Lerch Φ Asymptotics

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Abstract. We use a Mellin–Barnes integral representation for the Lerch transcendent $\Phi(z,s,a)$ to obtain large z asymptotic approximations. The simplest divergent asymptotic approximation terminates in the case that s is an integer. For non-integer s the asymptotic approximations consists of the sum of two series. The first one is in powers of $(\ln z)^{-1}$ and the second one is in powers of z^{-1}. Although the second series converges, it is completely hidden in the divergent tail of the first series. We use resummation and optimal truncation to make the second series visible.

Key words: Hurwitz–Lerch zeta function; analytic continuation; asymptotic expansions

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1 Introduction and summary

This paper is in memory of Richard Paris. He was an expert in special functions and asymptotics, especially the use of Mellin–Barnes integral representations. Most of the techniques that we use in this paper are all coming from his book [12] (co-authored by David Kaminski). Note that in [11] Richard discusses the large a asymptotics for the Lerch transcendent, and he uses a Mellin–Barnes integral representation that is with respect to a, whereas (1.5) is with respect to z.

The Lerch transcendent is defined via

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s},\qquad |z|<1, \tag{1.1}$$

and via analytic continuation (see (1.2)) elsewhere in the complex z plane. If s is not an integer then $|\arg a|<\pi$; if s is a positive integer then $a\neq 0,-1,-2,\dots$; if s is a non-positive integer then a can be any complex number. The Lerch transcendent has a branch-point at $z=1$, which follows from the expansion (see [3, Section 1.11 (8)])

$$\Phi(z,s,a)=\Gamma(1-s)z^{-a}(-\ln z)^{s-1}+z^{-a}\sum_{n=0}^{\infty}\frac{\zeta(s-n,a)}{n!}(\ln z)^n,$$

$|\ln z|<2\pi$, $s\neq 1,2,3,\dots$, $a\neq 0,-1,-2,\dots$, in which $\zeta(s,a)=\Phi(1,s,a)$ is the Hurwitz zeta function. The principle branch for $\Phi(z,s,a)$ is the sector $|\arg(1-z)|<\pi$.

In several publications the main tool to obtain asymptotic expansions is the integral representation

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}\mathrm{e}^{-ax}}{1-z\mathrm{e}^{-x}}\,\mathrm{d}x,\qquad \operatorname{Re}s>0,\quad \operatorname{Re}a>0. \tag{1.2}$$

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The small and large a asymptotics is discussed in [2, 4, 11] (and [6] for the case $z = 1$), and the asymptotics as $\operatorname{Re} s \to -\infty$ is discussed in [5]. In [4], the large z asymptotics is also discussed, and the authors obtain

$$\Phi(-z, s, a) \sim \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{A_n(z, s, a)}{(-z)^n} + \frac{(\ln z)^s}{z^a \Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n(s, a)}{(\ln z)^{n+1}} \tag{1.3}$$

as $z \to \infty$, with

$$A_n(z, s, a) = \frac{\Gamma(s, (a-n)\ln z) - \Gamma(s)}{(a-n)^s} = \frac{-(\ln z)^s}{s} M(s, s+1; (a-n)\ln z), \tag{1.4}$$

and

$$B_0(s, a) = \frac{\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)}{2},$$
$$B_n(s, a) = \frac{n!\binom{s-1}{n}}{2^{n+1}} \left(\zeta\left(n+1, \frac{a}{2}\right) - \zeta\left(n+1, \frac{a+1}{2}\right) \right),$$

$n = 1, 2, 3, \ldots$. The second representation for $A_n(z, s, a)$ in (1.4) is in terms of the Kummer M confluent hypergeometric function (see [10, equation (13.2.2)]), and is more convenient from an analytical/numerical point of view. In [4], they do truncate the two sums in (1.3) and give sharp error bounds for each truncated sum. It seems not possible to combine the two error bounds to obtain an approximation that is correct up to a combined error estimate of, say, $\mathcal{O}(z^{-N})$, especially because the second sum in (1.3) is divergent, and its terms completely dominate the combined asymptotics. The incomplete gamma function in the first series seems misplaced, because for each n we have $\frac{\Gamma(s,(a-n)\ln z)}{(a-n)^s(-z)^n} = \mathcal{O}\left(\frac{(\ln z)^s}{z^a}\right)$ as $z \to \infty$. Hence, it seems that the incomplete gamma function should be incorporated in the second series of (1.3). However, we do observe that the advantage of definition (1.4) is that in this way the $A_n(z, s, a)$ are bounded as $a \to n$.

