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A FUNCTIONAL CALCULUS FOR QUOTIENT BOUNDED OPERATORS

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Abstract. If $(X, \mathcal{P})$ is a sequentially locally convex space, then a quotient bounded operator $T \in Q_{\mathcal{P}}(X)$ is regular (in the sense of Waelbroeck) if and only if it is a bounded element (in the sense of Allan) of algebra $Q_{\mathcal{P}}(X)$. The classic functional calculus for bounded operators on Banach space is generalized for bounded elements of algebra $Q_{\mathcal{P}}(X)$.

1 Introduction

It is well-known that if X is a Banach space and $\mathcal{L}(X)$ is Banach algebra of bounded operators on X, then formula

$$f(T) = \frac{1}{2\pi i}\int_{\Gamma} f(z)\, R(z,T)\, dz,$$

(where f is an analytic function on some neighborhood of $\sigma(T)$, Γ is a closed rectifiable Jordan curve whose interior domain D is such that $\sigma(T) \subset D$, and f is analytic on D and continuous on $D \cup \Gamma$) defines a homomorphism $f \to f(T)$ from the set of all analytic functions on some neighborhood of $\sigma(T)$ into $L(X)$, with very useful properties.

Through this paper all locally convex spaces will be assumed Hausdorff, over complex field **C**, and all operators will be linear. If X and Y are topological vector spaces we denote by $L(X, Y)$ ($\mathcal{L}(X, Y)$) the algebra of linear operators (continuous operators) from X to Y.

Any family $\mathcal{P}$ of seminorms which generate the topology of locally convex space X (in the sense that the topology of X is the coarsest with respect to which all seminorms of $\mathcal{P}$ are continuous) will be called a calibration on X. A calibration $\mathcal{P}$ is characterized by the property that the collection of all sets

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$$S(p,\epsilon) = \{x \in X | p(x) < \epsilon\}, (p \in \mathcal{P}, \epsilon > 0),$$

constitute a neighborhoods sub-base at 0. A calibration on X will be principal if it is directed. The set of all calibrations for X is denoted by $\mathcal{C}(X)$ and the set of all principal calibration by $\mathcal{C}_0(X)$.

If $(X,\mathcal{P})$ is a locally convex algebra and each seminorms $p \in \mathcal{P}$ is submultiplicative then $(X,\mathcal{P})$ is locally multiplicative convex algebra or l.m.c.-algebra.

Any family of seminorms on a linear space is partially ordered by relation „ $\leq$”, where

$$p \leq q \Leftrightarrow p(x) \leq q(x),\ (\forall)\, x \in X.$$

A family of seminorms is preordered by relation „ $\prec$”, where

$$p \prec q \Leftrightarrow \text{there exists some } r > 0 \text{ such that } p(x) \leq rq(x), \text{ for all } x \in X.$$

If $p \prec q$ and $q \prec p$, we write $p \approx q$.

Definition 1. *Two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on a linear space are called Q-equivalent (denoted $\mathcal{P}_1 \approx \mathcal{P}_2$) provided:*

1. *for each $p_1 \in P_1$ there exists $p_2 \in \mathcal{P}_2$ such that $p_1 \approx p_2$;*
2. *for each $p_2 \in P_2$ there exists $p_1 \in \mathcal{P}_1$ such that $p_2 \approx p_1$.*

It is obvious that two Q-equivalent and separating families of seminorms on a linear space generate the same locally convex topology.

Definition 2. *If $(X,\mathcal{P})$, $(Y,\mathcal{Q})$ are locally convex spaces, then for each $p,q \in P$ the application $m_{pq} : L(X,Y) \to \mathbf{R} \cup \{\infty\}$, defined by*

$$m_{pq}(T) = \sup_{p(x)\neq 0} \frac{q(Tx)}{p(x)}, (\forall)\, T \in L(X,Y).$$

is called the mixed operator seminorm of T associated with p and q. When $X = Y$ and $p = q$ we use notation $\hat{p} = m_{pp}$.

