are isomorphic.

Remark 2.7. The matrix A of a ring $R(A, P)$ is responsible for the coefficients of the defining trinomials $g_i = g_{i,i+1,i+2}$. By rescaling variables we can always reduce to defining relations of the shape

$$T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \quad \lambda_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3}, \quad \ldots, \quad \lambda_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + T_r^{l_r}$$

with pairwise distinct $1 \neq \lambda_i \in \mathbb{C}^*$. In particular, in case of a single defining relation, there is no need to care about the coefficients. The matrix A is motivated by the geometry behind $R(A, P)$, see Remark 2.12.

We enter the second step, producing rational normal varieties X with torus action $\mathbb{T}^s \times X \to X$ of complexity one. Each of the resulting X comes embedded in a toric variety Z, is defined in homogeneous coordinates by the above trinomials $g_0, \ldots, g_{r-1}$ and the torus $\mathbb{T}^s$ acting on X is a subtorus of the acting torus $\mathbb{T}^{r+s}$ of Z. The original references are again [16, 19]; see also [4, Construction 3.4.3.6] as well as the more general [17, Constructions 3.5 and 6.13].

Construction 2.8. In the situation of Construction 2.2, assume $r \geq 2$ and that P is irredundant. Consider the common zero set of the defining relations of $R(A, P)$:

$$\bar{X} := V(g_I; \ I \in \mathfrak{J}) \subseteq \bar{Z} := \mathbb{K}^{n+m}.$$

Let Σ be any fan in $N = \mathbb{Z}^{r+s}$ having the columns of P as the primitive generators of its rays. Then $\hat{X} := \bar{X} \cap \hat{Z}$ and Construction 2.1 yield a commutative diagram

$$
\begin{array}{ccc}
\bar{X} & \subseteq & \bar{Z} \\
\cup\mathrm{I} & & \cup\mathrm{I} \\
\hat{X} & \longrightarrow & \hat{Z} \\
{\scriptstyle /\!/H}\downarrow{\scriptstyle p} & & {\scriptstyle p}\downarrow{\scriptstyle /\!/H} \\
X & \longrightarrow & Z,
\end{array}
$$

where $X := X(A, P, \Sigma) := p(\hat{X})$ is a non-toric, closed subvariety of the toric variety Z arising from Σ. Dimension, divisor class group and Cox ring of X are

$$\dim(X) = s + 1, \qquad \mathrm{Cl}(X) \cong K, \qquad \mathcal{R}(X) \cong R(A, P).$$

The subtorus $\mathbb{T}^s \subseteq \mathbb{T}^{r+s}$ of the acting torus of Z associated with the sublattice $\mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$ leaves X invariant and the induced T-action on X is of complexity one.

Example 2.9. We continue Example 2.3. Let Σ be the fan in $\mathbb{Z}^4$ having P as its generator matrix and the maximal cones

$$\mathrm{cone}(v_{02}, v_{11}, v_{21}, v_1), \quad \mathrm{cone}(v_{01}, v_{11}, v_{21}, v_1), \quad \mathrm{cone}(v_{01}, v_{02}, v_{21}, v_1),$$
$$\mathrm{cone}(v_{01}, v_{02}, v_{11}, v_1), \quad \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21}).$$

The associated toric variety Z is a four-dimensional fake weighted projective space with divisor class group

$$\mathrm{Cl}(Z) = K = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Moreover, $H = \mathbb{K}^* \times \{\pm 1\} \times \{\pm 1\}$ acts on $\bar{Z} = \mathbb{K}^5$ via the weights given by the columns of the degree matrix Q and Construction 2.8 becomes

$$
\begin{array}{ccccccc}
V\left(T_{01}T_{02} + T_{11}^4 + T_{21}^2\right) & = & \bar{X} & \subseteq & \bar{Z} & = & \mathbb{K}^5 \\
& & \cup\mathsf{I} & & \cup\mathsf{I} & & \cup\mathsf{I} \\
\bar{X} \setminus \{0\} & = & \hat{X} & \longrightarrow & \hat{Z} & = & \mathbb{K}^5 \setminus \{0\}. \\
& & {\scriptstyle /\!/ H}\downarrow{\scriptstyle p} & & {\scriptstyle p}\downarrow{\scriptstyle /\!/ H} & & \\
& & X & \longrightarrow & Z & &
\end{array}
$$

Theorem 2.10 (see [4, Theorem 4.4.1.6] and [17, Theorems 3.10 and 6.18]). *Every non-toric rational normal projective variety with a torus action of complexity one is equivariantly isomorphic to some $X(A, P, \Sigma)$ arising from Construction 2.2.*

Any variety $X = X(A, P, \Sigma)$ inherits many geometric properties from its ambient toric variety Z. A first observation concerns the restriction of the invariant divisors from Z to X; see [4, Proposition 3.2.4.5].

Remark 2.11. Consider $X = X(A, P, \Sigma)$ as in Construction 2.8. The columns v_{ij} and v_k of P define prime divisors $D_{ij} = V_Z(T_{ij})$ and $D_k = V_Z(T_k)$ on Z. The restrictions $D_{ij}^X = V_X(T_{ij})$ and $D_k^X = V_X(S_k)$ are prime divisors on X and in $\mathrm{Cl}(Z) = K = \mathrm{Cl}(X)$, we have

$$[D_{ij}] = \deg(T_{ij}) = \left[D_{ij}^X\right], \qquad [D_k] = \deg(T_k) = \left[D_k^X\right].$$

We recover the divisors D_{ij}^X as the components of the critical values $c_i \in \mathbb{P}_1$ of a certain quotient map; see [17, Proposition 3.16] for a general treatment.

Remark 2.12. Consider $X = X(A, P, \Sigma)$ as in Construction 2.8. Consider the open sets of points having finite isotropy groups with respect to the $\mathbb{T}^s$-action:

$$Z_0 = \{z \in Z; \ \mathbb{T}_z^s \text{ is finite}\}, \qquad X_0 = X \cap Z_0 = \{x \in X; \ \mathbb{T}_x^s \text{ is finite}\}.$$

Then $Z_0 \subseteq Z$ is invariant under the torus $\mathbb{T}^{r+s}$ and $X_0 \subseteq X$ is invariant under $\mathbb{T}^s$. Moreover, we have a commutative diagram

$$
\begin{array}{ccc}
X & \subseteq & Z \\
\cup\mathsf{I} & & \cup\mathsf{I} \\
X_0 & \longrightarrow & Z_0 \\
{\scriptstyle /\!/ \mathbb{T}^s}\downarrow{\scriptstyle \pi_X} & & {\scriptstyle \pi_Z}\downarrow{\scriptstyle /\!/ \mathbb{T}^s} \\
\mathbb{P}_1 & \longrightarrow & \mathbb{P}_r,
\end{array}
$$

where π_X and π_Z are categorical quotients with respect to the actions of $\mathbb{T}^s$ on X and Z respectively and π_Z is a toric morphism. Moreover, we obtain

$$\overline{\pi_X^{-1}(c_i)} = \bigcup_{j=1}^{n_i} D_{ij}^X \subseteq X, \qquad \overline{\pi_Z^{-1}(C_i)} = \bigcup_{j=1}^{n_i} D_{ij} \subseteq Z$$

with the toric divisors $C_0, \ldots, C_r \subseteq \mathbb{P}_r$ and the points $c_i \in \mathbb{P}_1$ having the i-th column of A as its homogeneous coordinates. Finally,

$$|\mathbb{T}^s_{x_{ij}}| = l_{ij}$$

holds for the order of the isotropy group $\mathbb{T}^s_{x_{ij}}$ of the action of the torus $\mathbb{T}^s$ at any general point $x_{ij} \in D^X_{ij}$.

The divisors from Remark 2.12 also allow an explicit presentation of an anticanonical divisor; see [4, Proposition 3.4.4.1].

Remark 2.13. Let $X = X(A, P, \Sigma)$ arise from Construction 2.8. Then the anticanonical divisor class of X is given as

$$-\mathcal{K}_X = \sum_{i,j} \deg(T_{ij}) + \sum_k \deg(S_k) - (r-1) \sum_{i=1}^{n_0} l_{0j} \deg(T_{0j}) \in K = \mathrm{Cl}(X).$$

In particular, due to $\deg(T_{ij}) = [D^X_{ij}]$ and $\deg(T_k) = [D^X_k]$, we have the following anticanonical divisor on X:

$$D^X_0 := \sum_{i,j} D^X_{ij} + \sum_k D^X_k - (r-1) \sum_{j=1}^{n_0} l_{0j} D^X_{0j}.$$

Example 2.14. For the variety X from Example 2.9, we can compute the anticanonical class as

$$-\mathcal{K}_X = \deg(T_{11}) + \deg(T_{21}) + \deg(S_1) = (4, \bar{0}, \bar{0}) \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = \mathrm{Cl}(X).$$

In particular, we see that the anticanonical class is ample and, consequently, X is a Fano variety.

Remark 2.15. Let $X = X(A, P, \Sigma)$ arise from Construction 2.8. We call $\sigma \in \Sigma$ an X-*cone* if the corresponding toric orbit $\mathbb{T}^{r+s} \cdot z_\sigma \subseteq Z$ meets $X \subseteq Z$. A cone $\sigma \in \Sigma$ is an X-cone if and only if one of the following holds:

   (i) σ is a *big cone*, that means $v_{0j_0}, \ldots, v_{rj_r} \in \sigma$ for some $j_0, \ldots, j_r$,

   (ii) σ is a *leaf cone*, that means $\sigma \subseteq \mathrm{cone}(v_{i1}, \ldots, v_{in_i}, v_1, \ldots, v_m)$ for some i.

Every X-cone $\sigma \in \Sigma$ defines an affine open subvariety $X_\sigma =: X \cap Z_\sigma$ in X by cutting down the corresponding affine toric chart $Z_\sigma \subseteq Z$. Note that X is covered by the X_σ, where σ runs through the X-cones of Σ.

Example 2.16. Consider again the variety X from Example 2.9. Then the fan Σ has exactly four maximal X-cones, namely

$$\mathrm{cone}(v_{02}, v_{11}, v_{21}, v_1), \qquad \mathrm{cone}(v_{01}, v_{11}, v_{21}, v_1),$$
$$\mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21}), \qquad \mathrm{cone}(v_{01}, v_{02}, v_1).$$

The first three are big cones, whereas the fourth one is a leaf cone. Thus, X is covered by four open affine subvarieties, given by the maximal X-cones of Σ.

Let us see how to detect Cartier divisors, that means locally principal Weil divisors, on a variety $X = X(A, P, \Sigma)$ in terms of the defining data.

Proposition 2.17. *Let $X = X(A, P, \Sigma)$ arise from Construction 2.8. Consider on Z and X the Weil divisors*

$$D = \sum a_{ij} D_{ij} + \sum a_k D_k, \qquad D^X = \sum a_{ij} D_{ij}^X + \sum a_k D_k^X,$$

in $K = \mathrm{Cl}(Z) = \mathrm{Cl}(X)$ the classes $\omega = [D] = [D^X]$, $\omega_{ij} = [D_{ij}] = [D_{ij}^X]$, $\omega_k = [D_k] = [D_k^X]$ and an X-cone $\sigma \in \Sigma$. Then the following statements are equivalent:

- *(i) The divisor D^X is Cartier on X_σ.*
- *(ii) The divisor D is Cartier on Z_σ.*
- *(iii) We have $D = \mathrm{div}(\chi^u)$ on Z_σ with a character χ^u of $\mathbb{T}^{r+s}$.*
- *(iv) There is $u \in \mathbb{Z}^{r+s}$ with $\langle u, v_{ij} \rangle = a_{ij}$ and $\langle u, v_k \rangle = a_k$ for all $v_{ij}, v_k \in \sigma$.*
- *(v) We have $\omega \in \langle \omega_{ij}, \omega_k; \ v_{ij}, v_k \notin \sigma \rangle$ in $K = \mathrm{Cl}(X)$.*

In particular, D is a Cartier divisor on X if and only if one of these conditions holds for all maximal X-cones $\sigma \in \Sigma$.

Proof. The equivalence of (i), (ii) and (v) follows from Proposition [4, Proposition 3.3.1.5]. The rest is basic toric geometry. ∎

A normal variety X is $\mathbb{Q}$-*factorial* if every Weil divisor D on X admits a Cartier multiple nD with $n \in \mathbb{Z}_{\geq 1}$.

Corollary 2.18. *A variety $X = X(A, P, \Sigma)$ as in Construction 2.8 is $\mathbb{Q}$-factorial if and only if each X-cone $\sigma \in \Sigma$ is simplicial.*

Now, recall that a variety is *Gorenstein* if its canonical class is Cartier. Combining Remark 2.13 and Proposition 2.17, we obtain the following characterization.

Corollary 2.19. *Consider $X = X(A, P, \Sigma)$ and let $D^X = \sum a_{ij} D_{ij}^X + \sum a_k D_k^X$ be an anticanonical divisor on X. Then X is Gorenstein if and only if for every maximal X-cone σ, there is a linear form $u \in \mathbb{Z}^{r+s}$ with*

$$\langle u, v_{ij} \rangle = a_{ij}, \qquad \langle u, v_k \rangle = a_k \qquad \text{for all} \quad v_{ij}, v_k \in \sigma.$$

Example 2.20. Consider again the variety X from Example 2.9 and the four maximal X-cones given in Example 2.16. Listed accordingly, we have linear forms

$$(2, 0, 1, -1), \quad (0, 0, 1, 1), \quad (-2, 2, -3, 0), \quad (-1, 1, 1, 0)$$

representing the anticanonical divisor D_0^X on the corresponding affine open subvarieties of X. In particular, X is Gorenstein.

If X is a $\mathbb{Q}$-factorial Fano variety of Picard number one, then, the divisor class group $\mathrm{Cl}(X)$ allows a positive splitting into a free cyclic part and its torsion part Γ, that means that we have an isomorphism

$$\mathrm{Cl}(X) \cong \mathbb{Z} \oplus \Gamma$$

such that for the anticanonical class $\omega_X = (w_X, \eta_X)$, the $\mathbb{Z}$-part w_X is positive. Note that in this setting the $\mathbb{Z}$-part of any divisor class $\omega = (w, \eta)$ does not depend on the particular choice of the splitting.

