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L^1-SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR IMPLICIT FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

Mouffak Benchohra and Mohammed Said Souid

Abstract. The aim of this paper is to present new results on the existence of solutions for a class of boundary value problem for fractional order implicit differential equations involving the Caputo fractional derivative. Our results are based on Schauder's fixed point theorem and the Banach contraction principle fixed point theorem.

1 Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [5, 15, 18, 19, 21]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [3, 4], Kilbas *et al.* [16], Lakshmikantham *et al.* [17], and the papers by Agarwal *et al* [1, 2], Benchohra *et al.* [6], and the references therein.

To our knowledge, the literature on integral solutions for fractional differential equations is very limited. El-Sayed and Hashem [14] studies the existence of integral and continuous solutions for quadratic integral equations. El-Sayed and Abd El Salam considered L^p-solutions for a weighted Cauchy problem for differential equations involving the Riemann-Liouville fractional derivative.

Motivated by the above papers, in this paper we deal with the existence of solutions for boundary value problem (BVP for short), for fractional order implicit differential equation

$$^cD^{\alpha}y(t) = f(t, y(t), {}^cD^{\alpha}y(t)),\ t \in J := [0, T],\ 1 < \alpha \leq 2, \tag{1.1}$$

$$y(0) = y_0,\ y(T) = y_T \tag{1.2}$$

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where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0,\ y_T \in \mathbb{R}$, and ${}^cD^\alpha$ is the Caputo fractional derivative.

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following section. In Section 3, we give two results, the first one is based on Schauder's fixed point theorem (Theorem 11) and the second one on the Banach contraction principle (Theorem 12). Some indications to nonlocal problems are given in Section 4. An example is given in Section 5 to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $L^1(J)$ denotes the class of Lebesgue integrable functions on the interval $J = [0, T]$, with the norm $\|u\|_{L^1} = \int_J |u(t)|dt$.

Definition 1. *. ([16, 20]). The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

where $\Gamma(.)$ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2. *. ([16, 20]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $h \in L^1([a, b], \mathbb{R}_+)$, is given by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α. If $\alpha \in (0, 1]$, then

$$(D_{a+}^\alpha h)(t) = \frac{d}{dt} I_{a+}^{1-\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^t (t-s)^{-\alpha} h(s)ds.$$

Definition 3. *. ([16]). The Caputo fractional derivative of order $\alpha > 0$ of function $h \in L^1([a, b], \mathbb{R}_+)$ is given by*

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n=[\alpha]+1$. *If* $\alpha\in(0,1]$, *then*

$$({}^cD^{\alpha}_{a+}h)(t)=I^{1-\alpha}_{a+}\frac{d}{dt}h(t)=\int_a^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{ds}h(s)ds.$$

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 4. *([16]). Let* $\alpha>0$, *then the differential equation*

$${}^cD^{\alpha}h(t)=0$$

has solution

$$h(t)=c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1},\ c_i\in\mathbb{R},\ i=0,1,2,...,n-1,\ n=[\alpha]+1.$$

Lemma 5. *([16]). Let* $\alpha>0$, *then*

$$I^{\alpha c}D^{\alpha}h(t)=h(t)+c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1},$$

for arbitrary $c_i\in\mathbb{R},\ i=0,1,2,...,n-1,\ n=[\alpha]+1$.

Proposition 6. *[16] Let* $\alpha,\ \beta>0$. *Then we have*

(i) $I^{\alpha}:\ L^1(J,\mathbb{R}_+)\to L^1(J,\mathbb{R}_+)$, *and if* $f\in L^1(J,\mathbb{R}_+)$, *then*

$$I^{\alpha}I^{\beta}f(t)=I^{\beta}I^{\alpha}f(t)=I^{\alpha+\beta}f(t).$$

(ii) *If* $f\in L^p(J,\mathbb{R}_+)$, $1\le p\le+\infty$, *then* $\|I^{\alpha}f\|_{L_p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L_p}$

(iii) *The fractional integration operator* I^{α} *is linear.*

The following theorems will be needed.

Theorem 7. *(Schauder fixed point theorem [12]) Let* E *a Banach space and* Q *be a convex subset of* E *and* $T:Q\longrightarrow Q$ *is compact, and continuous map. Then* T *has at least one fixed point in* Q.

