Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 15 (2020), 295 – 313

BOUNDEDNESS OF LITTLEWOOD-PALEY OPERATORS WITH VARIABLE KERNEL ON THE WEIGHTED HERZ-MORREY SPACES WITH VARIABLE EXPONENT

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Abstract. Let $\Omega \in L^{\infty}(\mathbb{R}^n) \times L^2(S^{n-1})$ be a homogeneous function of degree zero. In this article, we obtain some boundedness of the parameterized Littlewood−Paley operators with variable kernels on weighted Herz-Morrey spaces with variable exponent. As a supplement, the boundedness of fractional integral operators with variable kernel is also obtained on these spaces.

1 Introduction

It is well known that the boundedness of Littlewood−Paley operators on function spaces are one of the very important tools not only in harmonic analysis but also in potential theory and in partial differential equations (see [1],[2],[2],[23],[27],[33]) for details). In 2004, Ding, Lin and Shao [3] investigated the L^2-boundedness for a class of Marcinkiewicz integral operators with variable kernels μ_Ω and $\mu_{\Omega,s}$ related to the Littlewood−Paley function $\mu^*_{\Omega,\lambda}$ and the area integral g^*_λ. In 2006, the authors [4] have proved the L^p-boundedness of the Littlewood−Paley operators with variable kernels. In 2009, Xue and Ding [5] established the weighted estimate for Littlewood−Paley operators and their commutators.

In 1960, Hörmander [6] introduced the parameterized Littlewood−Paley operators for the first time. Now, let us recall the definitions of the parameterized Lusin area integral and the Littlewood-Paley g^*_λ function.

Let $S^{n-1}(n \geq 2)$ be the unit sphere in $\mathbb{R}^n$ with normalized Lebesgue measure $\mathrm{d}\sigma(x')$. Denote $\Omega(x,z) \in L^{\infty}(\mathbb{R}^n) \times L^r(S^{n-1})(r \geq 1)$ to be a homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x,z')\mathrm{d}\sigma(z') = 0, \qquad \text{for all} \quad x \in \mathbb{R}^n, \tag{1}$$

2020 Mathematics Subject Classification: 43A15; 43A85
Keywords: Parameterized Littlewood−Paley operators; variable kernel; weighted Herz space; Muckenhoupt weight; variable exponent.

where Ω satisfies the following conditions:

(i) For any $x, z \in \mathbb{R}^n$ and any $\lambda > 0$, one has $\Omega(x, \lambda z) = \Omega(x, z)$;

(ii) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} := \sup\limits_{r\geq 0, y\in\mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(rz'+y, z')|^r \mathrm{d}\sigma(z')\right)^{\frac{1}{r}} < \infty.$

The parameterized Littlewood−Paley operators $\mu^{\rho}_{\Omega,s}$ and $\mu^{*,\rho}_{\Omega,\lambda}$ with variable kernels, which are related to the Lusin area integral and the Littlewood−Paley g^*_λ function are defined by

$$\mu^{\rho}_{\Omega,s}(f)(x) = \left(\int\int_{\Gamma(x)} \left|\frac{1}{t^\rho}\int_{|y-z|\leq t} \frac{\Omega(y, y-z)}{|y-z|^{n-\rho}} f(z)\mathrm{d}z\right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}$$

and

$$\mu^{*,\rho}_{\Omega,\lambda}(f)(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^\rho}\int_{|y-z|\leq t} \frac{\Omega(y, y-z)}{|y-z|^{n-\rho}} f(z)\mathrm{d}z\right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x-y| < t \text{ and } \lambda > 1\}$.

In 2013, Wei and Tao [7] investigated the boundedness of parameterized Littlewood −Paley operators on weighted weak Hardy spaces. Lin and Xuan [8] established the boundedness for commutators of parameterized Littlewood−Paley operators and area integrals on weighted Lebesgue spaces $L^p(w)$.

On the other hand, the theory of the variable exponent function spaces has been rapidly developed after the work [9] where Kováčik and Rákosník have clarified fundamental properties of Lebesgue spaces with variable exponent. After that, many researchers have been interested in the theory of the variable exponent spaces (see [10]-[14]).

The generalization of the Muckenhoupt weights with variable exponent $A_{p(\cdot)}$ has been considered in ([15]-[18]). We note that the equivalence between the Muckenhoupt condition and the boundedness of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces with variable exponent were discussed in ([15], [16]). After that, Cruz-Uribe and Wang [19] proved the boundedness of some classical operators on weighted Lebesgue spaces with variable exponent $L^{p(\cdot)}(w)$.