Lemma 3 ([9])**.** *If $(X,\mathcal{P})$, $(Y,\mathcal{Q})$ are locally convex spaces and $T \in L(X,Y)$, then*

1. $m_{pq}(T) = \sup_{p(x)=1} q(Tx) = \sup_{p(x)\leq 1} q(Tx), (\forall)\, p \in P, (\forall)\, q \in Q;$
2. $q(Tx) \leq m_{pq}(T)\, p(x),\ (\forall)\, x \in X,$ *whenever* $m_{pq}(T) < \infty$.
3. $m_{pq}(T) = \inf\{M > 0\, | q(Tx) \leq Mp(x), (\forall)\, x \in X\},$ *whenever* $m_{pq}(T) < \infty$.

Definition 4. *An operator T on a locally convex space X is quotient bounded with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ if for every seminorm $p \in \mathcal{P}$ there exists some $c_p > 0$ such that*

$$p(Tx) \leq c_p p(x), (\forall)\, x \in X.$$

The class of quotient bounded operators with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ is denoted by $Q_{\mathcal{P}}(X)$. If X is a locally convex space and $\mathcal{P} \in C(X)$, then for every $p \in \mathcal{P}$ the application $\hat{p} : Q_{\mathcal{P}}(X) \to \mathbf{R}$ defined by

$$\hat{p}(T) = \inf\{ \ r > 0 \mid p(Tx) \leq r \ p(x) \, , (\forall) \, x \in X \ \},$$

is a submultiplicative seminorm on $Q_{\mathcal{P}}(X)$, satisfying $\hat{p}(I) = 1$. We denote by $\hat{\mathcal{P}}$ the family $\{ \ \hat{\mathcal{P}} \mid p \in \mathcal{P} \ \}$.

Lemma 5 ([8]). *If X is a sequentially complete convex space, then $Q_P(X)$ is a sequentially complete m-convex algebra for all $\mathcal{P} \in \mathcal{C}(X)$.*

Definition 6. *Let X be a locally convex space and $T \in Q_{\mathcal{P}}(X)$. We say that $\lambda \in \rho(Q_{\mathcal{P}}, T)$ if the inverse of $\lambda I - T$ exists and $(\lambda I - T)^{-1} \in Q_{\mathcal{P}}(X)$. Spectral sets $\sigma(Q_{\mathcal{P}}, T)$ are defined to be complements of resolvent sets $\rho(Q_{\mathcal{P}}, T)$.*

Let $(X, \mathcal{P})$ be a locally convex space and $T \in Q_{\mathcal{P}}(X)$. We have said that T is bounded element of the algebra $Q_{\mathcal{P}}(X)$ if it is bounded element in the sens of G.R.Allan [1], i.e. some scalar multiple of it generates a bounded semigroup. The class of bounded element of $Q_{\mathcal{P}}(X)$ is denoted by $(Q_{\mathcal{P}}(X))_0$.

Proposition 7 ([5]). *Let X is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$.*

1. *$Q_{\mathcal{P}}(X)$ is a unital subalgebra of the algebra of continuous linear operators on X*

2. *$Q_{\mathcal{P}}(X)$ is a unitary l.m.c.-algebra with respect to the topology determined by $\hat{\mathcal{P}}$*

3. *If $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$, then $Q_{\mathcal{P}'}(X) = Q_{\mathcal{P}}(X)$ and $\hat{\mathcal{P}} \approx \hat{\mathcal{P}}\,'$*

4. *The topology generated by $\hat{\mathcal{P}}$ on $Q_{\mathcal{P}}(X)$is finer than the topology of uniform convergence on bounded subsets of X*

Definition 8. *If $(X, \mathcal{P})$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ we denote by $r_{\mathcal{P}}(T)$ the radius of boundness of operator T in $Q_{\mathcal{P}}(X)$, i.e.*

$$r_{\mathcal{P}}(T) = \inf\{\alpha > 0 \ \mid \ \alpha^{-1}T \textit{ generates a bounded semigroup in } Q_{\mathcal{P}}(X)\}.$$

We have said that $r_{\mathcal{P}}(T)$ is the $\mathcal{P}$-spectral radius of the operator T.

