Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843 - 7265 (print)
Volume **3** (2008), 79 – 110

CONSIDERATIONS ON SOME ALGEBRAIC PROPERTIES OF FEYNMAN INTEGRALS

Lucian M. Ionescu

Abstract. Some algebraic properties of integrals over configuration spaces are investigated in order to better understand quantization and the Connes-Kreimer algebraic approach to renormalization.

In order to isolate the mathematical-physics interface to quantum field theory independent from the specifics of the various implementations, the sigma model of Kontsevich is investigated in more detail. Due to the convergence of the configuration space integrals, the model allows to study the Feynman rules independently, from an axiomatic point of view, avoiding the intricacies of renormalization, unavoidable within the traditional quantum field theory.

As an application, a combinatorial approach to constructing the coefficients of formality morphisms is suggested, as an alternative to the analytical approach used by Kontsevich. These coefficients are "Feynman integrals", although not quite typical since they do converge.

A second example of "Feynman integrals", defined as state-sum model, is investigated. Integration is understood here as formal categorical integration, or better as a duality structure on the corresponding category. The connection with a related TQFT is mentioned, supplementing the Feynman path integral interpretation of Kontsevich formula.

A categorical formulation for the Feynman path integral quantization is sketched, towards Feynman Processes, i.e. representations of dg-categories with duality, thought of as complexified Markov processes.

1 Introduction

Kontsevich's solution to the problem of deformation quantization of Poisson manifolds [23], contains deep algebraic structures related to the Connes-Kreimer algebraic approach to renormalization[8, 27]. On the mathematical side, the dg-coalgebra structure of graphs introduced in [14], leading to cohomology of Feynman graphs[15], is essentially a unification of Kontsevich's graph homology and Kreimer's graph coproduct.

On the physics side, the interpretation of Kontsevich's construction as a *Feynman Path Integral* (FPI) quantization, almost tautologically leading to deformation

2000 Mathematics Subject Classification:18G55, 81Q30, 81T18
Keywords: Feynman integrals; quantization; Hopf algebra; configuration spaces.
This work was supported by I.H.E.S.

quantization, was immediately given in [6].

In this article we analyze Kontsevich's construction, with emphasis on the integrals on configuration spaces which provide the coefficients of the star product, in order to *extract their homological algebra properties*, and push the physical interpretation towards an axiomatic formalism of FPI.

The main point, developed in subsequent papers, is the compatibility between Kontsevich's graph homology and Connes-Kreimer Hopf algebra structure of renormalization. It leads to a differential graded integration, generalizing the usual integration of forms.

Secondly, the construction is shown to be quite general, by replacing the angle form used by Kontsevich with any given closed form (Corollary 43), provided renormalization is used if the integrals are no longer convergent.

Third, the integration of forms used to define the configuration integrals is abstracted as a formal categorical integration, establishing the connection with topological quantum field theories.

The underlying goal of this article is to isolate the quantization process from the intricacies of renormalization. The later is a procedure for extracting the finite part from divergent configuration integrals, due in part to the use of an amorphic continuum model of the configuration space ("infrared divergencies"), and in part due to the lack of compactness of the state space ("ultra-violet divergencies").

Since Kontsevich construction uses a sigma model on the disk, leading to *convergent* configuration space integrals, it avoids renormalization. Therefore the considerations from this article apply to the interface to QFT, relying on assumptions like $\Delta_F(x, x) = 0$ (Remark 24). Further motivation for isolating the QFT interface from renormalization is provided in the concluding section.

The use of the operadic/PROP language, more adequate to use and making the connection with conformal field theory, string theory etc. [34], will be deferred to another forthcoming article [16]. Instead, we will include some of the details from [14], explaining the previous mentioned results from [17, 15].

The paper is organized as follows. The results on L_∞-morphisms expanded as a perturbation series over a class of "Feynman" graphs are reviewed in §2. Their coefficients satisfy a certain cocycle condition, representing modulo homotopy the cohomology class of the DG-coalgebra of Feynman graphs.

A mathematical interface to perturbative QFT is proposed supported by the findings of the next section.

A reader interested in the motivation for the above results may benefit from reading §3 first, where the integrals over configuration spaces used in[23] are studied, extracting some of their intrinsic properties, notably the "Forest Formula" 10 of the boundary of their compactification.

Feynman state spaces and configuration functors are defined (graph cohomology). Feynman rules defined as multiplicative Euler-Poincare maps are proved compatible with the Forest Formula via the (Feynman) integration pairing (Theorem 41).

The state-sum model from[23], and yielding an example of a generalized Feynman integral, is studied in Section 4. The corresponding TQFT provides the additional evidence towards its interpretation as a Feynman path integral, conform[6]. The properties of the generalized Feynman rule needed for the proof of the formality conjecture are identified.

Section 5 applies the previous results to formality of DGLAs. Kontsevich proof is translated into the language developed so far, providing the essential steps for a proof of the main theorem from Section 2.

Section 6 concludes with comments on the use of homotopical algebra towards the implementation of perturbative quantum field theory.

Before proceeding, we should mention that an alternative avenue to Kontsevich's solution to the Formality Theorem was provided by Tamarkin [31]. We will not pursue it since it *is* a more direct proof, being in the author's opinion, less "pedagogical" for the purposes of this article, of understanding the applications of graph complexes to QFT via FPI.

Acknowledgments I would like to express my gratitude for the excellent research conditions at I.H.E.S., where this project was conceived during the author's 2002 visit, under the influence of, and benefitting from stimulating discussions with Maxim Kontsevich.

Jim Stasheff's comments are greatly appreciated.

I also thank the reviewer for helping the author clarify the scope of the properties investigated, and for pointing to the connection with Gel'fand-Dickey brackets.

2 L_∞-morphisms as perturbation series

We will first recall from[17], the main concepts needed in the following sections, for the reader's convenience.

Given a graded map between L_∞-algebras represented as a *Feynman expansion* over a given class of graphs ("partition function"), the coefficients satisfy a certain cocycle equation in order to be an L_∞-morphism.

The goal is to understand the coefficients of formality morphisms and Kontsevich deformation quantization formula, as well as perturbative QFT (see §6 for details). The obstruction for a pre-L_∞-morphism[24] (p.142) to be a morphism is of cohomological nature, and we will point to its relation with renormalization.

2.1 Feynman graphs

A QFT defined via Feynman Path Integral quantization method is based on a graded class of *Feynman graphs*. For specific implementation purposes these can be 1-dimensional CW-complexes or combinatorial objects.

For definiteness we will consider the class of Kontsevich graphs $\Gamma \in \mathcal{G}_n$, the *admissible graphs* from [23], p.22.

Nevertheless we claim that the results are much more general, and suited for a generalization suited for an axiomatic approach; a Feynman graph will be thought off both as an object in a category of Feynman graphs (categorical point of view), as well as a cobordism between their *boundary vertices* (TQFT point of view). The *main assumption* the class of Feynman graphs needs to satisfy, is the *existence of subgraphs and quotients*.

While the concept of *subgraph* γ of Γ is clear (will be modeled after that of a subcategory), we will define the *quotient* of Γ by the subgraph γ as the graph Γ' obtained by collapsing γ (vertices and internal edges) to a vertex of the quotient (e.g. see[8], p.11).

Remark 1. *When γ contains "external legs", i.e. edges with 1-valent vertices belonging to the boundary of the Feynman graph when thought of as a cobordism, we will say that γ* meets the boundary *of Γ. In this case the vertex of the quotient obtained by collapsing γ will be part of the boundary (of Γ/γ) too. In other words the boundary of the quotient is the quotient of the boundary (compare [23], p.27).*

Formal definitions will be introduced elsewhere.

Definition 2. *A subgraph γ of $\Gamma \in \mathcal{G}$ is* normal *iff the corresponding* quotient *Γ/γ belongs to the same class of Feynman graphs $\mathcal{G}$.*

Definition 3. *An* extension *$\gamma \hookrightarrow \Gamma \twoheadrightarrow \gamma'$ in $\mathcal{G}$ is a triple (as displayed) determined by a subgraph γ of Γ, such that the quotient γ' is in $\mathcal{G}$. The extension is a* full extension *if γ is a* full subgraph*, i.e. together with two vertices of Γ contains all the corresponding connecting arrows (the respective "Hom").*

Edges will play the role of simple objects.

Definition 4. *A subgraph consisting of a single edge is called a* simple subgraph*.*

Example 5. *As a first example consider the class $\mathcal{G}_a$ of* admissible graphs *provided in[23]. Denote by $\mathcal{G}$ the larger class of graphs, including those for which edges from boundary points may point towards internal vertices (essentially all finite graph one-sided "cobordisms": $\emptyset \to [m]$). Then the normal subgraphs relative to the class $\mathcal{G}_a$ are precisely the subgraphs with no "bad-edge" ([23], p.27), i.e. those for which the quotient is still an admissible graph.*

Another example is the class of Feynman graphs of ϕ^3-theory. In this context a subgraph of a 3-valent graph collapses to a 3-valent vertex precisely when it is a normal subgraph in our sense.

There is a natural pre-Lie operation based on the operation of insertion of a graph at an internal vertex of another graph[27, 9] (addressed next). It is essentially a sum over extensions of two given graphs.

Definition 6. *The* extension product $\star : g \otimes g \to g$ *is the bilinear operation which on generators equals the sum over all possible extensions of one graph by the other one:*

$$\gamma' \star \gamma = \sum_{\gamma \to \Gamma \to \gamma'} \pm \Gamma. \tag{1}$$

It is essentially the "superposition of $Hom(\gamma, \gamma')$". As noted in[9], p.14, it is a pre-Lie operation, endowing g with a canonical Lie bracket (loc. cit. $\mathcal{L}_{FG}$).

The lack of an explicit sign intends to avoid the technical details and complications (see also [2]), to focus on the algebraic structures and the corresponding conceptual aspects. The sign is a generalization of the sign convention for the Gerstenhaber bracket [12], yielding the pre-Lie structure. The compatibility with the differential follows (see [11], p.18-19), as in any "pointed" DGLA, i.e. where the differential is given by the bracket with a special element of the Lie algebra.