Very recently, Mitsuo and Takahiro [20] introduced the weighted Herz spaces with variable exponent, and also studied the boundedness of fractional integrals on those spaces.

In this paper, by applying the theory of Banach function spaces and the variable exponent Muckenhoupt weight theory, we establish the boundedness of parameterized Littlewood− Paley operators with variable kernels on weighted Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,q}_{p(\cdot)}(w)$. The similar results for the boundedness of the fractional integral operators with variable kernel are also discussed. Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E| > 0$, χ_E means its characteristic function. We shall recall some definitions.

Definition 1 ([2]). *Let $p(\cdot): E \to [1,\infty)$ be a measurable function, the variable exponent Lebesgue space is defined by*

$$L^{p(\cdot)}(E) = \{f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0\}.$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) = \{f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset E\}.$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\eta > 0 : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

We denote $p_- = $ ess inf$\{p(x) : x \in E\}$, $p_+ = $ ess sup$\{p(x) : x \in E\}$, then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Definition 2 ([32]). *Let $p(\cdot): \mathbb{R}^n \to [1,\infty)$. A measurable function $p(\cdot)$ is said to be globally log-Höder continuous if it satisfies*

$$(1)\quad |p(x) - p(y)| \le \frac{1}{-\log(|x-y|)}, \qquad x, y \in \mathbb{R}^n, |x-y| \le 1/2;$$

$$(2)\quad |p(x) - p_\infty| \le \frac{1}{\log(e+|x|)}, \qquad x \in \mathbb{R}^n,$$

for some $p_\infty \ge 1$. The set of $p(\cdot)$ satisfying (1) and (2) is denoted by $LH(\mathbb{R}^n)$. We know that, if $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see[34]).

Definition 3 ([24]). *Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w is a weight function. The weighted Lebesgue spaces with variable exponent $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable function f such that $fw^{1/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm*

$$\|f\|_{L^{p(\cdot)}(w)} = \|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}}.$$

$p'(\cdot)$ is the conjugate of $p(\cdot)$ such that $\frac{1}{p'(\cdot)} + \frac{1}{p(\cdot)} = 1$. Next, we introduce the classical Muckenhoupt A_p weight.

Definition 4 ([21]). *Let $1 < p < \infty$, then $w \in A_p$ for every cube Q,*

$$\left(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\mathrm{d}x\right)^{p-1} \le C < \infty.$$

We say that $w \in A_1$ if it satisfies $Mw(x) \le w(x)$ for all $x \in \mathbb{R}^n$. The set of A_1 consists of all Muckenhoupt A_1 weights.

Definition 5 ([15],[19]). *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and a weight w, then $w \in A_{p(\cdot)}$ if*

$$\sup_{B:\text{ball}} |B|^{-1} \|w^{1/p(\cdot)}\chi_B\|_{L^{p(\cdot)}(w)} \|w^{-1/p(\cdot)}\chi_B\|_{L^{p'(\cdot)}(w)} < \infty.$$

Definition 6 ([19]). *Given $p_1(\cdot),\ p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $1/p_1(x) - 1/p_2(x) = \mu/n$ such that $0 < \mu < n$. Then $w \in A_{p_1(\cdot),p_2(\cdot)}$ if*

$$\|w\chi_B\|_{L^{p_2(\cdot)}} \|w^{-1}\chi_B\|_{L^{p_1'(\cdot)}} \le |B|^{\frac{n-\mu}{n}}$$

holds for all balls $B \in \mathbb{R}^n$.

Definition 7 ([19]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. We say that $(p(\cdot), w)$ is an M-pair if the maximal operator M is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$.*

Now, we need give the definition of weighted Herz space with variable exponent. For all $k \in \mathbb{Z}$, we denote $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \backslash B_{k-1}$, $\chi_k = \chi_{C_k}$.

Definition 8 ([20]). *Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q < \infty$, $\alpha \in \mathbb{R}$. The homogeneous weighted Herz space with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(w)$ is the collection of $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}, w)$ such that*

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(w)} := \left(\sum_{k=-\infty}^{\infty} 2^{\alpha q k} \|f\chi_k\|^q_{L^{p(\cdot)}(w)} \right)^{1/q} < \infty.$$

Definition 9. *Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $0 \le \lambda < \infty$, $0 < q < \infty$. The homogeneous weighted Herz-Morrey space with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w)$ is the collection of $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}, w)$ such that*

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(w)} := \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha q} \|f\chi_k\|^q_{L^{p(\cdot)}(w)} \right)^{1/q} < \infty.$$

Obviously, when $\lambda = 0$, then $M\dot{K}^{\alpha,0}_{q,p(\cdot)}(w) = \dot{K}^{\alpha,q}_{p(\cdot)}(w)$. Obviously, when $\lambda = 0$, then $M\dot{K}^{\alpha,0}_{q,p(\cdot)}(w) = \dot{K}^{\alpha,q}_{p(\cdot)}(w)$.