Proposition 9 ([8]). *Let X be a sequentially complete locally convex space and $\mathcal{P} \in C(X)$. If $T \in Q_{\mathcal{P}}(X)$, then $\mid \sigma(Q_{\mathcal{P}}, T) \mid = r_{\mathcal{P}}(T)$.*

Definition 10. *Let $\mathcal{P}$ be a calibration on X. A linear operator $T : X \to X$ is universally bounded on $(X, \mathcal{P})$ if exists a constant $c_0 > 0$ such that*

$$p(Tx) \leq c_0 p\,(x)\, , (\forall) \, x \in X.$$

Denote by $B_{\mathcal{P}}(X)$ the collection of all universally bounded operators on $(X, \mathcal{P})$. It is obvious that $Q_{\mathcal{P}}(X) \subset B_{\mathcal{P}}(X) \subset \mathcal{L}(X)$.

Lemma 11. *If $\mathcal{P}$ a calibration on X, then $B_{\mathcal{P}}(X)$ is a unital normed algebra with respect to the norm $\|\bullet\|_{\mathcal{P}}$ defined by*

$$\|T\|_{\mathcal{P}} = \inf\{M > 0 \mid \ p(Tx) \leq Mp(x), (\forall)\, x \in X, (\forall)\ p \in \mathcal{P}\}.$$

Corollary 12. *If $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in B_{\mathcal{P}}(X)$ we have*

$$\|T\|_{\mathcal{P}} = \sup\{m_{pp}(T) \mid p \in \mathcal{P}\}, (\forall)\, T \in B_{\mathcal{P}}(X).$$

Proposition 13 ([5])**.** *Let X be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. Then:*

1. *$B_{\mathcal{P}}(X)$ is a subalgebra of $\mathcal{L}(X)$;*
2. *$(B_{\mathcal{P}}(X), \|\bullet\|_{\mathcal{P}})$ is unitary normed algebra;*
3. *for each $\mathcal{P}' \in \mathcal{C}(X)$ with the property $\mathcal{P} \approx \mathcal{P}'$, we have*

$$B_{\mathcal{P}}(X) = B_{\mathcal{P}'}(X) \text{ and } \|\bullet\|_{\mathcal{P}} = \|\bullet\|_{\mathcal{P}'}.$$

Proposition 14 ([2])**.** *Let X be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. Then:*

1. *the topology given by the norm $\|\bullet\|_{\mathcal{P}}$ on the algebra $B_{\mathcal{P}}(X)$ is finer than the topology of uniform convergence;*
2. *if $(T_n)_n$ is a Cauchy sequences in $(B_{\mathcal{P}}(X), \|\bullet\|_{\mathcal{P}})$ which converges to an operator T, we have $T \in B_{\mathcal{P}}(X)$;*
3. *the algebra $(B_{\mathcal{P}}(X), \|\bullet\|_{\mathcal{P}})$ is complete if X is sequentially complete.*

Proposition 15 ([5])**.** *Let $(X, \mathcal{P})$ be a locally convex space. An operator $T \in Q_{\mathcal{P}}(X)$ is bounded in the algebra $Q_{\mathcal{P}}(X)$ if and only if there exists some calibration $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$ and $T \in B_{\mathcal{P}'}(X)$.*

Definition 16. *Let $(X, \mathcal{P})$ be a locally convex space and $T \in B_{\mathcal{P}}(X)$. We said that $\alpha \in \mathbf{C}$ is in resolvent set $\rho(B_{\mathcal{P}}, T)$ if there exists $(\alpha I - T)^{-1} \in B_{\mathcal{P}}(X)$. The spectral set $\sigma(B_{\mathcal{P}}, T)$ will be the complementary set of $\rho(B_{\mathcal{P}}, T)$.*

