

A SURVEY ON PROJECTIVELY EQUIVALENT REPRESENTATIONS OF FINITE GROUPS

Tania Luminița Costache

Abstract. The paper is a survey type article in which we present some results on projectively equivalent representations of finite groups.

1 Introduction

The theory of projective representations of finite groups was founded by I. Schur [25] and the notion of projective representation was suggested by the study of relations between linear representations of a group and its factor group over a central group. Schur has associated to every finite group G a finite abelian group M , called the multiplier of G , consisting of all equivalence classes of factor sets of projective representations of G . In the terminology of cohomology theory of groups, the multiplier M of G is in fact the second cohomology group $H^2(G, \mathbb{C}^*)$ for the multiplicative group of the complex number field \mathbb{C} under the trivial action of G on \mathbb{C}^* .

Curtis and Reiner [7] examined the structure of the multiplier group under the hypothesis that K is algebraically closed (Theorem 18).

The theory of projective representations of a finite group G suffers from the fact that the projective character is not in general a class function on G . However, this difficulty may be overcome by considering characters with respect to simple factor sets and in Theorem 16, Read [21] showed that given a factor set ω of G there is a simple factor set γ of G equivalent to it.

Linear representation of a finite group G over a field K can be interpreted as KG -modules. This interpretation permits the use of module theoretic language, in which many statements become more natural and their proofs simpler. The same situation prevails for projective representations in which the role of the group algebra KG plays the twisted group algebra $K^\omega G$ defined in [13] and Karpilovsky proved that the study of projective representations with the factor set ω is equivalent to the study of $K^\omega G$ -modules (Theorem 33). In [13], Karpilovsky observed that the

2020 Mathematics Subject Classification: 20C25 , 19C09 , 20B05 , 20C30, 20K01, 42A45

Keywords: projective representation, factor set or multiplier, finite group, second cohomology group, cyclic group, dihedral group, symmetric group

<http://www.utgjiu.ro/math/sma>

theory of projective representations for a fixed factor set ω and with respect to linear equivalency is analogous to the theory of linear representations. However, when it comes to the study of projective representations with respect to the projective equivalency, some of the parallelism with linear representations disappears (Theorem 34). Karpilovsky [13] found a fruitful way of linking $H^2(G, K^*)$ and twisted group algebras of G over K . A convenient way of doing this is to involve the notion of equivalence of twisted group algebras (Lemma 20).

Schur's theory of projective representations is founded on the fact that when $K = \mathbb{C}$ representation groups of G over K always exist. This fact allows to reduce the problem to determine all projective \mathbb{C} -representations of G to the determination of all linear \mathbb{C} -representations of a representation group G^* of G . For arbitrary fields, representation groups need not exist. For example, let K be an algebraic number field. Then the group $H^2(G, K^*)$ is infinite (Proposition 2.3.20, [13]) and hence a representation group of G over K cannot exist. In [13], Karpilovsky obtained Schur's results as a consequence of more general considerations (Lemma 38, Theorem 39, Corollary 40).

In [3] Brauer proved that every representation of a finite group G in the field \mathbb{C} of complex numbers is equivalent in \mathbb{C} to a representation of G in the field of the $|G|$ -th roots of unity and in [4] he improved this by replacing $|G|$ by the exponent of G . In [23], Reynolds considered the corresponding question for projective representations (Theorem 28, Theorem 29).

Schur [25] proved that for finite abelian groups a factor set is equivalent to the trivial one if and only if it is symmetric (Theorem 31) and Backhouse [2], cite Ba2 proved the same result for a large class of abelian groups. Backhouse [1] established a triviality criterion for the factor sets of finite groups (Theorem 41).

Tappe [27] proved a result on the number of irreducible projective representations of a finite group with respect to a given factor set and a group of linear characters acting on them (Theorem 56) and determined the number of classes of projectively equivalent representations and Osima's result ([18]) on the classes of linearly equivalent representations (Corollary 48, Corollary 49).

Morris [15] determined the number of inequivalent irreducible projective representations with factor set ω of $G = \mathbb{Z}_n \times \dots \times \mathbb{Z}_n$ (m copies) over \mathbb{C} in two special cases (Theorem 50, Theorem 51) :

- 1) when $\beta(i, j) = \varepsilon$, $1 \leq i < j \leq m$, where $\beta(i, j) = \omega(g_i, g_j)\omega^{-1}(g_j, g_i)$, $1 \leq i < j \leq m$, $g_i, g_j \in G$ and ε is a primitive n -th root of unity;
- 2) when n is even and $\beta(i, j) = -1$, $1 \leq i < j \leq m$.

Read [22] developed Clifford's theory [6] of inducing from normal subgroups of finite groups for projective representations (Theorem 56, Lemma 57).

In Section 4 we present some of Karpilovsky's results on irreducible projective representations of some classes of finite groups which are important in physics: direct products of cyclic groups of the same order, dihedral groups and symmetric groups (Theorem 58, Theorem 59, Theorem 60, Theorem 61).

2 Equivalent factor sets of finite groups

In what follows, G is a finite group and K^* is the multiplicative group of a field K .

Definition 1. ([25], [7],[23]) A map $\omega: G \times G \rightarrow K^*$ is a **factor set** (or **multiplier** or **2-cocycle**) of G (in K) if

$$i) \quad \omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z) \text{ for all } x, y, z \in G;$$

$$ii) \quad \omega(x, e) = \omega(e, x) = 1 \text{ for all } x \in G, \text{ where } e \text{ is the identity of } G.$$

Remark 2. ([23]) The factor sets of G in K form an abelian multiplicative group, where

$$(\omega\omega_1)(x, y) = \omega(x, y)\omega_1(x, y).$$

Definition 3. ([7], [23]) A **1-cochain** of G in K is a map $\mu: G \rightarrow K^*$ such that $\mu(e) = 1$.

Remark 4. ([23]) The 1-cochains of G in K form an abelian multiplicative group.

Definition 5. ([23]) The **coboundary** of a 1-cochain μ is the factor set $\delta\mu$ defined by

$$(\delta\mu)(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}.$$

Definition 6. ([25], [7], [23]) Two factor sets ω and ω_1 of G in K are called **equivalent** (or **cohomologous**) if there is a 1-cochain μ such that

$$\omega(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)$$

for all $x, y \in G$. We also can say that two factor sets are equivalent in K if their quotient is a coboundary.

This is an equivalence relation and the equivalence class containing the factor set ω will be denoted by $\{\omega\}$. The set of equivalence classes $\{\omega\}$ of factor sets form the **second cohomology group** $H^2(G, K^*)$ (or the **multiplier group** $M(G, K)$), which is a finite abelian multiplicative group.

Lemma 7. ([10]) Let ω and ω_1 be two equivalent factor sets of G . Then

$$\omega(x, y)\omega(y, x)^{-1} = \omega_1(x, y)\omega_1(y, x)^{-1}$$

for any $x, y \in G$ such that $xy = yx$.

Proof. Since ω and ω_1 are equivalent, there is a map $\mu: G \rightarrow \mathbb{C}^*$ such that

$$\omega(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)$$

$$\omega(y, x) = \mu(y)\mu(x)\mu(yx)^{-1}\omega_1(y, x)$$

So, $\omega(x, y)\omega(y, x)^{-1} = \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)(\mu(y)\mu(x)\mu(yx)^{-1}\omega_1(y, x))^{-1} =$
 $= \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)\mu(y)^{-1}\mu(x)^{-1}\mu(xy)\omega_1(y, x)^{-1} = \omega_1(x, y)\omega_1(y, x)^{-1} \quad \square$

Definition 8. ([21]) Let ω be a factor set of G . An element $a \in G$ is said to be ω -regular if $\omega(x, a) = \omega(a, x)$ for all $x \in C_G(a)$, where $C_G(a) = \{x \in G \mid xa = ax\}$ is the centralizer of a in G .

Remark 9. ([21]) If a is ω -regular, then a is ω_1 -regular for all factor sets ω_1 equivalent to ω .

Proof. Since a is ω -regular, we have $\omega(x, a) = \omega(a, x)$ for all $x \in C_G(a)$.

Since ω and ω_1 are equivalent, there is a map $\mu: G \rightarrow C^*$ such that

$$\omega(x, a) = \mu(x)\mu(a)\mu(xa)^{-1}\omega_1(x, a)$$

$$\omega(a, x) = \mu(a)\mu(x)\mu(ax)^{-1}\omega_1(a, x)$$

for $a \in G$ and $x \in C_G(a)$.

Hence $\mu(x)\mu(a)\mu(xa)^{-1}\omega_1(x, a) = \mu(a)\mu(x)\mu(ax)^{-1}\omega_1(a, x)$ for $a \in G$ and $x \in C_G(a)$. Then $\omega_1(x, a) = \omega_1(a, x)$ for $a \in G$ and $x \in C_G(a)$. Therefore, a is ω_1 -regular. \square

Definition 10. ([21]) Given a factor set ω of G , we define

$$f_\omega(x, a) = \omega(x, a)\omega(xax^{-1}, x)^{-1}$$

for all $a \in G$ ω -regular and for all $x \in G$.

Lemma 11. ([22]) Let ω be a factor set of G . Then

$$f_\omega(xy, z) = f_\omega(y, z)f_\omega(x, yzy^{-1})$$

for all $x, y, z \in G$.

Proof. Using Definition 10, the equality is equivalent with

$$\omega(y, z)\omega(yzy^{-1}, y)^{-1}\omega(x, yzy^{-1})\omega(xyzy^{-1}x^{-1}, x)^{-1} = \omega(xy, z)\omega(xyzy^{-1}x^{-1}, xy)^{-1}$$

We have to verify the following equation :

$$(2.1) \quad \omega(y, z)\omega(x, yzy^{-1})\omega(xyzy^{-1}x^{-1}, xy) = \omega(xy, z)\omega(yzy^{-1}, y)\omega(xyzy^{-1}x^{-1}, x)$$

$$\begin{aligned} \text{By Definition 1 i), we have : } & \omega(y, z)\omega(x, yzy^{-1})\omega(xyzy^{-1}x^{-1}, xy) = \\ & = \omega(y, z)\omega(x, yz)\omega(xyz, y^{-1})\omega(yz, y^{-1})^{-1}\omega(xyzy^{-1}x^{-1}, xy) = \\ & = \omega(x, yz)\omega(y, z)\omega(xyz, y^{-1})\omega(yz, y^{-1})^{-1}\omega(xyzy^{-1}x^{-1}, xy) \end{aligned}$$

The relation (2.1) becomes

$$\begin{aligned} & \omega(x, yz)\omega(y, z)\omega(xyz, y^{-1})\omega(yz, y^{-1})^{-1}\omega(xyzy^{-1}x^{-1}, xy) = \\ & = \omega(xy, z)\omega(yzy^{-1}, y)\omega(xyzy^{-1}x^{-1}, x) \end{aligned}$$

So,

$$(2.2) \quad \omega(x, yz)\omega(y, z)\omega(xyz, y^{-1})\omega(xyzy^{-1}x^{-1}, xy) =$$

$$= \omega(xy, z)\omega(yzy^{-1}, y)\omega(xzyy^{-1}x^{-1}, x)\omega(yz, y^{-1})$$

Applying Definition 1 i) in the left part of relation (2.1), we obtain:

$$\begin{aligned} \omega(x, y)\omega(xy, z)\omega(xyz, y^{-1})\omega(xzyy^{-1}x^{-1}, xy) &= \\ = \omega(xy, z)\omega(yzy^{-1}, y)\omega(xzyy^{-1}x^{-1}, x)\omega(yz, y^{-1}) \end{aligned}$$

So,

$$(2.3) \quad \omega(x, y)\omega(xyz, y^{-1})\omega(xzyy^{-1}x^{-1}, xy) = \omega(yzy^{-1}, y)\omega(xzyy^{-1}x^{-1}, x)\omega(yz, y^{-1})$$

Applying again Definition 1 i) in the left part of relation (2.1), we obtain:

$$\omega(x, y)\omega(xzyy^{-1}x^{-1}, x)\omega(xy, y^{-1}) = \omega(yzy^{-1}, y)\omega(xzyy^{-1}x^{-1}, x)\omega(yz, y^{-1})$$

Hence,

$$(2.4) \quad \omega(x, y)\omega(xy, y^{-1}) = \omega(yzy^{-1}, y)\omega(yz, y^{-1})$$

By Definition 1 i), the relation (2.1) becomes

$$\begin{aligned} \omega(x, yy^{-1})\omega(y, y^{-1}) &= \omega(yzy^{-1}, yy^{-1})\omega(y, y^{-1}) \iff \\ \iff \omega(x, e) = \omega(yzy^{-1}, e), \end{aligned}$$

□

Lemma 12. ([21]) *Let ω be a factor set of G and a an ω -regular element. Let $x, y \in G$ be such that $xax^{-1} = yay^{-1}$. Then*

$$f_\omega(x, a) = f_\omega(y, a).$$

Lemma 13. ([21]) *Let ω be a factor set of G and let a be ω -regular. If $f_\omega(x, a) = 1$ for all $x \in G$, then $f_\omega(x, yay^{-1}) = 1$ for all $x, y \in G$.*

Lemma 14. ([21]) *Let ω be a factor set of G and let $f_\omega(x, a) = 1$ for all $a \in G$ ω -regular and for all $x \in G$. Then $\omega(a, a^{-1}) = \omega(xax^{-1}, xa^{-1}x^{-1})$ for all ω -regular $a \in G$ and all $x \in G$.*

Definition 15. ([21]) *The factor set ω is called **simple** if :*

i) $f_\omega(x, a) = 1$ for all ω -regular $a \in G$ and all $x \in G$;

ii) $\omega(x, x^{-1}) = 1$ for all $x \in G$.

Theorem 16. ([21]) *Let ω be a factor set of G . Then there is a simple factor set γ of G equivalent to ω .*

Proof. We define $\mu: G \rightarrow K^*$ as follows :

Let $\{a_1, \dots, a_t\}$ be an ω -regular class of G and let $G = \bigcup_{i=1}^t x_i C_G(a_1)$ such that $a_i = x_i a_1 x_i^{-1}$ for $i = 1, \dots, t$. We call a_1 the representative element of the ω -regular class containing it.

Define $\mu(a_i) = f_\omega(x_i, a_1)$ for $i = 1, \dots, t$. Lemma 12 shows that μ is well defined for any choice of $\{x_1, \dots, x_t\}$. Similarly, we define μ on the other ω -regular classes.

Further, put $\mu(x) = 1$ if x is not ω -regular. Define

$$\beta(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}\omega(x, y)$$

for all $x, y \in G$. Then for all $z \in G$, we have $f_\beta(z, a_1) = 1$ and hence by Lemma 13, $f_\beta(z, a) = 1$ for all β -regular $a \in G$ and all $z \in G$.

Now we define $\eta(z) = \beta(z, z^{-1})^{-\frac{1}{2}}$ for all $z \in G$ and put

$$\gamma(x, y) = \eta(x)\eta(y)\eta(xy)^{-1}\beta(x, y)$$

for all $x, y \in G$.