Below, we will show that the large z asymptotics simplifies dramatically when s is an integer. It will take a considerable amount of work to obtain that result from (1.3).

It is very surprising that it is not well known that the Lerch transcendent has a simple Mellin–Barnes integral representation. This is a more obvious tool to obtain asymptotics. For the moment we take $\operatorname{Re} a > 0$. In this case, we have the Mellin–Barnes integral representation (see [13, equation (3.11)])

$$\Phi(-z, s, a) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\Gamma(1+t)\Gamma(-t) z^t}{(a+t)^s} \, \mathrm{d}t, \tag{1.5}$$

in which L is a contour from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ which crosses the real t axis somewhere in the interval $(\max(-\operatorname{Re} a, -1), 0)$. When we push this contour to the right we will get contributions from the poles of $\Gamma(-t)$ and this results in expansion (1.1). Integral representation (1.5) supplies an analytic continuation to the sector $|\arg z| < \pi$.

We write for the moment $z = \rho e^{\mathrm{i}\theta}$ and observe that for the dominant factor of the integrand in (1.5) we have $\left|\Gamma(1+t)\Gamma(-t) z^t\right| \sim 2\pi \rho^{\operatorname{Re} t} e^{(\mp\pi - \theta)\operatorname{Im} t}$ as $\operatorname{Im} t \to \pm\infty$. The factor $\rho^{\operatorname{Re} t}$ can be used to analytically continue integral representation (1.5). In the case $0 < \rho < 1$, we bent contour L to the right (say we go from $(1 - \mathrm{i})\infty$ to $(1 + \mathrm{i})\infty$) and with this choice for L integral representation (1.5) is also valid for $z \in (-1, 0)$. In the case that $\rho > 1$, we bent contour L to the left (say we go from $(-1-\mathrm{i})\infty$ to $(-1+\mathrm{i})\infty$) and with this choice for L integral representation (1.5) is valid across the z-branch-cut $(-\infty, -1)$.

To obtain a large z asymptotic expansion all that we have to do is to push L to the left. This is what they do in [13], but somehow they do miss the fact that there will also be contributions from the branch-point at $t = -a$.

In Section 2, we will study the large z asymptotics by pushing contour L to the left. The contributions of the poles of $\Gamma(1+t)$ will give us a simple convergent asymptotic expansion in powers of z^{-1} that can be expressed in terms of a Lerch transcendent. The contribution of the branch-point at $t=-a$ is more complicated. In the case that $s=S$ is an integer, this contribution is just a finite sum in powers of $1/\ln z$, and we obtain

$$\Phi(-z,s,a)=\frac{2\pi\mathrm{i}}{z^a}\sum_{n=0}^{S-1}\frac{b_n}{\Gamma(S-n)(\ln z)^{n-S+1}}-\sum_{n=1}^{\infty}\frac{(-z)^{-n}}{(a-n)^S}, \tag{1.6}$$

in which the first sum is zero in the case that S is a non-positive integer. The coefficients b_n are the Taylor coefficients of $\frac{\mathrm{i}}{2}\csc\pi(t-a)$ about $t=0$. This result has already been used in [7, Lemma 3].

