Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843 - 7265 (print)
Volume **3** (2008), 1 – 12

BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH FRACTIONAL ORDER

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Abstract. In this paper, we shall establish sufficient conditions for the existence of solutions for a first order boundary value problem for fractional differential equations.

1 Introduction

This paper deals with the existence of solutions for first order boundary value problems (BVP for short), for fractional order differential equations

$$^cD^\alpha y(t) = f(t, y(t)), \text{ for each } t \in J = [0, T], \quad 0 < \alpha < 1, \tag{1}$$

$$ay(0) + by(T) = c, \tag{2}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, is a continuous function, a, b, c are real constants with $a + b \neq 0$. Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [5, 11, 12, 13, 20, 21, 25, 27]). For noteworthy papers dealing with the integral operator and the arbitrary fractional order differential operator, see [8, 9]. There has been a significant development in fractional differential equations in recent years; see the monographs of Kilbas *et al* [16], Miller and Ross [22], Podlubny [27], Samko *et al* [29] and the papers of Delbosco and Rodino [4], Diethelm *et al* [5, 6, 7], El-Sayed [10], Kaufmann and Mboumi [14], Kilbas and Marzan [15], Mainardi [20], Momani and Hadid [23], Momani *et al* [24], Podlubny *et al* [28], Yu and Gao [31] and the references therein. Some results for fractional differential inclusions can be found in the book by Plotnikov *et al* [26].

Very recently some basic theory for the initial value problems of fractional differential equations involving Riemann-Liouville differential operator has been discussed

2000 Mathematics Subject Classification: 26A33, 34K05.
Keywords: Differential equation; Caputo fractional derivative; Fractional integral; Existence; Fixed point.

by Lakshmikantham and Vatsala [17, 18, 19]. Some existence results were given for the problem (1)-(2) with $\alpha = 1$ by Tisdell in [30].

In this paper, we present existence results for the problem (1)-(2). In Section 3, we give two results, one based on Banach fixed point theorem (Theorem 7) and another one based on Schaefer's fixed point theorem (Theorem 8). Some indications to nonlocal problems are given in Section 4. An example is given in Section 5 to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Definition 1. *([16, 27]). The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2. *([16, 27]). For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of h, is defined by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 3. *([16]). For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h, is defined by*

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.

3 Existence of Solutions

Let us start by defining what we mean by a solution of the problem (1)–(2).

Definition 4. *A function $y \in C^1([0,T],\mathbb{R})$ is said to be a solution of (1)–(2) if y satisfies the equation ${}^cD^\alpha y(t) = f(t, y(t))$ on J, and the condition $ay(0)+by(T) = c$.*

For the existence of solutions for the problem (1)–(2), we need the following auxiliary lemma:

Lemma 5. *[15]. Let $0 < \alpha < 1$ and let $h : [0,T] \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \tag{3}$$

if and only if y is a solution of the initial value problem for the fractional differential equation

$${}^cD^\alpha y(t) = h(t), \quad t \in [0,T], \tag{4}$$

$$y(0) = y_0. \tag{5}$$

As a consequence of Lemma 5 we have the following result which is useful in what follows.

Lemma 6. *Let $0 < \alpha < 1$ and let $h : [0,T] \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$\begin{array}{rcl} y(t) & = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ & - & \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds - c \right] \end{array} \tag{6}$$

if and only if y is a solution of the fractional BVP

$${}^cD^\alpha y(t) = h(t), \quad t \in [0,T], \tag{7}$$

$$ay(0) + by(T) = c. \tag{8}$$

Our first result is based on Banach fixed point theorem.

Theorem 7. *Assume that:*

(H1) There exists a constant $k > 0$ such that

$$|f(t,u) - f(t,\overline{u})| \le k|u - \overline{u}|, \ \textit{for each } t \in J, \textit{ and all } u, \ \overline{u} \in \mathbb{R}.$$

If

$$\frac{kT^{\alpha}\left(1+\frac{|b|}{|a+b|}\right)}{\Gamma(\alpha+1)}<1 \tag{9}$$

then the BVP (1)-(2) has a unique solution on $[0,T]$.

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the operator

$$F : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$$

defined by

$$\begin{array}{rcl} F(y)(t) & = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds \\ & - & \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s,y(s))ds - c\right]. \end{array} \tag{10}$$

Clearly, the fixed points of the operator F are solution of the problem (1)-(2). We shall use the Banach contraction principle to prove that F defined by (10) has a fixed point. We shall show that F *is a contraction*.