We show that $\gamma(z, z^{-1}) = 1$ for all $z \in G$.

$$\begin{aligned} \gamma(z, z^{-1}) &= \eta(z)\eta(z^{-1})\eta(zz^{-1})^{-1}\beta(z, z^{-1}) = \\ &= \beta(z, z^{-1})^{-\frac{1}{2}}\beta(z^{-1}, z)^{-\frac{1}{2}}[\beta(zz^{-1}, (zz^{-1})^{-1})^{-\frac{1}{2}}]^{-1}\beta(z, z^{-1}) = \\ &= \beta(z, z^{-1})^{-\frac{1}{2}}\beta(z^{-1}, z)^{-\frac{1}{2}}\beta(zz^{-1}, zz^{-1})^{\frac{1}{2}}\beta(z, z^{-1}) = \\ &= \beta(z, z^{-1})^{\frac{1}{2}}\beta(z^{-1}, z)^{-\frac{1}{2}}\beta(zz^{-1}, zz^{-1})^{\frac{1}{2}} = \\ &= \beta(z, z^{-1})^{\frac{1}{2}}[\beta(z, z^{-1})^{\frac{1}{2}}\beta(zz^{-1}, zz^{-1})^{-1}]^{-\frac{1}{2}} = \mu(z)^{\frac{1}{2}}\mu(z^{-1})^{\frac{1}{2}}\mu(zz^{-1})^{-\frac{1}{2}} = \\ &= \omega(z, z^{-1})^{\frac{1}{2}}[\mu(z^{-1})\mu(z)\mu(z^{-1}z)^{-1}\omega(z^{-1}, z)]^{\frac{1}{2}} \cdot \\ &\cdot [\mu(zz^{-1})\mu(zz^{-1})\mu(zz^{-1}zz^{-1})^{-1}\omega(zz^{-1}, zz^{-1})]^{\frac{1}{2}} = \\ &= \mu(z)^{\frac{1}{2}}\mu(z^{-1})^{\frac{1}{2}}\mu(e)^{-\frac{1}{2}}\omega(z, z^{-1})^{\frac{1}{2}}\mu(z^{-1})^{-\frac{1}{2}}\mu(z)^{-\frac{1}{2}} \\ &\mu(e)^{\frac{1}{2}}\omega(z^{-1}, z)^{-\frac{1}{2}}\mu(zz^{-1})^{-\frac{1}{2}}\mu(zz^{-1})^{-\frac{1}{2}}\mu(e)^{\frac{1}{2}}\omega(zz^{-1}, zz^{-1})^{\frac{1}{2}} = \\ &= \omega(z, z^{-1})^{\frac{1}{2}}\omega(z^{-1}, z)^{-\frac{1}{2}}\mu(e)^{-\frac{1}{2}}\omega(zz^{-1}, zz^{-1})^{\frac{1}{2}} = \omega(z, z^{-1})^{\frac{1}{2}}\omega(z^{-1}, z)^{-\frac{1}{2}}\omega(e, e)^{\frac{1}{2}} = \\ &= \omega(z, z^{-1})^{\frac{1}{2}}\omega(z^{-1}, z)^{-\frac{1}{2}}, \text{ by Definition 1 i).} \end{aligned}$$

$$\begin{aligned} \text{By Definition 1 i), } \omega(z, z^{-1})\omega(zz^{-1}, z) &= \omega(z, z^{-1}z)\omega(z^{-1}, z) \implies \\ \implies \omega(z, z^{-1})\omega(e, z) &= \omega(z, e)\omega(z^{-1}, z) \implies \omega(z, z^{-1}) = \omega(z^{-1}, z) \end{aligned}$$

Hence, $\gamma(z, z^{-1}) = 1$.

If a is ω -regular, by Lemma 14, follows that $f_\gamma(z, a) = 1$ for all $z \in G$, because

$$\begin{aligned} f_\gamma(z, a) &= \gamma(z, a)\gamma(zaz^{-1}, z)^{-1} = \\ &= \eta(z)\eta(a)\eta(za)^{-1}\beta(z, a)[\eta(zaz^{-1})\eta(z)\eta(zaz^{-1}z)^{-1}\beta(zaz^{-1}, z)]^{-1} = \\ &= \eta(a)\beta(z, a)\eta(zaz^{-1})^{-1}\beta(zaz^{-1}, z)^{-1} = \\ &= \beta(a, a^{-1})^{-\frac{1}{2}}\beta(z, a)\beta(zaz^{-1}, (zaz^{-1})^{-1})^{\frac{1}{2}}\beta(zaz^{-1}, z)^{-1} = \\ &= \beta(a, a^{-1})^{-\frac{1}{2}}\beta(z, a)\beta(zaz^{-1}, za^{-1}z^{-1})^{\frac{1}{2}}\beta(zaz^{-1}, z) = \\ &= \beta(z, a)\beta(zaz^{-1}, z)^{-1} = f_\beta(z, a) = 1, \text{ because, since } \beta \text{ is a factor set of } G \text{ and } \\ &f_\beta(z, a) = 1, \text{ we have } \beta(a, a^{-1}) = \beta(zaz^{-1}, za^{-1}z^{-1}) \text{ by Lemma 14.} \end{aligned}$$

By definition of γ results that γ is equivalent to ω .

Thus γ is a simple factor set equivalent to ω . □

Lemma 17. ([13]) (i) An element $x \in G$ is ω -regular if and only if it is β -regular for any factor set β equivalent with ω .

(ii) If $x \in G$ is ω -regular, then so is any power x^q , where q is a positive integer coprime with the order of x . In particular, if x is ω -regular, then so is x^{-1} .

(iii) If $x \in G$ is ω -regular, then so is any conjugate of x .

(iv) If $\text{char}K = p > 0$ and x is a p -element, then x is ω -regular.

Proof. (i) Let x be a ω -regular element and let β be an equivalent factor set with ω , i.e.

$$\beta(y, z) = \omega(y, z)\mu(y)\mu(z)\mu(yz)^{-1} \text{ for some } \mu: G \rightarrow K^*.$$

$$\begin{aligned} \text{Then for } y \in C_G(x), \beta(y, x) &= \omega(y, x)\mu(y)\mu(x)\mu(yx)^{-1} = \\ &= \omega(x, y)\mu(y)\mu(x)\mu(xy)^{-1} = \beta(x, y), \text{ proving that } x \text{ is } \omega\text{-regular.} \end{aligned}$$

The converse is obvious.

(ii) Let $y \in C_G(x^q)$. Then, since $C_G(x^q) = C_G(x)$ and since x is ω -regular, we have $\bar{y} \cdot \bar{x} = \bar{x} \cdot \bar{y}$.

Let $\lambda = \lambda(x) \in K^*$ be such that $x^q = \lambda \bar{x}^q$. Then $\overline{yx^q} = \lambda \bar{y} \bar{x}^q = \lambda \bar{x}^q \bar{y} = \overline{x^q y}$, so x^q is ω -regular.

(iii) Let $z \in G$ and let $y \in C_G(zxz^{-1})$. Then $z^{-1}yz \in C_G(x)$ and therefore $z^{-1}yz \cdot \bar{x} = \bar{x} \cdot z^{-1}yz$.

Since $z^{-1}yz = t\bar{z}^{-1} \cdot \bar{y} \cdot \bar{z}$ for some $t = t(y, z) \in K^*$, it follows that $\bar{z}^{-1} \cdot \bar{y} \cdot \bar{z} \cdot \bar{x} = \bar{x} \cdot \bar{z}^{-1} \cdot \bar{y} \cdot \bar{z}$.

Hence \bar{y} commutes with $\bar{z} \cdot \bar{x} \cdot \bar{z}^{-1}$. Again, since $\overline{z x z^{-1}} = \lambda \bar{z} \cdot \bar{x} \cdot \bar{z}^{-1}$ for some $\lambda = \lambda(x, z) \in K^*$, we see that \bar{y} commutes with $z x z^{-1}$. Thus $z x z^{-1}$ is also ω -regular.

(iv) We may assume that K is algebraically closed. Assume that $y \in C_G(x)$ and let $S = \langle y, x \rangle$. Then all Sylow l -subgroups of S with $l \neq p$ are cyclic. Applying Theorem 2.3.1 and 2.3.2 (iii), [13] together with Corollary 2.3.24, [13], we deduce that $\omega|_S$ is a coboundary. Hence $K^\omega S \cong KS$ is commutative, so $\bar{y} \cdot \bar{x} = \bar{x} \cdot \bar{y}$, proving the result. \square

Theorem 18. ([7]) *The second cohomology group $H^2(G, K^*)$ has finite order not divisible by the characteristic of K . The order of every element in $H^2(G, K^*)$ is a factor set of the order of G .*

Proof. First let $\{\omega\} \in H^2(G, K^*)$ and $n = [G : 1]$.

$$\text{For any } x \in G, \text{ define } \mu(x) = \prod_{y \in G} \omega(x, y).$$

Then from Definition 1 i), we have $\omega(x, y)^n = \frac{\mu(x)\mu(y)}{\mu(xy)}$ and it follows that $\{\omega\}^n = 1$. This proves the second statement of the theorem.

Let h be the order of $\{\omega\}$ in $H^2(G, K^*)$ and if $\text{char}K = p > 0$, write $h = p^a q$, where $a \geq 0$ and $p \nmid q$. Then there is a map $\beta: G \rightarrow K^*$ such that for all $x, y \in G$, we have

$$(2.5) \quad \omega(x, y)^h = \frac{\beta(x)\beta(y)}{\beta(xy)}$$

Since K is algebraically closed, K is a perfect field and we may extract p -th roots in K . We obtain $(\omega(x, y)^q)^{p^a} = \left(\frac{\beta(x)^{\frac{1}{p^a}} \beta(y)^{\frac{1}{p^a}}}{\beta(xy)^{\frac{1}{p^a}}} \right)^{p^a}$ and because p^a -th roots are unique in K , we have $\omega(x, y)^q = \frac{\beta(x)^{\frac{1}{p^a}} \beta(y)^{\frac{1}{p^a}}}{\beta(xy)^{\frac{1}{p^a}}}$ which contradicts our assumption that h is the order of $\{\omega\}$ unless $p^a = 1$. Therefore $p \nmid h$. Returning to (2.1), for each $x \in G$ we can find $\mu(x) \in K^*$ such that $\mu(x)^h = \beta(x)^{-1}$. Upon setting $\omega'(x, y) = \frac{\mu(x)\mu(y)}{\mu(xy)}\omega(x, y)$, we see that $(\omega'(x, y))^h = \frac{\beta(x)^{-1}\beta(y)^{-1}}{\beta(xy)^{-1}}\omega(x, y)^h = 1$.

We have proved that every class $\{\omega\} \in H^2(G, K^*)$ of order h contains a representative ω' whose values $\omega'(x, y)$ are h -th roots of 1 in K . Since $h|n$, it follows that there is at most a finite number of classes of factor sets and the order of $H^2(G, K^*)$ is finite. Furthermore, since the order of every element of $H^2(G, K^*)$ is not divisible by the characteristic of K , it follows that $\text{char}K \nmid [H^2(G, K^*) : 1]$ and the theorem is proved. \square

Let ω be a factor set. We denote by $K^\omega G$ the vector space over K with basis $\{\bar{x} \mid x \in G\}$ which is in one-to-one correspondence with G . We define multiplication in $K^\omega G$ distributively using for all $x, y \in G$, $\bar{x} \cdot \bar{y} = \omega(x, y)\overline{xy}$.

The finite dimensional K -algebra $K^\omega G$ is called the **twisted group algebra of G over K** . Note that if $\omega(x, y) = 1$ for all $x, y \in G$, then $K^\omega G \cong KG$.

Definition 19. ([13]) Let $K^\omega G$ and $K^\lambda G$ be twisted group algebras with bases $\{\bar{x} \mid x \in G\}$ and $\{\tilde{x} \mid x \in G\}$ respectively. We say that $K^\omega G$ and $K^\lambda G$ are **equivalent** if there is a K -algebra isomorphism $\psi: K^\omega G \rightarrow K^\lambda G$ and a map $t: G \rightarrow K^*$ such that for all $x \in G$, $\psi(\bar{x}) = t(x)\tilde{x}$.

Lemma 20. ([13]) The twisted group algebras $K^\omega G$ and $K^\lambda G$ are equivalent if and only if ω and λ are equivalent. Thus

- i) if ω is a coboundary, then $K^\omega G \cong KG$.
- ii) the map $\omega \mapsto K^\omega G$ induces a bijective correspondence between the elements of $H^2(G, K^*)$ and the equivalence classes of twisted group algebras of G over K .

Proof. Let an isomorphism $\psi: K^\omega G \rightarrow K^\lambda G$ and a map $t: G \rightarrow K^*$ such that $\psi(\bar{x}) = t(x)\tilde{x}$, where $\{\bar{x} \mid x \in G\}$ and $\{\tilde{x} \mid x \in G\}$ are bases of $K^\omega G$ and $K^\lambda G$ respectively. Then

$$\begin{aligned} t(xy)\overline{xy} &= \psi(\overline{xy}) = \psi(\omega(x, y)^{-1}\bar{x} \cdot \bar{y}) = \omega(x, y)^{-1}\psi(\bar{x})\psi(\bar{y}) = \\ &= \omega(x, y)^{-1}\mu(x)\mu(y)\tilde{x}\tilde{y} = \omega(x, y)^{-1}t(x)t(y)\lambda(x, y)\tilde{x}\tilde{y}, \text{ so} \end{aligned}$$

$$\omega(x, y) = \lambda(x, y)t(x)t(y)\mu(xy)^{-1},$$

proving that ω and λ are equivalent.

Conversely, suppose that ω and λ are equivalent, say

$$\omega(x, y) = \lambda(x, y)\mu(x)\mu(y)\mu(xy)^{-1}$$

for some 1-cochain $\mu: G \rightarrow K^*$. Then the map $\psi: K^\omega G \rightarrow K^\lambda G$, which is the extension of $\bar{x} \rightarrow \mu(x)\tilde{x}$ by K -linearity, is certainly a vector space isomorphism. To prove that ψ preserves multiplication, it is sufficient to check that $\psi(\bar{x}\cdot\bar{y}) = \psi(\bar{x})\psi(\bar{y})$ for all $x, y \in G$. Because

$$\begin{aligned} \psi(\bar{x}\cdot\bar{y}) &= \psi(\omega(x, y)\tilde{x}\tilde{y}) = \omega(x, y)\mu(xy)\tilde{x}\tilde{y} = \omega(x, y)\mu(xy)\lambda(x, y)^{-1}\tilde{x}\tilde{y} = \\ &= \mu(x)\mu(y)\tilde{x}\tilde{y} = \psi(\bar{x})\psi(\bar{y}), \end{aligned} \quad \square$$

3 Projectively equivalent representations

Definition 21. ([25], [7], [23], [13], [27]) Let G be a finite group with the identity e , K a field, V a finite dimensional vector space over K and $GL(V)$ the group of all automorphisms of V . A **projective representation** of G with the factor set ω is a map $\rho: G \rightarrow GL(V)$ such that

$$i) \quad \rho(x)\rho(y) = \omega(x, y)\rho(xy) \text{ for all } x, y \in G;$$

$$ii) \quad \rho(e) = 1_V.$$

If we identify $GL(V)$ with $GL(n, K)$, where $n = (V : K)$, the resulting map is called a projective matrix representation of G over K . We'll treat the terms "projective representation" and "projective matrix representation" as interchangeable.

A linear representation is a projective representation with $\omega(x, y) = 1$ for all $x, y \in G$.