In the case that s is not an integer, the first series in (1.6) does not terminate and it is a divergent series. We will use resummation and optimal truncation to obtain an approximation for $\Phi(-z,s,a)$ that is accurate up to order $\mathcal{O}\left(z^{-N-1}\right)$:

Theorem 1. *We take $|\arg(1+z)|\le\pi$, a and s bounded complex numbers, $\operatorname{Re}a>0$. Let N be a fixed integer with $N>\operatorname{Re}a$. Take $|z|$ large and let M be a positive integer such that $|M-s+1|\approx|(N+1-a)\ln z|$. Then*

$$\begin{aligned}\Phi(-z,s,a)=&\sum_{n=0}^{N}\frac{(-z)^n\Gamma(s,(a+n)\ln z)}{(a+n)^s\Gamma(s)}+\frac{2\pi\mathrm{i}}{z^a}\sum_{m=0}^{M-1}\frac{b_{m,N}}{\Gamma(s-m)(\ln z)^{m-s+1}}\\&+\frac{1}{\Gamma(s)}\sum_{n=1}^{N}\frac{A_n(z,s,a)}{(-z)^n}+\mathcal{O}\left(z^{-N-1}\right),\end{aligned}$$

as $z\to\infty$ uniformly with respect to $\arg z\in[-\pi,\pi]$. The $A_n(z,s,a)$ are defined in (1.4) and

$$\begin{aligned}b_{0,N}=&\frac{(-1)^N}{4\pi\mathrm{i}}\left(\psi\left(\frac{N+1+a}{2}\right)-\psi\left(\frac{N+2+a}{2}\right)\right.\\&\left.-\psi\left(\frac{N+1-a}{2}\right)+\psi\left(\frac{N+2-a}{2}\right)\right),\\b_{n,N}=&\frac{(-1)^N}{2^{n+2}\pi\mathrm{i}}\left(\zeta\left(n+1,\frac{N+2+a}{2}\right)-\zeta\left(n+1,\frac{N+1+a}{2}\right)\right.\\&\left.+(-1)^n\zeta\left(n+1,\frac{N+1-a}{2}\right)-(-1)^n\zeta\left(n+1,\frac{N+2-a}{2}\right)\right),\end{aligned}\tag{1.7}$$

$n=1,2,3,\ldots$.

We numerically verify this result in Table 1. Observe above that the M does depend on z. In the m-series we do take an optimal number of terms. Hence, its remainder will be 'exponentially-small'. In the proof below we do show that it is $\mathcal{O}\left(\mathrm{e}^{-(N+1)|x|}\right)$, with $x=\ln z$.

Often when one encounters divergent Poincaré asymptotic series, it is possible to convert it to a convergent factorial series. In the case that s is not an integer it would be convenient to replace the first series in (1.6) (which diverges) by a convergent factorial series and in that way we can also incorporate in our approximation the full second series of (1.6). We do create a convergent factorial-type series in Section 3. As usual with these types of series, the convergence is very slow, and it is not easy to obtain sharp error estimates.

2 Large z asymptotics

We start with $0<a<1$ and we can use analytic continuation afterwards. We take $|z|>1$ in the sector $|\arg(1+z)|\leq\pi$ and push L to the left and obtain

$$\Phi(-z,s,a)=B(z,s,a)+I(z,s,a). \tag{2.1}$$

The first term is the contribution of the branch-point at $t=-a$

$$B(z,s,a)=\frac{1}{2\pi\mathrm{i}}\int_{-(1+\mathrm{i})\infty}^{(-a+)}\frac{\Gamma(1+t)\Gamma(-t)z^t}{(a+t)^s}\,\mathrm{d}t, \tag{2.2}$$

in which the contour begins at $-(1+\mathrm{i})\infty$, circles $t=-a$ once in the positive direction, and returns to $-(1+\mathrm{i})\infty$. We did observe in the second paragraph below (1.5) that in the case $|z|>1$ the integrals converge across the z-branch cuts $\arg z=\pm\pi$. The second term in (2.1) is the sum of the residue contributions of the poles of $\Gamma(1+t)$

$$I(z,s,a)=-\mathrm{e}^{-\pi\mathrm{i}s}\sum_{n=1}^{\infty}\frac{(-z)^{-n}}{(n-a)^s}=\mathrm{e}^{-2\pi\mathrm{i}s}a^{-s}-\mathrm{e}^{-\pi\mathrm{i}s}\Phi(-1/z,s,-a). \tag{2.3}$$