Definition 10. *We say a kernel function $\Omega(x, z)$ satisfies the L^r-Dini condition $(r \ge 1)$, if*

$$\int_0^1 \frac{\omega_r(\delta)}{\delta}(1 + |\log\delta|^\sigma)\mathrm{d}\delta < \infty, \tag{2}$$

where $\omega_r(\delta)$ denotes the integral modulus of continuity of order r of Ω defined by

$$\omega_r(\delta) = \sup_{x \in \mathbb{R}^n, |\rho| < \delta} \left(\int_{S^{n-1}} |\Omega(x, \rho z') - \Omega(x, z')|^r \mathrm{d}\sigma(z') \right)^{\frac{1}{r}},$$

where ρ is the rotation in $\mathbb{R}^n$, $\|\rho\| = \sup_{z' \in S^{n-1}} \|\rho z' - z'\|$.

2 Preliminaries and notations

In order to prove our main theorems, we need the following Lemmas.

Lemma 11 ([23]). *Suppose that $X \subset \mathcal{M}$ is a Banach function space*

(1) *(The generalized Hölder's inequality) For all $f \in X$ and $g \in X'$, we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leq \|f\|_X \|g\|_X.$$

(2) *For all $f \in X$, we have*

$$\sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : \|g\|_{X'} \leq 1\right\} = \|f\|_X.$$

In particular, space $(X')' = X$.

As an application of the generalized Hölder's inequality above, we have the following Lemma.

Lemma 12. *Let X be a Banach function space, we have*

$$1 \leq \frac{1}{|B|}\|\chi_B\|_X \|\chi_B\|_{X'},$$

holds for all balls B.

Lemma 13 ([18]). *Let X be a Banach function space. If the Hardy-Littlewood maximal operator M is weakly bounded on X, that is*

$$\|\chi_{\{Mf>\lambda\}}\|_X \leq \lambda^{-1}\|f\|_X,$$

it holds for all $f \in X$ and $\lambda > 0$. Then we get

$$\sup_{B:\text{Ball}} \frac{1}{|B|}\|\chi_B\|_X \|\chi_B\|_{X'} < \infty.$$

Remark 14. ([24]) The weighted Banach function space $X(\mathbb{R}^n, W)$ is a Banach function space equipped the norm $\|f\|_{X(\mathbb{R}^n,W)} := \|fW\|_X$. The associated space of $X(\mathbb{R}^n, W)$ is a Banach function space and equals $X'(\mathbb{R}^n, W^{-1})$.

Remark 15. If $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, by comparing the definition of the weighted Banach function space with weighted variable Lebesgue space, we have

(1) If $X = L^{p_1(\cdot)}(\mathbb{R}^n)$ and $W = w$, then we obtain $L^{p_1(\cdot)}(\mathbb{R}^n, w) = L^{p_1(\cdot)}(w^{p_1(\cdot)})$.

(2) If $X = L^{p_1'(\cdot)}(\mathbb{R}^n)$ and $W = w^{-1}$, by Remark 14, then we obtain

$$L^{p_1'(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p_1'(\cdot)}(w^{-p_1'(\cdot)}) = (L^{p_1(\cdot)}(w^{p_1(\cdot)}))'.$$

Lemma 16 ([25]). *Suppose that X is a Banach space. Let M be bounded on the associated space X'. Then there exists a constant $0<\delta<1$ such that*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \le \left(\frac{|E|}{|B|}\right)^{\delta}$$

holds for all balls B and all measurable sets $E \subset B$.

Lemma 17 ([20]). *Suppose that $p_1(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w^{p_1(\cdot)} \in A_1$. Let M be a bounded on $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ and $L^{p_1'(\cdot)}(w^{-p_1'(\cdot)})$, then there exist constants $\delta_1, \delta_2 \in (0,1)$ such that*

$$\frac{\|\chi_S\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}{\|\chi_B\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \frac{\|\chi_S\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_B\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}$$

holds for all balls B and all measurable sets $S \subset B$.