Corollary 2.21. *Let $X = X(A, P, \Sigma)$ be $\mathbb{Q}$-factorial, Gorenstein, Fano and of Picard number one. Then, for every maximal X-cone σ, the $\mathbb{Z}$-parts w_{ij}, w_k of the generator degrees and w_X of the anticanonical class satisfy*

$$\gcd(w_{ij}, w_k; \ v_{ij} \notin \sigma, v_k \notin \sigma) \mid w_X.$$

Proof. As X is Gorenstein, the canonical class ω_X represents a Cartier divisor. Proposition 2.17 tells us that for every maximal X-cone σ, the ω_X lies in the subgroup of $\mathrm{Cl}(X) = K$ generated by the classes ω_{ij}, ω_k, where $v_{ij} \notin \sigma$, $v_k \notin \sigma$. Thus, the $\mathbb{Z}$-part w_X lies in the ideal of $\mathbb{Z}$ generated by the $\mathbb{Z}$-parts w_{ij}, w_k, where $v_{ij} \notin \sigma$, $v_k \notin \sigma$. The assertion follows. ∎

Finally, we discuss log terminality. Recall that given any resolution of singularities $\pi \colon X' \to X$ of a normal variety, we have the ramification formula

$$\mathcal{K}_{X'} - \pi^* \mathcal{K}_X = \sum_{i=1}^{r} a_i E_i$$

with canonical divisors on X' and X and the exceptional divisors $E_i, \ldots, E_r$. Then X is called *log terminal* if we have $a_i > -1$ for $i = 1, \ldots, r$.

We characterize log terminality of a given $\mathbb{Q}$-factorial Fano variety $X = X(A, P, \Sigma)$. A *platonic tuple* is a tuple $(l_0, \ldots, l_r)$ of positive integers such that after re-ordering the l_i decreasingly, we obtain a tuple $(a, b, c, 1, \ldots, 1)$ with (a, b, c) one of

$$(x, y, 1), \quad (y, 2, 2), \quad (5, 3, 2), \quad (4, 3, 2), \quad (3, 3, 2).$$

Proposition 2.22 (see [3, Theorem 3.13]). *A $\mathbb{Q}$-factorial Fano variety $X = X(A, P, \Sigma)$ has at most log terminal singularities if and only if for any X-cone $\sigma = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$ the exponents $l_{0j_0}, \ldots, l_{rj_r}$ form a platonic tuple.*

Example 2.23. For the variety $X = X(A, P, \Sigma)$ from Example 2.9 we have to consider the X-cones

$$\mathrm{cone}(v_{02}, v_{11}, v_{21}), \qquad \mathrm{cone}(v_{01}, v_{11}, v_{21}).$$

Both of them yield the exponent tuple $(1, 4, 2)$ which is platonic. Consequently, X is log terminal.

Log terminality leads to the following first constraints on the defining matrix P of our Fano varieties $X = X(A, P, \Sigma)$.

Proposition 2.24 (see [8, Lemma 5.2]). *Let $X = X(A, P, \Sigma)$ a non-toric, $\mathbb{Q}$-factorial, log terminal Fano threefold of Picard number one, where P is irredundant. Then, after suitable admissible operations, P fits into one of the following cases:*

   (i) *$m = 0$, $r = 2$ and $n = 5$, where $n_0 = n_1 = 2$, $n_2 = 1$,*

   (ii) *$m = 0$, $r = 3$ and $n = 6$, where $n_0 = n_1 = 2$, $n_2 = n_3 = 1$,*

  (iii) *$m = 0$, $r = 4$ and $n = 7$, where $n_0 = n_1 = 2$, $n_2 = n_3 = n_4 = 1$,*

  (iv) *$m = 0$, $r = 2$ and $n = 5$, where $n_0 = 3$, $n_1 = n_2 = 1$,*

   (v) *$m = 0$, $r = 3$ and $n = 6$, where $n_0 = 3$, $n_1 = n_2 = n_3 = 1$,*

  (vi) *$m = 1$, $r = 2$ and $n = 4$, where $n_0 = 2$, $n_1 = n_2 = 1$,*

 (vii) *$m = 1$, $r = 3$ and $n = 5$, where $n_0 = 2$, $n_1 = n_2 = n_3 = 1$,*

(viii) *$m = 2$, $r = 2$ and $n = 3$, where $n_0 = n_1 = n_2 = 1$.*

Remark 2.25. Every rational Gorenstein del Pezzo surface has at most canonical singularities and thus is in particular log terminal; see [20]. For Gorenstein Fano varieties of higher dimension even the latter property need not hold. For instance,

$$P = \begin{bmatrix} -3 & -1 & 3 & 1 & 0 \\ -3 & -1 & 0 & 0 & k \\ -4 & -1 & 1 & 0 & k \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

defines for each $k \geq 4$ a $\mathbb{Q}$-factorial Fano threefold $X = X(A, P, \Sigma)$ of Picard number one, which is not log terminal by Proposition 2.22. More explicitly, Σ consists of all pointed cones generated by columns of P and we have

$$X = V\left(T_{01}^3 T_{02} + T_{11}^3 T_{12} + T_{21}^k\right) \subseteq \mathbb{P}_{1, k-3, 1, k-3, 1},$$

$$[-\mathcal{K}_X] = k - 3 \in \mathbb{Z} = \mathrm{Cl}(X).$$

This example series shows moreover that the Gorenstein and Fano condition together are even in the specific setting of threefolds with an action of a two-dimensional torus not enough to guarantee finiteness in fixed dimension and Picard number.

3 Elaboration of a sample case

Proposition 2.24 divides the proof of the classification theorem into cases (i) to (viii). Here we give a treatment of case (vi) as a sample, showing all types of arguments of the full proof. The setting is the following.

Setting 3.1. Let $X = X(A, P, \Sigma)$ be a $\mathbb{Q}$-factorial Fano threefold of Picard number one with $r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$ and $m = 1$. Then

$$P = [v_{01}, v_{02}, v_{11}, v_{21}, v_1] = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} & 0 \\ d_{011} & d_{021} & d_{111} & d_{211} & d_{11} \\ d_{012} & d_{022} & d_{112} & d_{212} & d_{12} \end{bmatrix}$$

with pairwise different primitive columns $v_{01}, v_{02}, v_{11}, v_{21}$ and v_1 generating $\mathbb{Q}^4$ as a cone. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_{01} = \mathrm{cone}(v_{02}, v_{11}, v_{21}, v_1), \qquad \sigma_{02} = \mathrm{cone}(v_{01}, v_{11}, v_{21}, v_1),$$
$$\sigma_1 = \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21}), \qquad \tau_0 = \mathrm{cone}(v_{01}, v_{02}, v_1).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 = [w_{01}, w_{02}, w_{11}, w_{21}, w_1]$$

for the $\mathbb{Z}$-part of the degree matrix Q. Note that the vector $(w_{01}, w_{02}, w_{11}, w_{21}, w_1)$ is primitive in $\mathbb{Z}^5$ and generates $\ker(P)$.

Our first series of constraints arising from the log terminality and the Gorenstein property directly aims for entries of the defining matrix P.

Proposition 3.2. *Consider $X = X(A, P, \Sigma)$ as in Setting 3.1. Assume that X is non-toric, log terminal and Gorenstein.*

(i) *$D_0^X = (1 - l_{01})D_{01}^X + (1 - l_{02})D_{02}^X + D_{11}^X + D_{21}^X + D_1^X$ is an anticanonical divisor on X. In particular, the free part of the anticanonical divisor class is given by*

$$w_X = (1 - l_{01})w_{01} + (1 - l_{02})w_{02} + w_{11} + w_{21} + w_1.$$

(ii) *Admissible column and row operations turn the defining matrix P into the shape*

$$P = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} & 0 \\ 1 - l_{01} & 1 - l_{02} & d_{111} & d_{211} & 1 \\ d_{012} & d_{022} & d_{112} & d_{212} & 0 \end{bmatrix},$$

$$l_{01} \geq l_{02} \geq 1, \qquad l_{11} \geq l_{21} \geq 2,$$

$$0 \leq d_{211}, d_{212} < l_{21}, \qquad 0 \leq d_{012} < l_{01},$$

$$-\frac{w_X}{w_{02}} \leq d_{022} < l_{02} + \frac{w_X}{w_{02}},$$

where $w_{02} \mid w_X$ and the tuple of exponents (l_{01}, l_{11}, l_{21}) fits into precisely one of the following constellations:

$(1, x, y),$	$x \geq y > 1;$	$(2, z, 3),$	$3 \leq z \leq 5;$
$(y, 2, 2),$	$y \geq 2;$	$(3, z, 2),$	$3 \leq z \leq 5;$
$(2, y, 2),$	$y \geq 3;$	$(z, 3, 2),$	$4 \leq z \leq 5.$

Proof. For the first assertion, note that we have $r = 2$ and that the defining relation of the Cox ring is given as

$$g = T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}.$$

Thus, $\mu_X = \deg(g) = l_{01} \deg(T_{01}) + l_{02} \deg(T_{02})$ and Remark 2.13 shows that the anticanonical divisor D_0^X is as claimed.

We prove (ii). Swapping, if necessary, the first two columns of P, we achieve $l_{01} \geq l_{02}$. Similarly, exchanging the data of v_{11} and v_{21}, we ensure $l_{11} \geq l_{21}$. Since X is non-toric, we must have $l_{21} > 1$. As we assume X to be log terminal, we can apply Remark 2.22 to $\mathrm{cone}(v_{01}, v_{11}, v_{21})$, showing that (l_{01}, l_{11}, l_{21}) is as in the assertion.

We care about the entries of the (d, d') block of P. Since $v_1 \in \mathbb{Z}^4$ is primitive, we can apply a suitable unimodular 2×2 matrix from the left to the (d, d') block to ensure

$$d_{11} = 1, \qquad d_{12} = 0.$$

We begin to make use of the assumption that X is Gorenstein. First consider the X-cone $\tau_0 = \mathrm{cone}(v_{01}, v_{02}, v_1)$. Then Corollary 2.19 provides a linear form $u \in \mathbb{Z}^4$ such that

$$\langle u, v_{01} \rangle = 1 - l_{01}, \qquad \langle u, v_{02} \rangle = 1 - l_{02}, \qquad \langle u, v_1 \rangle = 1.$$

The last equation tells us in particular $u_3 = 1$. Plugging this into the first two equations yields

$$d_{011} = l_{01}(u_1 + u_2) - u_4 d_{012} + 1 - l_{01}, \qquad d_{021} = l_{02}(u_1 + u_2) - u_4 d_{022} + 1 - l_{02}.$$

Thus, adding the $(u_1 + u_2)$-fold of the first and the u_4-fold of the fourth row of P to the third one, we obtain

$$d_{011} = 1 - l_{01}, \qquad d_{021} = 1 - l_{02}.$$

Moreover, adding an appropriate multiple of the first row of P to the fourth one, we achieve

$$0 \leq d_{012} < l_{01}.$$

Now consider the maximal X-cone $\sigma_{02} = \text{cone}(v_{01}, v_{11}, v_{21}, v_1)$. Let $u \in \mathbb{Z}^4$ be a linear form representing D_0^X on $X_{\sigma_{02}}$ according to Corollary 2.19(iii). Then

$$0 = Q^0 \cdot P^* \cdot u = \sum \langle u, v_{ij} \rangle w_{ij} + \langle u, v_1 \rangle w_1 = w_X + (u_4 d_{022} - l_{02}(u_1 + u_2)) w_{02}.$$

In particular, we see that w_{02} divides w_X. Moreover, we must have $u_3 = 1$. We obtain

$$1 - l_{01} = \langle u, v_{01} \rangle = -l_{01} u_1 - l_{01} u_2 + 1 - l_{01} + u_4 d_{012}.$$

This merely means $l_{01}(u_1 + u_2) = u_4 d_{012}$. Plugging this into the previous equation yields

$$-l_{01} \frac{w_X}{w_{02}} = u_4 (d_{022} l_{01} - d_{012} l_{02}).$$

Thus, $(d_{022} l_{01} - d_{012} l_{02})$ divides $l_{01} \frac{w_X}{w_{02}}$. As a consequence, we can estimate d_{022} as follows:

$$l_{02} \frac{d_{012}}{l_{01}} - \frac{w_X}{w_{02}} \leq d_{022} \leq l_{02} \frac{d_{012}}{l_{01}} + \frac{w_X}{w_{02}}.$$

Combining this with $0 \leq d_{012} < l_{01}$, we arrive at the desired bounds for d_{022}. Finally, we achieve

$$0 \leq d_{211}, d_{212} < l_{21}$$

by adding suitable multiples of the difference of the first two rows of P to third and the fourth one. ∎

The second series of constraints shows that all entries of the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{02}, w_{11}, w_{21}, w_1]$ are bounded.

Proposition 3.3. *Let $X = X(A, P, \Sigma)$ be as in Setting 3.1. Assume that X is non-toric, log terminal and Gorenstein.*

(i) *Let α_{01}, α_{02} and β_1 be any three positive integers and consider the 5×5 matrix*

$$G := \begin{bmatrix} 1 - l_{01} - \alpha_{01} & 1 - l_{02} & 1 & 1 & 1 \\ 1 - l_{01} & 1 - l_{02} - \alpha_{02} & 1 & 1 & 1 \\ 1 - l_{01} & 1 - l_{02} & 1 & 1 & 1 - \beta_1 \\ -l_{01} & -l_{02} & l_{11} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} & 0 \end{bmatrix}.$$

Then G is of rank at least four. Moreover, $\det(G) = 0$ if and only if $\alpha_{01}, \alpha_{02}, \beta_1$ and $l_{01}, l_{02}, l_{11}, l_{21}$ satisfy the identity

$$\frac{1}{\beta_1} + \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \left(\frac{l_{01}}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} \right) \left(\frac{1}{l_{11}} + \frac{1}{l_{21}} - 1 \right) = 1.$$

(ii) *There are unique $\alpha_{01}, \alpha_{02}, \beta_1 \in \mathbb{Z}_{\geq 1}$ with $\alpha_{01}w_{01} = \alpha_{02}w_{02} = \beta_1 w_1 = w_X$ and the corresponding matrix G from (i) satisfies*

$$\ker(G) = \ker(P) = \mathbb{Z} \cdot (w_{01}, w_{02}, w_{11}, w_{21}, w_1).$$

(iii) *According to the possible constellations of (l_{01}, l_{11}, l_{02}) from Proposition 3.2(ii) we have the following bounds for the entries of the matrix G:*

(l_{01}, l_{11}, l_{21})	bounds for $(l_{01}, l_{02}, l_{11}, l_{21}, \alpha_{01}, \alpha_{02}, \beta_1)$
$(1, x, y)$	$(1, 1, 30, 8, 35, 4, 36)$
$(y, 2, 2)$	$(11, 6, 2, 2, 6, 6, 6)$
$(2, y, 2)$	$(2, 2, 18, 2, 28, 10, 30)$
$(2, z, 3)$	$(2, 2, 4, 3, 4, 4, 6)$
$(3, z, 2)$	$(3, 3, 5, 2, 12, 5, 24)$
$(z, 3, 2)$	$(5, 4, 3, 2, 14, 5, 18)$

Proof. We verify (i). In order to see that G is of rank at least four, we just compute the minor

$$G_{3,1} = \det \begin{bmatrix} 1 - l_{02} & 1 & 1 & 1 \\ 1 - l_{02} - \alpha_{02} & 1 & 1 & 1 \\ -l_{02} & l_{11} & 0 & 0 \\ -l_{02} & 0 & l_{21} & 0 \end{bmatrix} = \alpha_{02} l_{11} l_{21} \neq 0.$$

Moreover, suitably rearranging the equation $\det(G) = 0$, we arrive at the displayed identity on α_{01}, α_{02}, β_1 and l_{01}, l_{02}, l_{11}, l_{21}.