The infinite series representation of $I(z,s,a)$ is already a large z asymptotic expansion. Hence, all we need is an asymptotic expansion for $B(z,s,a)$. We modify integral representation (2.2) as

$$B(z,s,a)=z^{-a}\int_{-(1+\mathrm{i})\infty}^{(0+)}\mathrm{e}^{t\ln z}t^{-s}g(t)\,\mathrm{d}t, \tag{2.4}$$

with

$$g(t)=\frac{\mathrm{i}}{2\sin\pi(t-a)}=\lim_{N\to\infty}\frac{\mathrm{i}}{2\pi}\sum_{n=-N}^{N}\frac{(-1)^n}{t-a-n}. \tag{2.5}$$

We use this sum representation of $g(t)$ in (2.4) and obtain the expansion

$$B(z,s,a)=\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(-z)^n}{(a+n)^s\Gamma(s)}\Gamma(s,(a+n)\ln z), \tag{2.6}$$

in terms of the incomplete gamma function. This expansions converges slowly, but has no asymptotic property as $z\to\infty$.

To obtain a simple Poincaré asymptotic expansions in inverse powers of $\ln z$, we have to expand $g(t)$ about the origin. Let

$$g(t)=\sum_{n=0}^{\infty}b_nt^n,\qquad b_0=\frac{1}{2\mathrm{i}\sin\pi a},\qquad b_1=2\pi\mathrm{i}b_0^2\cos\pi a. \tag{2.7}$$

The reader can check that $g(t)g''(t)=2g'^2(t)+\pi^2g^2(t)$, from which we obtain the recurrence relation

$$\begin{aligned}(n+2)(n+1)b_0b_{n+2}=&\sum_{m=0}^{n}\left(2(m+1)(n-m+1)b_{m+1}b_{n-m+1}+\pi^2b_mb_{n-m}\right)\\&-\sum_{m=0}^{n-1}(m+2)(m+1)b_{m+2}b_{n-m},\qquad n\geq0.\end{aligned}$$

Hence, the computation of the coefficients is straightforward.

To obtain an asymptotic expansion for the right-hand side of (2.4), we will use Watson's lemma for loop integrals. See [9, Section 4.5.3]. We substitute (2.7) into (2.4) and obtain the asymptotic expansion

$$B(z,s,a) \sim \frac{2\pi\mathrm{i}}{z^a}\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(s-n)(\ln z)^{n-s+1}}, \tag{2.8}$$

as $z \to \infty$.

In the case that $s = S$ is an integer, the infinite series on the right-hand side of (2.8) terminates. If S is a non-positive integer, then $B(z,S,a) = 0$, and if S is a positive integer, we have

$$B(z,S,a) = \frac{2\pi\mathrm{i}}{z^a}\sum_{n=0}^{S-1}\frac{b_n}{\Gamma(S-n)(\ln z)^{n-S+1}}.$$

However, in the case that s is not an integer (2.8) is a divergent asymptotic expansion. For modestly large $|z|$, the $\ln z$ in this expansion is not very large, and the optimal number of terms will be small. In Section 3, we will obtain another type of asymptotic approximation for $B(z,s,a)$. Here we are going to combine the three expansions (2.8), (2.3) and (2.6).

First, we observe that when we take an optimal number of terms in (2.8) the smallest term will still be bigger than the first term in (2.3). The optimal number of terms is connected to the distance between the origin and the nearest pole of $g(t)$. By incorporating in the expansion the poles of $g(t)$ that are near the origin, we 'slow down' the divergence of (2.8).

Proof of Theorem 1. In (2.1), we write $\Phi(-z,s,a) = B(z,s,a)+I(z,s,a)$ and for $I(z,s,a)$ we have the convergent asymptotic expansion (2.3). Hence, we need an asymptotic approximation for $B(z,s,a)$ with the correct remainder estimate. Let N be a fixed positive integer and take

$$g_N(t) = g(t) - \frac{\mathrm{i}}{2\pi}\sum_{n=-N}^{N}\frac{(-1)^n}{t-a-n},$$

compare (2.5). Then

$$B(z,s,a) = \sum_{n=-N}^{N}\frac{(-z)^n}{(a+n)^s\Gamma(s)}\Gamma(s,(a+n)\ln z) + z^{-a}\int_{-(1+\mathrm{i})\infty}^{(0+)} \mathrm{e}^{t\ln z}t^{-s}g_N(t)\,\mathrm{d}t.$$