Let $H = T(g)$ be the tensor algebra with (reduced) coproduct:

$$\Delta\Gamma = \sum_{\gamma \to \Gamma \to \gamma'} \gamma \otimes \gamma', \tag{2}$$

where the sum is over all non-trivial subgraphs of Γ ("normal proper subobjects") such that collapsing γ to a vertex yields a graph from the given class $\mathcal{G}$ (compare with condition (7) [8], p.11).

Remark 7. *The two operations introduced are in a sense "opposite" to one another, since the coproduct unfolds a given graph into its constituents, while the product assembles two constituents in all possible ways. For the moment we will not dwell on the resulting algebraic structure.*

With the appearance of a Lie bracket and a comultiplication, we should be looking for a differential (towards a DG-structure).

Consider the graph homology differential [26], p.109:

$$d\Gamma = \sum_{e \in E_\Gamma} \pm \Gamma / \gamma_e, \tag{3}$$

where the sum is over the edges of Γ, γ_e is the one-edge graph, and Γ/γ_e is the quotient (forget about the signs for now).

Theorem 8. (H, d, Δ) *is a differential graded coalgebra.*

Proof. That it is a coalgebra results from [8], p.12. So all we need to prove is that d is a coderivation:

$$\Delta d = (d \otimes id + id \otimes d)\Delta.$$

Comparing the two sides (with signs omitted):

$$LHS = \sum_{e\in\Gamma}\sum_{\bar\gamma\subset\Gamma/e\to\bar\gamma'} \bar\gamma\otimes\bar{\gamma}' \tag{4}$$

$$= \sum_{e\in\Gamma}\Big(\sum_{e/e\in\bar\gamma\subset\Gamma/e\to\bar\gamma'} \bar\gamma\otimes\bar\gamma' + \sum_{e/e\notin\bar\gamma\subset\Gamma/e\to\bar\gamma'} \bar\gamma\otimes\bar\gamma'\Big), \tag{5}$$

and

$$RHS = \sum_{\gamma\subset\Gamma\to\gamma'}\Big(\sum_{e\in\gamma}\gamma/e\otimes\gamma' + \sum_{e\in\gamma'}\gamma\otimes\gamma'/e\Big) \tag{6}$$

$$= \sum_{e\in\Gamma}\Big(\sum_{e\in\gamma\subset\Gamma\to\gamma'}\gamma/e\otimes\gamma' + \sum_{e\notin\gamma\subset\Gamma\to\gamma'}\gamma\otimes\gamma'/e\Big), \tag{7}$$

with a correspondence uniquely defined by $e\in\gamma\to\bar\gamma$, i.e. $\bar\gamma=\gamma/e$ and $e\in\gamma'\to\bar\gamma'$, i.e. $\bar{\gamma'}=\gamma'/e$ respectively, concludes the proof. □

The boundary of the codimension 1 strata of the configuration spaces (see §3) suggests to consider its cobar construction $C(H) = T(s^{-1}\bar H)$ ([13], p.366, [29], p.171), where $\bar H$ denotes the augmentation ideal, and s^{-1} is an alternative notation for the suspension functor (see also [32]). Moreover, this is the natural set up for DG(L)A-infinity structures (e.g. [21]).

The total differential is $D = d + \bar\Delta$, where the "coalgebra part" $\bar\Delta$ is the graded derivation:

$$\bar\Delta\Gamma = \sum_{\gamma\to\Gamma\to\gamma'}\gamma\otimes\gamma', \tag{8}$$

corresponding to the reduced coproduct Δ given by equation 2.

Definition 9. *The cobar construction $(C(H),D)$ of the DG-coalgebra (H,d,Δ) of Feynman graphs is called the* Feynman cobar construction *on $\mathcal{G}$.*

Taking the homology of its dual $(Hom_{Calg}(C(H),k),\delta)$ relative some field k, with dual differential δ, yields $H^\bullet(H;k)$, the cohomology of the DG-coalgebra of Feynman diagrams $\mathcal{G}$. We will see in section 2.3 that it characterizes L_∞-morphisms represented as Feynman expansions.

2.2 Feynman-Taylor coefficients

Let $(g_1,Q^1_\bullet)$ and $(g_2,Q^2_\bullet)$ be L_∞-algebras, with coderivations $Q^i_\bullet$ of $C(g_i)$, $i=1,2$ ([23], p.12), and $f: g_1\to g_2$ a pre-L_∞ morphism ([23], p.11) with associated morphism of graded cocommutative coalgebras $\mathcal{F}_*: C(g_1[1])\to C(g_2[1])$, thought of as the Feynman expansion of a partition function:

$$\mathcal{F}_* = \sum\mathcal{F}_n,\quad \mathcal{F}_n(a) = \sum_{\Gamma\in\mathcal{G}_n} <\Gamma, a>,\quad a\in g_1^n,$$

where the "pairing" $< \, , \, >$ corresponds to a morphism $B : H \to Hom(g_1, g_2)$.

Definition 10. *A morphism $B : H \to Hom(g_1, g_2)$ is called* a generalized Feynman integral. *Its value $< \Gamma, a >$ will be called a* Feynman-Taylor coefficient.

Characters $W : H \to \mathbf{R}$ act on Feynman integrals:

$$U = W \cdot B, \quad U(\Gamma) = W(\Gamma)B(\Gamma), \Gamma \in \mathcal{G}.$$

An example of a generalized Feynman integral is U_Γ defined in [23], p.23, using the pairing between polyvector fields and functions on $\mathbb{R}^n$. An example of (pre)L_∞-morphisms associated with graphs is provided by $U_n = \sum_{\Gamma \in \mathcal{G}_n} W_\Gamma B_\Gamma$, the formality morphism of [23], p.24 (see §3 for more details).

Feynman integrals as pairings involving Feynman rules corresponding to propagators (the common value on all edges, e.g. W_Γ in [23], p.23) will be defined in Section 3.3 (Definition 30).

2.3 L_∞-morphisms

Before addressing the general case of L_∞-algebras, we will characterize formality morphisms of DGLAs (e.g. polyvector fields and polydifferential operators).

Theorem 11. *Let $(g_1, 0, [,]_{SN})$ and $(g_2, d_2, [,])$ be two DGLAs, and $f = W \cdot U : g_1 \to g_2$ a pre-L_∞-morphism as above. Then*

(i) $\delta W = [f, Q]$, where Q denotes the appropriate L_∞-structure.

(ii) f is an L_∞-morphism iff the character W is a cocycle of the DG-coalgebra of Feynman graphs: $\delta W = 0$.

The proof in the general case is essentially the proof from [23], which will be given in section 5 (see Theorem 56).

Definition 12. *A character W is called a* weight *if it is such a cocycle.*

We claim that the above result holds for arbitrary $L_\infty - algebras$. Moreover L_∞-morphisms can be expanded over a suitable class of Feynman graphs, and their moduli space corresponds to the cohomology group of the corresponding DG-coalgebra of Feynman graphs.

Theorem 13. *("Feynman-Taylor")*

Let $\mathcal{G}$ be a class of Feynman graphs and g_1, g_2 two L_∞-algebras.

In the homotopy category of L_∞-algebras, L_∞-morphisms correspond to the cohomology of the corresponding Feynman DG-coalgebra:

$$\mathcal{H}o(g_1, g_2) = H^\bullet(H; k).$$

The basic examples (formality morphisms) are provided by cocycles constructed using integrals over compactification of configuration spaces (periods ([25], p.26; see §3 and §5 for more details.)

Remark 14. *The initial motivation for the present approach was to find an algebraic construction for such cocycles. The idea consists in defining an algebraic version of the "configuration functor" $S : H \to C_\bullet(M)$, a "top" closed form $\omega : H \to \Omega^\bullet(M)$ and a pairing $< S, \omega >$. Their properties suggest the following framework, which will be detailed in section 3.3: a chain map $S : (H, d) \to (C_\bullet, \partial)$, and a cocycle ω in some dual cohomological complex $(C^\bullet(R)$:*

$$< \partial S, \omega >=< S, d\omega > (= 0),$$

so that the "Stokes theorem" holds. Then $W =< S, \omega >$ would be such a cocycle.

A physical interpretation will be suggested here, and investigated elsewhere.

2.4 A physical interpretation

Let H be the Hopf algebra of a class of Feynman graphs $\mathcal{G}$. If Γ is such a graph, then configurations are attached to its vertices, while momenta are attached to edges in the two dual representations (Feynman rules in position and momentum spaces).

This duality is *represented* by a pairing between a "configuration functor" (typically C_Γ, see §3.2), and a "Lagrangian" (e.g. ω determined by its value on an edge, i.e. by a propagator). Together with the pairing (typically integration) representing the action, they are thought of as part of the Feynman model of the *state space of a quantum system*.

Remark 15. *As already argued in [18], this "Feynman picture" is more general than the manifold based "Riemannian picture", since it models in a more direct way the observable aspects of quantum phenomena ("interactions" modeled by a class of graphs), without the assumption of a continuity (or even the existence) of the interaction or propagation process in an ambient "space-time", the later being clearly only an artificial model useful to relate with the classical physics, i.e. convenient for "quantization purposes".*

Definition 16. *An* action *on $\mathcal{G}$ ("S_{int}"), is a character $W : H \to \mathbf{R}$ which is a cocycle in the associated DG-coalgebra $(T(H^*), D)$.*

A source of such actions is provided by a morphism of complexes $S : H \to C_\bullet(M)$ ("configuration functor"), where M is some "space", $C_\bullet(M)$ is a complex ("configurations on M"), endowed with a pairing $\int : C_\bullet(M) \times \Omega^\bullet(M) \to \mathbf{R}$, where $\Omega^\bullet(M)$ is some dual complex ("forms on configuration spaces"), i.e. such that "Stokes theorem" holds:

$$< \partial S, \omega >=< S, d\omega > .$$

A Lagrangian *on the class $\mathcal{G}$ of Feynman graphs is a k-linear map $\omega : H \to \Omega^\bullet(M)$ associating to any Feynman graph Γ a closed volume form on $S(\Gamma)$ vanishing on the boundaries, i.e. for any subgraph $\gamma \to \Gamma$ (viewed as a subobject) meeting the boundary of $\Gamma : [s] \to [t]$ (viewed as a cobordism), $\omega(\gamma) = 0$.*

The associated action is $W =< S, \omega >$.