We show (ii). Applying Corollary 2.21 to the three maximal X-cones σ_{01}, σ_{02} and σ_1 we see that each of w_{01}, w_{02} and w_1 is a multiple of w_X and hence we obtain positive integers α_{01}, α_{02} and β_1 with

$$\alpha_{01}w_{01} = \alpha_{02}w_{02} = \beta_1 w_1 = (1 - l_{01})w_{01} + (1 - l_{02})w_{02} + w_{11} + w_{21} + w_1.$$

Moreover, by homogeneity of the defining relation g we have

$$l_{01}w_{01} + l_{02}w_{02} = l_{11}w_{11} = l_{21}w_{21}.$$

Now, G from (i) is the coefficient matrix of the corresponding system of linear equations. In particular, for any choice of α_{01}, α_{02} and β_1 the integral matrix G has kernel generated by the primitive vector $(w_{01}, w_{02}, w_{11}, w_{21}, w_1) \in \mathbb{Z}^5$.

We turn to (iii). We exemplarily treat the configuration $(l_{01}, l_{11}, l_{21}) = (1, x, y)$, where $x \geq y \geq 2$. In this situation the condition $\det(G) = 0$ reads as

$$\frac{1}{\beta_1} + \left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} \right) \left(\frac{1}{x} + \frac{1}{y} \right) = 1.$$

We may assume $\alpha_{01} \geq \alpha_{02}$. We directly see $\beta_1 \geq 2$. We arrive at 16 possible choices for α_{02} and y given by the conditions

$$1 \leq \alpha_{02} \leq 4, \qquad 2 \leq y \leq 8, \qquad \alpha_{02} \leq 2 \quad \text{or} \quad y = 2,$$

We establish preliminary bounds. First observe that the equation $\det(G) = 0$ can be resolved for each of the variables. For instance, we have

$$x = \frac{(\alpha_{01} + \alpha_{02})\beta_1 y}{\alpha_{01}\alpha_{02}\beta_1 y - \alpha_{01}\alpha_{02}y - \alpha_{01}\beta_1 - \alpha_{02}\beta_1}.$$

Plugging the above 16 choices for $a = \alpha_{02}$ and $b = \beta_1$ into this expression, we obtain more explicit presentations for the possible values of x:

$$
\begin{array}{cccc}
\dfrac{2+2a}{\left(1-\frac{2}{b}\right)a-1}, & \dfrac{4+2a}{\left(3-\frac{4}{b}\right)a-2}, & \dfrac{6+2a}{\left(5-\frac{6}{b}\right)a-3}, & \dfrac{8+2a}{\left(7-\frac{8}{b}\right)a-4}, \\[3mm]
\dfrac{3+3a}{\left(2-\frac{3}{b}\right)a-1}, & \dfrac{6+3a}{\left(5-\frac{6}{b}\right)a-2}, & \dfrac{4+4a}{\left(3-\frac{4}{b}\right)a-1}, & \dfrac{8+4a}{\left(7-\frac{8}{b}\right)a-2}, \\[3mm]
\dfrac{5+5a}{\left(4-\frac{5}{b}\right)a-1}, & \dfrac{10+5a}{\left(9-\frac{10}{b}\right)a-2}, & \dfrac{6+6a}{\left(5-\frac{6}{b}\right)a-1}, & \dfrac{12+6a}{\left(11-\frac{12}{b}\right)a-2}, \\[3mm]
\dfrac{14+7a}{\left(6-\frac{7}{b}\right)a-1}, & \dfrac{14+7a}{\left(13-\frac{14}{b}\right)a-2}, & \dfrac{8+8a}{\left(7-\frac{8}{b}\right)a-1}, & \dfrac{16+8a}{\left(15-\frac{16}{b}\right)a-2}.
\end{array}
$$

Each of these expressions can be maximized, having in mind $a, b, x \in \mathbb{Z}_{\geq 1}$. We arrive at $x \leq 48$, attained by the first expression with $b = 3$ and $a = 4$.

In a similar way, we obtain preliminary bounds for α_{01} and β_1. Now, running $\det(G) = 0$ with the preliminary bounds for the involved variables, we arrive at the bounds from the assertion. ∎

Besides constraints on the defining data, we also need criteria to decide computationally whether or not given defining data lead to isomorphic varieties. For this, we say that a defining matrix P as in Construction 2.8 has *ordered exponents* if we have

(i) $n_0 \geq \cdots \geq n_r$,

(ii) $l_{i1} \geq \cdots \geq l_{in_i}$ for each $i = 0, \ldots, r$ and

(iii) if $n_i = n_{i+1}$ then $l_{i1} \geq l_{i+1,1}$.

If P has ordered exponents, then we call the data $(n_0, \ldots, n_r, m)$ the *format* of P. Note that via admissible operations, we can always assume that P has ordered exponents.

Proposition 3.4. *Let (A, P, Σ) and (A', P', Σ') be as in Construction 2.8 such that the associated varieties X and X' are isomorphic to each other.*

(i) *There is an isomorphism $\varphi \colon X \to X'$ which is equivariant with respect to the torus actions.*

(ii) *If P and P' have ordered exponents, then they share the same format and for each i there is an i' with $n_{i'} = n_i$ and $(l_{i1}, \ldots, l_{in_i}) = (l'_{i'1}, \ldots, l'_{i'n_i})$ such that*

$$\langle \deg(T_{ij}); \ j = 1, \ldots, n_i \rangle \cong \langle \deg(T_{i'j}); \ j = 1, \ldots, n_i \rangle$$

holds for the subgroups in $\mathrm{Cl}(X)$ and $\mathrm{Cl}(X')$, respectively, generated by the corresponding degrees.

Proof. For the first assertion, observe that for any isomorphism $\varphi \colon X \to X'$ of varieties, we can install a torus action on X' making φ equivariant. Now, any torus action of complexity one on the non-toric X' corresponds to a maximal torus in the affine algebraic group $\mathrm{Aut}(X')$; see for instance [5, Theorem 2.1]. Thus, the assertion follows from the fact that any two maximal tori in an affine algebraic group are conjugate. The second assertion follows from the first one and the fact that any equivariant isomorphism respects the data described in Remark 2.12. ∎

This allows us in particular to associate with any $X \cong X(A, P, \Sigma)$ a *format* $(n_0, \ldots, n_r, m)$ by taking P with ordered exponents. Bringing together our constraints on the entries of P, the bounds on the entries of the matrix G and the distinction criterion just shown allows us to perform the classification computationally.

Corollary 3.5. *There is a list of* 155 *explicitly given matrices* P *of format* $(2, 1, 1, 1)$, *each of them defining a non-toric* $\mathbb{Q}$-*factorial, Gorenstein, log terminal Fano threefold* $X(A, P, \Sigma)$ *of Picard number one.*

Number of members P of the list according to divisor class group and exponent configuration.

	$\mathbb{Z}$	$\mathbb{Z} + \mathbb{Z}_2$	$\mathbb{Z} + \mathbb{Z}_3$	$\mathbb{Z} + \mathbb{Z}_4$	$\mathbb{Z} + \mathbb{Z}_5$	$\mathbb{Z} + \mathbb{Z}_6$	$\mathbb{Z} + \mathbb{Z}_2^2$	$\mathbb{Z} + \mathbb{Z}_2 + \mathbb{Z}_4$	$\mathbb{Z} + \mathbb{Z}_2^3$	sum
$(1, x, y)$	13	19	6	4	1	3	4	2		52
$(y, 2, 2)$		12		3			9		1	25
$(2, y, 2)$	13	24					8		1	46
$(2, z, 3)$	2		3			1				6
$(3, z, 2)$	7	6					1			14
$(z, 3, 2)$	5	6					1			12
sum	40	67	9	7	1	4	23	2	2	155

Distinct matrices from the list yield non-isomorphic varieties and every non-toric, $\mathbb{Q}$-*factorial, log terminal and Gorenstein Fano threefold of Picard number one of format* $(2, 1, 1, 1)$ *is isomorphic to an* $X = X(A, P, \Sigma)$ *with* P *from the list.*

Proof. Proposition 3.3 allows us to write down explicitly all possible matrices G and hence to determine all possible $Q^0 = [w_{01}, w_{02}, w_{11}, w_{21}, w_1]$ by computer. Now, recall that P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 3.2, all the remaining variables. So, we are left with a finite list of explicitly given possible defining matrices P. Checking for the necessary properties by means of [18] and reducing via Proposition 3.4 to data defining pairwise non-isomorphic varieties, we obtain the list presented in the assertion. ∎

The remaining cases from Proposition 2.24 are treated analogously to our sample case, using basically the same arguments in the details. We give a summarizing overview on the strategy. The explicit elaboration will be presented elsewhere.

Remark 3.6. Each of the cases (i) to (v) and (vii), $(viii)$ from Proposition 2.24 can be treated according to the following pattern.

- Establish the analogue of Setting 3.1 and Proposition 3.2. Observe that each row of the resulting matrix P admits at most one entry, which is not bounded by other entries of P.

- With each matrix P obtained so far associate a matrix G as in Proposition 3.3 by following the lines of the proof of Proposition 3.3(ii). Bound the entries of G, using that G is not of full rank as in the proof of Proposition 3.3(iii).

- Go through the possible values of G, compute $Q^0 = \ker(G) = \ker(P)$ explicitly and use this identity to determine the remaining unbounded values of P, one per row, as mentioned.

- From the resulting list of explicit matrices P remove those not defining a Gorenstein Fano variety and remove redundancies using Proposition 3.4.

In the present classification, we comfortably get along by establishing bounds for the integral solutions of the unit fraction identities such as $\det(G) = 0$ in Proposition 3.3 and naively going through the possibilities. For more extensive projects, for instance, higher Gorenstein indices or higher dimension, it turns out that establishing algorithms for the computation of specific unit fraction decompositions is considerably more efficient; see [6, 15].

4 Classification lists

Here we provide the detailed presentation of our classification result. Let us briefly recall the background. Each non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold X of Picard number one coming with an effective action of a two-dimensional torus is uniquely determined by its Cox ring. In particular, X can be encoded by the degree matrix Q, that means the list of degrees of the Cox ring generators in $\mathrm{Cl}(X)$ and the defining trinomial relations $g_0, \ldots, g_{r-1}$. For instance, our example variety X from Examples 2.3, 2.9, 2.16 and 2.20 is encoded by

$$Q = \begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}, \qquad g_0 = T_1 T_2 + T_3^4 + T_4^2,$$

where the columns of Q live in $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Indeed, the defining matrix P is determined up to admissible operations by Q, the format $(2,1,1,1)$ and the list of exponents of g_0. Alternatively, X is the hypersurface defined by g_0 in the fake weighted projective space $Z = \hat{Z}/H$, where $\hat{Z} = \mathbb{K}^5 \setminus \{0\}$ and the quasitorus H and its action on $\mathbb{K}^5$ are given by

$$H = \mathbb{K}^* \times \{\pm 1\} \times \{\pm 1\}, \qquad (t, \zeta, \eta) \cdot z = \left(t^2 \zeta \eta z_1, t^2 \zeta \eta z_2, t \eta z_3, t^2 \eta z_4, t z_5 \right).$$

We turn to the classification lists. Every non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold X of Picard number one coming with an effective action of a two-dimensional torus is isomorphic to precisely one of the listed varieties. Conversely, each of the listed data defines a non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold X of Picard number one coming with an effective action of a two-dimensional torus.

Each of the lists represents a possible divisor class group and format. Each variety in such a list is specified by its matrix Q of generator degrees and its defining trinomial relations; observe that we number the variables of the relation conventionally and not in accordance with the double-indexed enumeration of the columns of the associated defining matrix P. Besides the specifying data, we list the anticanonical self intersection. A file containing also the defining matrices P and further invariants is available at [7].

Classification list 4.1. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z}$ and format $(2, 2, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
10	$T_1^5 T_2 + T_3^3 T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \end{bmatrix}$	2	32	$T_1^4 T_2 + T_3^2 T_4^2 + T_5^3$	$\begin{bmatrix} 1 & 2 & 2 & 1 & 2 \end{bmatrix}$	6
11	$T_1^4 T_2^3 + T_3^3 T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 5 \end{bmatrix}$	2	33	$T_1^5 T_2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \end{bmatrix}$	6
14	$T_1^5 T_2 + T_3^3 T_4^2 + T_5^2$	$\begin{bmatrix} 2 & 4 & 4 & 1 & 7 \end{bmatrix}$	4	34	$T_1^4 T_2^2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \end{bmatrix}$	6
15	$T_1^3 T_2^2 + T_3^3 T_4^2 + T_5^2$	$\begin{bmatrix} 2 & 4 & 4 & 1 & 7 \end{bmatrix}$	4	35	$T_1^3 T_2 + T_3^2 T_4^2 + T_5^3$	$\begin{bmatrix} 2 & 6 & 3 & 3 & 4 \end{bmatrix}$	6
16	$T_1^3 T_2^2 + T_3^3 T_4^2 + T_5^2$	$\begin{bmatrix} 2 & 1 & 2 & 1 & 4 \end{bmatrix}$	4	36	$T_1^3 T_2^2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 1 & 6 & 6 & 3 & 5 \end{bmatrix}$	6
17	$T_1^5 T_2^3 + T_3^3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 4 \end{bmatrix}$	4	37	$T_1^3 T_2^2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 2 & 3 & 3 & 6 & 4 \end{bmatrix}$	6
18	$T_1^3 T_2 + T_3^3 T_4^2 + T_5^2$	$\begin{bmatrix} 4 & 2 & 4 & 1 & 7 \end{bmatrix}$	4	38	$T_1^{16} T_2 + T_3^2 T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 6 & 6 & 9 \end{bmatrix}$	6
31	$T_1^3 T_2 + T_3^3 T_4 + T_5^2$	$\begin{bmatrix} 3 & 1 & 3 & 1 & 5 \end{bmatrix}$	6	39	$T_1^{12} T_2^3 + T_3^2 T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 6 & 6 & 9 \end{bmatrix}$	6