We have

$$g_N(t) = \sum_{n=0}^{\infty} b_{n,N}t^n,$$

with

$$b_{n,N} = b_n + \frac{\mathrm{i}}{2\pi}\sum_{m=-N}^{N}\frac{(-1)^m}{(a+m)^{n+1}}. \tag{2.9}$$

A numerical more stable presentation for the $b_{n,N}$ is (1.7).

For large n, the $b_{n,N}$ will be of the size of the first omitted terms in (2.9), that is, for the case $0 < a < 1$, the term with $m = -N-1$. Thus $b_{n,N} \sim \frac{\mathrm{i}}{2\pi}(-1)^N(a-N-1)^{-n-1}$ as $n \to \infty$. We will estimate the remainder in

$$z^{-a}\int_{-(1+\mathrm{i})\infty}^{(0+)} \mathrm{e}^{t\ln z}t^{-s}g_N(t)\,\mathrm{d}t = \frac{2\pi\mathrm{i}}{z^a}\sum_{m=0}^{M-1}\frac{b_{m,N}}{\Gamma(s-m)(\ln z)^{m-s+1}} + R_M^{(B)}(z)$$

via the machinery in [8, Section 4], because the integrals are exactly of the same form as the ones discussed in that paper. We want the remainder after taking the optimal number of terms. Hence, M will be large. We have

$$\begin{aligned}\left|R_M^{(B)}(z)\right| &= \mathcal{O}\left(\frac{z^{-a}(\ln z)^{s-M}}{(N+1-a)^M}F^{(1)}\left((a-N-1)\ln z;\begin{matrix}M-s+1\\1\end{matrix}\right)\right)\\ &= \mathcal{O}\left(\frac{z^{-a}\Gamma(M-s+1)\sqrt{M-s+1}}{((N+1-a)\ln z)^{M-s+1}}\right),\end{aligned} \tag{2.10}$$

in which both M and z are large. For the first hyperterminant $F^{(1)}$ see [1, Appendix A]. The second equal sign in (2.10) follows from [1, Proposition B.1]. It follows that for the optimal $M = M_{\text{opt}}$, we have $|M_{\text{opt}} - s + 1| \sim |(N+1-a)\ln z|$. With this choice for M, we have

$$\left|R_M^{(B)}(z)\right| = \mathcal{O}\left(\frac{z^{-a}\Gamma(M-s+1)}{(M-s+1)^{M-s+\frac{1}{2}}}\right) = \mathcal{O}\left(z^{-a}\mathrm{e}^{s-M-1}\right) = \mathcal{O}\left(z^{-N-1}\right),$$

as $z \to \infty$.

Combining the results above, we have the approximation

$$\begin{aligned}\Phi(-z,s,a) = {}& \sum_{n=-N}^{N}\frac{(-z)^n\Gamma(s,(a+n)\ln z)}{(a+n)^s\Gamma(s)} + \frac{2\pi\mathrm{i}}{z^a}\sum_{m=0}^{M_{\text{opt}}-1}\frac{b_{m,N}}{\Gamma(s-m)(\ln z)^{m-s+1}}\\ &- \mathrm{e}^{-\pi\mathrm{i}s}\sum_{n=1}^{N}\frac{(-z)^{-n}}{(n-a)^s} + R_N(z),\\ = {}& \sum_{n=0}^{N}\frac{(-z)^n\Gamma(s,(a+n)\ln z)}{(a+n)^s\Gamma(s)} + \frac{2\pi\mathrm{i}}{z^a}\sum_{m=0}^{M_{\text{opt}}-1}\frac{b_{m,N}}{\Gamma(s-m)(\ln z)^{m-s+1}}\\ &+ \frac{1}{\Gamma(s)}\sum_{n=1}^{N}\frac{A_n(z,s,a)}{(-z)^n} + R_N(z),\end{aligned} \tag{2.11}$$

with $R_N(z) = \mathcal{O}\left(z^{-N-1}\right)$, as $z \to \infty$ uniformly with respect to $\arg z \in [-\pi, \pi]$. The second presentation in our main result (2.11) has the advantage that it is obvious that there are no issues when a approaches a positive integer. ■

In Table 1, we do illustrate that the implied constant in the order estimate (2.11) seems very reasonable in the full z-sector. Note that the final result in Table 1, is for z very close to the boundary of the sector $|\arg(1+z)| \le \pi$.