Definition 22. ([23], [9], [13], [27]) Let ρ_1 and ρ_2 be two projective representations of G , $\rho_i: G \rightarrow GL(V_i), i = 1, 2$. ρ_1 and ρ_2 are called **projectively equivalent** if there are a 1-cochain $\mu: G \rightarrow K^*$ and a vector space isomorphism $f: V_1 \rightarrow V_2$ such that

$$\rho_2(x) = \mu(x)f^{-1}\rho_1(x)f$$

for all $x \in G$.

If $\mu(x) = 1$ for all $x \in G$, then ρ_1 and ρ_2 are called **linearly equivalent**.

In terms of matrix representations, two representations $\rho_i: G \rightarrow GL(n, K), i = 1, 2$ are projectively equivalent if there are a 1-cochain $\mu: G \rightarrow K^*$ with $\mu(e) = 1$ and a matrix $P \in GL(n, K)$ such that $\rho_2(x) = \mu(x)P^{-1}\rho_1(x)P$ for all $x \in G$.

Remark 23. ([13]) If ω_1 is the factor set for ρ_1 and ω_2 is the factor set for ρ_2 , then the projective equivalence of ρ_1 and ρ_2 yields

$$\mu(xy)\omega_2(x, y) = \mu(x)\mu(y)\omega_1(x, y)$$

for all $x, y \in G$.

Proof. Since ρ_1 and ρ_2 are projectively equivalent, there are a 1-cochain $\mu: G \rightarrow K^*$ and a vector space isomorphism $f: V_1 \rightarrow V_2$ such that

$$\begin{aligned}\rho_2(x) &= \mu(x)f^{-1}\rho_1(x)f \\ \rho_2(y) &= \mu(y)f^{-1}\rho_1(y)f \\ \rho_2(xy) &= \mu(xy)f^{-1}\rho_1(xy)f\end{aligned}$$

for all $x, y \in G$.

Since ρ_1 and ρ_2 are projective representations projectively equivalent, we have $\omega_2(x, y)\rho_2(xy) = \rho_2(x)\rho_2(y) = \mu(x)f^{-1}\rho_1(x)f\mu(y)f^{-1}\rho_1(y)f = \mu(x)\mu(y)f^{-1}\rho_1(x)\rho_1(y)f = \mu(x)\mu(y)f^{-1}\omega_1(x, y)\rho_1(xy)f$.

Then $\omega_2(x, y)\mu(xy)f^{-1}\rho_1(xy)f = \mu(x)\mu(y)f^{-1}\omega_1(x, y)\rho_1(xy)f \implies \omega_2(x, y)\mu(xy) = \mu(x)\mu(y)\omega_1(x, y)$ □

Remark 24. ([23]) It is obvious that linearly equivalent projective representations have the same factor set.

Remark 25. ([13]) If ρ_1 and ρ_2 are projectively equivalent and have the same factor set, then the 1-cochain μ is a homeomorphism from G into K^* .

Proof. By Definition 22, there are a 1-cochain map $\mu: G \rightarrow K^*$ and a vector space isomorphism f such that $\mu(e) = 1$ and

$$\begin{aligned}\rho_2(x) &= \mu(x)f^{-1}\rho_1(x)f \\ \rho_2(y) &= \mu(y)f^{-1}\rho_1(y)f \\ \rho_2(xy) &= \mu(xy)f^{-1}\rho_1(xy)f\end{aligned}$$

for all $x, y \in G$.

Since ρ_2 is a projective representation with the factor set ω and ρ_1 and ρ_2 have the same factor set ω , we get that $\rho_2(x)\rho_2(y) = \omega_2(x, y)\rho_2(xy) = \omega_2(x, y)\mu(xy)f^{-1}\rho_1(xy)f = \mu(xy)f^{-1}\omega_2(x, y)\rho_1(xy)f = \mu(xy)f^{-1}\omega_1(x, y)\rho_1(xy)f = \mu(xy)f^{-1}\rho_1(x)\rho_1(y)f$

On the other hand,

$$\rho_2(x)\rho_2(y) = \mu(x)f^{-1}\rho_1(x)f\mu(y)f^{-1}\rho_1(y)f = \mu(x)\mu(y)f^{-1}\rho_1(x)\rho_1(y)f$$

Therefore, $\mu(xy)f^{-1}\rho_1(x)\rho_1(y)f = \mu(x)\mu(y)f^{-1}\rho_1(x)\rho_1(y)f \implies \mu(xy) = \mu(x)\mu(y)$ for all $x, y \in G$. □

Lemma 26. ([13]) (a) Let $\rho_i, i = 1, 2$ be two projective representations $V_i, i = 1, 2$ with the factor sets $\omega_i, i = 1, 2$. If ρ_1 is projectively (respectively, linearly) equivalent to ρ_2 , then ω_1 is equivalent (respectively, equal) to ω_2 . Furthermore, if $\omega_1 = \omega_2$, then any map $\mu: G \rightarrow K^*$ satisfying the relation in Definition 22 is a homomorphism.

(b) Let ρ_1 be a projective representation V with the factor set ω_1 . Then for any factor set ω_2 that is equivalent to ω_1 there is a projective representation ρ_2 V with the factor set ω_2 which is projectively equivalent to ρ_1 . In particular, if ω_1 is a coboundary, then ρ_1 is projectively equivalent to a linear representation.

Proof. (a) This assertion is proved in Remark 23 and Remark 25.

(b) Since ω_2 is equivalent to ω_1 , by Definition 6, there is a 1-cochain $\mu: G \rightarrow K^*$ such that $\mu(e) = 1$ and $\omega_2(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)$ for all $x, y \in G$.

For all $x \in G$, define $\rho_2: G \rightarrow GL(V)$ by $\rho_2(x) = \mu(x)\rho_1(x)$.

Since ρ_1 is a projective representation with the factor set ω_1 , we have $\rho_1(x)\rho_1(y) = \omega_1(x, y)\rho_1(xy)$ for all $x, y \in G$, that is $\omega_1(x, y) = \rho_1(x)\rho_1(y)\rho_1(xy)^{-1}$ for all $x, y \in G$.

So, $\omega_2(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}\rho_1(x)\rho_1(y)\rho_1(xy)^{-1} =$
 $= \mu(x)\rho_1(x)\mu(y)\rho_1(y)\mu(xy)^{-1}\rho_1(xy)^{-1} =$
 $= \rho_2(x)\rho_2(y)\rho_2(xy)^{-1}$. This means that ρ_2 is a projective representation with the factor set ω_2 . \square

Definition 27. ([23]) The **restriction** $\omega|_S$ of a factor set ω of G to a subgroup S is defined by restricting its arguments to S . If η is a factor set of a quotient group G/H in K , the **inflation** of η to G is the factor set $\text{inf } \eta$ of G in K defined by $(\text{inf } \eta)(x, y) = \eta(xH, yH)$.

Theorem 28. ([23]) Let H be a normal subgroup of a finite group G and let ω be a factor set of G/H in \mathbb{C} . Then there is a factor set η of G/H such that

- i) ω is equivalent to η in \mathbb{C} ;
- ii) the values of η are $|G|$ -th roots of unity;
- iii) if S is a subgroup of G , then every projective representation of S in \mathbb{C} with the factor set $(\text{inf } \eta)|_S$ or $(\text{inf } \eta)^{-1}|_S$ is linearly equivalent in \mathbb{C} to a projective representation of S in the field K of the $|G|$ -th roots of unity.

Proof. Let r be the order of the equivalence class $\{\text{inf } \omega\}$ of $\text{inf } \omega$ in $H^2(G, \mathbb{C}^*)$. Then this class also contains at least one factor set ε of G such that ε itself has order r in the multiplicative group of factor sets of G in \mathbb{C} (by Theorem 53.3, [7]). Here

$$(3.1) \quad \varepsilon = (\delta\mu)(\text{inf } \omega)$$

for some 1-cochain μ of G in \mathbb{C} .

Using an adaptation of an argument of Schur, [25], let A be the character group of the multiplicative cyclic group generated by ε ; A is cyclic of order r .

For any $x, y \in G$, let $a_{x,y} \in A$ be the character such that $a_{x,y}(\varepsilon^i) = \varepsilon(x, y)^i$. Then the ordered pairs $(a, x), a \in A, x \in G$ form a group G^* under the multiplication $(a, x)(b, y) = (aba_{x,y}, xy)$. If A^* consists of the pairs of form $(a, 1)$ and S^* consists of all pairs (a, s) with $s \in S$, then clearly $A^* \subseteq Z(G^*), A^* \cong A, G^*/A^* \cong G$ and $S^*/A^* \cong S$.

For any linear character λ of G^* , $\lambda(a, 1) = a^j(\varepsilon)$ for some j and for all $a \in A$. In particular, $\lambda(a_{x,y}, 1) = \varepsilon(x, y)^j$. Since $\lambda(1, x)\lambda(1, y) = \varepsilon(x, y)^j\lambda(1, xy)$, we have

$\{\inf \omega\}^j = \{\varepsilon\}^j = 1$, so that r divides j and $\lambda|A^* = 1$; since λ is arbitrary, $A^* \subseteq (G^*)'$.

To each projective representation ρ of S in \mathbb{C} with the multiplier $\omega|S$, there corresponds an ordinary representation π of S^* defined by $\pi(a, x) = a(\omega)\rho(x)$. By Theorem 3, [23] applied to G^* we see that π is equivalent to a representation of S^* in K , since $|G^* : A^*| = |G|$. Then ρ is linearly equivalent to a projective representation of S in K . The same holds true for projective representations with the factor set $\omega^{-1}|S$.

The order r of $\{\varepsilon\} = \{\inf \omega\}$ divides the order h of the class $\{\omega\}$ of ω in $H^2(G/H, \mathbb{C}^*)$. But h divides $|G : H|$ by [7]. Hence

$$(3.2) \quad \varepsilon^{|G:H|} = 1.$$

This proves the theorem in the case $H = 1$, by taking $\eta = \varepsilon$. But in general we must argue further, since ε may not be the inflation of a factor set of G/H .

Since $\omega(1, 1) = 1$, (3.1) implies that $(\delta\mu)|H = \varepsilon|H$.

By (3.1), $((\delta\mu)|H)^{|G:H|} = 1$. In other words, $(\mu|H)^{|G:H|}$ is a linear character of H . Therefore,

$$(3.3) \quad (\mu|H)^{|G|} = ((\mu|H)^{|G:H|})^{|H|} = 1.$$

For each $z \in G/H$, choose a representative $g_z \in g$ such that $g_z H = z$ with $g_1 = 1$.

A 1-cochain γ of G/H is defined by setting $\gamma(z) = \mu(g_z)$. We show that the factor set $\eta = (\delta\gamma)\omega$ of G/H satisfies conditions i), ii) and iii).

Condition i) holds by definition.

For the 1-cochain $\nu = (\inf \gamma)\mu^{-1}$ of G , whenever $h \in H$ and $z \in G/H$ we have $\nu(hg_z) = \gamma(z)\mu(hg_z)^{-1} = \mu(g_z)\mu(hg_z)^{-1}$.

But by (3.1), $\varepsilon(h, g_z) = (\delta\mu)(h, g_z)\omega(1, z) = \mu(h)\mu(g_z)\mu(hg_z)^{-1}$, so that $\nu(hg_z) = \mu(h)^{-1}\varepsilon(h, g_z)$.

By (3.1) and (3.1) both factors on the right are $|G|$ -th roots of unity. Hence

$$(3.4) \quad \nu^{|G|} = 1.$$

By (3.1) and the definitions of η and ν ,

$$(3.5) \quad \inf \eta = (\inf(\delta\gamma))(\inf \omega) = (\delta(\inf \gamma))(\delta\mu)^{-1}\varepsilon = (\delta\nu)\varepsilon$$

Then by (3.1) and (3.1), $\eta^{|G|} = 1$, which proves ii).

Corresponding to each projective representation τ of S with factor set $(\inf \eta)|S$, we can define a projective representation ρ with factor set $\varepsilon|S$ by writing $\rho(x) = \nu(x)^{-1}\tau(x)$, $x \in S$ by (3.1). We have shown that ρ is linearly equivalent to a projective representation over K . But for any matrix U over \mathbb{C} such that $U^{-1}\rho(x)U\rho$ lies in K for all $x \in S$, $U^{-1}\tau(x)U$ also lies in K , by (3.1).

This proves the part of iii) concerning $(\inf \eta)|S$; the rest of iii) follows from similar arguments. \square

This theorem implies that every projective representation of S in \mathbb{C} with factor set $(\inf \omega)|S$ is projectively equivalent in \mathbb{C} to a projective representation of S in K with factor set $(\inf \eta)|S$, which gives us the following theorem:

Theorem 29. ([23]) *Every projective representation ρ of G in \mathbb{C} is projectively equivalent in \mathbb{C} to a projective representation π of G in the field of the $|G|$ -th roots of unity. π can be chosen so that its factor set takes on only $|G|$ -th roots of unity as values and so that it is inflated from any quotient group G/H from which the factor set of ρ is inflated.*

Definition 30. ([13]) *The projective representation ρ on the space V is called **irreducible** if 0 and V are only subspaces of V which are sent into themselves by all the transformations $\rho(x), x \in G$.*

Theorem 31. ([25], [1]) *Let ω be a symmetric factor set of the abelian finite group G and suppose that there is at least one irreducible unitary projective representation of G with the factor set ω . Then ω is equivalent with the trivial factor set.*

Proof. Let ρ be an irreducible unitary projective representation of G with the factor set ω . Then

$$(3.6) \quad \rho(x)\rho(y) = \omega(x, y)\rho(xy)$$

for all $x, y \in G$.

Equation (3.1) holds if x and y interchanges and by hypothesis $\omega(x, y) = \omega(y, x)$ and $xy = yx$, it follows that $\rho(x)\rho(y) = \rho(y)\rho(x)$ for all $x, y \in G$. But ρ is irreducible and unitary, so from Schur's lemma $\rho(x)$ is equal to a scalar multiple $\lambda(x)$ of the identity. Then $\lambda(x)\lambda(y) = \omega(x, y)\lambda(xy)$, which means that ω is equivalent with the trivial factor set. \square

Let A be an associative K -algebra with basis $\{u_x | x \in G\}$ such that $u_x u_y = \omega(x, y)u_{xy}$, $x, y \in G$ and $\omega(x, y) \in K^*$. The associativity of A implies that for all $x, y, z \in G$, $\omega(x, y)\omega(xy, z) = \omega(y, z)\omega(x, yz)$. By making a diagonal change of basis, if necessary, we may assume that for all $x \in G$, $\omega(x, 1) = \omega(1, x) = 1$, so that ω is a factor set. Thus A is identifiable with $K^\omega G$.