A prototypical "configuration functor" is given by the compactification of configuration spaces $C_{n,m}$ described in [23] (see §3). The second condition for a Lagrangian emulates the vanishing on the boundary of the angle-form α (see [23], p.22). The coefficient $W(\Gamma)$ is then a *period* of the quadruple $(C_\Gamma, \partial C_\Gamma, \wedge_{k=1}^{|E_\Gamma|}\alpha(z_{i_k}, z_{j_k})$ ([29], p.24). A related formulation (effective periods) is given in [29], p.27.

3 Integrals over configuration spaces

The formality morphism U from [23] was constructed using ideas from string theory (loc. cit. p.1). The terms of the "n-point function" U_n are products of factors determined by the interaction term (1-form on the disk) and the kinetic part determining the propagator (Lagrangian "decouples").

With this interpretation in mind, we will investigate the properties of the L_∞-morphism and its coefficients W by analyzing the corresponding integrals on configuration spaces. The coefficients $W(\Gamma)$ of the terms of the n-point function U_n are expressed as integrals over configuration spaces of points of a closed form vanishing on the boundary.

We claim that the main property of the compactification of the configuration space (a manifold with corners), is the "Forest Formula" (reminiscent of renormalization), giving the decomposition of its boundary into disjoint strata. This formula is a consequence of the fact that "...open strata of $C_{n,m}$ are naturally isomorphic to products of manifolds of type $C_{n',m'}$ and $C_{n'}$" [23], p.19. The implications for the corresponding integrals and the properties of the integrands stated in [23] are translated in our language targeting a categorical and cohomological interpretation. Special consideration is given to the correspondence between the L_∞-morphism condition (F) and the coefficients c_Γ of the Feynman expansion (see [23], p.25).

3.1 Configuration spaces

Consider first the configuration space of n-points in a manifold M ("no boundary" case) denoted by $C_n(M)$. Then its compactification has the following structure:

$$\bar{C}_n(M) = \bigcup_{k=1}^{n-1} \bigcup_{k-forests} C_F,$$

where C_F is a certain bundle over the configuration space of the roots of the forest F with k trees ([26], p.106; [23], p.20). The codimension of a stratum equals k, the number of trees in the forest ([5], p.5280).

We will be interested in the codimension one strata, for which Stokes theorem holds (see [5], p.5281, (A3)). This relevant part of the boundary of the configuration space, denoted by $\partial\bar{C}_n(M)$, is a disjoint union of strata in one-to-one correspondence with proper subsets $S \subset \{1, 2, ..., n\} = [n]$ with cardinality at least two ([26], p.106):

$$\partial \bar{C}_n(M) = \bigcup_{[1] \subsetneqq S \subsetneqq [n]} \partial_S \bar{C}_n(M), \qquad \partial_S \bar{C}_n(M) \cong C_S \times C_{[n]/S}. \tag{9}$$

On the right, the "quotient" of $[n]$ by the "non-trivial subobject" S was preferred to the equivalent set with $n - |S| + 1$ elements ("in the category of pointed sets").

This formula involves a (reduced) coproduct structure, the same way Zimmerman forest formula does, in the (similar) context of regularized Feynman integrals and renormalization.

To extract the intrinsic properties of integrals over configuration spaces, we will follow the proof of the formality theorem [23], p.24, and record the relevant facts in our homological-physical interpretation: admissible graphs are "cobordisms" $\Gamma : \emptyset \to [m]$ when U_n is thought of as a state-sum model ([26], p.100; see Section 4). The graphs are also interpreted as "extensions" $\gamma \to \Gamma \to \gamma'$, when considering the associated Hopf algebra structure. The implementation of the concepts hinted above in quotation marks is scheduled for another article.

3.2 Boundary strata of codimension one

Let $C_{n,m}$ be the configuration space of n interior points and m boundary points in the manifold M with boundary ∂M (e.g. [23], p.6: upper half-plane $\mathcal{H}$). Its elements will be thought of as (geometric) "representations of cobordisms" (enabling degrees of freedom with constraints):

$$\{\emptyset \overset{[n]}{\to} [m]\} \xrightarrow{x} \{\emptyset \overset{M}{\to} \partial M\}.$$

$C_{n,m}$ will be also denoted by $C_{A,B}$, where A and B are the sets of internal respective boundary vertices, with n, respectively m elements. This notation will extend to graphs Γ, where $C_\Gamma = C_{A,B}$ will denote the space of states of the "cobordism" Γ (see above Remark 15), at the level of vertices $\Gamma^{(0)}$ (A/B the set of internal/boundary vertices).

Definition 17. *The* space of configurations of Γ *is* C_Γ*, the set of embeddings* σ *of the set* $\Gamma^{(0)}$ *of its vertices into the manifold with boundary* $(M, \partial M)$*, which respects the boundary ("source" and "target"), i.e. mapping* internal vertices *from* $[n]$ *to internal points of* M*, and boundary vertices from* $[m]$ *to boundary points from* ∂M*.*

If γ is a subgraph in Γ, then a configuration of Γ will induce by restriction a configuration on γ (functoriality of configuration spaces; see [5], p.5247).

Note that there is no canonical configuration induced on the corresponding quotient $\gamma' = \Gamma/\gamma$, and this is where the compactification plays an important role. Nevertheless the M-position of the vertex to which γ collapses will belong to the boundary of M iff the γ meets the boundary $[m]$ of Γ (see Remark 1). The properties of these two distinct cases ("type S1/S2"), will be treated below, and unified later on.

The codimension one boundary strata decomposes as follows ([23], p.22):

$$\partial \bar{C}_{n,m} = (\bigcup_{S_1} \partial_{S_1} \bar{C}_{n,m}) \cup (\bigcup_{S_1,S_2} \partial_{S_1,S_2} \bar{C}_{n,m}). \tag{10}$$

where S_1 and S_2 are subsets of points of $[n]$ and $[m]$.

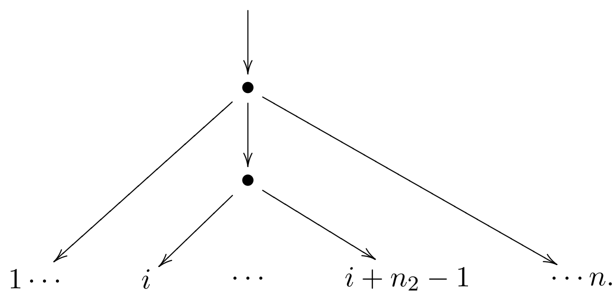
Before briefly mentioning the tree-description of Equation 9, we will reinterpret the above "definition" of the various portions of the boundary $\partial_{S_1,S_2}\bar{C}_{n,m}$.

Definition 18. *Let Γ be a graph with internal vertices $[n]$ and external vertices $[m]$ (cobordism $\Gamma : \emptyset \to [m]$, $[n] = \Gamma^{(0)}_{int}$). Then $\partial_{S_1,S_2}\bar{C}_\Gamma$ denotes the portion of the codimension one boundary of the compactification of C_Γ ($\subset \bar{C}_\Gamma$) corresponding to the above decomposition. If $S_2 = \emptyset$, it will also be denoted by $\partial_{S_1}\bar{C}_\Gamma$.*

For a full subgraph γ (Definition 3) determined by the sets of vertices S_1 and S_2, the codimension one stratum $\partial_{S_1,S_2}\bar{C}_\Gamma$ will also be denoted as $\partial_\gamma\bar{C}_\Gamma$.

Whether γ intersects the boundary or not ($S_2 = \emptyset$), $S = S_1 \cup S_2$ denotes the vertices which in the process of completion of the configuration spaces yield "Cauchy sequences" with M-coordinates getting closer and closer to one another (see [23], p.20 for additional details).

If $n_2 = |S_2| \geq 2$, the corresponding stratum can be alternatively labeled by the following tree:



It can be obtained by the insertion of the bottom line n_2-*corolla* [35], p.3, in a leaf of the top line corolla, yielding a term of the graph homology differential[35], p.4. Represented on an algebra, the operation becomes the $\circ_i$ Gerstenhaber composition ([12]; [19], p.4), yielding a term of the Hochschild differential.

Regarding codimension one strata, we will list the facts proved in [23], p.25-27, using our notation aiming to generalize the usual context of Feynman graphs to more general "cobordism categories"[16]. Recall that the graphs considered are the "admissible graphs": $\mathcal{G}_a$ as defined in Example 5. The results still hold for the class

$\mathcal{G}$, which is closed under taking subquotients (A Serre category seems to be the appropriate context for considering Feynman integrands as Euler-Poincare maps - see Definition 30).

Lemma 19. *("Type S_1") If γ_S is a non-trivial full subgraph in $\Gamma \in \mathcal{G}_a$ supported on S (set of vertices) and not intersecting the boundary $[m]$ of Γ ("vacuum fluctuation"), then it is normal:*

$$\gamma_S \hookrightarrow \Gamma \twoheadrightarrow \gamma',$$

with γ' the corresponding quotient, and the following "short exact sequence is split":

$$C_\gamma \hookrightarrow \partial_S \bar{C}_\Gamma \twoheadrightarrow C_{\gamma'},$$

i.e. $\partial_S \bar{C}_\Gamma = C_{\gamma_S} \times C_{\gamma'}$.

Proof. Note first that a set of vertices S determines uniquely a full subgraph γ_S. Any such subgraph is "normal" as a "subobject", i.e. the quotient is admissible, i.e. exists in the given class of Feynman graphs $\mathcal{G}_a$.