Theorem 8. *(Kolmogorov compactness criterion [12]) Let* $\Omega\subseteq L^p([0,T],\mathbb{R})$, $1\le p\le\infty$. *If*

(i) Ω *is bounded in* $L^p([0,T],\mathbb{R})$, *and*

(ii) $u_h\longrightarrow u$ *as* $h\longrightarrow 0$ *uniformly with respect to* $u\in\Omega$,

then Ω *is relatively compact in* $L^p([0,T],\mathbb{R})$,

where

$$u_h(t)=\frac{1}{h}\int_t^{t+h}u(s)ds.$$

3 Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem (1.1) − (1.2).

Definition 9. . *A function $y \in L^1(J, \mathbb{R})$ is said to be a solution of BVP (1.1) − (1.2) if y satisfies (1.1) and (1.2).*

For the existence of solutions for the problem (1.1) − (1.2), we need the following auxiliary lemma.

Lemma 10. . *Let $1 < \alpha \leq 2$ and let $x \in L^1(J, \mathbb{R})$. The boundary value problem (1.1) − (1.2) is equivalent to the integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, \tag{3.1}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right). \tag{3.2}$$

and $G(t,s)$ is the Green's function defined by

$$G(t,s) := \begin{cases} (t-s)^{\alpha-1} - \frac{t(T-s)^{\alpha-1}}{T}, & 0 \leq s \leq t \leq T, \\ \frac{-t(T-s)^{\alpha-1}}{T}, & 0 \leq t \leq s \leq T, \end{cases} \tag{3.3}$$

Proof. Let ${}^cD^\alpha y(t) = x(t)$ in equation (1.1), then

$$x(t) = f(t, y(t), x(t)) \tag{3.4}$$

and Lemma 5 implies that

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds.$$

From (1.2), a simple calculation gives

$$c_0 = y_0$$

and

$$c_1 = -\frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1)} x(s)ds + \frac{(y_T - y_0)}{T}$$

Hence we get equation (3.1).
Inversely, we prove that equation (3.1) satisfies the BVP (1.1) − (1.2).
Differentiating (3.1), we get

$${}^cD^\alpha y(t) = x(t) = f(t, y(t), {}^c D^\alpha y(t)).$$

By (3.1) and (3.3) we have

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}. \tag{3.5}$$

A simple calculation give $y(0) = y_0$ and $y(T) = y_T$. This complete the proof of the equivalent between the BVP (1.1)-(1.2) and the integral equation (3.1).
Let

$$G_0 := \max\{|G(t,s)|, (t,s) \in J \times J\},$$

and let us introduce the following assumptions:

(H1) $f : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in [0,T]$, for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$, for almost all $t \in [0,T]$.

(H2) There exist a positive function $a \in L^1[0,T]$ and constants, $b_i > 0; i = 1,2$ such that:

$$|f(t,u_1,u_2)| \leq |a(t)| + b_1|u_1| + b_2|u_2|, \forall (t,u_1,u_2) \in [0,T] \times \mathbb{R}^2.$$

Our first result is based on Schauder fixed point theorem.

Theorem 11. *Assume that the assumptions* $(H1) - (H2)$ *are satisfied. If*

$$\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2 < 1, \tag{3.6}$$

then the BVP (1.1) $-$ (1.2) *has at least one solution* $y \in L^1(J, \mathrm{I\!R})$.

Proof. Transform the problem (1.1)$-$(1.2) into a fixed point problem. Consider the operator

$$H : L^1(J, \mathrm{I\!R}) \longrightarrow L^1(J, \mathrm{I\!R})$$

defined by:

$$(Hx)(t) = f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right), \tag{3.7}$$

where G is given by (3.3). Let

$$r \geq \frac{b_1(|y_0| + |y_T|)T + \|a\|_{L_1}}{1 - \left(\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2\right)}.$$

Consider the set

$$B_r = \{x \in L^1([0,T], \mathrm{I\!R}) : \|x\|_{L_1} \leq r\}.$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we will show that $HB_r \subset B_r$, indeed, for each $x \in B_r$, from assumption **(H2)** and (3.6) we get

$$\begin{aligned}
\|Hx\|_{L_1} &= \int_0^T |Hx(t)|dt \\
&= \int_0^T \left| f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right)\right| dt \\
&\le \int_0^T \left[|a(t)| + b_1 \left| \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds - (\frac{t}{T} - 1)y_0 + \frac{t}{T}y_T \right| + b_2|x(t)| \right] dt \\
&\le \|a\|_{L_1} + \frac{b_1 G_0 T}{\Gamma(\alpha)} \|x\|_{L_1} + b_1(|y_0| + |y_T|)T + b_2\|x\|_{L_1} \\
&\le b_1(|y_0| + |y_T|)T + \|a\|_{L_1} + \left(\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2\right) r \\
&\le r.
\end{aligned}$$