Lemma 32. ([13]) *Let A be a K -algebra and let f be a map of G into the unit group of A which satisfies $f(x)f(y) = \omega(x, y)f(xy)$ ($x, y \in G$). Then the map $f^*: K^\omega G \rightarrow A$ defined by $f^*(\sum \alpha_x \bar{x}) = \sum \alpha_x f(x)$ is a homomorphism of K -algebras.*

Proof. Since f^* is the extension of $\bar{x} \rightarrow f(x)$ by K -linearity, f^* is a vector space homomorphism. To prove that f^* preserves multiplication it is sufficient to check it on the basis elements \bar{x} , $x \in G$. Since $f^*(\bar{x} \cdot \bar{y}) = f^*(\omega(x, y)\bar{xy}) = \omega(x, y)f(xy) = f(x)f(y) = f^*(\bar{x})f^*(\bar{y})$, the result follows. \square

Theorem 33. ([13]) *There is a bijective correspondence between projective representations of G with factor set ω and $K^\omega G$ -modules. This correspondence preserves sums and maps bijectively linearly equivalent (irreducible) representations into isomorphic (irreducible) modules.*

Proof. Let ρ be a projective representation of G with the factor set ω on the space V . Due to Lemma 32, we can define a homomorphism $f: K^\omega G \rightarrow \text{End}_K(V)$ by setting $f(\bar{x}) = \rho(x)$ and extending by linearity. Hence V becomes a $K^\omega G$ -module by setting $(\sum \alpha_x \bar{x})v = \sum \alpha_x \rho(x)v$ ($\alpha_x \in K, x \in G, v \in V$).

Conversely, given a $K^\omega G$ -module V and hence a homomorphism $f: K^\omega G \rightarrow \text{End}_K(V)$, define $\rho(x) = f(\bar{x})$. Then $\rho(x) \in GL(V)$ since \bar{x} is a unit of $K^\omega G$. Furthermore, $\rho(x)\rho(y) = f(\bar{x})f(\bar{y}) = f(\bar{x} \cdot \bar{y}) = f(\omega(x, y)\bar{xy}) = \omega(x, y)\rho(xy)$, so that ρ is a projective representation with the factor set ω on V . This sets up a bijective correspondence between projective representations with the factor set ω and $K^\omega G$ -modules.

A subspace W of V is invariant under all $\rho(x)$ if and only if W is a $K^\omega G$ -submodule of V . Hence the correspondence preserves sums and maps bijectively irreducible representations into irreducible modules.

We observe that a K -isomorphism $f: V_1 \rightarrow V_2$ of KG -modules is a $K^\omega G$ -isomorphism if and only if $\bar{x}f(v) = f(\bar{x}v)$ for all $x \in G, v \in V_1$.

Suppose that $\rho_i: G \rightarrow GL(V_i)$, $i = 1, 2$ are two projective representations with the factor set ω . Then ρ_1 is linearly equivalent to ρ_2 if and only if there is a K -isomorphism $f: V_1 \rightarrow V_2$ such that $\rho_2(x)f = f\rho_1(x)$ for all $x \in G$. The latter is equivalent to $\rho_2(x)f(v) = f\rho_1(x)v$ or to $\bar{x}f(v) = f(\bar{x}v)$ for all $g \in G, v \in V_1$. Thus two projective representations with the factor set ω are linearly equivalent if and only if the corresponding modules are isomorphic. \square

Theorem 34. ([13]) *Let m be the exponent of G/G' . Then the number $n(G, K)$ of projectively nonequivalent irreducible projective representations of G over K is finite if and only if the group $K^*/(K^*)^m \mu(K^*)$ is finite. In particular,*

- i) *the numbers $n(G, \mathbb{C}^*)$ and $n(G, \mathbb{R}^*)$ are finite;*
- ii) *the number $n(G, K^*)$ is infinite whenever K is an algebraic number field and $G \neq G'$.*

Proof. Due to Theorem 33, for any factor set ω there is an irreducible representation of G over K with this factor set. The same theorem also implies that the number of projectively nonequivalent irreducible projective representations with the factor set ω cannot exceed the number of nonisomorphic $K^\omega G$ -modules. Since the latter is finite, Lemma 26 implies that $n(G, K)$ is finite if and only if $H^2(G, K^*)$ is also.

The assertion (i) and (ii) can be deduced by Theorem 2.3.2, Proposition 2.3.20 (ii) and the remark preceding Proposition 2.3.20, [13]. \square

Definition 35. [13] Let ω be the factor set of G and α be the factor set of G/N , where N is a normal subgroup of G such that there is a K -algebra homomorphism $f: K^\omega G \rightarrow K^\alpha(G/N)$. Then f is called the **natural homomorphism** if for all $x \in G$, $\text{Supp}f(\bar{x}) = \overline{xN}$.

Of course, at the expense of replacing ω by an equivalent factor set, the latter equality can be strengthened by $f(\bar{x}) = \overline{xN}$. Indeed, write $f(\bar{x}) = \lambda(x)\overline{xN}$ for some $\lambda(x) \in K^*$. Then setting $\tilde{x} = \lambda(x)^{-1}\bar{x}$, we have $f(\tilde{x}) = \overline{xN}$. The latter, however, does not imply that the same can be achieved leaving ω unchanged and replacing α by an equivalent factor set. This is so since the values of $\lambda(x)$ need not be constant on the cosets of N .

Lemma 36. ([13]) Let N be a normal subgroup of G , let ω be a factor set of G and α a factor set of G/N . Then $K^\omega G$ admits the natural homomorphism $f: K^\omega G \rightarrow K^\alpha(G/N)$ if and only if ω is equivalent to $\text{inf } \alpha$.

Proof. Suppose that f is the natural homomorphism. By the foregoing, replacing ω by an equivalent factor set yields $f(\bar{x}) = \overline{xN}$. Thus ω is equivalent with $\text{inf } \alpha$, by Theorem 2.7 (i), [13].

Conversely, assume that ω is equivalent with $\text{inf } \alpha$, say $\omega = (\delta\lambda)\text{inf } \alpha$ for some $\lambda: G \rightarrow K^*$. Setting $\tilde{x} = \lambda(x)\bar{x}$, it follows that $\tilde{x}\tilde{y} = \text{inf } \alpha(x, y)\tilde{x}\tilde{y}$, $x, y \in G$. Applying Theorem 2.7 (i), [13], we deduce that the map $\tilde{x} \mapsto \overline{xN}$ extends to a K -algebra homomorphism $f: K^\omega G \rightarrow K^\alpha(G/N)$. Since $f(\bar{x}) = \lambda(x)^{-1}\overline{xN}$, we have $\text{Supp}f(\bar{x}) = \overline{xN}$ for all $x \in G$ as required. \square

Definition 37. ([13]) Let $1 \rightarrow A \rightarrow G^* \rightarrow G \rightarrow 1$ be a central group extension and let Γ be a linear representation of G^* on the vector space V . Assume that for all $a \in A$, $\Gamma(a)$ is a scalar multiple of the identity transformation. If μ is a section of f , then the map $\rho: G \rightarrow GL(V)$ defined by $\rho(x) = \Gamma(\mu(x))$ is easily seen to be a projective representation. We shall refer to ρ as a projective representation **lifted** to Γ . We shall also say that a projective representation ρ **can be lifted** to G^* if ρ is lifted to Γ for some linear representation Γ of G^* .

We denote the torsion subgroup of K^* by $t(K^*)$. We fix $c \in H^2(G, A)$ together with a finite central extension $E: 1 \rightarrow A \rightarrow G^* \rightarrow G \rightarrow 1$ associated with it. If μ is a section of f , then the map $\beta: G \times G \rightarrow A$ defined by $\beta(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}$ is a factor set whose cohomology class $\bar{\beta}$ coincides with c . Given $\chi \in \text{Hom}(A, K^*)$, the map $\lambda: G \times G \rightarrow K^*$ defined by $\lambda(x, y) = \chi(\beta(x, y))$ is a factor set such that $\bar{\lambda}$ is the image of χ under the transgression map $\text{Tra}: \text{Hom}(A, K^*) \rightarrow H^2(G, K^*)$ associated with c .

Expressed otherwise, $\text{Tra}(\chi) = \bar{\chi}(c)$, where $\bar{\chi}$ is the natural homomorphism $H^2(G, A) \rightarrow H^2(G, K^*)$ induced by χ .

Because A is finite, so is $\text{Hom}(A, K^*)$ and therefore $\text{Tra}(\chi) \in H^2(G, t(K^*))$. Consequently, we may regard the transgression map as a homomorphism

$$\tau: \text{Hom}(A, K^*) \rightarrow H^2(G, t(K^*)).$$

Lemma 38. ([13]) *A projective representation $\rho: G \rightarrow GL(V)$ with the factor set ω is projectively equivalent to one that can be lifted to G^* if and only if $\bar{\omega} \in \text{Im}\tau$.*

Proof. We may assume by Lemma 26 i) that ρ is lifted to a linear representation Γ of G^* . Then $\Gamma(a) = \chi(a) \cdot 1_A$, $a \in A$ for some $\chi: A \rightarrow K^*$ and there is a section μ of f such that $\rho(x) = \Gamma(\mu(x))$ for all $x \in G$.

Because Γ is a linear representation, it follows that $\chi \in \text{Hom}(A, K^*)$.

Setting $\beta(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}$, $x, y \in G$, we deduce that $\omega(x, y) = \rho(x)\rho(y)\rho(xy)^{-1} = \Gamma(\mu(x))\Gamma(\mu(y))\Gamma(\mu(xy))^{-1} = \Gamma(\mu(x)\mu(y))\Gamma(\mu(xy))^{-1} = \Gamma(\beta(x, y)\mu(xy))\Gamma(\mu(xy))^{-1} = \chi\beta(x, y)$ whence $\bar{\omega} \in \text{Im}\tau$.

Conversely, assume that $\bar{\omega} \in \text{Im}\tau$.

Owing to Lemma 26 i), we may assume that there is a section μ of f such that for $\beta(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}$, $\omega(x, y) = \chi(\beta(x, y))$ for some $\chi \in \text{Hom}(A, K^*)$, $x, y \in G$.

Now each element y of G^* can be uniquely written in the form $y = a\mu(x)$ with $a \in A, x \in G$.

Define a map $\Gamma: G \rightarrow GL(V)$ by $\Gamma(y) = \chi(a)\rho(x)$. Then $\rho(x) = \Gamma(\mu(x))$, each $\Gamma(a)$, $a \in A$ is a scalar multiple of the identity transformation and $y_i = a_i\mu(x_i)$, $i = 1, 2, a_i \in A, x_i \in G$, we have $\Gamma(y_1y_2) = \Gamma(a_1a_2\beta(x_1, x_2)\mu(x_1, x_2)) = \chi(a_1)\chi(a_2)\omega(x_1, x_2)\rho(x_1, x_2) = \chi(a_1)\chi(a_2)\rho(x_1)\rho(x_2) = \Gamma(y_1)\Gamma(y_2)$.

□

Theorem 39. ([13]) *There is a finite central extension $1 \rightarrow A \rightarrow G^* \rightarrow G \rightarrow 1$ such that any projective representation of G with the factor set ω , $\bar{\omega} \in H^2(G, t(K^*))$ is projectively equivalent to one that can be lifted to G^* . Furthermore, if each $c \in H^2(G, t(K^*))$ contains a factor set whose order is equal to the order of c , then $A \cong H^2(G, t(K^*))$.*

Proof. Because $H^2(G, t(K^*))$ is a finite abelian group (Theorem 2.3.22, [13]), it can be written in the form $H^2(G, t(K^*)) = \langle c_1 \rangle \times \langle c_2 \rangle \times \dots \times \langle c_m \rangle$.

Let ω_i be a factor set in c_i . Since G is finite and since the values of ω_i are roots of 1, ω_i is of finite order d_i , $1 \leq i \leq m$. Let A_i be the group of all d_i -th roots of 1 in K^* , let $A = A_1 \times \dots \times A_m$. We may always choose ω_i such that $A_i \cong \langle c_i \rangle$ in which case $A \cong H^2(G, t(K^*))$.

Let $\chi_i: H^2(G, A_i) \rightarrow H^2(G, t(K^*))$ be the homomorphism induced by the natural injection $\chi_i: A_i \rightarrow t(K^*)$. Since ω_i is an A_i -valued factor set, there is $\beta_i \in H^2(G, A_i)$ such that $\bar{\chi}_i(\beta_i) = c_i$, $1 \leq i \leq m$. Let $\beta \in H^2(G, A)$ be the image

of $\beta_1 \times \beta_2 \times \dots \times \beta_m$ under the natural isomorphism $\prod_{i=1}^m H^2(G, A_i) \rightarrow H^2(G, A)$ and let $E : 1 \rightarrow A \rightarrow G^* \rightarrow G \rightarrow 1$ be a central extension associated with β . Due to Lemma 38, the result will follow once we verify that the transgression map $\tau : \text{Hom}(A, K^*) \rightarrow H^2(G, t(K^*))$ associated with β is surjective. Let $\tilde{\chi}_i \in \text{Hom}(A, K^*)$ be defined by $(a_1, a_2, \dots, a_n) \mapsto \chi_i(a_i), a_i \in A_i$. Then we see that $\tau(\tilde{\chi}_i) = c_i, i \leq i \leq m$. Because the c_i generate $H^2(G, t(K^*))$, the result follows. \square

Corollary 40. ([13]) *The following conditions are equivalent:*

- (i) *there is a finite central extension $1 \rightarrow A \rightarrow G^* \rightarrow G \rightarrow 1$ such that any projective representation of G is projectively equivalent to one that can be lifted to G^* ;*
- (ii) $H^2(G, K^*) = H^2(G, t(K^*))$;
- (iii) $K^* = (K^*)^m t(K^*)$, where m is the exponent of G/G' .

Proof. That (i) implies (ii) is a consequence of Lemma 38 and the converse is true by Theorem 39.

To prove that (ii) is equivalent to (iii), set $A = K^*/t(K^*)$. By Theorem 2.3.21, [13], (ii) holds if and only if $H^2(G, A) = 1$. The latter, in view of Lemma 2.3.19, [13] is equivalent to $A = A^n$ for all n such that \mathbb{Z}_n is a direct factor of G/G' . Since \mathbb{Z}_m is a direct factor of G/G' and since $A^m \subseteq A^n$ for all $n|m$, we conclude that (ii) holds if and only if $A^m = A$. Because the latter is equivalent to (iii), the result follows. \square

Let \mathbb{T} denote the multiplicative group of all complex numbers of unit modulus.

Let $\omega : G \times G \rightarrow \mathbb{T}$ be a factor set of G . We define a group structure G^ω on the set $G \times \mathbb{T}$ by requiring that

$$(x, z_1)(y, z_2) = (xy, \omega(x, y)z_1z_2)$$

for all $x, y \in G$ and for all $z_1, z_2 \in \mathbb{T}$.

G^ω is a group having a central subgroup $\mathbb{T}^0 = \{(e, z) \mid z \in \mathbb{T}\}$ isomorphic to \mathbb{T} such that G^ω/\mathbb{T}^0 isomorphic to G , where e is the identity of G . G^ω is called the **central extension of G by ω** . It is straightforward to show that there is a natural isomorphism between G and a subgroup of G^ω if and only if ω belongs to the trivial cohomology class.

Theorem 41. ([1]) *Let G be a finite group and ω a factor set of G . Then ω is equivalent to the trivial factor set if and only if $(G^\omega)' \cap \mathbb{T}^0 = \{(e, 1)\}$, where $(G^\omega)'$ is the commutator subgroup of G^ω .*

Proof. Suppose that ω is equivalent to the trivial factor set. That is, for all $x, y \in G$, $\omega(x, y) = \frac{\mu(xy)}{\mu(x)\mu(y)}$, for some \mathbb{T} -valued function μ . We find that the commutator of (x, z_1) and (y, z_2) is given by

$$(x, z_1)(y, z_2)(x, z_1)^{-1}(y, z_2)^{-1} = (xyx^{-1}y^{-1}, \mu(xyx^{-1}y^{-1})).$$

Furthermore, if c_1 and c_2 are two commutators in G' , we have

$$(c_1, \mu(c_1))(c_2, \mu(c_2)) = (c_1c_2, \mu(c_1c_2)).$$

Thus G' and $(G^\omega)'$ are isomorphic under the map $(c, \mu(c)) \longleftrightarrow c$.