The rest of the statement is a translation of the corresponding one in loc. cit. p.25. □

If the subgraph intersects the boundary, then the quotient may be a non-admissible graph ([23], p.27: "bad-edge" sub case). In all cases of "type S_2" we have the following:

Lemma 20. *("Type S2") If γ_{S_1,S_2} is a non-trivial full subgraph in $\Gamma \in \mathcal{G}_a$ supported on internal vertices from S_1 and intersecting the boundary $[m]$ of Γ along S_2, then γ_{S_1,S_2} is normal in Γ viewed as an object of $\mathcal{G}$:*

$$\gamma_{S_1,S_2} \hookrightarrow \Gamma \twoheadrightarrow \gamma',$$

with γ' the corresponding quotient, and the following "s.e.s is split":

$$C_{\gamma_{S_1,S_2}} \hookrightarrow \partial_{S_1,S_2} \bar{C}_\Gamma \twoheadrightarrow C_{\gamma'},$$

i.e. $\partial_{S_1,S_2} \bar{C}_\Gamma = C_{\gamma_{S_1,S_2}} \times C_{\gamma'}$.

Proof. Note first that an arbitrary set of vertices S (here $S_1 \cup S_2$), internal or not, determines uniquely a full subgraph γ_S. Also recall that the boundary of the quotient of Γ is also a quotient: $\gamma' : \emptyset \to [m]/S_2$ (see Remark 1).

loc. cit. p.26. □

Remark 21. *If $n_2 = |S_1|$ and $m_2 = |S_2|$, then the condition $n_2 + m_2 < n + m$ is equivalent to $S \subsetneq \Gamma$, and $2n_2 + m_2 - 2 \geq 2$ is equivalent to $|S| \geq 2$ under the assumption that $S = S_1 \cup S_2$ intersects the boundary $[m]$.*

Remark 22. *The "bad edge" sub case (when the quotient is not an admissible graph), is eliminated at the level of integration, yielding zero integrals. Therefore from the point of view of the periods of a closed form, one may restrict to consider only "normal subobjects", i.e. considering non-trivial "extensions" within the given class of Feynman graphs realizing the given "object"* Γ.

It will be proved elsewhere that this is the natural coproduct to be considered in a suitable "category of Feynman graphs".

In view of the above remarks and with the notation from Definition 18, the two lemmas may be summed up as follows.

Proposition 23. *For any non-trivial full extension* $\gamma \hookrightarrow \Gamma \twoheadrightarrow \gamma'$ *in* $\mathcal{G}$*, the following "short exact sequence is split":*

$$C_\gamma \hookrightarrow \partial_\gamma \bar{C}_\Gamma \twoheadrightarrow C_{\gamma'},$$

i.e. $\partial_\gamma \bar{C}_\Gamma = C_\gamma \times C_{\gamma'}$.

Remark 24. *Note that from the point of view of integration, edges with the same source and target will yield zero integrals, since the "propagator" will be required to be zero on the diagonal:* $\Delta_F(x, x) = 0$. *Alternatively, considering equivalence classes of graphs with orientation ([7], p.2) would eliminate the graphs with one edge loops.*

One way or the other, all extensions may be considered.

3.3 Feynman state spaces

In this section we identify some intrinsic properties of integration of differential forms over the compactification of configuration spaces. These properties will lead to cohomological statements relating them to the cobar DG-algebra of Feynman graphs.

Let $\Gamma \in \mathcal{G}^l_{n,m}$ be a Feynman graph of *type* n, m and *degree* l, i.e. $\Gamma : \emptyset \to [m]$ with n internal vertices, m external legs and $2n + m - 2 + l$ edges. Then the compactification of the configuration space of Γ, $\bar{C}_\Gamma(M)$, is a manifold with corners, of dimension $k = 2n + m - 2$ ([23], p.18). Consider the rule $\Gamma \mapsto \omega(\Gamma)$, associating to a Feynman graph $\Gamma \in \mathcal{G}^l_{n,m}$ the differential form on $C_\Gamma(M)$, of codimension $-l$ corresponding to the "propagator" $\Delta_F(x, y) = d\phi(x, y)$, where $d\phi$ is the angle form on $\mathcal{H}$ as defined in [23], p.6. Note that the differential form $\omega(\Gamma)$ is closed and "vanishing on the boundary", i.e. $d\phi(x, y) = 0$ when $x \in \mathbf{R}$.

To have a non-trivial integration pairing with $\partial C_\Gamma(M)$, the codimension one strata of the boundary, $\omega(\Gamma)$ must have codimension one too, i.e. Γ should have $2n + m - 3$ edges ([23], p.25: $l = -1$).

The above C and ω, together with the integration pairing, can naturally be extended to H (see Section 2.1). We will ignore for the moment the natural categorical setup, where for instance H is the Grothendieck ring of a strict monoidal category etc.

Note also that to account for orientations (ignored all together with signs for the moment), the vertices/edges of the Feynman graphs must be labeled.

3.3.1 Feynman configurations

A "simplicial (co)homology" is considered, with models the class of graphs $\mathcal{G}$ (rather then trees, for now) to play the role of the family of standard simplices $\{\Delta_n\}_{n\in\mathbb{Z}}$. Of course, the natural thing to do to obtain a genuine configuration functor, would be to accept Feynman graphs for what they are: small categories. We will postpone "categorifying" the Hopf algebra of Feynman graphs (or rather discarding their decategorification), and therefore, in order to map it to the corresponding configurations, we will have to forget the internal structure of C_Γ, by considering the Grothendieck ring of the category of configuration spaces.

Definition 25. *The category of* configurations of Feynman graphs in M*, denoted* $\mathcal{C} = \mathcal{C}(\mathcal{G}, M)$*, is the additive strict monoidal category generated by the objects* $C_\Gamma(M)$ *with morphisms determined by equivalences of (labeled) Feynman graphs, and tensor product* $\times$ *corresponding to disjoint union of graphs.*

The above propagator $\Delta_F = d\phi$ is determined by a function $\phi : C_e(M) \to \mathbb{R}$ (element of $C_e(M)^*$: "Lagrangian"), where the simple object e (the edge) is assumed to belong to $\mathcal{G}$.

Consider the Grothendieck k-algebra of the above category, "graded" by $\mathcal{G}$:

$$C_\bullet(M) = \mathbb{R} \otimes K_0(\mathcal{C}).$$

Since H is the free (DG-co)algebra with generators Γ, the embedding of generators map C extends uniquely to H as a k-algebra homomorphism $S : H \to C_\bullet(M)$ (an isomorphism!):

$$S(\sum_i \Gamma_i) = \oplus_i c(\Gamma_i),$$

$$S(\Gamma_1 \cdot \Gamma_2) = c(\Gamma_1) \times c(\Gamma_2), \quad \Gamma_i \in \mathcal{G},$$

where (M) is tacitly understood, c_Γ denotes the isomorphism class of $C_\Gamma(M)$, and the alternative notation for addition and multiplication in the target space is meant to remind us about the categorical interpretation,

Now the category $\mathcal{C}$ has a sort of a cone functor represented by the compactification functor:

$$C_\Gamma(M) \hookrightarrow \bar{C}_\Gamma(M).$$

The codimension one boundary of the compactification of the configuration spaces has the following description:

$$\partial\bar{C}(\Gamma) \sum_{\gamma\hookrightarrow\Gamma\twoheadrightarrow\gamma'} \pm C(\gamma)\times C(\gamma'), \tag{11}$$

where the sum is restricted to proper extensions. It induces a derivation on the Grothendieck algebra defined on generators as follows:

$$\partial c_\Gamma = [\partial\bar{C}_\Gamma(M)],$$

called the *boundary map*. As wished for in Remark 14, we have the following.

Proposition 26. *$(C_\bullet(M), \times, \partial)$ is a DG-algebra.*

The following essential property is a consequence of the definitions and of Proposition 23.

Proposition 27. *The k-algebra morphism $S : T(H) \to C_\bullet$ extending the configuration space functor C is a chain map:*

$$\partial S(\Gamma) = S(\Delta\Gamma), \quad \Gamma\in H. \tag{12}$$

Proof. It follows from definitions:

$$\partial S(\Gamma) = [\partial\bar{C}_\Gamma] = [\oplus_{\gamma\to\Gamma\to\gamma'} C_\gamma\times C_{\gamma'}] \tag{13}$$
$$= \sum_{\gamma\to\Gamma\to\gamma'} S(\gamma)\times S(\gamma') = S(\Delta\Gamma). \tag{14}$$

Note that the right hand side involves the reduced coproduct defined by Equation 8. □

The above proposition is taken as a defining property.

Definition 28. *A* configuration functor *is a DG-algebra morphism*

$$S : (T(H), \otimes, \Delta) \to (C_\bullet, \times, \partial)$$

from the Feynman cobar construction to the DG-algebra of configuration spaces.

Since equation 11 will hold, a term $S(\gamma)\times S(\gamma')$, corresponding to the subgraph γ of Γ, will be denoted by $\partial_\gamma S(\Gamma)$ ("face boundary maps").

Remark 29. *The comparison with the simplicial case deserves some attention:*

$$\partial_\Gamma = \sum_{\gamma\to\Gamma\to\gamma'} \pm\partial_\gamma, \quad \partial_n = \sum_{i=1}^{n} \partial_i,$$

including the analogy with the Eilenberg-Zilber maps [10], p.55.

3.4 Feynman rules

Definition 30. *A* Feynman rule *is a multiplicative Euler-Poincare map (see [28], p.98), i.e. if $\gamma \hookrightarrow \Gamma \twoheadrightarrow \gamma'$ is an extension, then:*

$$\omega(\Gamma) = \omega(\gamma) \wedge \omega(\gamma'). \tag{15}$$

The (common) value on an edge (a simple object) is called a Feynman propagator, *and denoted by $\Delta_F = \omega(e)$.*

A Feynman integral *for the class of Feynman graphs $\mathcal{G}$, with Feynman rule $\omega(\Gamma)$ is the k-algebra morphism extending ω to H, with values in $C^\bullet(M) = \Omega(C_\bullet(M))$ (Lagrangian §2.4; compare [3], p.9):*

$$\omega(\Gamma_1 \cdot \Gamma_2) = \omega(\Gamma_1) \wedge \omega(\Gamma_2).$$

Of course a Feynman integral is also a generalized Feynman integral according to Definition 10.

To justify the last part of the above definition, recall that an Euler-Poincare map is determined by its values on simple objects, and therefore a Feynman integrand has a common value on every edge (isomorphic objects). Moreover, it descends on the corresponding Grothendieck algebra (a normalization is assumed: $\omega(pt) = 1$).