Lemma 18 ([26]). *Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies (1) and (2), $\lambda > 2$, $2\rho - n > 0$, $1 < p < \infty$. Then for all $f \in L^q(w)$ there exists $C > 0$ independent of f such that*

$$\|\mu^{\rho}_{\Omega,s} f\|_{L^q(w)} \le C\|f\|_{L^q(w)}$$

and

$$\|\mu^{*,\rho}_{\Omega,\lambda} f\|_{L^q(w)} \le C\|f\|_{L^q(w)}.$$

Lemma 19 ([19]). *Assume that for p_0, $1 < p_0 < \infty$ and every $w_0 \in A_{p_0}$,*

$$\int\limits_{\mathbb{R}^n} f(x)^{p_0} w_0(x)\mathrm{d}x \le \int\limits_{\mathbb{R}^n} g(x)^{p_0} w_0(x)\mathrm{d}x, \qquad (f,g) \in \mathcal{F}.$$

Then for any M-pair $(p(\cdot), w)$,

$$\|f\|_{L^{p(\cdot)}(w)} \le C\|g\|_{L^{p(\cdot)}(w)}, \qquad (f,g) \in \mathcal{F},$$

Lemma 19 holds for $p_0 = 1$ and the maximal operator is bounded on $L^{p'(\cdot)}(w^{-1})$. We know the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ (see [25]).

Combining Lemma 18 with Lemma 19, we obtain the following conclusion.

Corollary 20. *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})(r \ge 1)$ and $w \in A_{p(\cdot)}$. Then the parameterized Littlewood–Paley type operators $\mu^{\rho}_{\Omega,s}$ and $\mu^{*,\rho}_{\Omega,\lambda}$ with variable kernel are bounded on $L^{p(\cdot)}(w)$.*

3 Main Theorems and their proofs

In this section, we will prove the boundedness of the parameterized Littlewood−Paley operators with variable kernels on variable weighted Herz-Morrey spaces.

Theorem 21. *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $0 < q_1 \le q_2 < \infty$, $\lambda > 2$, $2\rho - n > 0$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies (1) and (2). If $w^{p_1(\cdot)} \in A_1$ and $-n\delta_1 < \alpha < n\delta_2$, where δ_1, δ_2 are the constants in Lemma 17, then the operator $\mu_{\Omega,s}$ is bounded from $M\dot{K}^{\alpha,\lambda}_{q_2,p_1(\cdot)}(w^{p_1(\cdot)})$ to $M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})$.*

Theorem 22. *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $0 < q_1 \le q_2 < \infty$, $\lambda > 2$, $2\rho - n > 0$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies (1) and (2). If $w^{p_1(\cdot)} \in A_1$ and $-n\delta_1 < \alpha < n\delta_2$, where δ_1, δ_2 are the constants in Lemma 17, then the operator $\mu^*_{\Omega,\lambda}$ is bounded from $M\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})$ to $M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})$.*

Remark 23. *As is well known that, $\mu^{\rho}_{\Omega,s} f(x) \le 2^{n\lambda} \mu^{*,\rho}_{\Omega,\lambda} f(x)$ (see [27], p.89). Therefore, we give only the proof of Theorem 22.*

Proof. Let $f \in M\dot{K}^{\alpha,\lambda}_{q_2,p_1(\cdot)}(w^{p_1(\cdot)})$. By the Jensen's inequality, we have

$$\begin{aligned}
\|\mu^{*,\rho}_{\Omega,\lambda} f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_2,p_1(\cdot)}(w^{p_1(\cdot)})} &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \left(\sum_{k=-\infty}^{k_0} 2^{\alpha q_2 k} \|(\mu^{*,\rho}_{\Omega,\lambda} f)\chi_k\|^{q_2}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1/q_2} \\
&\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \|(\mu^{*,\rho}_{\Omega,\lambda} f)\chi_k\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Denote $f_j := f\chi_j$ for each $j \in \mathbb{Z}$, then $f := \sum_{j=-\infty}^{\infty} f_j$, so we have

$$\begin{aligned}
\|\mu^{*,\rho}_{\Omega,\lambda} f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_2,p_1(\cdot)}(w^{p_1(\cdot)})} &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \|(\mu^{*,\rho}_{\Omega,\lambda} f)\chi_k\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{k-2} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\quad + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k-2}^{k+2} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\quad + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k+2}^{\infty} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&=: \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3.
\end{aligned}$$