3 A factorial series expansion

In this section, we will assume that $\left|1 - \mathrm{e}^{-2\pi\mathrm{i}a}\right| > 1$, which is the case when, for example, $a \in \left(\frac{1}{6}, \frac{5}{6}\right)$. We present $g(t)$ as

$$g(t) = \frac{1}{\mathrm{e}^{\pi\mathrm{i}(a-t)} - \mathrm{e}^{\pi\mathrm{i}(t-a)}} = \frac{\mathrm{e}^{\pi\mathrm{i}(a-t)}}{\left(\mathrm{e}^{2\pi\mathrm{i}a} - 1\right)\left(1 - \frac{1-\mathrm{e}^{-2\pi\mathrm{i}t}}{1-\mathrm{e}^{-2\pi\mathrm{i}a}}\right)}.$$

With the above assumption on a we guarantee that the geometric progression

$$g(t) = \mathrm{e}^{\pi\mathrm{i}(a-t)}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}an}}{\left(\mathrm{e}^{2\pi\mathrm{i}a} - 1\right)^{n+1}}\left(1 - \mathrm{e}^{-2\pi\mathrm{i}t}\right)^n, \tag{3.1}$$

	$z=5,\ M_{\text{opt}}=9$	$z=10,\ M_{\text{opt}}=13$
$\Phi(-z,s,a)$	1.3421782	1.0889334
approx (2.11)	1.3421692	1.0889332
$\lvert z^{N+1}R_N(z)\rvert$	0.140	0.158
	$z=100,\ M_{\text{opt}}=27$	$z=1000,\ M_{\text{opt}}=40$
$\Phi(-z,s,a)$	0.50810464209509	0.2350035297496389971
approx (2.11)	0.50810464209489	0.2350035297496389969
$\lvert z^{N+1}R_N(z)\rvert$	0.200	0.197
	$z=10\mathrm{i},\ M_{\text{opt}}=16$	$z=-10+0.01\mathrm{i},\ M_{\text{opt}}=22$
$\Phi(-z,s,a)$	0.98125249 − 0.54864116i	0.52526675 − 1.04285831i
approx (2.11)	0.98125270 − 0.54864133i	0.52526654 − 1.04285810i
$\lvert z^{N+1}R_N(z)\rvert$	0.269	0.297

Table 1. Approximation (2.11) for the case $a=0.3$, $s=\frac{3}{4}$ and $N=5$.

converges uniformly for t along the contour in (3.2). Using (3.1) in (2.4) gives us the expansion

$$B(z,s,a)=\frac{\mathrm{e}^{\pi\mathrm{i}a}}{z^a}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}an}}{\left(\mathrm{e}^{2\pi\mathrm{i}a}-1\right)^{n+1}}\int_{-\mathrm{i}\infty}^{(0+)}\mathrm{e}^{(\ln(z)-\pi\mathrm{i})t}t^{-s}\left(1-\mathrm{e}^{-2\pi\mathrm{i}t}\right)^n\,\mathrm{d}t, \tag{3.2}$$

which after taking $\tau=2\pi\mathrm{i}t$ and $x=\frac{1}{2}-\frac{\ln z}{2\pi\mathrm{i}}$ becomes

$$B(z,s,a)=\frac{\mathrm{e}^{\pi\mathrm{i}a}(2\pi\mathrm{i})^{s-1}}{z^a}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}an}}{\left(\mathrm{e}^{2\pi\mathrm{i}a}-1\right)^{n+1}}p_n(x,s), \tag{3.3}$$

with

$$p_n(x,s)=\int_{\infty}^{(0+)}\mathrm{e}^{-x\tau}\tau^{-s}\left(1-\mathrm{e}^{-\tau}\right)^n\,\mathrm{d}\tau. \tag{3.4}$$