Remark 17. *It is obvious that we have the following inclusions*

$$\sigma(T) \subset \sigma(Q_{\mathcal{P}}, T) \subset \sigma(B_{\mathcal{P}}, T).$$

Proposition 18. *Proposition If $(X, \mathcal{P}$ is a locally convex space and $T \in B_{\mathcal{P}}(X)$, then the set $\sigma(B_{\mathcal{P}}, T)$ is compact.*

Proposition 19. *Let $(X, \mathcal{P})$ be a locally convex space. Then an operator $T \in Q_{\mathcal{P}}(X)$ is regular if and only if $T \in (Q_{\mathcal{P}}(X))_0$.*

Proof. Assume that $T \in Q_{\mathcal{P}}(X)$ is bounded element of $Q_{\mathcal{P}}(X)$. It folows from proposition (15) that there is some calibration $\mathcal{P}\,' \in C(X)$ such that $\mathcal{P} \approx \mathcal{P}\,'$ ' and $T \in B_{\mathcal{P}\,'}(X)$. Moreover, $Q_{\mathcal{P}}(X)$=$Q_{\mathcal{P}'}(X)$.

If $\mid \lambda \mid > 2\,\|T\|_{\mathcal{P}}$, then Neumann series $\sum\limits_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$converges in $B_{\mathcal{P}\,'}(X)$ and its sum is $R\,(\lambda, T)$. This means that the operator $\lambda I - T$ is invertible in $Q_{\mathcal{P}}(X)$ for all $\mid \lambda \mid > 2\,\|T\,\|_{\mathcal{P}'}$. Moreover, for each $\epsilon > 0$ there exists an index $n_\epsilon \in \mathbf{N}$ such that

$$\left\| R\,(\lambda, T) - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} \right\|_{\mathcal{P}\,'} < \epsilon, (\forall)\, n \geq n_\epsilon,$$

which implies that for each $n \geq n_\epsilon$ we have

$$\|R\,(\lambda, T)\|_{\mathcal{P}\,'} \leq \left\| R\,(\lambda, T) - \sum_{k=0}^{n_\epsilon} \frac{T^k}{\lambda^{k+1}} \right\|_{\mathcal{P}\,'} + \left\| \sum_{k=0}^{n_\epsilon} \frac{T^k}{\lambda^{k+1}} \right\|_{P'} <$$

$$< \epsilon + \mid \lambda \mid^{-1} \sum_{k=0}^{n_\epsilon} \left\| \frac{T^k}{\lambda^k} \right\|_{\mathcal{P}\,'} < \epsilon + (2\,\|T\|_{\mathcal{P}\,'})^{-1} \sum_{k=0}^{n_\epsilon} 2^{-k} < \epsilon + (\|T\|_{\mathcal{P}\,'})^{-1}.$$

Since $\epsilon > 0$ is arbitrarily chosen, we have that

$$\|R\,(\lambda, T)\|_{\mathcal{P}\,'} < (\|T\|_{\mathcal{P}\,'})^{-1}, (\forall)\mid \lambda \mid > 2\,\|T\|_{\mathcal{P}\,'}$$

From definition of norm $\|\;\|_{\mathcal{P}}$ it follows that

$$\hat{p}\,(R\,(\lambda, T)) < (\|T\|_{\mathcal{P}\,'})^{-1},$$

for any $p \in \mathcal{P}\,'$ and for each $\mid \lambda \mid > 2\,\|T\|_{\mathcal{P}\,'}$, which means that the set

$$\{R\,(\lambda, T) \mid \; \mid \lambda \mid > 2\,\|T\|_{\mathcal{P}\,'}\}$$

is bounded in $Q_{\mathcal{P}}(X)$. Therefore, T is regular.