Let $x, y \in C([0,T],\mathbb{R})$. Then, for each $t \in J$ we have

$$\begin{array}{rcl} |F(x)(t)-F(y)(t)| & \le & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))| \\ & & +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds \\ & \le & \frac{k\|x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ & & +\frac{|b|k\|x-y\|_\infty}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}ds \\ & \le & \left[\frac{kT^{\alpha}\left(1+\frac{|b|}{|a+b|}\right)}{\alpha\Gamma(\alpha)}\right]\|x-y\|_\infty. \end{array}$$

Thus

$$\|F(x)-F(y)\|_\infty \le \left[\frac{kT^{\alpha}\left(1+\frac{|b|}{|a+b|}\right)}{\Gamma(\alpha+1)}\right]\|x-y\|_\infty.$$

Consequently by (9) F is a contraction. As a consequence of Banach fixed point theorem, we deduce that F has a fixed point which is a solution of the problem $(1)-(2)$. □

The second result is based on Schaefer's fixed point theorem.

Theorem 8. *Assume that:*

(H2) The function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(H3) There exists a constant $M > 0$ *such that*

$$|f(t,u)| \leq M \text{ for each } t \in J \text{ and all } u \in \mathbb{R}.$$

Then the BVP (1)-(2) has at least one solution on $[0,T]$.

Proof. We shall use Schaefer's fixed point theorem to prove that F defined by (10) has a fixed point. The proof will be given in several steps.

Step 1: *F is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,T],\mathbb{R})$. Then for each $t \in [0,T]$

$$\begin{aligned}
& |F(y_n)(t) - F(y)(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_n(s)) - f(s,y(s))|ds \\
& + \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}|f(s,y_n(s)) - f(s,y(s))|ds \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{s\in[0,T]}|f(s,y_n(s)) - f(s,y(s))|ds \\
& + \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}\sup_{s\in[0,T]}|f(s,y_n(s)) - f(s,y(s))|ds \\
\leq\ & \frac{\|f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))\|_\infty}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}ds + \frac{|b|}{|a+b|}\int_0^T (T-s)^{\alpha-1}ds\right] \\
\leq\ & \frac{\left(1+\frac{|b|}{|a+b|}\right)T^\alpha\|f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))\|_\infty}{\alpha\Gamma(\alpha)}.
\end{aligned}$$

Since f is a continuous function, we have

$$\|F(y_n) - F(y)\|_\infty \leq \frac{\left(1+\frac{|b|}{|a+b|}\right)T^\alpha\|f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))\|_\infty}{\Gamma(\alpha+1)} \to 0 \quad \textit{as } n \to \infty.$$

Step 2: *F maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$.*

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $y \in B_{\eta^*} = \{y \in C([0,T],\mathbb{R}) : \|y\|_\infty \leq \eta^*\}$, we have $\|F(y)\|_\infty \leq \ell$.

By (H3) we have for each $t \in [0,T]$,

$$\begin{aligned}
|F(y)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))|ds \\
&\quad + \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}|f(s,y(s))| + \frac{|c|}{|a+b|} \\
&\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{|b|M}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}ds + \frac{|c|}{|a+b|} \\
&\leq \frac{M}{\alpha\Gamma(\alpha)}T^\alpha + \frac{M|b|}{\alpha\Gamma(\alpha)|a+b|}T^\alpha + \frac{|c|}{|a+b|}.
\end{aligned}$$

Thus

$$\|F(y)\|_\infty \leq \frac{M}{\Gamma(\alpha+1)}T^\alpha + \frac{M|b|}{\Gamma(\alpha+1)|a+b|}T^\alpha + \frac{|c|}{|a+b|} := \ell.$$

Step 3: *F maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.*

Let $t_1, t_2 \in (0,T]$, $t_1 < t_2$, B_{η^*} be a bounded set of $C([0,T],\mathbb{R})$ as in Step 2, and let $y \in B_{\eta^*}$. Then

$$\begin{aligned}
|F(y)(t_2) - F(y)(t_1)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]f(s,y(s))ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds\Big| \\
&\leq \frac{M}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\leq \frac{M}{\Gamma(\alpha+1)}[(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha] + \frac{M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha \\
&\leq \frac{M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha + \frac{M}{\Gamma(\alpha+1)}(t_1^\alpha - t_2^\alpha).
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $F : C([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R})$ is continuous and completely continuous.

Step 4: *A priori bounds.*

Now it remains to show that the set

$$\mathcal{E} = \{y \in C(J,\mathbb{R}) : y = \lambda F(y) \ \text{ for some } \ 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}$, then $y = \lambda F(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$\begin{aligned} y(t) &= \lambda \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds \right. \\ &\quad \left. - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds - c \right] \right]. \end{aligned}$$

This implies by (H3) that for each $t \in J$ we have

$$\begin{aligned} |F(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y(s))| ds \\ &\quad + \frac{|b|}{\Gamma(\alpha)|a+b|} \int_0^T (T-s)^{\alpha-1} |f(s, y(s))| ds + \frac{|c|}{|a+b|} \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\quad + \frac{|b|M}{\Gamma(\alpha)|a+b|} \int_0^T (T-s)^{\alpha-1} ds + \frac{|c|}{|a+b|} \\ &\leq \frac{M}{\alpha\Gamma(\alpha)} T^\alpha + \frac{M|b|}{\alpha\Gamma(\alpha)|a+b|} T^\alpha + \frac{|c|}{|a+b|}. \end{aligned}$$

Thus for every $t \in [0, T]$, we have

$$\|F(y)\|_\infty \leq \frac{M}{\Gamma(\alpha+1)} T^\alpha + \frac{M|b|}{\Gamma(\alpha+1)|a+b|} T^\alpha + \frac{|c|}{|a+b|} := R.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1) − (2).