If we assume that $\operatorname{Re}s<1$, we can collapse the loop contour

$$p_n(x,s)=\left(\mathrm{e}^{-2\pi\mathrm{i}s}-1\right)\int_0^{\infty}\mathrm{e}^{-x\tau}\tau^{-s}\left(1-\mathrm{e}^{-\tau}\right)^n\,\mathrm{d}\tau.$$

Convergent expansion (3.3) is the main result of this section. It is a factorial-type expansion because $p_n(x,s)$ is a generalisation (a fractional integral) of

$$\int_0^{\infty}\mathrm{e}^{-x\tau}\left(1-\mathrm{e}^{-\tau}\right)^n\,\mathrm{d}\tau=B(n+1,x)=\frac{n!}{x(x+1)(x+2)\cdots(x+n)}.$$

We still have to discuss the large x asymptotic behaviour of $p_n(x,s)$, and an alternative method to evaluate this function.

We use the expansion

$$\tau^{-s}=\sum_{m=0}^{\infty}c_m\left(1-\mathrm{e}^{-\tau}\right)^{m-s},$$

which is the generating function for its coefficients c_m,[1] and obtain

$$
\begin{aligned}
p_n(x,s) &= \left(\mathrm{e}^{-2\pi\mathrm{i}s}-1\right)\sum_{m=0}^{\infty} c_m \int_0^{\infty} \mathrm{e}^{-x\tau}\left(1-\mathrm{e}^{-\tau}\right)^{n+m-s}\,\mathrm{d}\tau \\
&= \left(\mathrm{e}^{-2\pi\mathrm{i}s}-1\right)\sum_{m=0}^{\infty} c_m B(n+m-s+1,x) \\
&\sim \left(\mathrm{e}^{-2\pi\mathrm{i}s}-1\right)\Gamma(n-s+1)x^{s-n-1},
\end{aligned}
\tag{3.5}
$$

as $x \to \infty$. Hence, $p_0(x,s), p_1(x,s), p_2(x,s), \dots$ is definitely an asymptotic sequence. These infinite series are conditionally convergent!

The binomial expansion of $\left(1-\mathrm{e}^{-\tau}\right)^n$ in (3.4) will give us

$$
p_n(x,s) = \frac{-2\pi\mathrm{i}\mathrm{e}^{-\pi\mathrm{i}s}}{\Gamma(s)}\sum_{m=0}^{n}\binom{n}{m}(-1)^m(x+m)^{s-1},
\tag{3.6}
$$

a finite sum. For modestly large z, our x will not be large at all, and there is no issue computing the $p_n(x,s)$ via (3.6). In this case, (3.3) will still converge, but it can not be regarded an asymptotic expansion. However, in the case that x is large and comparing the terms in (3.6) with estimate (3.5) it follows that the terms in (3.6) are of the wrong size, hence cancellations. We can deal with these cancellations via the identity

$$
(1+\delta)^{\ell} = \sum_{k=0}^{n-1}\binom{\ell}{k}\delta^k + \frac{(-\ell)_n(-\delta)^n}{n!}\,{}_2F_1\left(\begin{matrix} n-\ell, 1 \\ n+1 \end{matrix}; -\delta\right).
$$

This leads to

$$
p_n(x,s) = \frac{-2\pi\mathrm{i}\mathrm{e}^{-\pi\mathrm{i}s}}{\Gamma(s-n)}x^{s-n-1}\sum_{m=0}^{n}\frac{(-1)^m m^n}{m!(n-m)!}\,{}_2F_1\left(\begin{matrix} n-s+1, 1 \\ n+1 \end{matrix}; -m/x\right).
$$

Now the terms are of the correct size.

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[1]We can identify c_m in terms of generalised Bernoulli coefficients: $c_m = (-1)^{m+1}s\frac{B_m^{(m-s)}}{(m-s)m!}$. See [10, equation (24.16.4)]).

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