Hence $(G^\omega)' \cap \mathbb{T}^0 = \{(e, 1)\}$.

Conversely, suppose $(G^\omega)' \cap \mathbb{T}^0 = \{(e, 1)\}$. Let π denote the natural epimorphism of G^ω onto the abelian factor group $G^\omega / (G^\omega)'$ and let $\pi_{\mathbb{T}^0}$ denote the restriction of π to \mathbb{T}^0 . By the first isomorphism theorem the image of \mathbb{T}^0 under $\pi_{\mathbb{T}^0}$ is isomorphic to $\mathbb{T}^0 / \text{Ker} \pi_{\mathbb{T}^0}$. Thus $\text{Im} \pi_{\mathbb{T}^0} \cong \mathbb{T}^0 / \text{Ker} \pi_{\mathbb{T}^0} = \mathbb{T}^0 / (\text{Ker} \pi) \cap \mathbb{T}^0 = \mathbb{T}^0 / (G^\omega)' \cap \mathbb{T}^0 = \mathbb{T}^0 / \{(e, 1)\} = \mathbb{T}^0$.

This shows that the abelian group $G^\omega / (G^\omega)'$ contains a subgroup isomorphic to \mathbb{T} , the circle group. Identifying this subgroup with \mathbb{T} , let us write $\pi(e, z) = z \in \mathbb{T}$.

We now claim that $G^\omega / (G^\omega)'$ has a linear character $\widehat{\chi}(z) = z$ for all $z \in \mathbb{T}$.

Finally χ defined by $\chi(x, z) = \widehat{\chi}[\{\pi(x, z)\}]$ is a linear character of G^ω , whereupon $\omega(x, y) = \frac{\lambda(x)\lambda(y)}{\lambda(xy)}$, where $\lambda(x) = \chi(x, 1)$ and ω is therefore equivalent to the trivial factor set. \square

Let G be a finite group, ω a factor set of G and K a field. In order to determine the number of orbits of the subgroup of the character group $\widehat{G} := H(G, K^*)$ on the classes of linearly equivalent irreducible representations with the factor set ω , we consider some definitions.

Definition 42. ([27]) *Let p be either zero or a prime number. A class of G is called p -regular if the order of the elements in this class is not divisible by p . A class is called (α, p) -regular if it is α -regular and p -regular.*

Let D be a finite group, A a group contained in the centre and in the commutator of D and $G = D/A$. For each $x \in G$ we choose an element $\bar{x} \in D$ such that $A\bar{x} = x$. This yields a map $r: G \times G \rightarrow A$, where $r(x, y)\bar{x}\bar{y} = \bar{xy}$. Let F be an algebraically closed field of characteristic $p \geq 0$ and $\widehat{A} = \text{Hom}(A, F^*)$ be the character group of A . For all $\lambda \in \widehat{A}$ we define $\lambda^r: G \times G \rightarrow F$ by $\lambda^r(x, y) = \lambda(r(x, y))$ and it is easy to see that λ^r is a factor set of G .

Let R be an irreducible linear F -representation of D . From Schur's Lemma we obtain $R(a) = \lambda(a)I$ for all $a \in A$, where $\lambda \in \widehat{A}$ and I is the identity matrix.

For all $x \in G$ we define

$$(3.7) \quad P(x) = R(\bar{x})$$

and it is easy to see that P is an irreducible projective representation of G with factor set λ^r . For all $a\bar{x} \in D$, where $a \in A$, we have $R(a\bar{x}) = \lambda(a)P(x)$.

On the other hand, if P is an arbitrary irreducible representation of G with factor set λ^r , then R is an irreducible linear representation of D .

Two irreducible linear representations $R_1 = \lambda_1 P_1$ and $R_2 = \lambda_2 P_2$ of D are equivalent if and only if $\lambda_1 = \lambda_2$ and P_1 and P_2 are linearly equivalent. The projective representations of G which are obtained by (3.1) are called **linearizable** in D (with respect to the map r). So we have a one-to-one correspondence between the equivalency classes of irreducible linear representations of D and the classes of linearly equivalent irreducible projective representations of G which can be linearized in D .

Let C be a subgroup of \widehat{G} , the character group of G . We denote by $\pi(G, C, D)$ the number of orbits of C on the set of classes of linearly equivalent irreducible projective F -representations of G which can be linearized in D (with respect to r). The group C can be regarded as a group of characters of G and D and we denote by $K(C, G)$ and $K(C, D)$ the intersections of the kernels of all $c \in C$. Hence, we have $K(C, D)/A = K(C, G)$.

By $p(C, D)$ we denote number of all p -regular classes (of conjugate elements) of D which are contained in $K(C, D)$.

For all $\lambda \in \widehat{A}$ we denote by $\pi(C, \lambda^r)$ the number of orbits of C on the set of all classes of linearly equivalent irreducible projective F -representations of G with factor set λ^r and $q(C, \lambda^r)$ denotes the number of (λ^r, p) -regular classes of G which are contained in $K(C, G)$.

Lemma 43. ([27]) *Let G, D and C as above. Then we have $\pi(G, C, D) = p(C, D)$.*

Let $x \in G$ and $\tilde{x} \in D$ such that $x = A\tilde{x}$ and let \tilde{x}^D denote the class of \tilde{x} . We define the following subgroup of A : $U(x) = \{a \mid a \in A, a\tilde{x} \in \tilde{x}^D\}$. It is obvious that $U(x)$ does not depend on the choice of \tilde{x} .

Lemma 44. ([27]) *Let $\lambda \in \widehat{A}$. Then x is λ^r -regular if and only if $U(x)$ is contained in the kernel of λ .*

Lemma 45. ([27]) *We have $p(C, D) = \sum q(C, \lambda^r)$, where the sum is taken over all $\lambda \in \widehat{A}$.*

Lemma 46. ([27]) *If λ_1 and λ_2 are faithful characters of A , we have $\pi(C, \lambda_1^r) = \pi(C, \lambda_2^r)$.*

Theorem 47. ([27]) *Let G be a finite group and F an algebraically closed field of characteristic $p \geq 0$. Let α be a factor set of G over F and C a subgroup of the character group \widehat{G} . Then the number of orbits of C on the set of all classes of linearly equivalent irreducible projective representations with factor set α coincides with the number of (α, p) -regular classes of G which are contained in the kernels of all characters in C .*

Proof. We can assume without loss of generality that $\alpha = \lambda^r$ for a $\lambda \in A$.

We prove the theorem by induction on the order of D . It is obvious that the irreducible projective representations of G with factor set λ^r can also be linearized in $D/\text{kernel}(\lambda)$. Thus, the induction hypothesis yields that λ is a faithful character of A , which implies that A is cyclic.

Let $\lambda = \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_s$ be the faithful characters of A and let $\sigma_1, \dots, \sigma_t$ be the non-faithful ones. By Lemma 44 we obtain for all $i = \overline{1, s}$, $\pi(C, \lambda_i^r) = \pi(C, \lambda^r)$.

Thus we have $\pi(G, C, D) = \sum_{j=1}^t \pi(C, \sigma_j^r) + \sum_{i=1}^s \pi(C, \lambda_i^r) = \sum_{j=1}^t \pi(C, \sigma_j^r) + s \cdot \pi(C, \lambda^r)$.

From Lemma 44 we obtain $q(C, \lambda_i^r) = q(C, \lambda^r)$ for all i . By Lemma 45 we have $p(C, D) = \sum_{j=1}^t q(C, \sigma_j^r) + \sum_{i=1}^s q(C, \lambda_i^r) = \sum_{j=1}^t q(C, \sigma_j^r) + s \cdot q(C, \lambda^r)$.

For the non-faithful characters σ_j we obtain by the induction hypothesis that $\pi(C, \sigma_j^r) = q(C, \sigma_j^r)$ holds. Hence we obtain by Lemma 43 that $\pi(C, \lambda^r) = q(C, \lambda^r)$ is valid. \square

Corollary 48. ([27]) *Let G be a finite group, F an algebraically closed field of characteristic $p \geq 0$ and α a factor set of G over F . Then the number of classes of linearly equivalent irreducible projective representations of G with factor set α equals the number of (α, p) -regular classes of G .*

Corollary 49. ([27]) *Let G and α be as in Corollary 48. Then the number of classes of projectively equivalent irreducible projective representations of G with factor set α coincides with the number of (α, p) -regular classes of G which are contained in the commutator subgroup of G .*

Let G be a finite abelian group of order n^m generated by m elements g_1, g_2, \dots, g_m of order n , i.e. $G \cong \mathbb{Z}_n \times \dots \times \mathbb{Z}_n$ (m copies), where \mathbb{Z}_n is a cyclic group of order n . Let π be a projective representation of G with factor set ω over \mathbb{C} .

Let $\mu(i) = \prod_{j=1}^{n-1} \omega(g_i^j, g_i)$, $i = \overline{1, m}$ and $\beta(i, j) = \omega(g_i, g_j) \omega^{-1}(g_j, g_i)$, $1 \leq i < j \leq m$.

It can be shown that the factor set ω can be chosen such that $\mu(i) = 1, i = \overline{1, m}$ and $\beta(i, j), 1 \leq i < j \leq m$ is an n -th root of unity.

Theorem 50. ([15]) *Let G be a finite abelian group of order n^m generated by g_1, g_2, \dots, g_m and let ω be a factor set of G over \mathbb{C} such that $\mu(i) = 1, i = \overline{1, m}$ and $\beta(i, j) = \varepsilon, 1 \leq i < j \leq m$, where ε is a primitive n -th root of unity. Then, if $m = 2\mu$ is even, G has only one inequivalent irreducible projective representation of degree n^μ and if $m = 2\mu + 1$ is odd, G has n inequivalent irreducible projective representations of degree n^μ .*

Proof. Let π be a projective representation of G with the factor set ω and let $\pi(g_i) = e_i, i = \overline{1, m}$. Then we must determine the number of elements $a = e_{s_1}^{\alpha_1} \dots e_{s_r}^{\alpha_r}$, where $1 \leq s_1 < s_2 < \dots < s_r \leq m, 0 \leq \alpha_i \leq n - 1, i = \overline{1, r}$ such that $e_i^{-1}ae_i = a, i = \overline{1, m}$.

In particular, we must have $e_{s_i}^{-1}ae_{s_i} = a, i = \overline{1, r}$.

It is easy to verify that $e_{s_i}^{-1}ae_{s_i} = \varepsilon^{(\alpha_1 + \dots + \alpha_{i-1} - \alpha_{i+1} - \dots - \alpha_r)}a, i = \overline{1, r}$ and so we

have $AX \equiv 0 \pmod{n}$, where A is the $r \times r$ matrix $\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & \dots & -1 & -1 \end{pmatrix}$ and

$$X^t = (\alpha_1, \dots, \alpha_r).$$

An easy calculation shows that

$$\det A = \begin{cases} 0, & \text{if } r \text{ is odd} \\ (-1)^\lambda, & \text{if } r = 2\lambda \text{ is even} \end{cases}$$

Thus, if r is even, the only solution is the trivial solution $\alpha_1 = \alpha_2 = \dots = \alpha_r = 0$.

On the other hand, if $r = 2\lambda + 1$ is odd, the above system of linear congruences reduces to $\alpha_1 \equiv -\alpha_2 \equiv \alpha_3 \equiv \dots \equiv -\alpha_{2\lambda} \equiv \alpha_{2\lambda+1} \pmod{n}$,

Thus, if $g_{s_1}^{\alpha_1} \dots g_{s_r}^{\alpha_r}$ is an ω -regular element, it can only take the form $g_{s_1}^i g_{s_2}^{-i} \dots g_{s_{r-1}}^{-i} g_{s_r}^i, i = \overline{0, n-1}$. But we must also have that when $s \neq s_i, e_s^{-1}ae_s = a$.

If $s \neq s_i, i = \overline{1, r}$ and we put $s_0 = 1, s_{r+1} = m$, then $s > s_j$ and $s < s_{j+1}$ for some $0 \leq j \leq r + 1$ and $e_s^{-1}ae_s = \varepsilon^{(\alpha_1 + \dots + \alpha_j - \alpha_{j+1} - \dots - \alpha_r)}a = \varepsilon^{\pm i}a$, for some $0 \leq i \leq n - 1$. That is, if $r < m, i = 0$. Thus, it follows that if $m = 2k$ is even, 1 is the only ω -regular element and if $m = 2k + 1$ is odd, the ω -regular elements are given by $g_1^i g_2^{-i} \dots g_{2k}^{-i} g_{2k+1}^i, i = \overline{0, n-1}$.

When $m = 2\mu$ is even, G has only one inequivalent irreducible projective representation, whose degree must be n^μ .

When $m = 2\mu + 1$ is odd, the n inequivalent projective representations have the same degree which is n^μ . □

Theorem 51. ([15]) *Let G be as in Theorem 50, $n = 2\nu$ and the factor set ω satisfies $\mu(i) = 1, i = \overline{1, m}$ and $\beta(i, j) = -1, 1 \leq i < j \leq m$. Then, if $m = 2\mu$ is even, G has ν^m inequivalent irreducible projective representations of degree 2^μ and if $m = 2\mu + 1$ is odd, G has $2\nu^m$ inequivalent irreducible projective representations of degree 2^μ .*

Proof. The proof is similar to the proof of Theorem 50, replacing ε by -1. Again, in this case, let $\pi(g_i) = e_i, i = \overline{1, m}$.

It is easily verified that if r is even $g_{s_1}^{\alpha_1} \dots g_{s_r}^{\alpha_r}$ is ω -regular if and only if $\alpha_i \equiv 0 \pmod{2}$, that is $\alpha_i = 0, 2, 4, \dots, n - 2$.

If r is odd and $a = e_{s_1}^{\alpha_1} \dots e_{s_r}^{\alpha_r}, e_{s_i}^{-1}ae_{s_i} = a, i = \overline{1, r}$ implies that $\alpha_1 \equiv \alpha_2 \equiv \dots \equiv \alpha_r \pmod{2}$.

If $s \neq s_i$, for any $i = \overline{1, r}$, then $e_s^{-1}ae_s = a$ implies that $\alpha_1 + \dots + \alpha_r \equiv 0 \pmod{2}$ or $r\alpha_r \equiv 0 \pmod{2}$.

Since r is odd, this means that $\alpha_r \equiv 0 \pmod{2}$. Thus, if r is odd and $r < m$, $g_{s_1}^{\alpha_1} \dots g_{s_r}^{\alpha_r}$ is ω -regular if and only if $\alpha_i = 0, 2, 3, \dots, n-2$.