Then $HB_r \subset B_r$. Assumption **(H1)** implies that H is continuous. Now, we will show that H is compact, this is HB_r is relatively compact. Clearly HB_r is bounded in $L^1(J, \mathbb{R})$, i.e condition **(i)** of Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \longrightarrow (Hx)$ in $L^1(J, \mathbb{R})$ for each $x \in B_r$.
Let $x \in B_r$, then we have

$$\begin{aligned}
& \|(Hx)_h - (Hx)\|_{L^1} \\
= & \int_0^T |(Hx)_h(t) - (Hx)(t)|dt \\
= & \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Hx)(s)ds - (Hx)(t) \right| dt \\
\le & \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Hx)(s) - (Hx)(t)|ds \right) dt \\
\le & \int_0^T \Big(\frac{1}{h} \int_t^{t+h} \Big| f(s, \frac{1}{\Gamma(\alpha)} \int_0^T G(s,\tau)x(\tau)d\tau + y_0 + \frac{(y_T - y_0)s}{T}, x(s)) \\
& -f(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)) \Big| ds \Big) dt.
\end{aligned}$$

Since $x \in B_r \subset L^1(J, \mathbb{R})$ and assumption **(H2)** that implies $f \in L^1(J, \mathbb{R})$, then we have

$$\begin{aligned}
& \frac{1}{h} \int_t^{t+h} \Big| f(s, \frac{1}{\Gamma(\alpha)} \int_0^T G(s,\tau)x(\tau)d\tau + y_0 + \frac{(y_T - y_0)s}{T}, x(s)) \\
-f(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)) \Big| ds \longrightarrow 0, \ as\ h \longrightarrow 0,\ t \in J.
\end{aligned}$$

Hence

$$(Hx)_h \longrightarrow (Hx) \text{ uniformly as } h \longrightarrow 0.$$

Then by Kolmogorov compactness criterion, HB_r is relatively compact. As a consequence of Schauder's fixed point theorem the BVP (1.1) – (1.2) has at least one solution in B_r.

The following result is based on the Banach contraction principle.

Theorem 12. *Assume that **(H1)** and the following condition hold.*

(H3) *There exist constants $k_1, k_2 > 0$ such that*

$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \leq k_1|x_1-x_2|+k_2|y_1-y_2|, \ t \in [0,T], \ x_1,x_2,y_1,y_2 \in \mathbb{R}.$$

If

$$\frac{k_1 T G_0}{\Gamma(\alpha)} + k_2 < 1, \tag{3.8}$$

then the BVP (1.1) – (1.2) has a unique solution $y \in L^1([0,T],\mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H defined by (3.7) has a fixed point. Let $x, y \in L^1(J,\mathbb{R})$, and $t \in J$. Then we have,

$$\begin{aligned}
|(Hx)(t)-(Hy)(t)| &= \Bigg| f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T-y_0)t}{T}, x(t)\right) \\
&\quad - f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)y(s)ds + y_0 + \frac{(y_T-y_0)t}{T}, y(t)\right)\Bigg|. \\
&\leq \frac{k_1}{\Gamma(\alpha)}\int_0^T |G(t,s)(x(s)-y(s))|ds + k_2|x(t)-y(t)| \\
&\leq \frac{k_1 G_0}{\Gamma(\alpha)}\int_0^T |x(s)-y(s)|ds + k_2|x(t)-y(t)|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|(Hx)-(Hy)\|_{L_1} &\leq \frac{k_1 T G_0}{\Gamma(\alpha)}\|x-y\|_{L_1} + k_2\int_0^T |x(t)-y(t)|dt \\
&\leq \frac{k_1 T G_0}{\Gamma(\alpha)}\|x-y\|_{L_1} + k_2\|x-y\|_{L_1} \\
&\leq \left(\frac{k_1 T G_0}{\Gamma(\alpha)} + k_2\right)\|x-y\|_{L_1}.
\end{aligned}$$

Consequently by (3.8) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point which is a solution of the problem (1.1) – (1.2).