Now suppose that $T \in Q_{\mathcal{P}}(X)$ is regular, but it is not bounded in $Q_{\mathcal{P}}(X)$. By proposition (9) this means that

$$\mid \sigma(Q_{\mathcal{P}}, T) \mid = r_{\mathcal{P}}(T) = \infty,$$

which contradicts the assumption we have made. Therefore, T is bounded element of $Q_{\mathcal{P}}(X)$. □

2 A functional calculus

In this section we assume that X will be sequentially complete locally convex space. We show that we can develop a functional for bounded elements of algebra $Q_{\mathcal{P}}(X)$, where $\mathcal{P} \in C(X)$.

Definition 20. *Let $(X, \mathcal{P})$ be a locally convex space. The Waelbroeck resolvent set of an operator $T \in Q_{\mathcal{P}}(X)$, denoted by $\rho_W(Q_{\mathcal{P}}, T)$, is the subset of elements of $\lambda_0 \in \mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, for which there exists a neighborhood $V \in \mathbf{V}_{(\lambda_0)}$ such that:*

1. *the operator $\lambda I - T$ is invertible in $Q_{\mathcal{P}}(X)$ for all $\lambda \in V \backslash \{\infty\}$*
2. *the set $\{ \ (\lambda I - T)^{-1} \mid \lambda \in V \backslash \{\infty\} \ \}$ is bounded in $Q_{\mathcal{P}}(X)$.*

The Waelbroeck spectrum of T, denoted by $\sigma_W(Q_{\mathcal{P}}, T)$, is the complementary set of $\rho_W(Q_{\mathcal{P}}, T)$ in $\mathbf{C}_\infty$. It is obvious that $\sigma(Q_{\mathcal{P}}, T) \subset \sigma_W(Q_{\mathcal{P}}, T)$.

Definition 21. *Let $(X, \mathcal{P})$ be a locally convex space. An operator $T \in Q_{\mathcal{P}}(X)$ is regular if $\infty \notin \sigma_W(Q_{\mathcal{P}}, T)$, i.e. there exists some $t > 0$ such that:*

1. *the operator $\lambda I - T$ is invertible in $Q_{\mathcal{P}}(X)$, for all $| \lambda | > t$*
2. *the set $\{ \ R(\lambda, T) \mid | \lambda | > t\}$ is bounded in $Q_{\mathcal{P}}(X)$.*

Let $\mathcal{P} \in C(X)$ be arbitrary chosen and $D \subset C$ a relatively compact open set.

Lemma 22. *Let $p \in \mathcal{P}$ then the application $| f |_{p,D}$: $\mathcal{O}(D, Q_{\mathcal{P}}(X)) \to \mathbf{R}$ given by,*

$$| f |_{p,D} = \sup_{z \in D} p(f(z)), (\forall) f \in \mathcal{O}(D, Q_{\mathcal{P}}(X)),$$

is a submultiplicative seminorm on $\mathcal{O}(D, Q_{\mathcal{P}}(X))$.

If we denote by $\sigma_{\mathcal{P},D}$ the topology defined by the family $\{ \ | f |_{p,D} \mid p \in \mathcal{P}\}$ on $\mathcal{O}(D, Q_{\mathcal{P}}(X))$, then $(\mathcal{O}(D, Q_{\mathcal{P}}(X)), \sigma_{\mathcal{P},D})$ is a l.m.c.-algebra.

Let $K \subset \mathbf{C}$ be a compact set arbitrary chosen. We define the set

$$\mathcal{O}(K, Q_{\mathcal{P}}(X)) = \cup \{\mathcal{O}(D, Q_{\mathcal{P}}(X)) | \ D \subset C \text{ is relatively compact open } \}$$

We need the following lemma from complex analysis.