Proposition 31. *To any given propagator Δ_F there is a unique extension to a Feynman rule.*

Proof. Apply equation 15 for the case of simple subgraphs (edges). □

3.5 Feynman integrals

Definition 32. *The integration of forms $\omega(\Gamma)$ over the corresponding configuration space $S(\Gamma)$ extends bilinearly, yielding a functional on H:*

$$W(X) = \int_{S(X)} \omega(X), \quad X \in H.$$

This will be called a Feynman integral *(action on $\mathcal{G}$ - see §2.4, Definition 16, conform Theorem 33 below).*

As expected W is a character.

Theorem 33. *By extending C and ω as algebra homomorphisms, the natural pairing induced by integration:*

$$W(X) = \int_{S(X)} \omega(X), \quad X \in H,$$

yields a character of the Hopf algebra of Feynman graphs.

Proof. This is essentially "Fubini theorem". Indeed, on generators $\Gamma_i \in \mathcal{G}, i = 1, 2$:

$$W(\Gamma_1 \cdot \Gamma_2) = \int_{S_{(\Gamma_1 \cdot \Gamma_2)}} \omega(\Gamma_1 \cdot \Gamma_2) = \int_{C_{\Gamma_1} \times C_{\Gamma_2}} \omega(\Gamma_1) \wedge \omega(\Gamma_2) = W(\Gamma_1) \cdot W(\Gamma_2).$$

□

More important is the relation with the boundary map of the configuration functor (considered next), as it will be shown later on. It leads to the cohomological properties of the Feynman integrals.

3.6 Cohomological properties of Feynman integrals

As a consequence of the previous Theorem 33, the integrals over the codimension one boundary match the codifferential of the cobar construction:

$$\int_{\partial \bar{C}} = W \circ D.$$

It follows that the character W associated to the configuration functor S and propagator Δ_F is a cocycle of the cobar construction of the Hopf algebra of Feynman diagrams. This result will be used to prove the claim from Section 2, characterizing L_∞-morphisms: modulo equivalence they correspond to cohomology classes of the DG-coalgebra H.

Although the statements hold when the triple $(C, \bar{C}, \partial)$ is replaced by any configuration functor S (as the proofs show), to fix the ideas, assume S extends the configuration functor C and that $\omega = d\phi$ is the angle form as in [23]. Fix a Feynman graph $\Gamma \in \mathcal{G}^{-1}_{n,m}$, so that $\omega(\Gamma)$ is a codimension one form on the corresponding configuration space $\bar{C}_\Gamma(M)$.

¿From the previous results we deduce the following translation of the statements from (6.4.1, 6.4.2) [23], p.25, regarding the integrals over the codimension one strata.

Proposition 34. *Let $(C, \bar{C}, \partial)$ be a configuration functor, ω a Feynman rule with values in the algebra of differential forms $\Omega(C_\bullet(M))$, and W the Feynman integral corresponding to the natural pairing defined by integration.*

For any full extension $\gamma \hookrightarrow \Gamma \twoheadrightarrow \gamma'$:

$$\int_{\partial_\gamma S(\Gamma)} \omega(\Gamma) = W(\gamma) \cdot W(\gamma'). \tag{16}$$

Proof. By Proposition 23 $\partial_\gamma \bar{C}_\Gamma = C_\gamma \times C_{\gamma'}$. Since ω is an Euler-Poincare map (Definition 30), the claim follows by "Fubini theorem":

$$\int_{\partial_\gamma \bar{C}_\Gamma} \omega(\Gamma) \int_{C_\gamma \times C_{\gamma'}} \omega(\Gamma) \int_{C_\gamma \times C_{\gamma'}} \omega(\gamma) \wedge \omega(\gamma') W(\gamma) \cdot W(\gamma').$$

□

Remark 35. *Note that the statement holds also when γ' is not admissible ("bad-edge" case), due to the fact that the "propagator" $\Delta_F(x,y)$ vanishes on the boundary.*

Regarding the relation with the condition corresponding to L_∞-algebra morphisms ((F) from [23], p.24; see Section 5), note that some of the integrals over the codimension one boundary strata vanish, the remaining ones matching the coefficient c_γ of U_Γ in the Feynman expansion of (F).

$$``\int_{\partial\bar{C}} \omega = c''.$$

In order to distinguish various portions of the boundary $\partial\bar{C}$, corresponding via Equation 12 to portions of the reduced comultiplication of the Hopf algebra H, we will introduce the following.

Notation 36.

$$\Delta_e = \sum_{e\hookrightarrow\Gamma\twoheadrightarrow\Gamma/e} \pm\Gamma/e\otimes e. \tag{17}$$

When restricted to internal edges, the above sum will be denoted by Δ_e^{int}. The corresponding portion of the sum in the graph homology differential 3 will be denoted by d^{int}.

In general, when considering internal subgraphs, i.e. with their boundary consisting of internal vertices, or subgraphs meeting the boundary, the following notation will be used:

$$\Delta_i = \sum_{\Gamma_2\hookrightarrow\Gamma\twoheadrightarrow\Gamma_1,\ \Gamma_2\cap\partial\Gamma=\emptyset} \pm\Gamma_1\otimes\Gamma_2. \tag{18}$$

$$\Delta_b = \sum_{\Gamma_2\hookrightarrow\Gamma\twoheadrightarrow\Gamma_1,\ \Gamma_2\cap\partial\Gamma\neq\emptyset} \pm\Gamma_1\otimes\Gamma_2. \tag{19}$$

Δ_{i-e} and Δ_{b-e} refer to sums over extensions where Γ_2 is not an edge (all extensions are assumed to be proper).

3.6.1 Type S1 terms.

First recall that any full subgraph not meeting the boundary ∂M is a normal subgraph. The integrals over the codimension one strata corresponding to such a subgraph γ of type $(n,0)$, i.e. not intersecting the boundary of M (and yielding type S_1 terms), with $n\geq 3$ vanish (see 6.4.1.2. [23], p25).

The other terms ($n\leq 2$) correspond to full subgraphs γ consisting of one *internal edge* of Γ, i.e. simple subgraphs not meeting the boundary. The corresponding terms total the "internal part" of the graph homology differential 3.

Proposition 37. *For any Feynman graph* $\Gamma \in \mathcal{G}$*:*

$$\sum_{e \in \Gamma^{(1)}_{int}} \int_{\partial_e \bar{C}_\Gamma} \omega(\Gamma) W(d^{int}\Gamma),$$

where e *is a simple subgraphs of* Γ*, without boundary.*

Proof. Let γ_e denote such a subgraph corresponding to the internal edge $e \in \Gamma^{(1)}$. Applying Proposition 34 yields:

$$\sum_{e \in \Gamma^{(1)}} \int_{\partial_{\gamma_e} \bar{C}_\Gamma} \omega(\Gamma) = \sum_{e \in \Gamma^{(1)}} W(\gamma_e) \cdot W(\Gamma/\gamma_e) = coef. \cdot W(\sum_{e \in \Gamma^{(1)}} \pm \Gamma/\gamma_e) = coef. \cdot W(d\Gamma),$$

where the coefficient depends only on the propagator Δ_F. With the appropriate normalization, we have:

$$W(\gamma_e) = \int_{\bar{C}_{2,0}(M)} d\phi_e = \pm 1$$

The sign is given by the labels of the two vertices of the internal edge, corresponding to the convention for the orientation of graph homology. □

To emphasize the relation with the comultiplication from Section 2, the above results may be restated as follows.

Corollary 38.

$$W \circ \Delta^{int}_e = W \circ d^{int}, \quad W \circ \Delta_{i-e} = 0.$$

3.6.2 Type S2 terms.

Subgraphs Γ_2 which do meet the boundary of Γ may produce quotients which are not admissible graphs ($\Gamma_1 \notin \mathcal{G}_a$).

Lemma 39. *Feynman integrals over codimension one strata corresponding to non-normal subgraphs vanish.*

Proof. See [23], p.27, the "bad edge" case 6.4.2.2. □

The remaining terms corresponding to normal proper subgraphs meeting the boundary $[m]$ of $\Gamma \in \mathcal{G}_a$ yield a "forest formula" corresponding to the coproduct Δ_b of $\mathcal{G}$.

Proposition 40. *For a Feynman graph* $\Gamma \in \mathcal{G}_a$*:*

$$\sum_{\gamma \hookrightarrow \Gamma \twoheadrightarrow \gamma' \ in \ \mathcal{G}} \int_{\partial_\gamma \bar{C}_\Gamma} \omega(\Gamma) W(\Delta_b \Gamma), \tag{20}$$

where the proper normal subgraph γ *meets non-trivially the boundary of* Γ*.*

Proof. The formula follows from definitions and from the multiplicative property of W (Proposition 34), in the same way as for Proposition 37. □

Putting together the two types of terms, S_1 and S_2, and independent of the vanishing of some of the terms, we obtain the following "Forest Formula".

Theorem 41. *For any graph* $\Gamma \in \mathcal{G}$*:*

$$\int_{\partial \bar{C}_\Gamma} \omega(\Gamma) = W(\Delta\Gamma). \tag{21}$$

Remark 42. *The above result holds for an arbitrary configuration functor* S *and Feynman integrand* ω*.*

So far we did not need $\omega(\Gamma)$ to be a closed form. With this additional assumption, using Stokes Theorem (duality at the level of a general configuration functor - see Remark 14), the closed form produces a cocycle.

Corollary 43. *If the "Lagrangian"* ω *is a closed form then the corresponding Feynman integral* W *is a cocycle.*

Proof.

$$(\delta W)(\Gamma) = W(\Delta\Gamma) \overset{T41}{=} \int_{\partial \bar{C}_\Gamma} \omega(\Gamma) \overset{Stokes}{=} \int_{\bar{C}_\Gamma} d\omega(\Gamma) = 0.$$

□

The main property of the Feynman integrals W, the Forest Formula, may be interpreted in a manner relevant to renormalization, as follows.

Note first that W is obtained as a "cup product/convolution":

$$S * \omega = \int \circ (S \otimes \omega) \circ c : H \to \mathbf{R}, \tag{22}$$

where $c : H \to H \otimes H$ denotes the natural group-like comultiplication of H.