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
40	$T_1^8 T_2^5 + T_3^2 T_4 + T_5^2$	$[1\ 2\ 6\ 6\ 9]$	6	88	$T_1^2 T_2 + T_3^2 T_4 + T_5^2$	$[3\ 2\ 3\ 2\ 4]$	12
41	$T_1^7 T_2^4 + T_3^2 T_4 + T_5^2$	$[2\ 1\ 6\ 6\ 9]$	6	89	$T_1^{21} T_2 + T_3 T_4 + T_5^3$	$[1\ 3\ 12\ 12\ 8]$	12
42	$T_1^4 T_2 + T_3^2 T_4 + T_5^2$	$[3\ 2\ 6\ 2\ 7]$	6	90	$T_1^{18} T_2^2 + T_3 T_4 + T_5^3$	$[1\ 3\ 12\ 12\ 8]$	12
52	$T_1^4 T_2 + T_3^3 T_4^3 + T_5^2$	$[1\ 2\ 1\ 1\ 3]$	8	91	$T_1^{12} T_2^4 + T_3 T_4 + T_5^3$	$[1\ 3\ 12\ 12\ 8]$	12
53	$T_1^4 T_2^3 + T_3^3 T_4 + T_5^2$	$[1\ 2\ 2\ 4\ 5]$	8	92	$T_1^{10} T_2 + T_3 T_4 + T_5^4$	$[1\ 2\ 6\ 6\ 3]$	12
54	$T_1^{11} T_2 + T_3^2 T_4 + T_5^2$	$[1\ 1\ 4\ 4\ 6]$	8	93	$T_1^9 T_2^5 + T_3 T_4 + T_5^3$	$[1\ 3\ 12\ 12\ 8]$	12
55	$T_1^9 T_2^3 + T_3^2 T_4 + T_5^2$	$[1\ 1\ 4\ 4\ 6]$	8	94	$T_1^9 T_2 + T_3 T_4 + T_5^6$	$[1\ 3\ 6\ 6\ 2]$	12
56	$T_1^8 T_2 + T_3^2 T_4 + T_5^2$	$[1\ 2\ 4\ 2\ 5]$	8	95	$T_1^7 T_2^3 + T_3 T_4 + T_5^3$	$[3\ 1\ 12\ 12\ 8]$	12
57	$T_1^7 T_2^5 + T_3^2 T_4 + T_5^2$	$[1\ 1\ 4\ 4\ 6]$	8	96	$T_1^6 T_2^3 + T_3 T_4 + T_5^4$	$[1\ 2\ 6\ 6\ 3]$	12
58	$T_1^4 T_2^3 + T_3^2 T_4 + T_5^2$	$[1\ 2\ 4\ 2\ 5]$	8	97	$T_1^5 T_2^2 + T_3 T_4 + T_5^4$	$[2\ 1\ 6\ 6\ 3]$	12
65	$T_1^3 T_2 + T_3^2 T_4 + T_5^5$	$[1\ 2\ 2\ 1\ 1]$	10	98	$T_1^3 T_2^2 + T_3 T_4 + T_5^{12}$	$[2\ 3\ 6\ 6\ 1]$	12
66	$T_1^2 T_2 + T_3^2 T_4 + T_5^5$	$[2\ 1\ 2\ 1\ 1]$	10	99	$T_1^3 T_2 + T_3 T_4 + T_5^3$	$[4\ 3\ 3\ 12\ 5]$	12
67	$T_1^{15} T_2 + T_3 T_4 + T_5^5$	$[1\ 5\ 10\ 10\ 4]$	10	100	$T_1^2 T_2 + T_3 T_4 + T_5^2$	$[3\ 4\ 4\ 6\ 5]$	12
68	$T_1^{10} T_2^2 + T_3 T_4 + T_5^5$	$[1\ 5\ 10\ 10\ 4]$	10	106	$T_1^4 T_2 + T_3^3 T_4^2 + T_5^2$	$[1\ 4\ 2\ 1\ 4]$	16
69	$T_1^5 T_2^3 + T_3 T_4 + T_5^5$	$[1\ 5\ 10\ 10\ 4]$	10	107	$T_1^3 T_2 + T_3^3 T_4^2 + T_5^2$	$[2\ 8\ 4\ 1\ 7]$	16
70	$T_1^5 T_2 + T_3 T_4 + T_5^5$	$[2\ 5\ 5\ 10\ 3]$	10	108	$T_1^3 T_2 + T_3^3 T_4 + T_5^2$	$[1\ 1\ 1\ 1\ 2]$	16
77	$T_1^3 T_2 + T_3^3 T_4^3 + T_5^2$	$[2\ 6\ 3\ 1\ 6]$	12	109	$T_1^3 T_2 + T_3^2 T_4 + T_5^4$	$[1\ 1\ 1\ 2\ 1]$	16
78	$T_1^3 T_2 + T_3^3 T_4^2 + T_5^2$	$[2\ 12\ 4\ 3\ 9]$	12	110	$T_1^7 T_2 + T_3^2 T_4 + T_5^2$	$[1\ 1\ 2\ 4\ 4]$	16
79	$T_1^5 T_2 + T_3^2 T_4 + T_5^3$	$[1\ 4\ 4\ 1\ 3]$	12	111	$T_1^5 T_2^3 + T_3^2 T_4 + T_5^2$	$[1\ 1\ 2\ 4\ 4]$	16
80	$T_1^{10} T_2 + T_3^2 T_4 + T_5^2$	$[1\ 2\ 3\ 6\ 6]$	12	112	$T_1^4 T_2 + T_3^2 T_4 + T_5^2$	$[1\ 2\ 2\ 2\ 3]$	16
81	$T_1^7 T_2 + T_3^2 T_4 + T_5^2$	$[2\ 4\ 3\ 12\ 9]$	12	113	$T_1^8 T_2 + T_3 T_4 + T_5^4$	$[1\ 4\ 4\ 8\ 3]$	16
82	$T_1^6 T_2^3 + T_3^2 T_4 + T_5^2$	$[1\ 2\ 3\ 6\ 6]$	12	114	$T_1^7 T_2 + T_3 T_4 + T_5^4$	$[1\ 1\ 4\ 4\ 2]$	16
83	$T_1^5 T_2^2 + T_3^2 T_4 + T_5^2$	$[2\ 4\ 3\ 12\ 9]$	12	115	$T_1^6 T_2 + T_3 T_4 + T_5^8$	$[1\ 2\ 4\ 4\ 1]$	16
84	$T_1^5 T_2^2 + T_3^2 T_4 + T_5^2$	$[2\ 1\ 3\ 6\ 6]$	12	116	$T_1^5 T_2^3 + T_3 T_4 + T_5^4$	$[1\ 1\ 4\ 4\ 2]$	16
85	$T_1^4 T_2 + T_3^2 T_4 + T_5^2$	$[4\ 2\ 3\ 12\ 9]$	12	117	$T_1^3 T_2^2 + T_3 T_4 + T_5^8$	$[2\ 1\ 4\ 4\ 1]$	16
86	$T_1^3 T_2^3 + T_3^2 T_4 + T_5^2$	$[2\ 4\ 3\ 12\ 9]$	12	126	$T_1^4 T_2 + T_3^3 T_4^2 + T_5^2$	$[1\ 6\ 2\ 2\ 5]$	18
87	$T_1^2 T_2 + T_3^2 T_4 + T_5^3$	$[4\ 1\ 4\ 1\ 3]$	12	127	$T_1^5 T_2 + T_3^3 T_4 + T_5^2$	$[1\ 1\ 1\ 3\ 3]$	18

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
128	$T_1^3 T_2 + T_3^3 T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 3 & 1 & 1 & 3 \end{bmatrix}$	18	164	$T_1^2 T_2 + T_3^2 T_4 + T_5^6$	$\begin{bmatrix} 1 & 4 & 2 & 2 & 1 \end{bmatrix}$	24
129	$T_1^8 T_2 + T_3^2 T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 2 & 6 & 5 \end{bmatrix}$	18	165	$T_1^2 T_2 + T_3^2 T_4 + T_5^4$	$\begin{bmatrix} 3 & 2 & 1 & 6 & 2 \end{bmatrix}$	24
130	$T_1^4 T_2^3 + T_3^2 T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 2 & 6 & 5 \end{bmatrix}$	18	166	$T_1^2 T_2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	24
131	$T_1^{15} T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 9 & 6 & 18 & 8 \end{bmatrix}$	18	167	$T_1^{22} T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 8 & 6 & 24 & 15 \end{bmatrix}$	24
132	$T_1^{11} T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 6 & 6 & 4 \end{bmatrix}$	18	168	$T_1^{16} T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 6 & 12 & 9 \end{bmatrix}$	24
133	$T_1^{10} T_2^2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 6 & 6 & 4 \end{bmatrix}$	18	169	$T_1^{13} T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 3 & 4 & 12 & 8 \end{bmatrix}$	24
134	$T_1^8 T_2^4 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 6 & 6 & 4 \end{bmatrix}$	18	170	$T_1^{12} T_2^3 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 6 & 12 & 9 \end{bmatrix}$	24
135	$T_1^7 T_2^5 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 6 & 6 & 4 \end{bmatrix}$	18	171	$T_1^9 T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 6 & 3 & 12 & 5 \end{bmatrix}$	24
136	$T_1^7 T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 2 & 3 & 6 & 3 \end{bmatrix}$	18	172	$T_1^8 T_2^5 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 6 & 12 & 9 \end{bmatrix}$	24
137	$T_1^6 T_2^2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 9 & 6 & 18 & 8 \end{bmatrix}$	18	173	$T_1^7 T_2^4 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 2 & 1 & 6 & 12 & 9 \end{bmatrix}$	24
138	$T_1^6 T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 2 & 9 & 3 & 18 & 7 \end{bmatrix}$	18	174	$T_1^7 T_2^3 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 3 & 4 & 12 & 8 \end{bmatrix}$	24
139	$T_1^5 T_2^2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 2 & 3 & 6 & 3 \end{bmatrix}$	18	175	$T_1^6 T_2^3 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 8 & 6 & 24 & 15 \end{bmatrix}$	24
140	$T_1^5 T_2 + T_3 T_4 + T_5^6$	$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 \end{bmatrix}$	18	176	$T_1^6 T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 3 & 8 & 2 & 24 & 13 \end{bmatrix}$	24
141	$T_1^4 T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 2 & 1 & 3 & 6 & 3 \end{bmatrix}$	18	177	$T_1^5 T_2 + T_3 T_4 + T_5^4$	$\begin{bmatrix} 1 & 3 & 2 & 6 & 2 \end{bmatrix}$	24
142	$T_1^3 T_2 + T_3 T_4 + T_5^9$	$\begin{bmatrix} 2 & 3 & 3 & 6 & 1 \end{bmatrix}$	18	178	$T_1^5 T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 2 \end{bmatrix}$	24
143	$T_1^2 T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 2 & 2 & 3 & 3 & 2 \end{bmatrix}$	18	179	$T_1^5 T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 3 & 1 & 4 & 12 & 8 \end{bmatrix}$	24
145	$T_1^3 T_2 + T_3^3 T_4 + T_5^2$	$\begin{bmatrix} 2 & 10 & 5 & 1 & 8 \end{bmatrix}$	20	180	$T_1^4 T_2^2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 2 \end{bmatrix}$	24
146	$T_1^{26} T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 4 & 10 & 20 & 15 \end{bmatrix}$	20	181	$T_1^4 T_2 + T_3 T_4 + T_5^6$	$\begin{bmatrix} 1 & 2 & 2 & 4 & 1 \end{bmatrix}$	24
147	$T_1^{18} T_2^3 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 4 & 10 & 20 & 15 \end{bmatrix}$	20	182	$T_1^4 T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 6 & 4 & 6 & 5 \end{bmatrix}$	24
148	$T_1^{10} T_2^5 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 4 & 10 & 20 & 15 \end{bmatrix}$	20	183	$T_1^4 T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 3 & 3 & 3 \end{bmatrix}$	24
149	$T_1^7 T_2^2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 4 & 1 & 10 & 20 & 15 \end{bmatrix}$	20	184	$T_1^4 T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 3 & 2 & 2 & 12 & 7 \end{bmatrix}$	24
150	$T_1^3 T_2^2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 4 & 5 & 2 & 20 & 11 \end{bmatrix}$	20	185	$T_1^3 T_2^2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 6 & 3 & 12 & 5 \end{bmatrix}$	24
151	$T_1^2 T_2 + T_3 T_4 + T_5^4$	$\begin{bmatrix} 5 & 2 & 2 & 10 & 3 \end{bmatrix}$	20	186	$T_1^3 T_2 + T_3 T_4 + T_5^3$	$\begin{bmatrix} 1 & 3 & 3 & 3 & 2 \end{bmatrix}$	24
160	$T_1^4 T_2 + T_3^3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 6 & 3 & 1 & 5 \end{bmatrix}$	24	187	$T_1^2 T_2 + T_3 T_4 + T_5^8$	$\begin{bmatrix} 3 & 2 & 2 & 6 & 1 \end{bmatrix}$	24
161	$T_1^4 T_2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 1 & 2 & 1 & 4 & 2 \end{bmatrix}$	24	190	$T_1^3 T_2 + T_3^3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 15 & 5 & 3 & 9 \end{bmatrix}$	30
162	$T_1^3 T_2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 1 & 12 & 6 & 3 & 5 \end{bmatrix}$	24	191	$T_1^2 T_2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 1 & 10 & 5 & 2 & 4 \end{bmatrix}$	30
163	$T_1^2 T_2^2 + T_3^2 T_4 + T_5^3$	$\begin{bmatrix} 1 & 2 & 1 & 4 & 2 \end{bmatrix}$	24	192	$T_1^{13} T_2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 5 & 3 & 15 & 9 \end{bmatrix}$	30