4 Nonlocal problems

This section is devoted to some existence and uniqueness results for the following class of nonlocal problems

$$^{c}D^{\alpha}y(t) = f(t, y, ^{c}D^{\alpha}y(t)), \ t \in J := [0, T], \ 1 < \alpha \leq 2, \tag{4.1}$$

$$y(0) = g(y), \ y(T) = y_T \tag{4.2}$$

where $g : L^1(J, \mathbb{R}) \to \mathbb{R}$ a continuous function. The nonlocal condition can be applied in physics with better effect than the classical initial condition $y(0) = y_0$. For example, $g(y)$ may be given by

$$g(y) = \sum_{i=1}^{p} c_i y(t_i).$$

where $c_i, i = 1, 2, ..., p$ are given constants and $0 < ... < t_p < T$. Nonlocal conditions were initiated by Byszewski [9] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [10, 11], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.
Let us introduce the following condition on the function g.

(H4) There exists a constant $\tilde{k} > 0$ such that

$$|g(y) - g(\tilde{y}| \leq \tilde{k}|y - \tilde{y}|, \ for \ each \ y, \tilde{y} \in L^1(J, \mathbb{R}).$$

Theorem 13. *Assume that the assumptions **(H1)**,**(H3)**,**(H4)** are satisfied. If*

$$\frac{2k_1 T^{\alpha}}{\Gamma(\alpha+1)} + k_1\tilde{k} + k_2 < 1, \tag{4.3}$$

then the BVP (4.1) − (4.2) *has a unique solution* $y \in L^1(J, \mathbb{R})$.

Transform the problem (4.1)−(4.2) into a fixed point problem. Consider the operator

$$\tilde{H} : L^1(J, \mathbb{R}) \longrightarrow L^1(J, \mathbb{R})$$

defined by:

$$(\tilde{H}x)(t) \quad = \quad f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds + g(y) + \frac{(y_T - g(y))t}{T}, x(t)\right). \tag{4.4}$$

Proof. Clearly, the fixed points of the operator $\tilde{H}$ are solution of the problem (4.1) − (4.2). We can easily show the $\tilde{H}$ is a contraction.

5 Example

Let us consider the following boundary value problem,

$$ {}^{c}D^{\alpha}y(t) = \frac{e^{-t}}{(e^{t}+6)(1+|y(t)|+|{}^{c}D^{\alpha}y(t)|)}, \ t \in J := [0,1], \ 1 < \alpha \leq 2, \tag{5.1} $$

$$ y(0) = 1, \ y(1) = 2. \tag{5.2} $$

Set

$$ f(t,y,z) = \frac{e^{-t}}{(e^{t}+6)(1+y+z)}, \ (t,y,z) \in J \times [0,+\infty) \times [0,+\infty). $$

Let $y, z \in [0,+\infty)$ and $t \in J$. Then we have

$$ \begin{aligned} |f(t,y_1,z_1) - f(t,y_2,z_2)| &= \left| \frac{e^{-t}}{e^{t}+6}\left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right)\right| \\ &\leq \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^{t}+6)(1+y_1+z_1)(1+y_2+z_2)} \\ &\leq \frac{e^{-t}}{(e^{t}+6)}(|y_1-y_2|+|z_1-z_2|) \\ &\leq \frac{1}{7}|y_1-y_2| + \frac{1}{7}|z_1-z_2|. \end{aligned} $$

Hence the condition **(H3)** holds with $k_1 = k_2 = \frac{1}{7}$. We shall check that condition (3.8) is satisfied with $T = 1$. Indeed9

$$ \frac{k_1 T G_0}{\Gamma(\alpha)} + k_2 = \frac{G_0}{7\Gamma(\alpha)} + \frac{1}{7} < 1. \tag{5.3} $$

Then by Theorem 12, the problem (5.1) – (5.2) has a unique integrable solution on $[0,1]$ for values of α satisfying condition (5.3).

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Mouffak Benchohra
Laboratory of Mathematics,
University of Sidi Bel Abbès,
PO Box 89, Sidi Bel Abbès 22000, Algeria.
and
Department of Mathematics,
King AbdulazizUniversity
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
E-mail: benchohra@univ-sba.dz

Mohammed Said Souid
Département de Science Economique,
Université de Tiaret,
Tiaret, Algérie.
E-mail: souimed2008@yahoo.com

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