Definition 44. *The dual DG-algebra of* Feynman characters*:*

$$H^* = \{w : H \to \mathbf{R} |\ w \ character\ \}$$

is called the convolution algebra *of* $\mathcal{G}$*. The corresponding differential is given by:*

$$\delta w = w \circ \Delta, \quad w \in T(H)^*.$$

Theorem 45. δ *is a derivation with respect to the "convolution" of the configuration functor* S *and the Lagrangian form* ω*:*

$$\delta(S * \omega) = (\partial S) * \omega + S * (d\omega).$$

Proof. Let $W = S * \omega$ (Equation 22). Interpreting the left hand side of Equation 21 according to the above definition, and using Theorem 41 yields:

$$(\delta W)(\Gamma) = W(\Delta\Gamma) = \int_{\partial S(\Gamma)} \omega(\Gamma).$$

Since a Lagrangian is a closed form, $d\omega = 0$, the second term is zero, and the equality is proved. □

The implications to the deformation point of view to renormalization will be considered elsewhere.

4 State sum models and Feynman rules

In the previous section a Feynman rule with propagator $\Delta_F = d\phi$ paired via integration with a configuration functor $(C, \bar{C}, \partial)$ produced the Feynman integral $W : H \to \mathbb{R}$.

In this section a typical generalized Feynman rule is considered, yielding the generalized Feynman integral $U : H \to Hom(T, D)$ of[23] (see Definition 10). It is a typical "state-sum model" (state model [26], p.100; see also [33], p.345), having a Feynman path integral interpretation as already noted in [6].

4.1 A state-sum model on graphs

We will review the construction mostly keeping the original notation.

For each graph $\Gamma \in \mathcal{G}_a$, a function $\Phi =< U_\Gamma(\gamma), f >$ will be defined, where $\gamma = \gamma_1 \otimes ...\gamma_n$, $f = f_1 \otimes ... \otimes f_m$ and $<,>$ denotes the natural evaluation pairing.

States of a graph have two conceptually distinct groups of data: associating polyvector fields to internal vertices and appropriate functions to boundary points.

First chose a basis $\{\partial_i\}_{i=1..d}$ for the algebra of vector fields. A *coloring* of the labeled graph $\Gamma \in \mathcal{G}_{n,m}$ (vertices are ordered), is a map:

$$I : \Gamma^{(1)} \to \{1, ..., d\}.$$

A *basic state* of the graph Γ is the following association corresponding to a coloring of its edges:

$$\phi^0(v) = \gamma_v,\ v \in \Gamma^{(0)}, \qquad \phi^1(e) = \partial_{I(e)},\ e \in \Gamma^{(1)}.$$

Remark 46. *It is customary to implement ϕ^0 via an ordering of the vertices of Γ, obtaining the map $U_\Gamma : T^n \to Hom(A^m, A)$.*

We preferred this more cumbersome notation (e.g. [22], p.28) because it reveals the true nature of a state-sum: a 2-functor when interpreted categorically ([16]).

Then Φ is the sum over all basic states corresponding to a fixed choice of ϕ^0 ($\gamma' s$ and $f' s$), of the corresponding "amplitude" (to be defined shortly):

$$\Phi = \sum_{all\ basic\ states\ \phi\ of\ \Gamma} \Phi_\phi.$$

Remark 47. *The true amplitude of the process would involve a sum over all states, when the values of ϕ^0 varies on the internal vertices while the state of the boundary f is fixed. The sum over values on the edges amounts to a contraction process (traces etc.).*

Now Φ_ϕ (Φ_I of [23], p.23) is a product over the contributions $\Phi_\phi(v)$ over the vertices of Γ, n internal and m boundary type.

For an internal vertex v:

$$\Phi_\phi(v) = (\prod_{e\in In(v)} \phi(e))\psi_v, \quad \psi_v =< \gamma_v, \bigotimes_{e\in Out(v)} dx^{I(e)} >,$$

where $In(v)$ ($Out(v)$) denotes the set of incoming (outgoing) edges of the vertex v, and the shorthand notation $\phi = \phi^{(1)}$ was used since ϕ^0 is fixed within this state-sum.

4.2 The amplitude interpretation

Towards an "propagation amplitude" interpretation, replace the evaluation pairing with the inner product (,) such that the above basis $\{\partial_i\}_{i=1..n}$ be orthonormal. Also collect the "in" and "out" products, introducing the following terminology.

Definition 48. *For any basic state ϕ of the graph Γ:*

$$\phi_{In} = \prod_{e\in In(v)} \phi(e), \quad \phi_{Out} = \bigotimes_{e\in Out(v)} \phi(e),$$

are called the In *and* Out *states of the scattering process at the vertex v.*

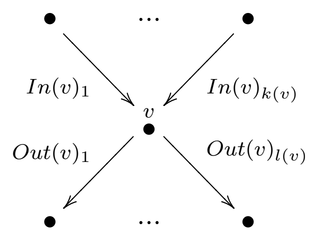
Proposition 49.

$$-\Phi_\phi(v) = (\phi_{Out}(v), ad_{\phi(v)}(\psi_{In}(v)))$$

is the scattering amplitude*:*

$$(\phi_{Out}|ad_\phi|\phi_{In})_{|v}$$

of the elementary process at the internal vertex v:



Proof. Here $ad_X(Y) = [X, Y]$ is the commutation bracket on differential operators. Therefore if X and Y commute, then $[X, fY] = X(f)Y$.

Now to retrieve the appropriate component, use the above inner product:

$$\Phi_\phi(v) = \phi_{In}(\psi_v) = ([\psi_{In}(v), \gamma_v], \phi_{Out}(v)).$$

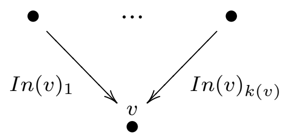
This can be put in the form of a propagation amplitude (| |), establishing the above claim. ☐

In a similar manner, for boundary vertices we have the following

Proposition 50. *If $v \in \partial\Gamma$ is a boundary vertex, then:*

$$\Phi_\phi(v) =< \phi_{In}(v), \phi(v) >,$$

is the "expectation value" of the process:



where < , > denotes the natural evaluation pairing between polyvector fields and functions.

4.3 A TQFT interpretation

The properties of the generalized Feynman (path) integral U may be viewed as consequences from the generalized TQFT implemented via the above state-sum model. We will only sketch some points related to this TQFT interpretation[16].

Interpret graphs as cobordisms and extensions as composition of cobordisms determined by the insertion vertex and the order of matching the external legs. If v is an internal vertex of Γ_1 for instance, the insertion of Γ_2 at the vertex v (with the additional data σ regarding the vertex matching order), precisely corresponds to the composition of the corresponding cobordisms:

$$\Gamma_1 \circ_v^\sigma \Gamma_2 = [\Gamma_1 - v] \circ [\Gamma_2], \qquad \emptyset \overset{[\Gamma_2]}{\to} [k], \quad [k] \overset{[\Gamma_1 - v]}{\to} [m],$$

where k is the valency of v, and $\Gamma_1 - v$ is the graph obtained by cutting the vertex v out ($\Gamma_1 - v$ will have both an "In" and an "Out" boundary).

In this context, the Euler-Poincare property of a Feynman rule generalizes in the present context (states on graphs, i.e. graph cohomology) to a "propagator property":

$$K(In, Out) = \sum_{states\ \phi} K(In, \phi)K(\phi, Out).$$

Since here the propagator is essentially $< g|U_\Gamma(\phi)|f >$ (if Γ has both an In and Out boundary), one may chose to consider extensions at the level of states (graph cohomology):

$$(\Gamma_2, \phi_2) \hookrightarrow (\Gamma, \phi) \twoheadrightarrow (\Gamma_1, \phi_1).$$

Here ϕ_2 and ϕ_1 are determined as restrictions of ϕ to Γ_2 and $\Gamma_1 - v$, while the state of the vertex $v \in \Gamma_1$ is the "effective state" of Γ_2:

$$\phi_1(v) = U_{\Gamma_2}(\phi_2)$$

(see operation $\bullet$ and Lemma 53 below).

The following basic property of U is expected (generalized Euler-Poincare map / propagator property).

Proposition 51. *If* $(\Gamma_2, \phi_2) \hookrightarrow (\Gamma, \phi) \twoheadrightarrow (\Gamma_1, \phi_1)$*, then:*

$$U_\Gamma(\phi) = U_{\Gamma_1}(\phi_1) \circ U_{\Gamma_2}(\phi_2).$$

4.4 The generalized Feynman rule

Returning to our main objective, we still have to prove that:

Proposition 52. *U is a pre-Lie morphism:*

$$U_{\Gamma_1 \star_b \Gamma_2} = U_{\Gamma_1} \circ U_{\Gamma_2},$$

where the extensions defining the product $\star_b$ correspond to subgraphs intersecting the boundary.

Proof. The above claim is a consequence of the more basic fact regarding insertions of appropriate graphs at a vertex. The "Gerstenhaber-like" compositions from the above right hand sides are typical for this TQFT gluing/composition operations, as sketched in Section 4.3. One would then establish the claim at the level of the corresponding TQFT, using the propagator property (Proposition 51). □

It is not clear for the moment the role of the above restriction to boundary meeting extensions (discarding the "vacuum fluctuations").

Note also the following relation with the operation $\bullet$ on polyvector fields.

Lemma 53. *If $\Gamma' = \Gamma/e$ is obtained by collapsing an edge of Γ, then:*

$$U_{\Gamma'} \circ U_e(\gamma) = \sum_{i \neq j} U_{\Gamma'}((\gamma_i \bullet \gamma_j) \wedge ...).$$

Proof. Again this is a consequence of the above "propagator property" (Proposition 51) where the state on the collapsed edge is:

$$\phi(v) = \gamma_i \bullet \gamma_j.$$

□

The following consequence is claimed (see also [23], 6.4.1.1., p.25).

Corollary 54. *For all* $\Gamma' \in \mathcal{G}$*:*

$$U_{\Gamma' \star e}(\gamma) = \sum_{i \neq j} U_{\Gamma'}((\gamma_i \bullet \gamma_j) \wedge ...).$$

Perhaps one can avoid involving the pre-Lie operation, and remain at the level of Lie algebras/UEAs (L_∞-algebras).