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
193	$T_1^7 T_2 + T_3 T_4 + T_5^3$	$[1\ 5\ 2\ 10\ 4]$	30	221	$T_1^3 T_2 + T_3^2 T_4 + T_5^2$	$[4\ 2\ 1\ 12\ 7]$	36
194	$T_1^3 T_2^3 + T_3 T_4 + T_5^2$	$[1\ 5\ 3\ 15\ 9]$	30	222	$T_1^{10} T_2 + T_3 T_4 + T_5^2$	$[1\ 4\ 2\ 12\ 7]$	36
195	$T_1^2 T_2^2 + T_3 T_4 + T_5^3$	$[1\ 5\ 2\ 10\ 4]$	30	223	$T_1^7 T_2 + T_3 T_4 + T_5^2$	$[1\ 1\ 2\ 6\ 4]$	36
196	$T_1 T_2 + T_3 T_4 + T_5^5$	$[2\ 3\ 2\ 3\ 1]$	30	224	$T_1^5 T_2^3 + T_3 T_4 + T_5^2$	$[1\ 1\ 2\ 6\ 4]$	36
203	$T_1^5 T_2 + T_3^2 T_4 + T_5^2$	$[1\ 1\ 1\ 4\ 3]$	32	225	$T_1^3 T_2^2 + T_3 T_4 + T_5^2$	$[4\ 1\ 2\ 12\ 7]$	36
204	$T_1^4 T_2 + T_3^2 T_4 + T_5^2$	$[2\ 2\ 1\ 8\ 5]$	32	226	$T_1^3 T_2 + T_3 T_4 + T_5^4$	$[1\ 1\ 1\ 3\ 1]$	36
205	$T_1^3 T_2^3 + T_3^2 T_4 + T_5^2$	$[1\ 1\ 1\ 4\ 3]$	32	227	$T_1 T_2 + T_3 T_4 + T_5^2$	$[3\ 1\ 2\ 2\ 2]$	36
206	$T_1^3 T_2^2 + T_3^2 T_4 + T_5^2$	$[2\ 2\ 1\ 8\ 5]$	32	230	$T_1^3 T_2 + T_3 T_4 + T_5^5$	$[1\ 2\ 1\ 4\ 1]$	40
207	$T_1^2 T_2 + T_3^2 T_4 + T_5^2$	$[1\ 2\ 1\ 2\ 2]$	32	231	$T_1^2 T_2 + T_3 T_4 + T_5^5$	$[2\ 1\ 1\ 4\ 1]$	40
208	$T_1^{11} T_2 + T_3 T_4 + T_5^2$	$[1\ 1\ 4\ 8\ 6]$	32	232	$T_1^5 T_2 + T_3 T_4 + T_5^2$	$[3\ 7\ 1\ 21\ 11]$	42
209	$T_1^9 T_2^3 + T_3 T_4 + T_5^2$	$[1\ 1\ 4\ 8\ 6]$	32	233	$T_1^4 T_2 + T_3 T_4 + T_5^3$	$[2\ 7\ 1\ 14\ 5]$	42
210	$T_1^8 T_2 + T_3 T_4 + T_5^2$	$[1\ 2\ 2\ 8\ 5]$	32	234	$T_1^2 T_2 + T_3 T_4 + T_5^7$	$[2\ 3\ 1\ 6\ 1]$	42
211	$T_1^7 T_2^5 + T_3 T_4 + T_5^2$	$[1\ 1\ 4\ 8\ 6]$	32	235	$T_1^5 T_2 + T_3 T_4 + T_5^3$	$[1\ 4\ 1\ 8\ 3]$	48
212	$T_1^4 T_2^3 + T_3 T_4 + T_5^2$	$[1\ 2\ 2\ 8\ 5]$	32	236	$T_1^2 T_2 + T_3 T_4 + T_5^3$	$[4\ 1\ 1\ 8\ 3]$	48
213	$T_1^3 T_2 + T_3 T_4 + T_5^2$	$[1\ 1\ 2\ 2\ 2]$	32	237	$T_1 T_2 + T_3 T_4 + T_5^3$	$[1\ 2\ 1\ 2\ 1]$	48
214	$T_1^2 T_2 + T_3 T_4 + T_5^4$	$[1\ 2\ 2\ 2\ 1]$	32	238	$T_1 T_2 + T_3 T_4 + T_5^2$	$[1\ 3\ 1\ 3\ 2]$	48
215	$T_1^2 T_2 + T_3 T_4 + T_5^2$	$[1\ 4\ 2\ 4\ 3]$	32	241	$T_1^5 T_2 + T_3 T_4 + T_5^2$	$[1\ 1\ 1\ 5\ 3]$	50
217	$T_1^6 T_2 + T_3^2 T_4 + T_5^2$	$[1\ 2\ 1\ 6\ 4]$	36	242	$T_1^3 T_2^3 + T_3 T_4 + T_5^2$	$[1\ 1\ 1\ 5\ 3]$	50
218	$T_1^5 T_2 + T_3^2 T_4 + T_5^2$	$[2\ 4\ 1\ 12\ 7]$	36	243	$T_1^7 T_2 + T_3 T_4 + T_5^2$	$[1\ 3\ 1\ 9\ 5]$	54
219	$T_1^3 T_2^2 + T_3^2 T_4 + T_5^2$	$[2\ 4\ 1\ 12\ 7]$	36	244	$T_1^3 T_2 + T_3 T_4 + T_5^2$	$[3\ 1\ 1\ 9\ 5]$	54
220	$T_1^3 T_2^2 + T_3^2 T_4 + T_5^2$	$[2\ 1\ 1\ 6\ 4]$	36	245	$T_1 T_2 + T_3 T_4 + T_5^2$	$[1\ 1\ 1\ 1\ 1]$	54

Classification list 4.2. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z}$ and format $(2, 2, 1, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
30	$T_1^3 T_2 + T_3 T_4 + T_5^3,\ \lambda T_3 T_4 + T_5^3 + T_6^2$	$[1\ 3\ 3\ 3\ 2\ 3]$	6
76	$T_1^2 T_2 + T_3 T_4 + T_5^3,\ \lambda T_3 T_4 + T_5^3 + T_6^2$	$[1\ 4\ 2\ 4\ 2\ 3]$	12
189	$T_1 T_2 + T_3 T_4 + T_5^3,\ \lambda T_3 T_4 + T_5^3 + T_6^2$	$[1\ 5\ 1\ 5\ 2\ 3]$	30

Classification list 4.3. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z}$ and format $(3, 1, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
3	$T_1^5 T_2^3 T_3^2 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 6 \end{bmatrix}$	2	51	$T_1^2 T_2^2 T_3 + T_4^5 + T_5^2$	$\begin{bmatrix} 2 & 1 & 4 & 2 & 5 \end{bmatrix}$	8
4	$T_1^4 T_2^3 T_3^2 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 4 & 6 \end{bmatrix}$	2	63	$T_1 T_2 T_3 + T_4^7 + T_5^3$	$\begin{bmatrix} 1 & 10 & 10 & 3 & 7 \end{bmatrix}$	10
5	$T_1^5 T_2^5 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 6 \end{bmatrix}$	2	64	$T_1 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 2 & 5 & 5 & 4 & 6 \end{bmatrix}$	10
6	$T_1^5 T_2^3 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 4 & 6 \end{bmatrix}$	2	75	$T_1^3 T_2^3 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 3 & 1 & 12 & 8 & 12 \end{bmatrix}$	12
7	$T_1^5 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 2 & 1 & 1 & 4 & 6 \end{bmatrix}$	2	104	$T_1^2 T_2^2 T_3 + T_4^7 + T_5^2$	$\begin{bmatrix} 1 & 4 & 8 & 2 & 7 \end{bmatrix}$	16
8	$T_1^4 T_2^3 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 2 & 1 & 1 & 4 & 6 \end{bmatrix}$	2	105	$T_1^2 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 3 \end{bmatrix}$	16
9	$T_1^3 T_2^2 T_3 + T_4^5 + T_5^2$	$\begin{bmatrix} 2 & 1 & 2 & 2 & 5 \end{bmatrix}$	2	121	$T_1^5 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 6 & 1 & 4 & 6 \end{bmatrix}$	18
13	$T_1^2 T_2^2 T_3 + T_4^7 + T_5^2$	$\begin{bmatrix} 4 & 1 & 4 & 2 & 7 \end{bmatrix}$	4	122	$T_1^4 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 6 & 4 & 6 \end{bmatrix}$	18
25	$T_1^5 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 2 & 1 & 6 & 6 & 9 \end{bmatrix}$	6	123	$T_1^3 T_2^3 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 6 & 4 & 6 \end{bmatrix}$	18
26	$T_1^4 T_2^4 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 2 & 1 & 6 & 6 & 9 \end{bmatrix}$	6	124	$T_1^2 T_2^2 T_3 + T_4^5 + T_5^2$	$\begin{bmatrix} 1 & 2 & 6 & 2 & 5 \end{bmatrix}$	18
27	$T_1^4 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 6 & 2 & 6 & 9 \end{bmatrix}$	6	125	$T_1 T_2 T_3 + T_4^7 + T_5^2$	$\begin{bmatrix} 1 & 9 & 18 & 4 & 14 \end{bmatrix}$	18
28	$T_1^3 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 2 & 3 & 3 & 4 & 6 \end{bmatrix}$	6	144	$T_1^2 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 4 & 1 & 20 & 10 & 15 \end{bmatrix}$	20
29	$T_1^2 T_2 T_3 + T_4^7 + T_5^2$	$\begin{bmatrix} 3 & 2 & 6 & 2 & 7 \end{bmatrix}$	6	158	$T_1^4 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 12 & 2 & 6 & 9 \end{bmatrix}$	24
46	$T_1^5 T_2^3 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 4 & 6 \end{bmatrix}$	8	159	$T_1^2 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 2 & 1 & 12 & 6 & 9 \end{bmatrix}$	24
47	$T_1^4 T_2^4 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 4 & 6 \end{bmatrix}$	8	188	$T_1 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 5 & 30 & 12 & 18 \end{bmatrix}$	30
48	$T_1^4 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \end{bmatrix}$	8	201	$T_1^3 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 8 & 1 & 4 & 6 \end{bmatrix}$	32
49	$T_1^3 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 4 & 1 & 4 & 6 \end{bmatrix}$	8	202	$T_1^2 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 8 & 4 & 6 \end{bmatrix}$	32
50	$T_1^3 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \end{bmatrix}$	8	240	$T_1 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 10 & 4 & 6 \end{bmatrix}$	50

Classification list 4.4. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z}$ and format $(2, 1, 1, 1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
1	$T_1^5 T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 2 & 1 & 4 & 6 & 1 \end{bmatrix}$	2	20	$T_1^3 T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 2 & 6 & 6 & 9 & 1 \end{bmatrix}$	6
2	$T_1^3 T_2^2 + T_3^5 + T_4^2$	$\begin{bmatrix} 2 & 2 & 2 & 5 & 1 \end{bmatrix}$	2	21	$T_1^3 T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 3 & 3 & 4 & 6 & 2 \end{bmatrix}$	6
12	$T_1^2 T_2 + T_3^4 + T_4^3$	$\begin{bmatrix} 4 & 4 & 3 & 4 & 1 \end{bmatrix}$	4	22	$T_1^2 T_2 + T_3^4 + T_4^3$	$\begin{bmatrix} 3 & 6 & 3 & 4 & 2 \end{bmatrix}$	6
19	$T_1^3 T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 2 & 3 & 4 & 6 & 3 \end{bmatrix}$	6	23	$T_1^2 T_2 + T_3^9 + T_4^2$	$\begin{bmatrix} 6 & 6 & 2 & 9 & 1 \end{bmatrix}$	6

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
24	$T_1^2T_2+T_3^7+T_4^2$	$\begin{bmatrix}6 & 2 & 2 & 7 & 3\end{bmatrix}$	6	119	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}2 & 2 & 2 & 3 & 3\end{bmatrix}$	18
43	$T_1^5T_2+T_3^3+T_4^2$	$\begin{bmatrix}1 & 1 & 2 & 3 & 1\end{bmatrix}$	8	120	$T_1T_2+T_3^5+T_4^4$	$\begin{bmatrix}2 & 18 & 4 & 5 & 9\end{bmatrix}$	18
44	$T_1^3T_2+T_3^5+T_4^2$	$\begin{bmatrix}2 & 4 & 2 & 5 & 1\end{bmatrix}$	8	152	$T_1T_2+T_3^8+T_4^3$	$\begin{bmatrix}2 & 22 & 3 & 8 & 11\end{bmatrix}$	22
45	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix}4 & 2 & 2 & 5 & 1\end{bmatrix}$	8	153	$T_1^3T_2+T_3^3+T_4^2$	$\begin{bmatrix}1 & 3 & 2 & 3 & 3\end{bmatrix}$	24
59	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix}5 & 10 & 4 & 10 & 1\end{bmatrix}$	10	154	$T_1^3T_2+T_3^3+T_4^2$	$\begin{bmatrix}2 & 12 & 6 & 9 & 1\end{bmatrix}$	24
60	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}5 & 2 & 4 & 6 & 5\end{bmatrix}$	10	155	$T_1T_2+T_3^9+T_4^2$	$\begin{bmatrix}6 & 12 & 2 & 9 & 1\end{bmatrix}$	24
61	$T_1T_2+T_3^5+T_4^4$	$\begin{bmatrix}10 & 10 & 4 & 5 & 1\end{bmatrix}$	10	156	$T_1T_2+T_3^7+T_4^2$	$\begin{bmatrix}2 & 12 & 2 & 7 & 3\end{bmatrix}$	24
62	$T_1T_2+T_3^5+T_4^3$	$\begin{bmatrix}5 & 10 & 3 & 5 & 2\end{bmatrix}$	10	157	$T_1T_2+T_3^3+T_4^2$	$\begin{bmatrix}3 & 3 & 2 & 3 & 1\end{bmatrix}$	24
71	$T_1^4T_2+T_3^3+T_4^2$	$\begin{bmatrix}3 & 12 & 8 & 12 & 1\end{bmatrix}$	12	197	$T_1^4T_2+T_3^3+T_4^2$	$\begin{bmatrix}1 & 8 & 4 & 6 & 1\end{bmatrix}$	32
72	$T_1^2T_2+T_3^9+T_4^2$	$\begin{bmatrix}3 & 12 & 2 & 9 & 4\end{bmatrix}$	12	198	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix}1 & 8 & 2 & 5 & 2\end{bmatrix}$	32
73	$T_1T_2+T_3^8+T_4^3$	$\begin{bmatrix}12 & 12 & 3 & 8 & 1\end{bmatrix}$	12	199	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}1 & 4 & 2 & 3 & 4\end{bmatrix}$	32
74	$T_1T_2+T_3^5+T_4^3$	$\begin{bmatrix}3 & 12 & 3 & 5 & 4\end{bmatrix}$	12	200	$T_1T_2+T_3^5+T_4^2$	$\begin{bmatrix}2 & 8 & 2 & 5 & 1\end{bmatrix}$	32
101	$T_1^4T_2+T_3^3+T_4^2$	$\begin{bmatrix}1 & 2 & 2 & 3 & 2\end{bmatrix}$	16	216	$T_1^2T_2+T_3^7+T_4^2$	$\begin{bmatrix}1 & 12 & 2 & 7 & 4\end{bmatrix}$	36
102	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}2 & 2 & 2 & 3 & 1\end{bmatrix}$	16	228	$T_1T_2+T_3^7+T_4^3$	$\begin{bmatrix}1 & 20 & 3 & 7 & 10\end{bmatrix}$	40
103	$T_1T_2+T_3^4+T_4^3$	$\begin{bmatrix}4 & 8 & 3 & 4 & 1\end{bmatrix}$	16	229	$T_1T_2+T_3^3+T_4^2$	$\begin{bmatrix}1 & 5 & 2 & 3 & 5\end{bmatrix}$	40
118	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix}2 & 6 & 2 & 5 & 1\end{bmatrix}$	18	239	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}1 & 10 & 4 & 6 & 1\end{bmatrix}$	50