Hence, if m is even, the ω -regular are $g_1^{\alpha_1} \dots g_m^{\alpha_m}$, where $\alpha_i = 0, 2, \dots, n-2$, $i = \overline{1, m}$, that is, there are $\left(\frac{n}{2}\right)^m$ ω -regular classes. If m is odd, the ω -regular elements are $g_1^{\alpha_1} \dots g_m^{\alpha_m}$, where $\alpha_i = 0, 2, \dots, n-2$, $i = \overline{1, m}$ and $\alpha_i = 1, 3, \dots, n-1$, $i = \overline{1, m}$, that is, there are $2\left(\frac{n}{2}\right)^m$ ω -regular classes.

Since all irreducible projective representations of an abelian group (with a fixed factor set) have the same degree, it is easily verified that these representations have the degrees given in the statement of the theorem. \square

Definition 52. ([22]) Let ω be a factor set of G and ρ an irreducible projective representation of a normal subgroup H of G with the factor set ω . Define

$$\rho^{(g)}(h) = f_\omega(g, h)\rho(ghg^{-1})$$

for all $g \in G, h \in H$.

Lemma 53. ([22]) $\rho^{(g)}$ is an irreducible projective representation of H with the factor set ω .

Proof. Applying Definition 52, Definition 1 i), Definition 10 and the fact that ρ is a projective representation, we obtain

$$\begin{aligned} \rho^{(g)}(h)\rho^{(g)}(h_1) &= f_\omega(g, h)\rho(ghg^{-1})f_\omega(g, h_1)\rho(gh_1g^{-1}) = \\ &= f_\omega(g, h)f_\omega(g, h_1)\omega(ghg^{-1}, gh_1g^{-1})\rho(ghg^{-1}gh_1g^{-1}) = \\ &= f_\omega(g, h)f_\omega(g, h_1)\omega(ghg^{-1}, gh_1g^{-1})\rho(ghh_1g^{-1}) = \\ &= \omega(g, h)\omega(ghg^{-1}, g)^{-1}\omega(g, h_1)\omega(gh_1g^{-1}, g)^{-1}\omega(ghg^{-1}, gh_1g^{-1})\rho(ghh_1g^{-1}) = \\ &= \omega(g, h)\omega(ghg^{-1}, g)^{-1}\omega(g, h_1)\omega(gh_1g^{-1}, g)^{-1}\omega(ghg^{-1}, gh_1g^{-1}g)\omega(gh_1g^{-1}, g) \cdot \\ &\cdot \omega(ghg^{-1}gh_1g^{-1}, g)^{-1}\rho(ghh_1g^{-1}) = \omega(g, h)\omega(ghg^{-1}, g)^{-1}\omega(g, h_1)\omega(gh_1g^{-1}, g)^{-1} \cdot \\ &\cdot \omega(ghg^{-1}, gh_1)\omega(gh_1g^{-1}, g)\omega(ghh_1g^{-1}, g)^{-1}\rho(ghh_1g^{-1}) = \\ &= \omega(g, h)\omega(ghg^{-1}, g)^{-1}\omega(g, h_1)\omega(ghg^{-1}, gh_1)\omega(ghh_1g^{-1}, g)^{-1}\rho(ghh_1g^{-1}) = \\ &= \omega(g, h)\omega(ghg^{-1}, g)^{-1}\omega(ghg^{-1}, g)\omega(gh, h_1)\omega(ghh_1g^{-1}, g)^{-1}\rho(ghh_1g^{-1}) = \\ &= \omega(g, h)\omega(gh, h_1)\omega(ghh_1g^{-1}, g)^{-1}\rho(ghh_1g^{-1}) = \\ &= \omega(g, hh_1)\omega(h, h_1)\omega(ghh_1g^{-1}, g)^{-1}\rho(ghh_1g^{-1}) = \\ &= \omega(h, h_1)f_\omega(g, hh_1)\rho(ghh_1g^{-1}) = \omega(h, h_1)\rho^{(g)}(hh_1) \\ \rho^{(g)}(1_H) &= f_\omega(g, 1_H)\rho(g1_Hg^{-1}) = f_\omega(g, 1_H)\rho(gg^{-1}) = f_\omega(g, 1_H)\rho(1_G) = \\ &= f_\omega(g, 1_H)I = \omega(g, 1_H)\omega(g1_Hg^{-1}, g)^{-1}I = \omega(gg^{-1}, g)^{-1}I = I \end{aligned}$$

Thus $\rho^{(g)}$ is a projective representation of H with the factor set ω , which is irreducible since ρ is irreducible. \square

Definition 54. ([22]) Two irreducible projective representations ρ_1 and ρ_2 of H are *conjugate* if $\rho_2 \simeq \rho_1^{(g)}$ for some $g \in G$, where " \simeq " denotes linear equivalence of representations.

Lemma 55. ([22]) Let ρ be an irreducible projective representation of H with the factor set ω and let $I_\rho = \{g \in G \mid \rho^{(g)} \simeq \rho\}$. I_ρ is a group called **the inertia group** of ρ and $F_\rho = I_\rho/H$ is called **the inertia factor** of ρ .

Theorem 56. ([22]) Let ρ, ω, I_ρ be as in Lemma 55 Then there is an irreducible projective representation $\tilde{\rho}$ of I_ρ with some factor set σ such that

- (i) $\tilde{\rho}(g)\rho(h)\tilde{\rho}(g)^{-1} = \rho^{(g)}(h)$;
- (ii) $\tilde{\rho}(h) = \rho(h)$;
- (iii) $\rho(h)\tilde{\rho}(g) = \omega(h, g)\tilde{\rho}(hg)$, for all $g \in I_\rho, h \in H$.

Proof. By definition of I_ρ , there is some matrix $\rho'(g)$ such that $\rho'(g)\rho(h)\rho'(g)^{-1} = \rho^{(g)}(h)$ for all $g \in I_\rho, h \in H$.

Let $g_1, g_2 \in G$. We have $\rho'(g_1g_2)\rho(h)\rho'(g_1g_2)^{-1} = \rho^{(g_1g_2)}(h) = \rho'(g_1)\rho'(g_2)\rho(h)(\rho'(g_1)\rho'(g_2))^{-1} = \rho'(g_1)\rho'(g_2)\rho(h)\rho'(g_2)^{-1}\rho'(g_1)^{-1}$ for all $h \in H$.

So $\rho'(g_1g_2)^{-1}\rho'(g_1)\rho'(g_2)\rho(h) = \rho(h)\rho'(g_1g_2)^{-1}\rho'(g_1)\rho'(g_2)$.

By Schur lemma, since ρ is irreducible, there is an element $\sigma'(g, g_1) \neq 0$ in \mathbb{C} such that $\rho'(g_1g_2)^{-1}\rho'(g_1)\rho'(g_2) = \sigma'(g, g_1)I \implies \rho'(g_1)\rho'(g_2) = \sigma'(g, g_1)\rho'(g_1g_2)$, where σ' is some factor set of I_ρ .

Since $e \in H$, we may take $\rho'(e) = I$, and we have proved that ρ' is a projective representation of G .

Now choose a set $\{x_1, x_2, \dots, x_n\}$ of right coset representatives of H in I_ρ and define

$$(3.8) \quad \tilde{\rho}(hx_i) = \omega(h, x_i)^{-1}\rho(h)\rho'(x_i) \quad (\tilde{\rho}(h) = \rho(h))$$

for all $h \in H, i = 1, \dots, n$.

It is easy to check that $\tilde{\rho}$ is a projective representation of I_ρ with some factor set σ satisfying (i) and (ii). The fact that $\tilde{\rho}$ is irreducible follows as in the linear case ([6])

Further, by (3.1), $\tilde{\rho}(h(h_1x_i)) = \tilde{\rho}(hh_1x_i) = \omega(hh_1, x_i)^{-1}\rho(hh_1)\rho'(x_i) = \omega(hh_1, x_i)^{-1}\omega(h, h_1)^{-1}\rho(h)\rho(h_1)\rho'(x_i) = \rho(h)\omega(h, h_1)^{-1}\omega(hh_1, x_i)^{-1}\rho(h_1)\rho'(x_i) = \rho(h)\omega(h, h_1x_1)^{-1}\omega(h_1, x_i)^{-1}\rho(h_1)\rho'(x_i) = \omega(h, h_1x_1)^{-1}\rho(h)\tilde{\rho}(h_1x_i)$.

We obtain $\rho(h)\tilde{\rho}(g) = \omega(h, g)\tilde{\rho}(hg)$ for all $g \in I_\rho, h \in H$. □

Lemma 57. ([22]) If ω and σ are the factor sets defined in Theorem 56, the factor set $\omega\sigma^{-1}$ of I_ρ satisfies $\omega\sigma^{-1}(hg, h_1g_1) = \omega\sigma^{-1}(gg_1)$ for all $g, g_1 \in I_\rho, h, h_1 \in H$.

Proof. By Theorem 56, (iii), $\tilde{\rho}(hg)\tilde{\rho}(h_1g_1) = \sigma(hg, h_1g_1)\tilde{\rho}(hgh_1g_1) = \sigma(hg, h_1g_1)\omega(hgh_1g^{-1}, gg_1)^{-1}\rho(hgh_1g^{-1})\tilde{\rho}(gg_1)$

But $\tilde{\rho}(hg)\tilde{\rho}(h_1g_1) = \omega(h, g)^{-1}\rho(h)\tilde{\rho}(g)\omega(h_1, g_1)^{-1}\rho(h_1)\tilde{\rho}(g_1) = \omega(h, g)^{-1}\omega(h_1, g_1)^{-1}\rho(h)\tilde{\rho}(g)\rho(h_1)\tilde{\rho}(g_1) = \omega(h, g)^{-1}\omega(h_1, g_1)^{-1}\rho(h)\rho^{(g)}(h_1)\tilde{\rho}(g)\tilde{\rho}(g_1) =$

$$\begin{aligned}
&= \omega(h, g)^{-1} \omega(h_1, g_1)^{-1} \rho(h) f_\omega(g, h_1) \rho(gh_1g^{-1}) \tilde{\rho}(g) \tilde{\rho}(g_1) = \\
&= \omega(h, g)^{-1} \omega(h_1, g_1)^{-1} f_\omega(g, h_1) \rho(h) \rho(gh_1g^{-1}) \tilde{\rho}(g) \tilde{\rho}(g_1) = \\
&= \omega(h, g)^{-1} \omega(h_1, g_1)^{-1} f_\omega(g, h_1) \omega(h, gh_1g^{-1}) \rho(hgh_1g^{-1}) \sigma(g, g_1) \tilde{\rho}(gg_1) = \\
&= \sigma(g, g_1) \omega(h, g)^{-1} \omega(h_1, g_1)^{-1} f_\omega(g, h_1) \omega(h, gh_1g^{-1}) \rho(hgh_1g^{-1}) \tilde{\rho}(gg_1) \\
&\quad \text{Thus } \sigma(g, g_1) \omega(h, g)^{-1} \omega(h_1, g_1)^{-1} f_\omega(g, h_1) \omega(h, gh_1g^{-1}) = \\
&= \sigma(hg, h_1g_1) \omega(hgh_1g^{-1}, gg_1)^{-1} \implies \\
&\implies \sigma(g, g_1) \omega(h, g)^{-1} \omega(h_1, g_1)^{-1} \omega(g, h_1) \omega(gh_1g^{-1}, g)^{-1} \omega(h, gh_1g^{-1}) = \\
&= \sigma(hg, h_1g_1) \omega(hgh_1g^{-1}, gg_1) \implies
\end{aligned}$$

$$\begin{aligned}
(3.9) \quad &\omega(hgh_1g^{-1}, gg_1) \sigma(hg, h_1g_1)^{-1} \omega(g, h_1) \omega(h, gh_1g^{-1}) = \\
&\omega(h, g) \omega(h_1, g_1) \omega(gh_1g^{-1}, g) \sigma(g, g_1)^{-1}
\end{aligned}$$

By Definition 1, $\omega(h, gh_1g^{-1}) \omega(hgh_1g^{-1}, gg_1) = \omega(h, gh_1g^{-1}gg_1) \omega(gh_1g^{-1}, gg_1) =$
 $= \omega(h, gh_1g_1) \omega(gh_1g^{-1}, gg_1)$

$$\begin{aligned}
&\text{So, by Definition 1, (3.1) becomes } \omega(g, h_1) \omega(h, gh_1g_1) \omega(gh_1g^{-1}, gg_1) \sigma(hg, h_1g_1)^{-1} = \\
&= \omega(h, g) \omega(h_1, g_1) \omega(gh_1g^{-1}, g) \sigma(g, g_1)^{-1} \implies \\
&\implies \omega(g, h_1) \omega(h, g) \omega(hg, h_1g_1) \omega(g, h_1g_1)^{-1} \omega(gh_1g^{-1}, gg_1) \sigma(hg, h_1g_1)^{-1} = \\
&= \omega(h, g) \omega(h_1, g_1) \omega(gh_1g^{-1}, g) \sigma(g, g_1)^{-1} \implies \\
&\implies \omega(g, h_1) \omega(g, h_1g_1)^{-1} \omega(gh_1g^{-1}, gg_1) [\omega(hg, h_1g_1) \sigma(hg, h_1g_1)]^{-1} = \\
&= \omega(h_1, g_1) \omega(gh_1g^{-1}, g) \sigma(g, g_1)^{-1} \implies
\end{aligned}$$

$$\begin{aligned}
(3.10) \quad &\omega(g, h_1) \omega(g, h_1g_1)^{-1} \omega(gh_1g^{-1}, gg_1) \omega(g, g_1) \omega(hg, h_1g_1) \sigma(hg, h_1g_1)^{-1} = \\
&= \omega(h_1, g_1) \omega(gh_1g^{-1}, g) \omega(g, g_1) \sigma(g, g_1)^{-1}
\end{aligned}$$

To verify the relation in Lemma 57, it is sufficient to show that

$$(3.11) \quad \omega(g, h_1) \omega(g, h_1g_1)^{-1} \omega(gh_1g^{-1}, gg_1) \omega(g, g_1) = \omega(h_1, g_1) \omega(gh_1g^{-1}, g),$$

since (3.1) holds.

Relation (3.1) is equivalent with $\omega(g, h_1) \omega(gh_1g^{-1}, gg_1) \omega(g, g_1) =$
 $= \omega(h_1, g_1) \omega(g, h_1g_1) \omega(gh_1g^{-1}, g) \iff \omega(g, h_1) \omega(gh_1g^{-1}, g) \omega(gh_1g^{-1}g, g_1) =$
 $= \omega(h_1, g_1) \omega(g, h_1g_1) \omega(gh_1g^{-1}, g) \iff \omega(g, h_1) \omega(gh_1g^{-1}, g) \omega(gh_1, g_1) =$
 $= \omega(h_1, g_1) \omega(g, h_1g_1) \omega(gh_1g^{-1}, g) \iff \omega(g, h_1) \omega(gh_1, g_1) = \omega(h_1, g_1) \omega(g, h_1g_1)$, which
is true by Definition 1 i). \square

4 Projectively equivalent representations of particular groups

4.1 Direct products of cyclic groups of the same order

Let G be a cyclic group of order n . It is a consequence of Theorem 2.3.1, [13] that $H^2(G, \mathbb{C}^*) = 1$. Hence by Lemma 26 (ii) every projective representation of G is projectively equivalent to a linear representation. The following is a direct proof of

this fact. Let ω be a factor set of G in \mathbb{C} and let ρ be a projective representation of G with the factor set ω . If x is a generator for G , then $\rho(x)^n = \mu(x)I$, where I is the identity matrix and $\mu(x) = \omega(x, x)\omega(x^2, x) \dots \omega(x^{n-1}, x)$. For each $x \in G$, fix $\lambda(x) \in \mathbb{C}^*$ such that $\lambda(x)^n = \mu(x)^{-1}$ and define $\rho'(x) = \lambda(x)\rho(x)$ for all $x \in G$. Then $(\rho'(x))^n = \lambda(x)^n \rho(x)^n = \mu(x)^{-1} \mu(x)I = I$, so ρ' is a linear representation of G which is projectively equivalent to ρ . In what follows, G will denote the direct product $\langle x_1 \rangle \times \dots \times \langle x_m \rangle$ of m cyclic groups of order n .