First, we consider F_2. Using Corollary 20 and $-2 \leq k-j \leq 2$, it is easy to get

$$\begin{aligned}
F_2 &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k-2}^{k+2} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left(\sum_{j=k-2}^{k+2} 2^{\alpha(k-j)} 2^{\alpha j} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Now we need to consider $\mu^{*,\rho}_{\Omega,\lambda} f_j$. Applying the Minkowski inequality, we conclude that

$$\begin{aligned}
|\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)| &= \left(\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \frac{1}{t^\rho} \int_{|y-z|\leq t} \frac{\Omega(y,y-z)}{|y-z|^{n-\rho}} f_j(z) \mathrm{d}z \right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\leq \int_{\mathbb{R}^n} f_j(z) \left(\int_0^\infty \int_{|y-z|\leq t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}} \frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}} \right)^{\frac{1}{2}} \mathrm{d}z \\
&\leq \int_{\mathbb{R}^n} f_j(z) \left(\int_0^{|x-z|} \int_{|y-z|\leq t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}} \frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}} \right)^{\frac{1}{2}} \mathrm{d}z \\
&+ \int_{\mathbb{R}^n} f_j(z) \left(\int_{|x-z|}^\infty \int_{|y-z|\leq t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}} \frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}} \right)^{\frac{1}{2}} \mathrm{d}z.
\end{aligned}$$

For $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ and $2\rho - n > 0$, the following inequality holds

$$\begin{aligned}
\int_{|y-z|\leq t} \frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}} \mathrm{d}y &\leq \int_{s^{n-1}} \int_0^t \frac{|\Omega(sy'+z,y')|^2}{s^{2n-2\rho}} s^{n-1} \mathrm{d}s\mathrm{d}\sigma(y') \\
&\leq \|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})} t^{2\rho-n}.
\end{aligned}$$

Since $|x-z| \leq |x-y| + |y-z| \leq |x-y| + t$. For $\lambda > 2$, taking $0 < \delta < (\lambda-2)n$, so we have

$$\begin{aligned}
&\int_0^{|x-z|}\int_{|y-z|\le t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}}\\
&\le \int_0^{|x-z|}\int_{|y-z|\le t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n-2n-\delta}\frac{1}{|x-z|^{2n+\delta}}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho-n-\delta+1}}\\
&\le \frac{1}{|x-z|^{2n+\delta}}\int_0^{|x-z|}\int_{|y-z|\le t}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho-n-\delta+1}}\\
&\le \frac{\|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})}}{|x-z|^{2n+\delta}}\int_0^{|x-z|}t^{\delta-1}\mathrm{d}t\\
&\le C|x-z|^{-2n}.
\end{aligned}$$

If we take $1<\lambda_1<2$, then $\lambda_1 n-n>0$ and $\lambda_1 n-2n<0$, we have

$$\begin{aligned}
&\int_{|x-z|}^{\infty}\int_{|y-z|\le t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}}\\
&\le \int_{|x-z|}^{\infty}\int_{|y-z|\le t}|x-z|^{-\lambda_1 n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho-\lambda_1 n+n+1}}\\
&\le \int_{|x-z|}^{\infty}|x-z|^{-\lambda_1 n}\int_{|y-z|\le t}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-\lambda_1 n}}\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\\
&\le \int_{|x-z|}^{\infty}|x-z|^{-\lambda_1 n}\int_{s^{n-1}}\int_0^t\frac{|\Omega(y',(y-z)')|^2}{s^{2n-\lambda_1 n}}s^{n-1}\mathrm{d}s\mathrm{d}\sigma(y')\frac{\mathrm{d}t}{t^{n+1}}\\
&\le C\|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})}|x-z|^{-\lambda_1 n}\int_{|x-z|}^{\infty}t^{\lambda_1 n-2n-1}\mathrm{d}t\\
&\le C|x-z|^{-2n}.
\end{aligned}$$

Combining the above two estimates, we obtain

$$\mu^*_{\Omega,\lambda}(f)(x)\le\int_{\mathbb{R}^n}\frac{|f(z)|}{|x-z|^n}\mathrm{d}z. \tag{3}$$

Next, we consider F_1. Noting that for $x\in A_k$, $z\in A_j$ and $j\le k-2$, then $|x-z|\sim|x|$. By the virtue of the generalized Hölder's inequality, we have

$$\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)\le C2^{-kn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}.$$

Applying Lemma 13 and Lemma 17, we take $\|.\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$ for each side, we have

$$\begin{aligned}
&\|\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C2^{-kn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C2^{-kn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\\
&\leq C\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}\\
&\leq C2^{(j-k)n\delta_2}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