Classification list 4.5. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2, 2, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
252	$T_1^4T_2^2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}1 & 2 & 2 & 1 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1}\end{bmatrix}$	2	273	$T_1^4T_2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix}2 & 2 & 1 & 4 & 5 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1}\end{bmatrix}$	4
253	$T_1^3T_2^2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}2 & 1 & 2 & 1 & 4 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1}\end{bmatrix}$	2	274	$T_1^3T_2^2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix}2 & 2 & 1 & 4 & 5 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0}\end{bmatrix}$	4
254	$T_1^4T_2^4+T_3^3T_4+T_5^2$	$\begin{bmatrix}1 & 1 & 2 & 2 & 4 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1}\end{bmatrix}$	2	275	$T_1^8T_2^4+T_3^2T_4+T_5^2$	$\begin{bmatrix}1 & 1 & 4 & 4 & 6 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1}\end{bmatrix}$	4
271	$T_1^4T_2+T_3^3T_4^3+T_5^2$	$\begin{bmatrix}1 & 2 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0}\end{bmatrix}$	4	276	$T_1^4T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2 & 1 & 4 & 2 & 5 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1}\end{bmatrix}$	4
272	$T_1^4T_2^2+T_3^3T_4+T_5^2$	$\begin{bmatrix}2 & 1 & 2 & 4 & 5 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1}\end{bmatrix}$	4	291	$T_1^{10}T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1 & 2 & 3 & 6 & 6 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1}\end{bmatrix}$	6

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
292	$T_1^8T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 3 & 6 & 6 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	6	332	$T_1^2T_2 + T_3^2T_4^2 + T_5^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 3 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
293	$T_1^6T_2^3 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 3 & 6 & 6 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	6	333	$T_1^6T_2^2 + T_3T_4 + T_5^4$	$\begin{bmatrix} 1 & 1 & 4 & 4 & 2 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	8
294	$T_1^5T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 2 & 1 & 3 & 6 & 6 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	6	334	$T_1^4T_2^2 + T_3T_4 + T_5^8$	$\begin{bmatrix} 1 & 2 & 4 & 4 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	8
295	$T_1^4T_2^4 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 3 & 6 & 6 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	6	335	$T_1^4T_2^2 + T_3T_4 + T_5^4$	$\begin{bmatrix} 1 & 4 & 4 & 8 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	8
296	$T_1^2T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 3 & 2 & 3 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	6	350	$T_1^4T_2 + T_3^3T_4 + T_5^2$	$\begin{bmatrix} 1 & 6 & 3 & 1 & 5 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	12
320	$T_1^4T_2 + T_3^3T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 4 & 2 & 1 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	8	351	$T_1^2T_2^2 + T_3^2T_4 + T_5^3$	$\begin{bmatrix} 1 & 2 & 1 & 4 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	12
321	$T_1^3T_2 + T_3^3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	8	352	$T_1^2T_2 + T_3^2T_4 + T_5^6$	$\begin{bmatrix} 2 & 2 & 1 & 4 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	12
322	$T_1^3T_2 + T_3^2T_4 + T_5^4$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	8	353	$T_1^2T_2 + T_3^2T_4 + T_5^4$	$\begin{bmatrix} 1 & 6 & 3 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	12
323	$T_1^7T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	8	354	$T_1^2T_2 + T_3^2T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	12
324	$T_1^6T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	8	355	$T_1^{10}T_2^4 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 6 & 12 & 9 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	12
325	$T_1^6T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8	356	$T_1^9T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 6 & 3 & 12 & 5 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	12
326	$T_1^5T_2^3 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	8	357	$T_1^8T_2^2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 2 & 1 & 6 & 12 & 9 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	12
327	$T_1^4T_2^4 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	8	358	$T_1^5T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 2 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	12
328	$T_1^4T_2^4 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	8	359	$T_1^4T_2^4 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 3 & 4 & 12 & 8 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	12
329	$T_1^4T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	8	360	$T_1^4T_2^2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 2 & 3 & 2 & 12 & 7 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	12
330	$T_1^2T_2^2 + T_3^2T_4 + T_5^4$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8	361	$T_1^4T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 3 & 3 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	12
331	$T_1^2T_2 + T_3^2T_4^2 + T_5^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 3 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	8	362	$T_1^3T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 3 & 3 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	12

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
363	$T_1^3T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 3 & 3 & 3 & 2 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	12	391	$T_1^2T_2 + T_3T_4 + T_5^4$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16
364	$T_1^2T_2^2 + T_3T_4 + T_5^4$	$\begin{bmatrix} 1 & 3 & 2 & 6 & 2 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	12	392	$T_1^2T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 4 & 2 & 4 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16
381	$T_1^5T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	16	400	$T_1^6T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 6 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	18
382	$T_1^4T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	16	401	$T_1^4T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 6 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	18
383	$T_1^4T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	16	402	$T_1^3T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 2 & 1 & 1 & 6 & 4 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	18
384	$T_1^4T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 2 & 2 & 1 & 8 & 5 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	16	403	$T_1^4T_2^4 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 6 & 4 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	18
385	$T_1^3T_2^3 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	16	405	$T_1^3T_2 + T_3T_4 + T_5^5$	$\begin{bmatrix} 1 & 2 & 1 & 4 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	20
386	$T_1^3T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 2 & 2 & 1 & 8 & 5 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	16	408	$T_1^5T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 4 & 1 & 8 & 3 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	24
387	$T_1^2T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	16	409	$T_1T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	24
388	$T_1^8T_2^4 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 8 & 6 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	16	410	$T_1T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 2 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	24
389	$T_1^4T_2^2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 2 & 1 & 2 & 8 & 5 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	16	411	$T_1T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	24
390	$T_1^3T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16				

Classification list 4.6. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2, 2, 1, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
269	$T_1^2T_2 + T_3T_4 + T_5^4,$ $\lambda T_3T_4 + T_5^4 + T_6^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	4
270	$T_1^2T_2 + T_3T_4 + T_5^2,$ $\lambda T_3T_4 + T_5^2 + T_6^2$	$\begin{bmatrix} 2 & 2 & 2 & 4 & 3 & 3 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4
349	$T_1T_2 + T_3T_4 + T_5^4,$ $\lambda T_3T_4 + T_5^4 + T_6^2$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	12

Classification list 4.7. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(3, 1, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
248	$T_1^3 T_2^3 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2	312	$T_1^2 T_2^2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	8
249	$T_1^3 T_2^2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2	313	$T_1^2 T_2^2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 2 & 4 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	8
250	$T_1^3 T_2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 2 & 1 & 1 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2	314	$T_1^2 T_2 T_3 + T_4^8 + T_5^2$	$\begin{bmatrix} 1 & 2 & 4 & 1 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	8
251	$T_1^2 T_2^2 T_3 + T_4^8 + T_5^2$	$\begin{bmatrix} 2 & 1 & 2 & 1 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2	315	$T_1^2 T_2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	8
263	$T_1^4 T_2^4 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 4 & 6 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4	316	$T_1^2 T_2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	8
264	$T_1^4 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	4	317	$T_1^2 T_2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 3 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
265	$T_1^3 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	4	318	$T_1^4 T_2^2 T_3 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	8
266	$T_1^2 T_2^2 T_3 + T_4^6 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4	319	$T_1^3 T_2^2 T_3 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 4 & 4 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	8
267	$T_1^2 T_2^2 T_3 + T_4^5 + T_5^2$	$\begin{bmatrix} 2 & 1 & 4 & 2 & 5 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4	337	$T_1 T_2 T_3 + T_4^8 + T_5^2$	$\begin{bmatrix} 1 & 5 & 10 & 2 & 8 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	10
268	$T_1^2 T_2 T_3 + T_4^{10} + T_5^2$	$\begin{bmatrix} 2 & 2 & 4 & 1 & 5 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4	347	$T_1^2 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 2 & 1 & 12 & 6 & 9 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	12
288	$T_1^2 T_2^2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 2 & 1 & 6 & 3 & 6 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	6	348	$T_1 T_2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 3 & 12 & 4 & 8 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	12
289	$T_1^4 T_2^2 T_3 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 3 & 2 & 6 & 6 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	6	380	$T_1^2 T_2^2 T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 8 & 4 & 6 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	16
290	$T_1 T_2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 2 & 3 & 3 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	6	399	$T_1 T_2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	18
311	$T_1^3 T_2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 4 & 1 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	8				

Classification list 4.8. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
246	$T_1^3T_2^2 + T_3^4 + T_4^2$	$\begin{bmatrix} 2 & 1 & 2 & 4 & 1 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	2	285	$T_1^5T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 2 & 1 & 6 & 6 & 3 \\ \bar{0} & \bar{1} & \bar{1} & 0 & 0 \end{bmatrix}$	6
247	$T_1^3T_2^2 + T_3^4 + T_4^2$	$\begin{bmatrix} 2 & 1 & 2 & 4 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	2	286	$T_1^3T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 2 & 3 & 6 & 6 & 1 \\ \bar{0} & \bar{1} & \bar{1} & 0 & 0 \end{bmatrix}$	6
256	$T_1^5T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	4	287	$T_1T_2 + T_3^6 + T_4^4$	$\begin{bmatrix} 6 & 6 & 2 & 3 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	6
257	$T_1^4T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 4 & 4 & 6 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	4	297	$T_1^4T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 2 & 2 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	8
258	$T_1^4T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	4	298	$T_1^3T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	8
259	$T_1^4T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	4	299	$T_1^3T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	8
260	$T_1^2T_2^2 + T_3^5 + T_4^2$	$\begin{bmatrix} 4 & 1 & 2 & 5 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	4	300	$T_1^2T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 2 & 1 & 2 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	8
261	$T_1^2T_2 + T_3^{12} + T_4^2$	$\begin{bmatrix} 4 & 4 & 1 & 6 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	4	301	$T_1^2T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 2 & 1 & 2 & 3 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	8
262	$T_1^2T_2 + T_3^{10} + T_4^2$	$\begin{bmatrix} 4 & 2 & 1 & 5 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	4	302	$T_1^2T_2 + T_3^8 + T_4^2$	$\begin{bmatrix} 2 & 4 & 1 & 4 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	8
277	$T_1^3T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 2 & 6 & 3 & 6 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	6	303	$T_1^2T_2 + T_3^8 + T_4^2$	$\begin{bmatrix} 2 & 4 & 1 & 4 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	8
278	$T_1^2T_2 + T_3^{12} + T_4^2$	$\begin{bmatrix} 3 & 6 & 1 & 6 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	6	304	$T_1^2T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	8
279	$T_1^2T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 3 & 6 & 2 & 6 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	6	305	$T_1^2T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	8
280	$T_1^2T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 3 & 6 & 2 & 6 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	6	306	$T_1^2T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 2 & 2 & 2 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	8
281	$T_1^2T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 3 & 2 & 2 & 4 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	6	307	$T_1^7T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 4 & 4 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	8
282	$T_1^2T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 3 & 2 & 2 & 4 & 3 \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	6	308	$T_1^6T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 4 & 4 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
283	$T_1^{10}T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 6 & 6 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	6	309	$T_1^5T_2^3 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 4 & 4 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	8
284	$T_1^6T_2^3 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 6 & 6 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	6	310	$T_1^3T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 2 & 1 & 4 & 4 & 1 \\ \bar{0} & \bar{1} & \bar{1} & 0 & 0 \end{bmatrix}$	8

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
336	$T_1T_2 + T_3^6 + T_4^4$	$\begin{bmatrix} 2 & 10 & 2 & 3 & 5 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	10	374	$T_1T_2 + T_3^{12} + T_4^2$	$\begin{bmatrix} 4 & 8 & 1 & 6 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16
340	$T_1^3T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 3 & 2 & 3 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	12	375	$T_1T_2 + T_3^{10} + T_4^2$	$\begin{bmatrix} 2 & 8 & 1 & 5 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16
341	$T_1^4T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 3 & 3 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	12	376	$T_1T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 2 & 4 & 1 & 3 & 4 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16
342	$T_1T_2 + T_3^{18} + T_4^2$	$\begin{bmatrix} 6 & 12 & 1 & 9 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	12	377	$T_1T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 2 & 4 & 1 & 3 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16
343	$T_1T_2 + T_3^{16} + T_4^2$	$\begin{bmatrix} 4 & 12 & 1 & 8 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	12	378	$T_1T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16
344	$T_1T_2 + T_3^{10} + T_4^2$	$\begin{bmatrix} 4 & 6 & 1 & 5 & 6 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	12	379	$T_1T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16
345	$T_1T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 3 & 3 & 2 & 3 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	12	393	$T_1^2T_2 + T_3^8 + T_4^2$	$\begin{bmatrix} 1 & 6 & 1 & 4 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	18
346	$T_1T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 6 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	12	394	$T_1^2T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 6 & 2 & 4 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	18
365	$T_1^4T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 8 & 4 & 6 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	16	395	$T_1^2T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 6 & 2 & 4 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	18
366	$T_1^2T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 1 & 4 & 1 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	16	396	$T_1^2T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 3 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	18
367	$T_1^2T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 1 & 4 & 1 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	16	397	$T_1T_2 + T_3^8 + T_4^2$	$\begin{bmatrix} 2 & 6 & 1 & 4 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	18
368	$T_1^2T_2 + T_3^5 + T_4^2$	$\begin{bmatrix} 1 & 8 & 2 & 5 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16	398	$T_1T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	18
369	$T_1^2T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16	404	$T_1T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 5 & 2 & 3 & 5 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	20
370	$T_1^2T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16	406	$T_1T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 3 & 1 & 2 & 3 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	24
371	$T_1^2T_2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 4 & 2 & 3 & 4 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	16	407	$T_1T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 3 & 1 & 2 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	24
372	$T_1^3T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16	412	$T_1T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	32
373	$T_1^2T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	16				

Classification list 4.9. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2, 1, 1, 1, 1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
255	$T_1T_2 + T_3^3 + T_4^2,$ $\lambda T_3^3 + T_4^2 + T_5^2$	$\begin{bmatrix} 4 & 2 & 2 & 3 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	4

Classification list 4.10. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(1, 1, 1, 2)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
338	$T_1^3 + T_2^2 + T_3^2$	$\begin{bmatrix} 2 & 3 & 3 & 6 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	12	339	$T_1^3 + T_2^2 + T_3^2$	$\begin{bmatrix} 2 & 3 & 3 & 3 & 1 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	12

Classification list 4.11. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2, 2, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
480	$T_1^4T_2^2 + T_3^2T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2	495	$T_1^2T_2 + T_3^2T_4^2 + T_5^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4
491	$T_1^6T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	4	501	$T_1^3T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 3 & 3 & 3 & 2 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	6
492	$T_1^4T_2^4 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4	518	$T_1^4T_2^2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
493	$T_1^2T_2^2 + T_3^2T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	4	520	$T_1T_2 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	12
494	$T_1^2T_2^2 + T_3^2T_4 + T_5^4$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	4				

Classification list 4.12. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2, 2, 1, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
479	$T_1^2T_2^2 + T_3T_4 + T_5^2,$ $\lambda T_3T_4 + T_5^2 + T_6^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2