We fix a factor set ω of G in \mathbb{C} and put

$$\mu(i) = \prod_{j=1}^{n-1} \omega(x_i^j, x_i), \quad i = \overline{1, m}$$

$$\beta(i, j) = \omega(x_i, x_j)\omega^{-1}(x_j, x_i), \quad 1 \leq i < j \leq m$$

Replacing ω by an equivalent factor set, if necessary, we may assume that $\mu(i) = 1$ for all $i = \overline{1, m}$ and that $\beta(i, j)$ is an n -th root of 1.

We introduce the following notations:

If n is odd, let P and Q be the $n \times n$ matrices defined by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \varepsilon & 0 & 0 & \dots & 0 \\ 0 & 0 & \varepsilon^2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \varepsilon^{n-1} \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\text{If } n \text{ is even, let } P \text{ be defined as above and } Q = \begin{pmatrix} 0 & \delta & 0 & 0 & \dots & 0 \\ 0 & 0 & \delta^3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \delta^{2n-3} \\ \delta^{2n-1} & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where δ is a primitive $2n$ -th root of 1 such that $\delta^2 = \varepsilon$. Put

$$R = \begin{cases} P^{n-1}Q, & \text{if } n \text{ is odd} \\ \delta P^{n-1}Q, & \text{if } n \text{ is even} \end{cases}$$

If m is even, say $m = 2k$, set

$$E_i = R \otimes R \otimes \dots \otimes R \otimes P \otimes I \otimes \dots \otimes I$$

$$E_{k+i} = R \otimes R \otimes \dots \otimes R \otimes Q \otimes I \otimes \dots \otimes I, \quad i = \overline{1, k}$$

where P and Q appear in the i -th position and each tensor product has k factors (here I denotes the identity matrix).

If m is odd, say $m = 2k + 1$, let

$$E_{2k+1} = R \otimes R \otimes \dots \otimes R \quad (k \text{ factors})$$

Theorem 58. ([13]) Let ω be a factor set.

- (i) If $m = 2k$ is even, then the map $\rho: G \rightarrow GL(n^k, \mathbb{C})$ defined by $\rho(x_i) = E_i, i = \overline{1, m}$ is the only up to linear equivalency irreducible projective representation of G with the factor set ω .
- (ii) If $m = 2k + 1$ is odd, then G has precisely n irreducible linearly nonequivalent projective representations with the multiplier ω , namely

$$\begin{cases} G \longrightarrow GL(n^k, \mathbb{C}) \\ x_i \longmapsto \varepsilon^j E_i \end{cases}$$

$$(j = \overline{0, n-1}, i = \overline{1, 2k+1})$$

Proof. A straightforward calculation shows that ρ and ρ_j , where $\rho_j(x_i) = \varepsilon^j E_i, j = \overline{0, n-1}$ are projective representations of G with factor set ω for the cases (i) and (ii).

Furthermore, the representations ρ_j are inequivalent. Indeed, for all $j = \overline{0, n-1}$,

$$\begin{aligned} (\varepsilon^j E_1)^{n-1} (\varepsilon^j E_{k+1}) (\varepsilon^j E_2)^{n-1} (\varepsilon^j E_{k+2}) \dots (\varepsilon^j E_k)^{n-1} (\varepsilon^j E_{2k}) (\varepsilon^j E_{2k+1})^{n-1} = \\ = \varepsilon^{j(n-1)} I_{n^k} = \varepsilon^{-j} I_{n^k}. \end{aligned}$$

Thus the element $\overline{x_1^{n-1} x_{k+1} \overline{x_2^{n-1} x_{k+2} \dots x_k^{n-1} x_{2k} x_{2k+1}^{n-1}}$ of $\mathbb{C}^\omega G$ if n is odd (respectively, $\delta^k \overline{x_1^{n-1} x_{k+1} \dots x_{2k+1}^{n-1}}$ if n is even) is represented by a distinct scalar matrix for each $j = \overline{0, n-1}$, which shows that the ρ_j are linearly inequivalent.

We now claim that in the case (i), 1 is the only ω -regular element of G , while in case (ii) there are exactly n such elements. Once this is established the result will follow in view of these considerations:

Suppose that in case (i), 1 is the only ω -regular element of G . By Theorem 3.6.7, [13] we conclude that $\mathbb{C}^\omega G \cong M_d(\mathbb{C})$ for some $d \geq 1$. Comparing the \mathbb{C} -dimensions of both sides yields $\mathbb{C}^\omega G \cong M_{n^k}(\mathbb{C})$ which establishes the case $m = 2k$.

Suppose that in case (ii) there are exactly n ω -regular elements of G . By Theorem 7.9.5 (i), [13], all irreducible projective representations of G with the factor set ω are projectively equivalent and hence have the same dimension. By Theorem 3.6.7, [13], $\mathbb{C}^\omega G \cong M_{n^k}(\mathbb{C}) \times \dots \times M_{n^k}(\mathbb{C})$ (n factors) and therefore we are left to sustain our claim.

Let ρ be a projective representation of G with the factor set ω and let $\rho(x_i) = e_i, i = \overline{1, m}$. Then we must determine the number of elements $a = e_{k_1}^{\omega_1} \dots e_{k_r}^{\omega_r}$, where $1 \leq k_1 < k_2 < \dots < k_r \leq m, 0 \leq \omega_i \leq n-1, i = \overline{1, r}$ such that $e_i^{-1} a e_i = a, i = \overline{1, m}$.

In particular, we must have $e_{k_i}^{-1} a e_{k_i} = a, i = \overline{1, r}$.

A straightforward calculation shows that $e_{k_i}^{-1} a e_{k_i} = \varepsilon^{(\omega_1 + \dots + \omega_{i-1} - \omega_{i+1} - \dots - \omega_r)} a, i = \overline{1, r}$ and thus we must have $AX \equiv 0 \pmod{n}$, where A is the $r \times r$ matrix

$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & \dots & -1 & -1 \end{pmatrix}$ and $X^t = (\omega_1, \dots, \omega_r)$. An easy calculation shows that

$$\det A = \begin{cases} 0, & \text{if } r \text{ is odd} \\ (-1)^\lambda, & \text{if } r = 2\lambda \text{ is even} \end{cases}$$

It therefore follows that if r is even, then the only solution is the trivial solution $\omega_1 = \omega_2 = \dots = \omega_r = 0$. On the other hand, if $r = 2\lambda + 1$ is odd, the system of linear congruences above reduces to $\omega_1 \equiv -\omega_2 \equiv \omega_3 \equiv \dots - \omega_{2\lambda} \equiv \omega_{2\lambda+1} \pmod{n}$.

The conclusion is that if $x_{k_1}^{\omega_1} \dots x_{k_r}^{\omega_r}$ is a ω -regular element, it must be of the form $x_{k_1}^i x_{k_2}^{-i} \dots x_{k_{r-1}}^{-i} x_{k_r}^i, i \in \overline{0, n-1}$.

But for $k \neq k_i$ we must also have $e_k^{-1} a e_k = a$.

If $k \neq k_i$ for $i = \overline{1, r}$ and we put $k_0 = 1, k_{r+1} = m$, then $k > k_j$ and $k < k_{j+1}$ for some $j = \overline{0, r+1}$ and $e_k^{-1} a e_k = \varepsilon^{(\omega_1 + \dots + \omega_{i-1} - \omega_{i+1} - \dots - \omega_r)} a = \varepsilon^{\pm i} a$ for suitable $i \in \overline{0, n-1}$. In other words, if $r < m$, then $i = 0$. Consequently, if $m = 2k$ is even, 1 is the only ω -regular element and if $m = 2k + 1$ is odd, the ω -regular elements of G are given by $x_1^i x_2^{-i} \dots x_{2k}^{-i} x_{2k+1}^i, i \in \overline{0, n-1}$.

Thus we have sustained our claim and the result follows. □

In what follows G will denote the group in Theorem 58 and $n = 2k$ is even.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and for $i \in \overline{1, t}$, let $M_{2i-1} = D \otimes D \otimes \dots \otimes D \otimes B \otimes A \otimes \dots \otimes A$

$M_{2i} = D \otimes D \otimes \dots \otimes D \otimes C \otimes A \otimes \dots \otimes A$

$M_{2t+1} = D \otimes D \otimes \dots \otimes D \otimes D \otimes D \otimes \dots \otimes D$ be tensor products, where B and C are in the i -th position and each product has t factors. If $m = 2t$ and ε is a primitive $n = 2k$ -th root of 1, for $i \in \overline{1, m}$ and $0 \leq \lambda_i < k$, define $\rho_{(\lambda_1, \dots, \lambda_m)}(g_i) = \varepsilon^{\lambda_i} M_i$.

Theorem 59. ([13]) *The following properties hold:*

(i) *If $m = 2t$ is even, then G has precisely k^m linearly nonequivalent irreducible projective representations with factor set ω , namely $G \mapsto GL(2^t, \mathbb{C})$,*

$g_i \mapsto \rho_{(\lambda_1, \dots, \lambda_m)}(g_i), 0 \leq \lambda_i < k, i \in \overline{1, m}$.

(ii) *If $m = 2t + 1$ is odd, then G has precisely $2k^m$ linearly nonequivalent irreducible projective representations with the factor set ω , namely $G \mapsto GL(2^t, \mathbb{C})$,*

$g_i \mapsto \pm \rho_{(\lambda_1, \dots, \lambda_m)}(g_i), 0 \leq \lambda_i < k, i \in \overline{1, m}$.

Proof. An easy calculation shows that the formulas of (i) and (ii) define projective representations with the factor set ω of G and that these representations are linearly inequivalent. Applying the argument employed in the proof of Theorem 58, it is sufficient to show that if m is even, then there are exactly $(\frac{n}{2})^m$ ω -regular elements in G , while in the case where m is odd the number of such elements is $2(\frac{n}{2})^m$.

Let ρ be a projective representation of G with the factor set ω and let $\rho(g_i) = e_i, i = \overline{1, m}$. A straightforward calculation shows that if r is even, then $g_{k_1}^{\alpha_1} \dots g_{k_r}^{\alpha_r}$ is ω -regular if and only if $\alpha_i \equiv 0 \pmod{2}$, that is $\alpha_i = \overline{0, n-2}$. If r is odd and $a = e_{k_1}^{\alpha_1} \dots e_{k_r}^{\alpha_r}$, then $e_{k_i}^{-1} a e_{k_i} = a, i = \overline{1, r}$ implies that $\alpha_i \equiv \alpha_2 \equiv \dots \equiv \alpha_r \pmod{2}$.

If $s \neq k_i$ for any $i = \overline{1, r}$, then $e_s^{-1} a e_s = a$ implies that $\alpha_1 + \dots + \alpha_r \equiv 0 \pmod{2}$ or $r\alpha_r \equiv 0 \pmod{2}$.

Because r is odd, the latter implies that $\alpha_r \equiv 0 \pmod{2}$. Consequently, if r is odd and $r < m$, then $g_{k_1}^{\alpha_1} \dots g_{k_r}^{\alpha_r}$ is ω -regular if and only if $\alpha_1 = 0, 2, 3, \dots, n-2$. Hence, if m is even, the ω -regular elements are $g_1^{\alpha_1} \dots g_m^{\alpha_m}$, where $\alpha_i = 0, 2, \dots, n-2, i = \overline{1, m}$, proving that there are exactly $(\frac{n}{2})^m$ ω -regular elements in G .

Finally, if m is odd, the ω -regular elements are $g_1^{\alpha_1} \dots g_m^{\alpha_m}$, where $\alpha_i = 0, 2, \dots, n-2, i = \overline{1, m}$ and $\alpha_i = 1, 3, \dots, n-1$, proving that there are exactly $2(\frac{n}{2})^m$ ω -regular elements in G . \square

4.2 Dihedral groups

Let D_n be the dihedral group of order $2n$ defined by $D_n = \langle a, b \mid a^n = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle$. Let ε be a primitive n -th root of 1 and let $\omega: D_n \times D_n \rightarrow \mathbb{C}^*$ be defined by $\omega(a^i, a^j b^k) = 1$ and $\omega(a^i b, a^j b^k) = \varepsilon^j$ for all $i, j = \overline{0, n-1}$ and $k = 0, 1$.

If n is even, for each $r \in \{0, \dots, \frac{n}{2}\}$ put $A_r = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^{1-r} \end{pmatrix}, B_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and let $\rho_r: D_n \rightarrow GL(2, \mathbb{C})$ be defined by $\rho_r(a^i b^j) = A_r^i B_r^j, i, j = \overline{0, n-1}$.

Theorem 60. ([13])

- (i) If n is odd, then every projective representation of D_n in \mathbb{C} is projectively equivalent to a linear representation.
- (ii) Suppose that n is even. Then for a factor set ω of D_n , the following properties hold:
 - (a) $\rho_1, \rho_2, \dots, \rho_{\frac{n}{2}}$ are all linearly nonequivalent irreducible projective representations of D_n with the factor set ω .
 - (b) There are exactly $\frac{n}{2}$ ω -regular classes of D_n , namely $\{1\}, \{a, a^{-1}\}, \{a^2, a^{-2}\}, \dots, \{a^{\frac{n}{2}-1}, a^{-\frac{n}{2}+1}\}$.
 - (c) The elements $z_0 = \overline{1}$ and $z_i = \overline{a^i} + \varepsilon^i \overline{a^{-i}}, 1 \leq i \leq \frac{n}{2} - 1$ constitute a \mathbb{C} -basis for the center of $\mathbb{C}^\omega D_n$.
 - (d) Every irreducible projective representation of D_n is either projectively equivalent to a linear representation or projectively equivalent to a projective representation ρ_i with the factor set ω for a suitable $i \in \{1, \dots, \frac{n}{2}\}$.
- (iii) For any factor set β of D_n in \mathbb{C} either $\mathbb{C}^\beta D_n \cong \mathbb{C} D_n$ or n is even and $\mathbb{C}^\beta D_n \cong \mathbb{C}^\omega D_n$

Proof. (i) This is a consequence of Proposition 4.6.4, [13].

Let β be a factor set of D_n and let ρ be a projective representation of D_n with the factor set β . Then there are $\lambda, \mu, \nu \in \mathbb{C}^*$ such that $(\rho(a))^n = \lambda I, (\rho(b))^2 = \mu I, \rho(b)\rho(a)\rho(b)^{-1} = \nu\rho(a)^{-1}$.