Remark 55. *If* $B_\Gamma = U_\Gamma(\alpha \wedge ... \wedge \alpha)$ *where* α *is the Poisson structure [23], p.28, then* B_Γ *is a Feynman integral corresponding to the propagator* α*.*

5 Applications

As a first application of the previous formalism, we interpret Kontsevich formality between the two DGLAs $T = T_{poly}(\mathbb{R}^d)$ and $D = D_{poly}(\mathbb{R}^d)$ of [23]. We will prove that the L_∞-condition (F) from [23], p.24:

$$(F1) \qquad \sum_{i \neq j} \pm U_{n-1}((\gamma_i \bullet \gamma_j) \wedge ... \wedge \gamma_n) \tag{23}$$

$$(F2) \quad + \sum_{k+l=n} 1/(k!l!) \sum_{\sigma \in \Sigma_n} \pm U_k \circ U_l(\gamma_\sigma) = 0, \tag{24}$$

follows in a direct way from the fact that U is a generalized Feynman integral, and it preserves the pre-Lie composition of Feynman graphs (Definition 6), as claimed in the previous section. This will essentially yield a proof of the general result of §2.3 (Theorem 11).

Theorem 56. *(i)* $[Q, U] = (\delta W)U$*, where* Q *denotes the appropriate* L_∞*-structure;*
(ii) U *is an* L_∞*-algebra morphism iff* $\delta W = 0$*.*

Proof. We will prove (i), since (ii) becomes clear after recalling that U is an L_∞-morphism iff $[Q, U] = 0$ (see [21]) or (F) holds true (see [23]). Here $[Q, U] = Q^1 \circ U \pm U \circ Q^2$ (see §§2.2 and [21], 8,9,12), and $\delta W(\Gamma) = W(\Delta\Gamma) = W(d^{int}\Gamma) + W(\Delta_b\Gamma)$. Instead of $[Q, U]$ we will refer to its alternative form (F).

As stated in [23], the L_∞-algebra condition (F) corresponds to $W(d^{int}\Gamma)$ (first line - 6.4.1.1, p.25) and $W(\Delta_b\Gamma)$ (second line - 6.4.2.1., p.26). Since the other integrals vanish, the sum of the two contributions equals $W(\Delta)$.

Indeed, we will compute the coefficients c_Γ, and prove that:

$$c_\Gamma = \delta W(\Gamma). \tag{25}$$

Recall that $U = \sum_n U_n$ and $U_n = \sum_{m\geq 0} \sum_{\Gamma\in G^{-1}_{n,m}} W_\Gamma U_\Gamma$, where $G^{-1}_{n,m}$ is the set of graphs with n internal vertices, m external vertices and $2n+m-2$ edges. Therefore $U_\Gamma : T^n \to D_m$, and $U_\Gamma(\gamma) : A^m \to A$, where $A = C^\infty(M)$.

Substitute the above Feynman expansion in equation (F), to obtain:

$$\sum_{\Gamma'\in G_{n-1,m}} \pm W_{\Gamma'} \sum_{i\neq j} U_{\Gamma_1}((\gamma_i \bullet \gamma_j)\wedge ...) \tag{F1}$$

$$+ \sum_{\Gamma_1\in G_{k,m},\Gamma_2\in G_{l,m}} \pm W_{\Gamma_1} W_{\Gamma_2} U_{\Gamma_1} \overset{\circ}{\wedge} U_{\Gamma_2} = 0, \tag{F2}$$

where the result of alternating the Gerstenhaber composition was denoted by:

$$U_{\Gamma_1} \overset{\circ}{\wedge} U_{\Gamma_2}(\gamma) 1/(k!l!) \sum_{\sigma\in\Sigma_n} U_{\Gamma_1}(\gamma_{\sigma(1)}\wedge ...) \circ U_{\Gamma_2}(\gamma_{\sigma(k+1)}\wedge ...).$$

In order to compare it with our claim (Equation 25):

$$c_\Gamma = \sum_{\Gamma_1\to\Gamma\to\Gamma_2} \pm W_{\Gamma_1} W_{\Gamma_2}, \tag{26}$$

use the above lemmas and rearrange the sums. The first line (F1) transforms as follows:

$$\begin{aligned}
\sum_{\Gamma'\in G_{n-1,m}} W_{\Gamma'} \sum_{i\neq j} U_{\Gamma'}((\gamma_i\bullet\gamma_j)\wedge ...) &= \sum_{\Gamma'} W_{\Gamma'} U_{\Gamma'\star e}(\gamma), \quad \textit{by Corollary 54} && (27)\\
&= \sum_{e\hookrightarrow\Gamma\twoheadrightarrow\Gamma'^{(1)}_{int}} W_{\Gamma'} U_\Gamma(\gamma) \quad \textit{by Definition 6} && (28)\\
&= \sum_{\Gamma\in G_{n,m}} \Big(\sum_{e\hookrightarrow\Gamma\twoheadrightarrow\Gamma'^{(1)}_{int}} \pm W_{\Gamma/e}) U_\Gamma(\gamma) && (29)\\
&= \sum_{\Gamma\in G_{n,m}} W_{d^{int}\Gamma} U_\Gamma(\gamma) \quad \textit{by Equation 3}. && (30)
\end{aligned}$$

To transform the second line (F2), postpone the application of the alternation operator $\wedge$:

$$\sum_{\Gamma_1\in G_{k,m},\Gamma_2\in G_{l,m}} \pm W_{\Gamma_1}W_{\Gamma_2}U_{\Gamma_1}\circ U_{\Gamma_2} = \sum_{\Gamma_1,\Gamma_2} \pm W_{\Gamma_1}W_{\Gamma_2}U_{\Gamma_1\star_b\Gamma_2} \quad \textit{by Proposition 52} \tag{31}$$

$$= \sum_{\Gamma_1,\Gamma_2} W_{\Gamma_1}W_{\Gamma_2} \sum_{\Gamma_1\hookrightarrow\Gamma\twoheadrightarrow\Gamma_2,\ \Gamma_1\cap\partial\Gamma\neq\emptyset} \pm U_\Gamma \quad \sim \textit{Definition 6} \tag{32}$$

$$= \sum_{\Gamma_1,\Gamma_2} \sum_{\Gamma_1\hookrightarrow\Gamma\twoheadrightarrow\Gamma_2,\ \Gamma_1\cap\partial\Gamma\neq\emptyset} \pm W_{\Gamma_1}W_{\Gamma_2}U_\Gamma \tag{33}$$

$$= \sum_{\Gamma\in G_{n,m}} W(\Delta_b\Gamma)U_\Gamma, \tag{34}$$

where Propositions 34 and Proposition 40 were used. Adding the two expressions yields the nonzero terms from the right hand side of Equation 26. □

Remark 57. *Working with the above equalities after applying the alternation operator amounts to proving the statements at the level of Lie algebras, avoiding the pre-Lie operations.*

Remark 58. *The equation (i) from the Theorem 56 may be interpreted as:*

$$ad_Q(U) = (\delta W)U,$$

i.e. that pre-L_∞-morphisms which are Feynman expansions are solutions of an eigenvalue problem: U is an eigenvector corresponding to the eigenvalue δW. The kernel consists of L_∞-morphisms.

The formality is obtained as a corollary.

Corollary 59. *The pre-L_∞-morphism $U : T \to D$ is an L_∞-morphism.*

Proof. By Corollary 43 the Feynman integral W determined by the configuration functor $(C, \bar{C}, \partial)$ is a cocycle. To conclude apply the above theorem. □

As a second application we envision an algebraic/combinatorial Feynman integral (closer in spirit to the "book keeping" of a Gaussian expansion) based on the Hopf algebra of forests ("labels of the boundary of S_Γ" - see [23], p.20). More generally, the problem (to be investigate elsewhere) is to find suitable examples of pairings with Lagrangians, yielding such cocycles. Then the corresponding system of weights would provide a formula for the star-product of a Poisson manifold [23].

6 Conclusions and further developments

The intent of the present article is to isolate some of the algebraic properties of the Feynman integral which are *independent of renormalization*. The infinities of QFT are of two types: "infrared", due to the continuum nature of the ambient space-time used as a configuration space for the system quantized, and "ultra-violet" divergencies due to the non-compactness of the corresponding state space (energy-momenta). The first "problem" is expected to be cured by a reformulation of QFT as a graded theory by scale, similar in nature to the Haar multi-resolution analysis. The adequate framework seems to be that of L/A-infinity algebras as used by Kontsevich to prove the Formality Theorem. Moreover, this result is, in the author's opinion, prototypical of what a resolution of a manifold is, or rather its Poisson algebra of observables [20].

The second "problem" is due to the starting point in quantization, the classical framework of initial value problems characteristic of *pointwise physics*. In contrast, quantization aims to describe interactions as input-output processes, for which the categorical language is mandatory from the mathematical side, and for which the Feynman approach is the standard: the *physics of processes* (Markov, Feynman etc.). The non-compactness of the momentum-energy state space and the need for compactification of the corresponding configuration space bears a similarity with the compactification of Euclidean plane geometry as a representative of the conformal geometry on its compactification, the Riemann sphere. The author thinks that the algebraic renormalization of Connes-Kreimer contains an underlying *categorification* of physics, via the Birkhoff decomposition (factorization) [20]. The connections with the BV-approach [4, 1, 30] will also be investigated.

In conclusion, it is important to be able to formulate an axiomatic interface to QFT of the Feynman Path Integral type. Here the "label" QFT is used generically, not just referring to the Feynman graphs as models of the external geometric data for the quantum interaction processes. Separating the interface from a specific implementation dependent on a particular language used (e.g. distributions), allows to implement the mathematical model for the Feynman path integral quantization using the mathematical tools of homotopical algebra:

$$\int \mathcal{D}\gamma \; e^{S[\gamma]} \quad = \quad \sum_n \sum_\gamma U_n(\gamma).$$

The left hand side is a conceptual framework which need not be implemented using analytical tools (integrals, measures, etc.), but most likely with algebraic tools, e.g. state sum models yielding TQFTs etc..