Classification list 4.13. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,2,1,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
478	$T_1T_2 + T_3T_4 + T_5^2,$ $\lambda_1 T_3T_4 + T_5^2 + T_6^2,$ $\lambda_2 T_5^2 + T_6^2 + T_7^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar0 & \bar0 & \bar0 & \bar0 & \bar1 & \bar1 & \bar0 \\ \bar1 & \bar1 & \bar1 & \bar1 & \bar1 & \bar0 & \bar0 \end{bmatrix}$	2

Classification list 4.14. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
477	$T_1^2 T_2^2 T_3 + T_4^6 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ \bar0 & \bar0 & \bar0 & \bar1 & \bar1 \\ \bar1 & \bar0 & \bar0 & \bar0 & \bar1 \end{bmatrix}$	2	490	$T_1^2 T_2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar0 & \bar1 & \bar1 & \bar0 & \bar0 \\ \bar0 & \bar1 & \bar1 & \bar1 & \bar1 \end{bmatrix}$	4
489	$T_1^2 T_2^2 T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 2 & 4 \\ \bar1 & \bar0 & \bar0 & \bar1 & \bar0 \\ \bar0 & \bar0 & \bar0 & \bar1 & \bar1 \end{bmatrix}$	4				

Classification list 4.15. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(3,1,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
476	$T_1 T_2 T_3 + T_4^2 + T_5^2,$ $\lambda T_4^2 + T_5^2 + T_6^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 \\ \bar0 & \bar0 & \bar0 & \bar1 & \bar1 & \bar0 \\ \bar0 & \bar0 & \bar0 & \bar0 & \bar1 & \bar1 \end{bmatrix}$	2

Classification list 4.16. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
474	$T_1^4 T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ \bar0 & \bar1 & \bar0 & \bar1 & \bar0 \\ \bar1 & \bar0 & \bar0 & \bar1 & \bar0 \end{bmatrix}$	2	481	$T_1^3 T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar1 & \bar1 & \bar0 & \bar0 & \bar0 \\ \bar0 & \bar0 & \bar1 & \bar1 & \bar0 \end{bmatrix}$	4
475	$T_1^2 T_2^2 + T_3^6 + T_4^2$	$\begin{bmatrix} 2 & 1 & 1 & 3 & 1 \\ \bar0 & \bar1 & \bar1 & \bar0 & \bar0 \\ \bar0 & \bar0 & \bar1 & \bar1 & \bar0 \end{bmatrix}$	2	482	$T_1^2 T_2^2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar0 & \bar0 & \bar1 & \bar1 & \bar0 \\ \bar0 & \bar1 & \bar1 & \bar0 & \bar0 \end{bmatrix}$	4

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
483	$T_1^2T_2^2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar 1 & \bar 1 & \bar 1 & \bar 1 & \bar 0 \\ \bar 0 & \bar 1 & \bar 1 & \bar 0 & \bar 0 \end{bmatrix}$	4	511	$T_1^3T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \\ \bar 0 & \bar 0 & \bar 1 & \bar 0 & \bar 1 \end{bmatrix}$	8
484	$T_1^2T_2^2 + T_3^3 + T_4^2$	$\begin{bmatrix} 2 & 1 & 2 & 3 & 2 \\ \bar 1 & \bar 0 & \bar 0 & \bar 1 & \bar 0 \\ \bar 1 & \bar 0 & \bar 0 & \bar 0 & \bar 1 \end{bmatrix}$	4	512	$T_1^2T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar 1 & \bar 0 & \bar 1 & \bar 0 & \bar 0 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \end{bmatrix}$	8
485	$T_1^2T_2 + T_3^8 + T_4^2$	$\begin{bmatrix} 2 & 4 & 1 & 4 & 1 \\ \bar 1 & \bar 0 & \bar 1 & \bar 0 & \bar 0 \\ \bar 1 & \bar 0 & \bar 0 & \bar 1 & \bar 0 \end{bmatrix}$	4	513	$T_1^2T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar 0 & \bar 0 & \bar 0 & \bar 1 & \bar 1 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \end{bmatrix}$	8
486	$T_1^2T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar 1 & \bar 0 & \bar 0 & \bar 0 & \bar 1 \\ \bar 1 & \bar 0 & \bar 0 & \bar 1 & \bar 0 \end{bmatrix}$	4	514	$T_1^2T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar 0 & \bar 0 & \bar 0 & \bar 1 & \bar 1 \\ \bar 1 & \bar 0 & \bar 0 & \bar 0 & \bar 1 \end{bmatrix}$	8
487	$T_1^6T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 4 & 4 & 2 \\ \bar 0 & \bar 1 & \bar 1 & \bar 0 & \bar 0 \\ \bar 0 & \bar 1 & \bar 0 & \bar 0 & \bar 1 \end{bmatrix}$	4	515	$T_1^2T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \\ \bar 1 & \bar 0 & \bar 1 & \bar 0 & \bar 0 \end{bmatrix}$	8
488	$T_1^4T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 4 & 4 & 1 \\ \bar 0 & \bar 1 & \bar 1 & \bar 0 & \bar 0 \\ \bar 1 & \bar 1 & \bar 0 & \bar 0 & \bar 0 \end{bmatrix}$	4	516	$T_1T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 2 & 4 & 1 & 3 & 4 \\ \bar 1 & \bar 1 & \bar 0 & \bar 0 & \bar 0 \\ \bar 1 & \bar 1 & \bar 0 & \bar 1 & \bar 1 \end{bmatrix}$	8
499	$T_1^4T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 3 & 3 & 3 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \\ \bar 0 & \bar 0 & \bar 1 & \bar 0 & \bar 1 \end{bmatrix}$	6	517	$T_1T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar 1 & \bar 1 & \bar 0 & \bar 0 & \bar 0 \\ \bar 1 & \bar 1 & \bar 1 & \bar 1 & \bar 0 \end{bmatrix}$	8
500	$T_1^2T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 2 & 1 & 3 & 3 & 3 \\ \bar 0 & \bar 0 & \bar 0 & \bar 1 & \bar 1 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \end{bmatrix}$	6	519	$T_1T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 3 & 1 & 2 & 3 \\ \bar 1 & \bar 1 & \bar 0 & \bar 0 & \bar 0 \\ \bar 1 & \bar 1 & \bar 0 & \bar 1 & \bar 1 \end{bmatrix}$	12
509	$T_1^2T_2 + T_3^6 + T_4^2$	$\begin{bmatrix} 1 & 4 & 1 & 3 & 1 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \\ \bar 1 & \bar 0 & \bar 0 & \bar 1 & \bar 0 \end{bmatrix}$	8	523	$T_1T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \\ \bar 0 & \bar 0 & \bar 1 & \bar 0 & \bar 1 \end{bmatrix}$	16
510	$T_1^2T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ \bar 0 & \bar 0 & \bar 1 & \bar 0 & \bar 1 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \end{bmatrix}$	8				

Classification list 4.17. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
473	$T_1T_2 + T_3^4 + T_4^2,$ $\lambda T_3^4 + T_4^2 + T_5^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 2 & 1 \\ \bar 0 & \bar 0 & \bar 1 & \bar 0 & \bar 1 & \bar 0 \\ \bar 0 & \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \end{bmatrix}$	2	508	$T_1T_2 + T_3^2 + T_4^2,$ $\lambda T_3^2 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar 1 & \bar 1 & \bar 1 & \bar 1 & \bar 0 & \bar 0 \\ \bar 0 & \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \end{bmatrix}$	8
507	$T_1T_2 + T_3^2 + T_4^2,$ $\lambda T_3^2 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 & \bar 0 \\ \bar 0 & \bar 0 & \bar 0 & \bar 1 & \bar 1 & \bar 0 \end{bmatrix}$	8				

Classification list 4.18. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(1,1,1,2)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
496	$T_1^6 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 3 & 3 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	6	505	$T_1^2 + T_2^2 + T_3^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
497	$T_1^3 + T_2^2 + T_3^2$	$\begin{bmatrix} 2 & 3 & 3 & 3 & 1 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	6	506	$T_1^2 + T_2^2 + T_3^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
498	$T_1^2 + T_2^2 + T_3^2$	$\begin{bmatrix} 3 & 3 & 3 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	6	521	$T_1^2 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	16
502	$T_1^4 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 2 & 2 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8	522	$T_1^2 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	16
503	$T_1^4 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	8	524	$T_1^2 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 1 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	18
504	$T_1^4 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8				

Classification list 4.19. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,2,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
533	$T_1^2 T_2^2 + T_3^2 T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	2

Classification list 4.20. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
532	$T_1^2 T_2^2 + T_3^4 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	2	537	$T_1^2 T_2^2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	4

Classification list 4.21. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2, 1, 1, 1, 1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
536	$T_1 T_2 + T_3^2 + T_4^2,$ $\lambda T_3^2 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	4

Classification list 4.22. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(1, 1, 1, 2)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
534	$T_1^4 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	4
535	$T_1^2 + T_2^2 + T_3^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	4
538	$T_1^2 + T_2^2 + T_3^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	8

Classification list 4.23. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2, 2, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
529	$T_1^2 T_2^2 + T_3 T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{3} & \bar{2} & \bar{0} & \bar{1} \end{bmatrix}$	4

Classification list 4.24. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2, 2, 1, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
528	$T_1 T_2 + T_3 T_4 + T_5^2,$ $\lambda T_3 T_4 + T_5^2 + T_6^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{3} & \bar{1} & \bar{3} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	4

Classification list 4.25. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
527	$T_1T_2T_3 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{1} & 0 & \bar{1} & \bar{1} & \bar{1} \\ \bar{2} & 0 & \bar{0} & 3 & \bar{1} \end{bmatrix}$	4

Classification list 4.26. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
526	$T_1T_2 + T_3^4 + T_4^4$	$\begin{bmatrix} 2 & 2 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ 3 & \bar{1} & 3 & 0 & 3 \end{bmatrix}$	4	530	$T_1T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \\ \bar{2} & 0 & \bar{1} & \bar{3} & 0 \end{bmatrix}$	8

Classification list 4.27. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and format $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
531	$T_1T_2T_3 + T_4^3 + T_5^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & 0 & \bar{1} & 0 & 0 \\ \bar{5} & \bar{5} & \bar{2} & \bar{4} & \bar{0} \end{bmatrix}$	2

Classification list 4.28. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(2,2,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
424	$T_1^3T_2 + T_3^3T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 3 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{2} & \bar{1} & \bar{0} \end{bmatrix}$	6	434	$T_1T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{2} & 0 & \bar{2} & 0 & \bar{1} \end{bmatrix}$	12
425	$T_1^9T_2^3 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 1 & 6 & 6 & 4 \\ \bar{0} & \bar{1} & 0 & 0 & \bar{2} \end{bmatrix}$	6	437	$T_1^7T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 3 & 1 & 9 & 5 \\ \bar{0} & \bar{1} & \bar{1} & 0 & \bar{2} \end{bmatrix}$	18
426	$T_1^3T_2^3 + T_3T_4 + T_5^3$	$\begin{bmatrix} 1 & 2 & 3 & 6 & 3 \\ \bar{0} & \bar{1} & 0 & 0 & \bar{2} \end{bmatrix}$	6	438	$T_1T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	18
432	$T_1^{10}T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 4 & 2 & 12 & 7 \\ \bar{0} & \bar{1} & \bar{1} & 0 & \bar{2} \end{bmatrix}$	12	439	$T_1T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{0} \end{bmatrix}$	18
433	$T_1^7T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 6 & 4 \\ \bar{0} & \bar{1} & \bar{1} & 0 & \bar{2} \end{bmatrix}$	12				

Classification list 4.29. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(2, 2, 1, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
423	$\begin{array}{c} T_1T_2 + T_3T_4 + T_5^3, \\ \lambda T_3T_4 + T_5^3 + T_6^3 \end{array}$	$\begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 1 \\ \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{0} \end{bmatrix}$	6

Classification list 4.30. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(3, 1, 1, 0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
414	$T_1^2T_2^2T_3 + T_4^3 + T_5^3$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$	2	422	$T_1T_2T_3 + T_4^3 + T_5^3$	$\begin{bmatrix} 1 & 2 & 6 & 3 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$	6
415	$T_1^2T_2T_3 + T_4^3 + T_5^3$	$\begin{bmatrix} 2 & 1 & 1 & 2 & 2 \\ \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{0} \end{bmatrix}$	2	429	$T_1T_2T_3 + T_4^3 + T_5^3$	$\begin{bmatrix} 1 & 1 & 4 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$	8
416	$T_1T_2T_3 + T_4^9 + T_5^3$	$\begin{bmatrix} 1 & 4 & 4 & 1 & 3 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	4	430	$T_1T_2T_3 + T_4^3 + T_5^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	8
421	$T_1^3T_2^3T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 6 & 4 & 6 \\ \bar{1} & \bar{0} & \bar{0} & \bar{2} & \bar{0} \end{bmatrix}$	6				

Classification list 4.31. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(2, 1, 1, 1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
413	$T_1^2T_2^2 + T_3^3 + T_4^3$	$\begin{bmatrix} 2 & 1 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	2	428	$T_1^2T_2 + T_3^3 + T_4^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	8
418	$T_1T_2 + T_3^{12} + T_4^3$	$\begin{bmatrix} 6 & 6 & 1 & 4 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	6	431	$T_1T_2 + T_3^{12} + T_4^3$	$\begin{bmatrix} 2 & 10 & 1 & 4 & 5 \\ \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	10
419	$T_1T_2 + T_3^9 + T_4^3$	$\begin{bmatrix} 3 & 6 & 1 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$	6	435	$T_1T_2 + T_3^9 + T_4^3$	$\begin{bmatrix} 1 & 8 & 1 & 3 & 4 \\ \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16
420	$T_1T_2 + T_3^3 + T_4^3$	$\begin{bmatrix} 3 & 3 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{2} & \bar{0} & \bar{1} \end{bmatrix}$	6	436	$T_1T_2 + T_3^3 + T_4^3$	$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ \bar{1} & \bar{2} & \bar{2} & \bar{0} & \bar{2} \end{bmatrix}$	16
427	$T_1^2T_2 + T_3^3 + T_4^3$	$\begin{bmatrix} 1 & 4 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	8				

Classification list 4.32. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(1,1,1,2)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
417	$T_1^3 + T_2^3 + T_3^2$	$\begin{bmatrix} 2 & 2 & 3 & 3 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{2} \end{bmatrix}$	6

Classification list 4.33. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(2,2,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
525	$T_1T_2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{0} \\ \bar{2} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	6

Classification list 4.34. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2,2,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
452	$T_1^5T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{0} & \bar{2} & \bar{3} & \bar{0} & \bar{1} \end{bmatrix}$	8	455	$T_1^{10}T_2^2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 8 & 6 \\ \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{3} \end{bmatrix}$	8
453	$T_1^3T_2^3 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{2} & \bar{0} & \bar{3} & \bar{0} & \bar{1} \end{bmatrix}$	8	456	$T_1^6T_2^2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 2 & 8 & 5 \\ \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{3} \end{bmatrix}$	8
454	$T_1^2T_2 + T_3^2T_4 + T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ \bar{0} & \bar{2} & \bar{3} & \bar{0} & \bar{1} \end{bmatrix}$	8	457	$T_1^2T_2^2 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{3} & \bar{0} & \bar{2} & \bar{1} \end{bmatrix}$	8