On the other side, we have the following fact:

$$\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &= 2^{-j\alpha}\left(2^{j\alpha q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{1/q_1}\\
&\leq C2^{-j\alpha}\left(\sum_{n=-\infty}^{j}2^{n\alpha q_1}\|f_n\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{1/q_1}\\
&= C2^{j(\lambda-\alpha)}\left(2^{-\lambda j}\left(\sum_{n=-\infty}^{j}2^{n\alpha q_1}\|f_n\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{1/q_1}\right)\\
&\leq C2^{j(\lambda-\alpha)}\|f\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Thus, by using condition $\alpha < \lambda + n\delta_2$, it follows that

$$\begin{aligned}
\mathrm{F}_1 &= \sup_{k_0\in z}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}\|(\mu^{*,\rho}_{\Omega,\lambda}f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1}\\
&\leq C\sup_{k_0\in z}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_2}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1}\\
&\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_2}2^{j(\lambda-\alpha)}\right)^{q_1}\\
&\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{\lambda q_1 k}\left(\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha-\lambda-n\delta_2)}\right)^{q_1}\\
&\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda q_1}\left(\sum_{k=-\infty}^{k_0}2^{\lambda q_1 k}\right).\\
&\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Finally, we estimate F_3. Noting that for $x \in A_k$, $y \in A_j$ and $j \geq k+2$, then $|y-x| \sim |y|$. By (3) and the virtue of the generalized Hölder's inequality, we show that

$$\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x) \leq C 2^{-jn} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}.$$

Applying Lemma 13 and Lemma 17, we can take $\|.\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$ for each side, we have

$$\begin{aligned}
&\|\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C 2^{-jn} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C 2^{-jn} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))} \|\chi_{B_j}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))}\\
&\leq C \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \frac{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))}}{\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))}}\\
&\leq C 2^{(k-j)n\delta_1} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Then, by using condition $\alpha > \lambda - n\delta_1$, it follows that

$$\begin{aligned}
F_3 &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k+2}^{\infty} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1}\\
&\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_1} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1}\\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_1} 2^{j(\lambda-\alpha)} \right)^{q_1}\\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\lambda q_1 k} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda+n\delta_1)} \right)^{q_1}\\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \left(\sum_{k=-\infty}^{k_0} 2^{\lambda q_1 k} \right)\\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

This completes the proof of Theorem 22. □

4 Fractional integral operator with variable kernel on $M\dot{K}^{\alpha,q}_{p(\cdot)}(w^{p(\cdot)})$ spaces

Let $0<\mu<n$ and $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})$ satisfies (i)-(ii), the fractional integral operator with variable kernel $T_{\Omega,\mu}$ is defined by

$$T_{\Omega,\mu}f(y)=\int_{\mathbb{R}^n}\frac{\Omega(y,y-z)}{|y-z|^{n-\mu}}f(z)\mathrm{d}z.$$

In this section, we will prove some main results of this paper. We begin two preliminary Lemmas.

Lemma 24 ([29],[30]). *Let $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})(r\geq 1)$ and $w^{r'}(x)\in A(p/r',q/r')$. If $0<\mu<n$, $1\leq r'<p<\frac{n}{\mu}$, $\frac{1}{p}-\frac{1}{q}=\frac{n}{\mu}$, then we have*

$$\left(\int_{\mathbb{R}^n}[T_{\Omega,\mu}f(x)w(x)]^q\mathrm{d}x\right)^{\frac{1}{q}}\leq\left(\int_{\mathbb{R}^n}|f(x)w(x)|^p\mathrm{d}x\right)^{\frac{1}{p}}.$$

Lemma 25 ([19]). *Assume that $1<p_0\leq q_0<\infty$ and every $w_0\in A_{p_0,q_0}$,*

$$\left(\int_{\mathbb{R}^n}f(x)^{q_0}w_0(x)^{q_0}\mathrm{d}x\right)^{\frac{1}{q_0}}\leq C\left(\int_{\mathbb{R}^n}g(x)^{p_0}w_0(x)^{p_0}\mathrm{d}x\right)^{\frac{1}{p_0}},\qquad (f,g)\in\mathcal{F}.$$

Given $p(\cdot),q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{1}{p_0}-\frac{1}{q_0}.$$

Define $\sigma\geq 1$ by $\frac{1}{\sigma}=\frac{1}{p_0}-\frac{1}{q_0}$. If $w\in A_{p(\cdot),q(\cdot)}$ and $(q(\cdot)/\sigma,w^\sigma)$ is an M-pair then

$$\|f\|_{L^{q(\cdot)}(w)}\leq C\|g\|_{L^{p(\cdot)}(w)},\qquad (f,g)\in\mathcal{F},$$

Lemma 25 holds for $p_0=1$ and the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}(w^{-q_0})$. Izuki and Noi [20] proved the Hardy−Littlewood maximal operator M is bounded on $L^{q(\cdot)/\sigma}(w^{\sigma.q(\cdot)/\sigma})$ and $L^{(q(\cdot)/\sigma)'}(w^{-\sigma(q(\cdot)/\sigma)'})$, then $(q(\cdot)/\sigma,w^\sigma)$ is an M-pair when $w\in A_{p(\cdot),q(\cdot)}$ and $p(\cdot),q(\cdot)\in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$.