□

Remark 9. *Our results for the BVP (1)-(2) are applied for initial value problems ($a = 1,\ b = 0$), terminal value problems ($a = 0,\ b = 1$) and anti-periodic solutions ($a = 1,\ b = 1,\ c = 0$).*

4 Nonlocal problems

This section is devoted to some existence and uniqueness results for the following class of nonlocal problems

$$^cD^\alpha y(t) = f(t, y(t)), \text{ for each } t \in J = [0, T], \quad 0 < \alpha < 1, \tag{11}$$

$$y(0) + g(y) = y_0, \tag{12}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, is a continuous function, and $g : C([0,T],\mathbb{R}) \to \mathbb{R}$ is a continuous function. The nonlocal condition can be applied in physics with better effect than the classical initial condition $y(0) = y_0$. For example, $g(y)$ may be given by

$$g(y) = \sum_{i=1}^{p} c_i y(t_i)$$

where c_i, $i = 1,\ldots,p$, are given constants and $0 < t_1 < \ldots < t_p \leq T$. Nonlocal conditions were initiated by Byszewski [1] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [2, 3], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. Let us introduce the following set of conditions on the function g.

(H4) There exists a constant $\bar{M} > 0$ such that

$$|g(y)| \leq \bar{M} \text{ for each } u \in C([0,T],\mathbb{R}).$$

(H5) There exists a constant $\bar{k} > 0$ such that

$$|g(y) - g(\overline{y})| \leq \bar{k}|y - \overline{y}|, \text{ for each } y,\ \overline{y} \in C([0,T],\mathbb{R}).$$

Theorem 10. *Assume that (H1), (H2), (H5) hold. If*

$$\bar{k} + \frac{kT^\alpha}{\Gamma(\alpha+1)} < 1 \tag{13}$$

then the nonlocal problem (11)-(12) has a unique solution on $[0,T]$.

Proof. Transform the problem (11)–(12) into a fixed point problem. Consider the operator

$$\tilde{F} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$$

defined by

$$\tilde{F}(y)(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))ds.$$

Clearly, the fixed points of the operator $\tilde{F}$ are solution of the problem (11)-(12). We can easily show the $\tilde{F}$ is a contraction. □

Theorem 11. *Assume that (H2)-(H4) hold. Then the nonlocal problem (11)-(12) has at least one solution on* $[0,T]$.

5 An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem,

$$ {}^cD^{\alpha}y(t) = \frac{e^{-t}|y(t)|}{(9+e^t)(1+|y(t)|)}, \quad t \in J := [0,1], \quad \alpha \in (0,1], \tag{14}$$

$$ y(0) + y(1) = 0. \tag{15}$$

Set

$$ f(t,x) = \frac{e^{-t}x}{(9+e^t)(1+x)}, \quad (t,x) \in J \times [0,\infty). $$

Let $x, y \in [0,\infty)$ and $t \in J$. Then we have

$$ \begin{aligned} |f(t,x) - f(t,y)| &= \frac{e^{-t}}{(9+e^t)}\left|\frac{x}{1+x} - \frac{y}{1+y}\right| \\ &= \frac{e^{-t}|x-y|}{(9+e^t)(1+x)(1+y)} \\ &\leq \frac{e^{-t}}{(9+e^t)}|x-y| \\ &\leq \frac{1}{10}|x-y|. \end{aligned} $$

Hence the condition $(H1)$ holds with $k = \frac{1}{10}$. We shall check that condition (9) is satisfied for appropriate values of $\alpha \in (0,1]$ with $a = b = T = 1$. Indeed

$$ \frac{3k}{2\Gamma(\alpha+1)} < 1 \Leftrightarrow \Gamma(\alpha+1) > \frac{3k}{2} = 0,15. \tag{16}$$

Then by Theorem 7 the problem (14)-(15) has a unique solution on $[0,1]$ for values of α satisfying condition (16). For example

- If $\alpha = \frac{1}{5}$ then $\Gamma(\alpha+1) = \Gamma\left(\frac{6}{5}\right) = 0.92$ and

$$ \frac{3k}{2}\frac{1}{\Gamma(\alpha+1)} = \frac{0.15}{0.92} = 0.1630434 < 1. $$

- If $\alpha = \frac{2}{3}$ then $\Gamma(\alpha+1) = \Gamma\left(\frac{5}{3}\right) = 0.89$ and

$$ \frac{3k}{2}\frac{1}{\Gamma(\alpha+1)} = \frac{0.15}{0.89} = 0.1685393 < 1. $$

Acknowledgement. The authors are grateful to Prof. Tzanko Donchev for his remarks.

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