Classification list 4.35. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2,2,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
451	$T_1T_2 + T_3T_4 + T_5^2,$ $\lambda T_3T_4 + T_5^2 + T_6^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{3} & \bar{3} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	8

Classification list 4.36. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
443	$T_1T_2T_3 + T_4^{10} + T_5^2$	$\begin{bmatrix} 1 & 3 & 6 & 1 & 5 \\ \bar{3} & \bar{3} & \bar{2} & \bar{0} & \bar{2} \end{bmatrix}$	6	449	$T_1T_2T_3 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{3} & \bar{3} & \bar{0} & \bar{3} & \bar{1} \end{bmatrix}$	8
444	$T_1T_2T_3 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 2 & 3 & 3 & 3 \\ \bar{3} & \bar{2} & \bar{3} & \bar{2} & \bar{0} \end{bmatrix}$	6	450	$T_1T_2T_3 + T_4^2 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{3} & \bar{1} & \bar{2} & \bar{3} & \bar{1} \end{bmatrix}$	8
448	$T_1^2T_2^2T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 8 & 4 & 6 \\ \bar{1} & \bar{0} & \bar{0} & \bar{2} & \bar{3} \end{bmatrix}$	8				

Classification list 4.37. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
440	$T_1T_2 + T_3^8 + T_4^4$	$\begin{bmatrix} 4 & 4 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	4	446	$T_1^2T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{0} & \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	8
441	$T_1^3T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 3 & 3 & 3 & 2 \\ \bar{0} & \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	6	447	$T_1T_2 + T_3^4 + T_4^4$	$\begin{bmatrix} 2 & 2 & 1 & 1 & 2 \\ \bar{1} & \bar{3} & \bar{3} & \bar{0} & \bar{3} \end{bmatrix}$	8
442	$T_1T_2 + T_3^8 + T_4^4$	$\begin{bmatrix} 2 & 6 & 1 & 2 & 3 \\ \bar{2} & \bar{2} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	6	458	$T_1T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{3} & \bar{1} & \bar{0} \end{bmatrix}$	16
445	$T_1^3T_2 + T_3^2 + T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	8				

Classification list 4.38. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and format $(2,2,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
462	$T_1^3T_2^3 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 5 & 3 \\ \bar{3} & \bar{0} & \bar{4} & \bar{0} & \bar{2} \end{bmatrix}$	10

Classification list 4.39. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and format $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
459	$T_1T_2T_3 + T_4^5 + T_5^5$	$\begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ \bar{2} & \bar{4} & \bar{4} & \bar{4} & \bar{0} \end{bmatrix}$	2	461	$T_1T_2T_3 + T_4^3 + T_5^2$	$\begin{bmatrix} 1 & 1 & 10 & 4 & 6 \\ \bar{1} & \bar{0} & \bar{0} & \bar{2} & \bar{3} \end{bmatrix}$	10

Classification list 4.40. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and format $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
460	$T_1T_2 + T_3^5 + T_4^5$	$\begin{bmatrix} 1 & 4 & 1 & 1 & 2 \\ \bar{2} & \bar{3} & \bar{4} & \bar{0} & \bar{4} \end{bmatrix}$	8

Classification list 4.41. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and format $(2,2,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
469	$T_1^4T_2^4 + T_3T_4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 6 & 4 \\ \bar{1} & \bar{0} & \bar{4} & \bar{0} & \bar{5} \end{bmatrix}$	6

Classification list 4.42. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and format $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
465	$T_1T_2T_3 + T_4^3 + T_5^3$	$\begin{bmatrix} 1 & 1 & 4 & 2 & 2 \\ \bar{3} & \bar{0} & \bar{0} & \bar{5} & \bar{1} \end{bmatrix}$	4	468	$T_1T_2T_3 + T_4^4 + T_5^2$	$\begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ \bar{2} & \bar{0} & \bar{0} & \bar{5} & \bar{1} \end{bmatrix}$	6
466	$T_1T_2T_3 + T_4^3 + T_5^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{5} & \bar{5} & \bar{4} & \bar{0} \end{bmatrix}$	4				

Classification list 4.43. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and format $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
463	$T_1^2T_2 + T_3^3 + T_4^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{3} & \bar{5} & \bar{1} & \bar{0} \end{bmatrix}$	4	467	$T_1T_2 + T_3^4 + T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 3 \\ \bar{2} & \bar{4} & \bar{0} & \bar{3} & \bar{3} \end{bmatrix}$	6
464	$T_1T_2 + T_3^6 + T_4^6$	$\begin{bmatrix} 2 & 4 & 1 & 1 & 2 \\ \bar{2} & \bar{4} & \bar{5} & \bar{0} & \bar{5} \end{bmatrix}$	4	470	$T_1T_2 + T_3^3 + T_4^3$	$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ \bar{5} & \bar{1} & \bar{4} & \bar{0} & \bar{4} \end{bmatrix}$	8

Classification list 4.44. Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ and format $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$	ID	relations	gd-matrix	$-\mathcal{K}^3$
471	$T_1T_2T_3 + T_4^4 + T_5^4$	$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ \bar{3} & \bar{7} & \bar{6} & \bar{6} & \bar{0} \end{bmatrix}$	2	472	$T_1T_2T_3 + T_4^6 + T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 1 & 3 \\ \bar{2} & \bar{0} & \bar{0} & \bar{3} & \bar{5} \end{bmatrix}$	4

5   Hilbert–Poincaré series

Here we present the Hilbert–Poincaré series of our Fano varieties. Recall that the Hilbert–Poincaré series of a finitely generated $\mathbb{Z}_{\geq 0}$-graded $\mathbb{K}$-algebra $A = \oplus_k A_k$ is the formal power series

$$\mathrm{HP}_A(t) := \sum_{k \geq 0} \dim_{\mathbb{K}}(A_k) t^k.$$

Assume that $f_1, \ldots, f_r \in A$ are homogeneous of degrees $w_1, \ldots, w_r$ respectively and generate A as an algebra. Then there is a polynomial $q_A \in \mathbb{Z}[t]$ such that

$$\mathrm{HP}_A(t) = \frac{q_A(t)}{\prod_{i=1}^r (1 - t^{w_i})}.$$

Given a Fano variety X, we associate with it the Hilbert–Poincaré series $\mathrm{HP}_X(t)$ of its anti-canonical ring A_X and we define the corresponding polynomial $q_X(t)$ with respect to a minimal system of homogeneous generators of the anticanonical ring A_X.

Proposition 5.1. *The following table lists for each possible pair (g, c) of genus and codimension the classification IDs from Section 4 of the varieties X attaining (g, c) and the cancelled presentation of the associated Hilbert–Poincaré series together with its first eight terms.*

(g, c)	$\mathrm{HP}_X(t)$	IDs
$(2, 1)$	$\dfrac{1 + t^3}{1 - t^4}$ $1 + 4t + 10t^2 + 21t^3 + 39t^4 + 66t^5 + 104t^6 + 155t^7 + \cdots$	2, 9, 10, 11, 246, 251, 252, 253, 254, 459, 471, 473, 474, 475, 477, 478, 479, 480, 531, 532, 533
$(2, 2)$	$\dfrac{1 + t^3}{1 - t^4}$ $1 + 4t + 10t^2 + 21t^3 + 39t^4 + 66t^5 + 104t^6 + 155t^7 + \cdots$	1, 3, 4, 5, 6, 7, 8, 247, 248, 249, 250, 413, 414, 415, 476
$(3, 1)$	$\dfrac{1 + t + t^2 + t^3}{1 - t^4}$ $1 + 5t + 15t^2 + 35t^3 + 69t^4 + 121t^5 + 195t^6 + 295t^7 + \cdots$	12, 416, 440, 472, 529
$(3, 2)$	$\dfrac{1 + t + t^2 + t^3}{1 - t^4}$ $1 + 5t + 15t^2 + 35t^3 + 69t^4 + 121t^5 + 195t^6 + 295t^7 + \cdots$	13, 14, 15, 16, 17, 18, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 463, 464, 465, 466, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 526, 527, 528, 534, 535, 536, 537
$(4, 2)$	$\dfrac{1 + 2t + 2t^2 + t^3}{1 - t^4}$ $1 + 6t + 20t^2 + 49t^3 + 99t^4 + 176t^5 + 286t^6 + 435t^7 + \cdots$	19, 21, 22, 28, 30, 32, 33, 34, 35, 36, 37, 280, 282, 287, 290, 417, 418, 419, 420, 421, 423, 424, 425, 426, 442, 443, 467, 468, 469, 501, 525

(g, c)	$\mathrm{HP}_X(t)$	IDs
$(4, 4)$	$$\frac{1 + 2t + 2t^2 + t^3}{1 - t^4}$$ $1 + 6t + 20t^2 + 49t^3 + 99t^4 + 176t^5 + 286t^6 + 435t^7 + \cdots$	20, 23, 24, 25, 26, 27, 29, 31, 38, 39, 40, 41, 42, 277, 278, 279, 281, 283, 284, 285, 286, 288, 289, 291, 292, 293, 294, 295, 296, 422, 441, 444, 496, 497, 498, 499, 500
$(5, 3)$	$$\frac{1 + 3t + 3t^2 + t^3}{1 - t^4}$$ $1 + 7t + 25t^2 + 63t^3 + 129t^4 + 231t^5 + 377t^6 + 575t^7 + \cdots$	301, 317, 321, 322, 325, 330, 332, 333, 334, 335, 448, 451, 452, 453, 454, 455, 456, 457, 503, 516, 517, 530
$(5, 6)$	$$\frac{1 + 3t + 3t^2 + t^3}{1 - t^4}$$ $1 + 7t + 25t^2 + 63t^3 + 129t^4 + 231t^5 + 377t^6 + 575t^7 + \cdots$	43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 297, 298, 299, 300, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 318, 319, 320, 323, 324, 326, 327, 328, 329, 331, 427, 428, 429, 430, 445, 446, 447, 449, 450, 460, 470, 502, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 518, 538
$(6, 4)$	$$\frac{1 + 4t + 4t^2 + t^3}{1 - t^4}$$ $1 + 8t + 30t^2 + 77t^3 + 159t^4 + 286t^5 + 468t^6 + 715t^7 + \cdots$	59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 336, 337, 431, 461, 462
$(7, 5)$	$$\frac{1 + 5t + 5t^2 + t^3}{1 - t^4}$$ $1 + 9t + 35t^2 + 91t^3 + 189t^4 + 341t^5 + 559t^6 + 855t^7 + \cdots$	71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 432, 433, 434, 519, 520
$(9, 7)$	$$\frac{1 + 7t + 7t^2 + t^3}{1 - t^4}$$ $1 + 11t + 45t^2 + 119t^3 + 249t^4 + 451t^5 + 741t^6 + 1135t^7 + \cdots$	101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 435, 436, 458, 521, 522, 523

(g,c)	$\mathrm{HP}_X(t)$	IDs
$(10,8)$	$$\frac{1+8t+8t^2+t^3}{1-t^4}$$ $1+12t+50t^2+133t^3+279t^4+506t^5+832t^6+1275t^7+\cdots$	118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 437, 438, 439, 524
$(11,9)$	$$\frac{1+9t+9t^2+t^3}{1-t^4}$$ $1+13t+55t^2+147t^3+309t^4+561t^5+923t^6+1415t^7+\cdots$	144, 145, 146, 147, 148, 149, 150, 151, 404, 405
$(12,10)$	$$\frac{1+10t+10t^2+t^3}{1-t^4}$$ $1+14t+60t^2+161t^3+339t^4+616t^5+1014t^6+1555t^7+\cdots$	152
$(13,11)$	$$\frac{1+11t+11t^2+t^3}{1-t^4}$$ $1+15t+65t^2+175t^3+369t^4+671t^5+1105t^6+1695t^7+\cdots$	153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 406, 407, 408, 409, 410, 411
$(16,14)$	$$\frac{1+14t+14t^2+t^3}{1-t^4}$$ $1+18t+80t^2+217t^3+459t^4+836t^5+1378t^6+2115t^7+\cdots$	188, 189, 190, 191, 192, 193, 194, 195, 196
$(17,15)$	$$\frac{1+15t+15t^2+t^3}{1-t^4}$$ $1+19t+85t^2+231t^3+489t^4+891t^5+1469t^6+2255t^7+\cdots$	197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 412
$(19,17)$	$$\frac{1+17t+17t^2+t^3}{1-t^4}$$ $1+21t+95t^2+259t^3+549t^4+1001t^5+1651t^6+2535t^7+\cdots$	216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227
$(21,19)$	$$\frac{1+19t+19t^2+t^3}{1-t^4}$$ $1+23t+105t^2+287t^3+609t^4+1111t^5+1833t^6+2815t^7+\cdots$	228, 229, 230, 231
$(22,20)$	$$\frac{1+20t+20t^2+t^3}{1-t^4}$$ $1+24t+110t^2+301t^3+639t^4+1166t^5+1924t^6+2955t^7+\cdots$	232, 233, 234
$(25,23)$	$$\frac{1+23t+23t^2+t^3}{1-t^4}$$ $1+27t+125t^2+343t^3+729t^4+1331t^5+2197t^6+3375t^7+\cdots$	235, 236, 237, 238

(g, c)	$\mathrm{HP}_X(t)$	IDs
$(26, 24)$	$\dfrac{1+24t+24t^2+t^3}{1-t^4}$ $1+28t+130t^2+357t^3+759t^4+1386t^5+2288t^6+3515t^7+\cdots$	239, 240, 241, 242
$(28, 26)$	$\dfrac{1+26t+26t^2+t^3}{1-t^4}$ $1+30t+140t^2+385t^3+819t^4+1496t^5+2470t^6+3795t^7+\cdots$	243, 244, 245

Proof. Observe that the anticanonical ring is the Veronese subalgebra of the Cox ring associated to the subgroup generated by the anticanonical class. Thus, we can use the Cox ring data from the classification lists in Section 4 to compute a minimal system of generators and the associated relations. This provides us in particular with genus and codimension. Moreover, it allows us to compute the Hilbert–Poincaré series; we used the computer algebra system Singular. ∎

Corollary 5.2. *The Hilbert–Poincaré series of a non-toric, $\mathbb{Q}$-factorial, log-terminal, Gorenstein, Fano threefold X of Picard number one with an effective action of a two-dimensional torus only depends on the genus g of X and can be explicitly written down as*

$$\mathrm{HP}_X(t) = \frac{1 + (g-2)(t+t^2) + t^3}{(1-t)^4}.$$

Acknowledgements

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