From Lemma 24 and Lemma25, we have the following conclusion.

Corollary 26. *If $p_1(\cdot)\in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$, $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})(r\geq 1)$, $0<\mu<n/p_1^+$ and $1/p_1(\cdot)-1/p_2(\cdot)=\mu/n$. Then for $w\in A_{p_1(\cdot),p_2(\cdot)}$, the fractional integral operator with variable kernel $T_{\Omega,\mu}$ is bounded from $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{p_2(\cdot)})$.*

Theorem 27. *Let* $0 \le \lambda < \infty$, $0 < q_1 \le q_2 < \infty$, $p_2(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ *and* $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})(r \ge 1)$. *If* $w^{p_2(\cdot)} \in A_1$ *and* $\lambda - n\delta_1 < \alpha < n\delta_2 - \frac{n}{r} + \lambda$, *where* δ_1, δ_2 *are the constants in Lemma 17. Define* $p_2(\cdot)$ *by* $1/p_1(\cdot) - 1/p_2(\cdot) = \mu/n$, *then the operator* $T_{\Omega,\mu}$ *is bounded from* $M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})$ *to* $M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})$.

Proof. Let $f \in M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})$. By the Jensen's inequality, we have

$$\begin{aligned}
\|T_{\Omega,\mu} f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \left(\sum_{k=-\infty}^{k_0} 2^{\alpha q_2 k} \|(T_{\Omega,\mu} f)\chi_k\|^{q_2}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1/q_2} \\
&\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \|(T_{\Omega,\mu} f)\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

Denote $f_j := f\chi_j$ for each $j \in z$, then $f := \sum_{j=-\infty}^{\infty} f_j$, so we have

$$\begin{aligned}
\|T_{\Omega,\mu} f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \|(T_{\Omega,\mu} f)\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{k-2} \|(T_{\Omega,\mu} f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\
&\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k-2}^{k+2} \|(T_{\Omega,\mu} f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\
&\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k+2}^{\infty} \|(T_{\Omega,\mu} f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\
&=: \mathrm{H}_1 + \mathrm{H}_2 + \mathrm{H}_3.
\end{aligned}$$

First, we consider H_2. Using Corollary 26 and $-2 \le k - j \le 2$, it is easy to get

$$\begin{aligned}
\mathrm{H}_2 &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=k-2}^{k+2} \|(T_{\Omega,\mu} f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left(\sum_{j=k-2}^{k+2} 2^{\alpha(k-j)} 2^{\alpha j} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\le C \|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Now we turn to estimate of H_1. Noting that $j \le k-2$ for every $j,k \in \mathbb{Z}$, then we have the following inequality by applying the generalized Hölder's inequality

$$|T_{\Omega,\mu} f_j| \le C2^{-k(n-\mu)} \|\Omega(y, y-z)\|_{L^r(\mathbb{R}^n)} \|f_j(z)\|_{L^{r'}(\mathbb{R}^n)}.$$

By virtue of the generalized Hölder's inequality, we have

$$\|f_j(z)\|_{L^{r'}} \le |B_j|^{-\frac{1}{r}} \|f_j w\|_{L^{p_1(\cdot)}} \|w^{-1}\chi_j\|_{L^{p_1'(\cdot)}}. \tag{4}$$

for any $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$, we get by (4)

$$|T_{\Omega,\mu} f_j| \le C2^{-k(n-\mu)} 2^{(k-j)\frac{n}{r}} \|f_j w\|_{L^{p_1(\cdot)}} \|w^{-1}\chi_j\|_{L^{p_1'(\cdot)}}. \tag{5}$$

By 5 and taking $\|.\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$ for each side, we have

$$\begin{aligned}
&\|(T_{\Omega,\mu} f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\le C2^{-k(n-\mu)} 2^{(k-j)\frac{n}{r}} \|f_j w\|_{L^{p_1(\cdot)}} \|w^{-1}\chi_j\|_{L^{p_1'(\cdot)}} \|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\le C2^{-k(n-\mu)} 2^{(k-j)\frac{n}{r}} \|f_j w\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_{B_k}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}$$