Lemma 23. *For each compact set $K \subset C$ and each relatively compact open set $D \supset K$ there exists some open set G such that:*

1. *$K \subset G \subset \overline{G} \subset D$;*
2. *G has a finite number of conex components $(G_i)_{i=\overline{1,n}}$, the closure of which are pairwise disjoint;*

3. *the boundary ∂G_i of $G_i, i = \overline{1,n}$, consists of a finite positive number of closed rectifiable Jordan curves $(\Gamma_{ij})_{j=\overline{1,m_i}}$, no two of which intersect;*

4. *$K \cap \Gamma_{ij} = \Phi$, for each $i = \overline{1,n}$ and every $j = \overline{1,m_i}$.*

Definition 24. *Let K and D be like in the previous lemma. An open set G is called Cauchy domain for pair (K,D) if it has the properties 1-4 of the previous lemma. The boundary*

$$\Gamma = \cup_{i=\overline{1,n}} \cup_{j=\overline{1,m_i}} \Gamma_{ij}$$

of G is called Cauchy boundary for pair (K,D).

Using some results from I. Colojoara [3] we can develop a functional calculus for bounded elements of locally m-convex algebra $Q_{\mathcal{P}}(X)$.

Theorem 25. *If $\mathcal{P} \in C_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$, then for each relatively compact open set $D \supset \sigma_W(Q_{\mathcal{P}}, T)$, the application $F_{T,D} : O(D) \to Q_{\mathcal{P}}(X)$ defined by*

$$F_{T,D}(f) = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z,T)\, dz, (\forall) f \in \mathcal{O}(D),$$

where Γ is a Cauchy boundary for pair $(\sigma_W(Q_{\mathcal{P}}, T), D)$, is a unitary continuous homomorphism. Moreover,

$$F_{T,D}(z) = T,$$

where z is the identity function.

Like in the Banach case we make the following notation $f(T) = F_{T,D}(f)$.

The following theorem represents the analogous of spectral mapping theorem for Banach spaces.

Theorem 26. *If $\mathcal{P} \in \mathcal{C}_0(X)$, $T \in (Q_{\mathcal{P}}(X)\ )_0$ and f is a holomorphic function on an open set $D \supset \sigma_W(Q_{\mathcal{P}}, T)$, then*

$$\sigma_W(Q_{\mathcal{P}}, f(T)) = f(\sigma_W(Q_{\mathcal{P}}, T)).$$

Theorem 27. *Assume that $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$. If f is holomorphic function on the open set $D \supset \sigma_W(Q_{\mathcal{P}}, T)$ and $g \in O(D_g)$, such that $D_g \supset f(D)$, then $(g \circ f)(T) = g(f(T))$.*

Lemma 28. *Assume that $P \in C_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$. If f is holomorphic function on the open set $D \supset \sigma_W(Q_{\mathcal{P}}, T)$ and $f(\lambda) = \sum_{k=0}^{k=\infty} \alpha_k \lambda^k$ on D, then $f(T) = \sum_{k=0}^{k=\infty} \alpha_k T^k$.*

Corollary 29. *If* $\mathcal{P} \in \mathcal{C}_0(X)$ *and* $T \in (Q_{\mathcal{P}}(X))_0$*, then* $\exp T = \sum_{k=0}^{k=\infty} \frac{T^k}{k!}$.

Theorem 30. *Let* $\mathcal{P} \in \mathcal{C}_0(X)$ *and* $T \in (Q_{\mathcal{P}}(X))_0$*. If* D *is an open relatively compact set which contains the set* $\sigma_W(Q_{\mathcal{P}}, T), f \in \mathcal{O}(D)$ *and* $S \in (Q_{\mathcal{P}}(X))_0$*, such that* $r_{\mathcal{P}}(S) <$*dist*$(\sigma_W(Q_{\mathcal{P}}, T), C \backslash D)$*, and* $TS = ST$*, then we have*

1. $\sigma_W(Q_{\mathcal{P}}, T + S) \subset D$;
2. $f(T+S) = \sum_{n \geq 0} \frac{f^{(n)}(T)}{n!} S^n$.

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