The L_∞-algebra approach is a direct implementation of the RHS. It is aimed to represent a resolution of the partition function, based on a resolution of the algebra of observables of an ambient space-time, if present. Recall that the main role of the Formality Theorem, in order to solve the quantization problem (star-products), is to allow to deform the Poisson algebra by deforming the "resolution" (the other side of the quasi-isomorphism).

Therefore the study of a sigma-model which is convergent, and does not require renormalization, allows to pursue the above long-term goal.

Returning to the above "equation", one can defend the above strategy by claiming that the RHS is conceptually closer to the spirit of quantum theory focusing on describing correlations. The philosophy sketched in [18] reinterprets the concept of space-time as a receptacle of interactions/transitions between states, and adequately modeled by "categories with Lagrangians", while the LHS comes from the traditional "manifold approach to space-time" trying to force integrals in the sense of analysis "converge".

The former philosophy can be implemented by defining a "Feynman category" to be essentially a "generalized cobordism category"[16], with actions as functors (see [18]). Cobordism categories and TQFTs, tangles, operads, PROPs, and various other graphical calculi can be restated in terms of Generalized Cobordism Categories and their representations.

Glossary

The following table is a (tentative) dictionary of selected terms and notations used. The symbol $\sim$ is used to denote merely a relation.

Notation	Mathematics	Physics
$\mathcal{G}$	Gen. Cobord. Cat.	Class of Feynman graphs
Γ	Object	Feynman graph
g	Lie algebra of PIs	Feynman graphs + insertions
$H(\mathcal{G})$	Grothendieck DG-coalgebra	Feynman graphs
$\mathcal{F}_\bullet$	L_∞-morphism	Partition function
$\mathcal{F}_n$	n-th derivative	Green function
W	Weight / cocycle	$\sim$ Feynman integral
U	pre-Lie algebra morphism	$\sim$ Feynman integral
$H(H;k)$	L_∞-morphisms	Feynman expansions
S	Configuration functor	Configuration spaces
$\omega(\Gamma)$	Closed top form on $S(\Gamma)$	Interaction Lagrangian
$W =< S, \omega >$	Pairing	$\sim$ Action

References

[1] M. Alexandrov, M. Konstsevich, A. Schwarz, and O. Zaboronsky, *The geometry of the master equation and topological quantum field theory.* Internat. J. Modern Phys. A **12** (1997), no. 7, 1405–1429. MR1432574(98a:81235). Zbl 1073.81655. hep-th/9502010.

[2] D. Arnal, D. Manchon, and M. Masmoudi, *Choix des signes pour la formalite de M. Kontsevich*, Pacific J. Math., **203**(1):23–66, 2002. MR1895924(2003k:53123). Zbl 1055.53066. math/0003003.

[3] S. Axelrod and I.M. Singer, *Chern-Simons perturbation theory II.* MR1258919(95b:58163). Zbl 0889.53053. hep-th/9304087 v1.

[4] I. Batalin and G. Vilkovisky, *Gauge algebra and quantization*, Phys. Lett. **102B**, 27 (1981). MR0616572(82j:81047).

[5] R. Bott and C. Taubes, *On self-linking of knots*, J. Math. Phys. **35** (10), October 1994, 5247-5287. MR1295465(95g:57008). Zbl 0863.57004.

[6] A. S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*. Comm. Math. Phys. **212** (2000), no. 3, 591–611. MR1779159(2002b:53141). Zbl 1038.53088. math.QA/9902090.

[7] J. Conant and K. Voghtmann, *Infinitezimal operations on graphs and graph homology*, Math. Ann. **327** (2003), no. 3, 545–573. MR2021029(2004m:17026). Zbl 1092.17014. math.QA/0111198 v2.

[8] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem I*, Comm. Math. Phys. **210** (2000), no. 1, 249–273. MR1748177(2002f:81070). Zbl 1032.81026. hep-th/9912092.

[9] A. Connes and D. Kreimer, *Insertion and Elimination: the doubly infinite Lie algebra of Feynman graphs*, Ann. Henri Poincaré **3** (2002), no. 3, 411–433. MR1915297(2003j:81118). Zbl 1033.81061. hep-th/0201157.

[10] J. F. Davis and P. Kirk, *Lecture notes in algebraic topology*, Graduate studies in mathematics Vol. **35**, American Mathematical Society, Providence, RI, 2001. xvi+367 pp. ISBN: 0-8218-2160-1. MR1841974(2002f:55001). Zbl 1018.55001.

[11] D. Fiorenza and L. M. Ionescu, *Graph complexes in deformation quantization*, Lett. Math. Phys. **73** (2005), no. 3, 193–208. MR2188293(2007d:53153). Zbl 1101.53058. math/0505410 v2.

[12] M. Gerstenhaber and S. Schack, *Algebraic cohomology and deformation theory*, *Deformation theory of algebras and structures and applications*, M. Hazewinkel and M. Gerstenhaber (eds.), 11-264, 1988, Kluwer Academic Publishers. MR0981619(90c:16016). Zbl 0676.16022.

[13] V. K. A. M. Gugenheim, L. A. Lambe, and J. D. Stasheff, *Perturbation theory in differential homological algebra II*, Ill. J. Math. Vol. **35**, No.3, Fall 1991, p.357-373. MR1103672(93e:55018). Zbl 0727.55012.

[14] L. M. Ionescu, *Perturbative quantum field theory and integrals on configuration spaces*, hep-th/0307062.

[15] L. M. Ionescu, *Cohomology of Feynman graphs and perturbative Quantum Field Theory*, Focus on Quantum Field Theory, Nova Publishers Inc., O. Kovras (editor), ISBN: 1-59454-126-4, 2004, 17 pages. math.QA/0506142.

[16] L. M. Ionescu, *The Feynman Legacy*, math.QA/0701069.

[17] L. M. Ionescu, *Perturbative Quantum Field Theory and L-infinity Algebras*, Advances in Topological Quantum Field Theory, Proceedings of the NATO ARW on New Techniques in Topological Quantum Field Theory, editor J. Bryden, Kluwer Academic Publishers, 2004, 243–252. MR2147421(2006h:81088).

[18] L. M. Ionescu, *Remarks on quantum theory and noncommutative geometry*, Int. J. Pure and Applied Math., Vol. **11**, No.4, 2004, pp.363-376. math.HO/0006024.

[19] L. M. Ionescu, *Non-associative algebras: a Framework for Differential Geometry*, Int. J. Math. Math. Sci., Vol. **2003**, No.60. math.DG/9910016.

[20] L. M. Ionescu, *Connections between Kontsevich's Formality Theorem and Connes-Kreimer renormalization*, in preparation.

[21] B. Keller, *An introduction to $A-\infty$-algebras*, Homology Homotopy Appl. **3** (2001), no. 1, 1–35 (electronic). MR1854636(2004a:18008a). Zbl 0989.18009. math.RA/9910179.

[22] V. Kodiyalam and V. S. Sunder, *Topological quantum field theories from subfactors*, Research Notes in Mathematics Series, Chapman & Hall/CRC, 2001. MR1870810(2003a:57024). Zbl 1050.46038.

[23] M. Kontsevich, *Deformation quantization of Poisson manifolds I*, First European Congress of Mathematics, Vol. II (Paris, 1992), 97–121, Progr. Math., **120**, Birkhäuser, Basel, 1994. MR1341841(96h:57027). Zbl 1058.53065. q-alg/9709040

[24] M. Kontsevich, *Formality conjecture, Deformation theory and symplectic geometry*, (Ascona, 1996), 139–156, Math. Phys. Stud., 20, Kluwer Acad. Publ., Dordrecht, 1997. MR1480721(98m:58044).

[25] M. Kontsevich, *Operads and Motives in Deformation Quantization*, Moshé Flato (1937–1998). Lett. Math. Phys. **48** (1999), no. 1, 35–72. MR1718044(2000j:53119). Zbl 0945.18008. math.QA/9904055.

[26] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, Joseph, A. (ed.) et al., First European congress of mathematics (ECM), Paris, France,

July 6-10, 1992. Volume II: Invited lectures (Part 2). Basel: Birkhäuser. Prog. Math. **120**, 97-121 (1994). MR1341841(96h:57027). Zbl 0872.57001.

[27] D. Kreimer, *Structures in Feynman Graphs - Hopf algebras and symmetries*, Graphs and patterns in mathematics and theoretical physics, 43–78, Proc. Sympos. Pure Math., **73**, Amer. Math. Soc., Providence, RI, 2005. MR2131011(2006b:81178). Zbl 1088.81077. hep-th/0202110

[28] S. Lang, *Algebra, Addison-Wesley series in mathematics*, Addison-Wesley Publishing Co., 1965. MR1878556(2003e:00003). Zbl 0984.00001.

[29] M. Markl, *A cohomology theory for A(m)-algebras and applications*, JPAA **83** (1992) 141-175. MR1191090(94a:18012). Zbl 0801.55004.

[30] V. Rivasseau, *An introduction to renormalization*, Seminaire Poincare, Vol. **2** (2002), 1–29. MR2169912(2006f:81118). Zbl 1052.81072.

[31] D. Tamarkin, *Another proof of M. Kontsevich formality theorem*, math/9803025.

[32] D. Tanre, *Homotopie rationnelle*, L.N.M. **1025**, 1983. MR0764769(86b:55010). Zbl 0594.55013.

[33] V. Turaev, *Quantum invariants of knots and 3-manifolds*, Walter de Gruyer, Berlin - Ney York, 1994. MR1292673(95k:57014). Zbl 0812.57003.

[34] A. A. Voronov, *Topological field theories, string backgrounds and homotopy algebras*, Differential geometric methods in theoretical physics (Ixtapa-Zihuatanejo, 1993). Adv. Appl. Clifford Algebras **4** (1994), Suppl. 1, 167–178. MR1337702(96k:81245). Zbl 0864.57035. hep-th/9401023.

[35] A. A. Voronov, *Lecture 9: The A_∞ operad and A_∞ algebras*, http://www.math.umn.edu/˜voronov/8390/.

Lucian M. Ionescu
Illinois State University, Mathematics Department
School Street,
Normal, IL 61790-4520, USA.
e-mail: lmiones@ilstu.edu
http://www.ilstu.edu/˜lmiones/