Noting that $\chi_{B_k}(x) \le C2^{-k\mu} T_{\Omega,\mu}(\chi_{B_k})(x)$, by the Corollary 26, we obtain

$$\begin{aligned}
\|\chi_{B_k}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C2^{-k\mu} \|T_{\Omega,\mu}(\chi_{B_k})\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\le C2^{-k\mu} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

From this and using Lemma 13 and Lemma 17, we have

$$\begin{aligned}
&\|(T_{\Omega,\mu} f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\le C2^{-k(n-\mu)} 2^{(k-j)\frac{n}{r}} \|f_j w\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} 2^{-k\mu} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\le C2^{(k-j)\frac{n}{r}} \|f_j w\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} 2^{-kn} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\le C2^{(k-j)\frac{n}{r}} \|f_j w\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\le C2^{(k-j)\frac{n}{r}} \|f_j w\|_{L^{p_1(\cdot)}} \frac{\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}} \\
&\le C2^{(k-j)(\frac{n}{r}-n\delta_2)} \|f_j w\|_{L^{p_1(\cdot)}}.
\end{aligned}$$

On the other side, we have the following fact:

$$\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &= 2^{-j\alpha}\left(2^{j\alpha q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{1/q_1}\\
&\le C2^{-j\alpha}\left(\sum_{n=-\infty}^{j} 2^{n\alpha q_1}\|f_n\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{1/q_1}\\
&= C2^{j(\lambda-\alpha)}\left(2^{-\lambda j}\left(\sum_{n=-\infty}^{j} 2^{n\alpha q_1}\|f_n\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{1/q_1}\right)\\
&\le C2^{j(\lambda-\alpha)}\|f\|_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Thus, by using condition $\alpha < \lambda + n\delta_2 - \frac{n}{r}$, it follows that

$$\begin{aligned}
\mathrm{H}_1 &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}\|(T_{\Omega,\mu}f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}2^{(k-j)(\frac{n}{r}-n\delta_2)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1}\\
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}2^{(k-j)(\frac{n}{r}-n\delta_2)}2^{j(\lambda-\alpha)}\right)^{q_1}\\
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{\lambda q_1 k}\left(\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha-\lambda-n\delta_2+\frac{n}{r})}\right)^{q_1}\\
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\left(\sum_{k=-\infty}^{k_0} 2^{\lambda q_1 k}\right).\\
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Finally, we estimate U_3. For every $j,k\in\mathbb{Z}$ with $j\ge k+2$ and applying the generalized Hölder's inequality assures that

$$|T_{\Omega,\mu}f_j| \le C2^{-j(n-\mu)}\|\Omega(y,y-z)\|_{L^r(\mathbb{R}^n)}\|f_j(z)\|_{L^{r'}(\mathbb{R}^n)}.$$

For any $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})$, we get by (4)

$$|T_{\Omega,\mu}f_j| \le C2^{-j(n-\mu)}\|f_jw\|_{L^{p_1(\cdot)}}\|w^{-1}\chi_j\|_{L^{p_1'(\cdot)}}.$$

from this and applying Lemma 17 and taking $\|.\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$ for each side, we have

$$\begin{aligned}
&\|T_{\Omega,\mu}f_j(x)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C2^{-j(n-\mu)}\|f_jw\|_{L^{p_1(\cdot)}}\|w^{-1}\chi_j\|_{L^{p_1'(\cdot)}}\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C2^{-j(n-\mu)}\|f_jw\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C2^{-j(n-\mu)}\|f_jw\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\frac{\|\chi_{B_k}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_{B_j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}\\
&\le C2^{-j(n-\mu)}2^{n\delta_1(k-j)}\|f_jw\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}
\end{aligned}$$

From the definition of $A_{p_1(\cdot),p_2(\cdot)}$ (Definition 6), we get

$$\begin{aligned}
&\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le \|w^{-1}\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|w\chi_{B_j}\|_{L^{p_2(\cdot)}}\\
&\le |B_j|^{\frac{n-\mu}{n}}\\
&= 2^{j(n-\mu)}.
\end{aligned}$$

Then, we obtain that

$$\|T_{\Omega,\mu}f_j(x)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \le C2^{(k-j)n\delta_1}\|f_jw\|_{L^{p_1(\cdot)}}.$$

The rest of the proof is the same as the proof of F_3 in Theorem 22, it is not difficult to obtain

$$H_3 \le C\|f\|^{q_1}_{M\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.$$

Thus, we omit the details there. Then the proof of Theorem 27 is finished. □

Acknowledgement. The author wishes to express profound gratitude to the reviewers for their useful comments on the manuscript.

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