Replacing $\rho(a)$ by $\lambda^{-\frac{1}{n}}\rho(a)$ and $\rho(b)$ by $\mu^{-\frac{1}{2}}\rho(b)$ if necessary, we may assume that

$$(4.1) \quad (\rho(a))^n = I, (\rho(b))^2 = I, \rho(b)\rho(a)\rho(b)^{-1} = \nu\rho(a)^{-1}$$

Raising the third equality to the n -th power and taking into account the first and the second equalities, we deduce that $\nu^n = 1$. Hence $\nu = \varepsilon^m$ for some positive integer m . If k is a positive integer, then replacement of $\rho(a)$ by $\varepsilon^{-k}\rho(a)$ does not change the first two equalities of (4.1) and the third becomes

$$(4.2) \quad \rho(b)\rho(a)\rho(b)^{-1} = \varepsilon^{m-2k}\rho(a)^{-1}$$

Since n is odd, $\langle \varepsilon^2 \rangle = \langle \varepsilon \rangle$ and hence $\varepsilon^m = \varepsilon^{2k}$ for some $k \geq 1$. Choosing the k above, we finally obtain from (4.1) and (4.2), $(\rho(a))^n = I, (\rho(b))^2 = I, \rho(b)\rho(a)\rho(b)^{-1} = \rho(a)^{-1}$ and the desired assertion follows.

(ii) Observe that if m of (4.1) is even, then ρ is projectively equivalent to a linear representation. In case m is odd, ν is a primitive n -th root of 1 and hence we may replace ν by ε in (4.1). Thus if ρ is not projectively equivalent to a linear representation, then up to projective equivalency, the matrices $\rho(a)$ and $\rho(b)$ satisfy the following equalities:

$$(4.3) \quad (\rho(a))^n = I, (\rho(b))^2 = I, \rho(b)\rho(a)\rho(b)^{-1} = \varepsilon\rho(a)^{-1}$$

In particular, if we put $\rho(a) = A_r$ and $\rho(b) = B_r$, then a straightforward calculation shows that A_r and B_r satisfy (4.1); hence for each $r \in \{1, \dots, \frac{n}{2}\}$, ρ_r is a projective representation and in fact an easy calculation shows that each ρ_r is a projective representation with the factor set ω .

Observe that $a^{\frac{n}{2}} \in Z(D_n)$ and that $\omega(a^{\frac{n}{2}}, b) = 1$ and $\omega(b, a^{\frac{n}{2}}) = \varepsilon^{\frac{n}{2}} \neq 1$. Hence $a^{\frac{n}{2}}$ is not ω -regular, so ω is not a coboundary. It follows that all irreducible projective representations with the factor set ω of D_n have degree ≥ 2 and therefore each ρ_r is irreducible. By looking at the restriction ρ_r to $\langle a \rangle$, it follows that $\rho_1, \dots, \rho_{\frac{n}{2}}$ are

linearly nonequivalent. Since $\sum_{i=1}^{\frac{n}{2}} (deg \rho_i)^2 = 2n = |D_n|$, we conclude that $\rho_1, \dots, \rho_{\frac{n}{2}}$ are all linearly nonequivalent irreducible projective representations of D_n with the factor set ω , proving (a).

By Theorem 6.7, [13], D_n has exactly $\frac{n}{2}$ ω -regular classes. Note that for each $i \in \{1, \dots, \frac{n}{2} - 1\}$, $ba^i b^{-1} = a^{-i} \neq a^i$ for otherwise $a^{2i} = 1$, so $(a^i)^2 = 1$, contrary to the fact that $a^{\frac{n}{2}}$ is the only element of order 2 in $\langle a \rangle$. It follows that for each

$i \in \{1, \dots, \frac{n}{2} - 1\}$, $C(a^i) = \langle a \rangle$. Since for all $x \in \langle a \rangle$, $\omega(a^i, x) = \omega(x, a^i) = 1$ each a^i , $i \in \{1, \dots, \frac{n}{2}\}$ is ω -regular, proving (b).

It follows from (a) that $\dim_{\mathbb{C}} Z(\mathbb{C}^\omega D_n) = \frac{n}{2}$ and therefore we need only verify that for all $i \in \{1, \dots, \frac{n}{2} - 1\}$, $b(\bar{a}^i + \varepsilon^i \bar{a}^{-i})\bar{b}^{-1} = \bar{a}^i + \varepsilon^i \bar{a}^{-i}$. The latter being a consequence of the equalities $(\bar{a})^i = \bar{a}^i$ and $\bar{b} \cdot \bar{a}^i \cdot \bar{b}^{-1} = \varepsilon^i \bar{a}^{-i}$ ($0 \leq i \leq n - 1$), (c) follows.

Finally, (d) is a consequence of the isomorphism $H^2(D_n, \mathbb{C}^*) \cong \mathbb{Z}_2$ (which may be deduced from (4.1) or from Proposition 4.6.4, [13]), Lemma 26, (ii) and (a).

(iii) Suppose that β is a factor set of D_n in \mathbb{C} such that $\mathbb{C}^\beta D_n \not\cong \mathbb{C} D_n$. Then β is not a coboundary and by (i), n is even. Since $H^2(D_n, \mathbb{C}^*) \cong \mathbb{Z}_2$ and ω is not a coboundary, we conclude that ω is cohomologous to β . The desired conclusion follows by appealing to Lemma 20, (ii). □

4.3 Symmetric groups

Let S_n be the symmetric group of degree n . Then S_n is generated by the transpositions

$t_1 = (12), t_2 = (23), \dots, t_{n-1} = (n-1 \ n)$ with the defining relations $t_i^2 = (t_j t_{j+1})^3 = (t_r t_s)^2 = 1, 1 \leq i \leq n - 1, 1 \leq j \leq n - 2, r \leq s - 2$. The last equality can be rewritten as $t_r t_s = t_s t_r$.

Theorem 61. [13]

(i) If $n < 4$, then every projective representation of S_n is projectively equivalent to a linear representation.

(ii) Suppose that $n \geq 4$. Then the following properties hold:

(a) Every projective representation of S_n is projectively equivalent to a representation ρ satisfying

$$(4.4) \quad \rho(t_i)^2 = \lambda I, (\rho(t_j)\rho(t_{j+1}))^3 = \lambda I, \rho(t_r)\rho(t_s) = \lambda \rho(t_s)\rho(t_r)$$

where $\lambda = \pm 1$. In the case $\lambda = 1$, ρ is a linear representation of S_n .

(b) For each partition $n = n_1 + n_2 + \dots + n_k$ of n with $n_1 > n_2 > \dots > n_k > 0$ there is a projective representation ρ such that

$$\text{deg} \rho = 2^{\lfloor \frac{n-k}{2} \rfloor} \frac{n!}{n_1! n_2! \dots n_k!} \prod_{1 \leq i < j \leq k} \frac{n_i - n_j}{n_i + n_j},$$

where $\lfloor \frac{n-k}{2} \rfloor$ is the largest integer $\leq \frac{n-k}{2}$. Furthermore, ρ satisfies (4.1) for $\lambda = -1$ and ρ is not projectively equivalent to a linear representation.

Proof. (i) If $n < 4$, then all Sylow p -groups of S_n are cyclic and hence by Corollary 2.3.25, [13], $H^2(S_n, \mathbb{C}^*) = 1$. The desired conclusion follows by Lemma 26, (ii).

(ii) Let Γ be a projective representation of S_n and let $\Gamma(t_i) = A_i$. Then there are a_i, b_j and c_{rs} in \mathbb{C}^* such that

$$(4.5) \quad A_i^2 = a_i I, 1 \leq i \leq n - 1$$

$$(4.6) \quad (A_j A_{j+1})^3 = b_j I, 1 \leq j \leq n - 2$$

$$(4.7) \quad A_r A_s = c_{rs} A_s A_r, r \leq s - 2$$

From (4.1) $A_r A_s A_r^{-1} = c_{rs} A_s$ and squaring both sides gives

$$(4.8) \quad c_{rs}^2 = 1.$$

Let $t_r = (r \ r + 1)$ and $t_s = (s \ s + 1)$, where $s > r + 1$. Then $r, r + 1, s, s + 1$ are four distinct elements. Let $t'_r = (r' \ r' + 1)$, $t'_s = (s' \ s' + 1)$ and let $t \in S_n$ be defined by $t = \begin{bmatrix} \cdots & r & r + 1 & \cdots & s & s + 1 & \cdots \\ \cdots & r' & r' + 1 & \cdots & s' & s' + 1 & \cdots \end{bmatrix}$. Then $tt_r t^{-1} = t_r$ and $tt_s t^{-1} = t'_s$ and so setting $\Gamma(t) = A$, we obtain

$$(4.9) \quad AA_r A^{-1} = c A_r, AA_s A^{-1} = d A_s$$

for some $c, d \in \mathbb{C}^*$. From (4.1) it follows that $AA_r A^{-1} AA_s A^{-1} = c_{rs} AA_s A^{-1} AA_r A^{-1}$. Invoking (4.1), we therefore deduce that $cd A_r A_s = c d c_{rs} A_s A_r$ or $A_r A_s = c_{rs} A_s A_r$, which in view of (4.1) implies that $c_{rs} = c_{r's'}$.

Thus by (4.1), all the c_{rs} are equal to $\lambda = \pm 1$. From (4.1) it follows that $A_j A_{j+1} A_j = b_j A_{j+1}^{-1} A_j^{-1} A_{j+1}^{-1}$ and squaring both sides gives

$$(4.10) \quad A_j A_{j+1} A_j^2 A_{j+1} A_j = b_j^2 A_{j+1}^{-1} A_j^{-1} A_{j+1}^{-2} A_j^{-1} A_{j+1}^{-1}, a_j^2 a_{j+1} = b_j^2 (a_j a_{j+1}^2)^{-1}, b_j^2 = a_j^3 a_{j+1}^3$$

Since we may replace each A_i by its scalar multiple, the elements a_i of \mathbb{C}^* can be chosen arbitrarily. Setting $a_1 = a_2 = \dots = a_{n-1} = \lambda$ it follows from (4.1) that $b_j = \pm 1$. Replacing Γ by a projectively equivalent representation ρ such that $\rho(t_1) = A_1, \rho(t_2) = \lambda b_1 A_2, \rho(t_3) = b_1 b_2 A_3, \rho(t_4) = \lambda b_1 b_2 b_3 A_4, \dots$ it follows that $\rho(t_i)^2 = \lambda I, (\rho(t_j) \rho(t_{j+1}))^3 = \lambda I, \rho(t_r) \rho(t_s) = \lambda \rho(t_s) \rho(t_r)$ proving (ii) (a). For the proof of (ii) (b) we refer to the original paper of Schur [26]. □

References

[1] N.B. Backhouse, *Projective representations of space groups, II: Factor systems*, Quart. J. Math. Oxford (2), **21** (1970), p.277-295. [MR0281803](#). [Zbl 0208.03603](#).

- [2] N.B. Backhouse, C.J. Bradely, *Projective representations of space groups, I: Translation groups*, Quart. J. Math. Oxford (2), **21** (1970), p.203-222. [MR0260890](#). [Zbl 0194.33701](#).
- [3] R. Brauer, *On the representation of a group of order g in the field of the g -th roots of unity*, Amer. J. Math., vol. **67** (1945), p. 461-471. [MR0014085](#). [Zbl 0061.03901](#).
- [4] R. Brauer, *Applications of induced characters*, Amer. J. Math., vol. **69** (1947), p. 709-716. [MR0022851](#).
- [5] J.F. Carinena, *Representation groups for semiunitary projective representations of finite groups*, J. of Math. Phys. **20** (11), November 1979, p.2168-2177. [MR0550691](#) [Zbl 0439.20006](#).
- [6] A.H. Clifford, *Representations induced in an invariant subgroup*, Ann. Math., vol. 38, No. **3** (1937), p.533-550. [MR1503352](#). [Zbl 0017.29705](#).
- [7] C. Curtis, I. Reiner, *Representations theory of groups and associative algebras*, Pure and Applied Math. XI, Interscience, New-York (1962). [MR1013113](#).
- [8] J.R. Durbin, K.B. Farmer, *On projective representations of finite wreath products*, Math. of Computation, vol. 31, No. **138** (1977), p.527-535. [MR0453855](#). [Zbl 0357.20005](#).
- [9] K.B. Farmer, *A survey of projective representation theory of finite groups*, Nieuw Archief Voor Wiskunde (3),XXVI (1978), p.292-308. [MR0498823](#). [Zbl 0381.20008](#).
- [10] Shin-ichiro Ihara, T. Yokonuma, *On the second cohomology groups (Schur-multipliers) of finite reflection groups*, J. Fac. Sci. Univ. Tokyo. Sect. 1, XI (1965), p.155-171. [MR0190232](#). [Zbl 0136.28802](#).
- [11] N. Iwahori, H. Matsumoto, *Several remarks on projective representations of finite groups*, J. Fac. Sci. Univ. Tokyo. Sect. 2, **10** (1964), p.129-146. [MR0180607](#). [Zbl 0125.01504](#).
- [12] G. Karpilovsky, *On linearizable irreducible projective representations of finite groups*, J. Math. Soc. Japan, vol. 28, No. **3** (1976), p.541-549. [MR0422399](#). [Zbl 0316.20007](#).
- [13] G. Karpilovsky, *Projective representations of finite groups*, Marcel-Dekker (1985). [MR0788161](#). [Zbl 0568.20016](#).
- [14] A. Kleppner, *Multipliers on abelian groups*, Math. Annalen **158** (1965), p.11-34. [MR0174656](#) [Zbl 0135.06604](#)

Surveys in Mathematics and its Applications **15** (2020), 425 – 458

<http://www.utgjiu.ro/math/sma>

- [15] A.O. Morris, *Projective representations of abelian groups*, J. London Math. Soc. (2), **7** (1973), p.235-238. [MR0327885](#). [Zbl 0272.20007](#).
- [16] A.O. Morris, M. Saeed-ul-Islam, E. Thomas, *Some projective representations of finite abelian groups*, Glasgow Math. J. **29** (1987), p.197-203. [MR0901666](#). [Zbl 0631.20010](#).
- [17] H.N. NG, *Faithful irreducible projective representations of metabelian groups*, J. of Algebra, **38** (1976), p.8-28. [MR0432749](#). [Zbl 0357.20004](#).
- [18] M. Osima, *On the representations of groups of finite order*, Math. J. Okayama Univ., **1** (1952), p.33-61. [MR0049894](#). [Zbl 0047.02802](#).
- [19] R. Quinlan, *Generic central extensions and projective representations of finite groups*, Representation Theory, An Electronic J. of Amer. Math. Soc., vol. 5 (2001), p.129-146. [MR1835002](#). [Zbl 0986.20009](#).
- [20] E.W. Read, *On projective representations of finite reflection groups of type B_l and D_l* , J. London Math. Soc. (2), **10** (1975), p.129-142. [MR0367047](#). [Zbl 0307.20008](#).
- [21] E.W. Read, *Projective characters of Weyl group of type F_4* , J. London Math. Soc. (2), **8** (1974), p.83-93. [MR0340401](#). [Zbl 0279.20010](#).
- [22] E.W. Read, *The projective representations of the generalized symmetric group*, J. of Algebra **46** (1977), p.102-133. [MR0457547](#). [Zbl 0357.20007](#).
- [23] W.F. Reynolds, *Projective representations of finite groups in cyclotomic fields*, J. Math. **9** (1965), p.191-198. [MR0174631](#). [Zbl 0134.03201](#).
- [24] P. Rudra, *On projective representations of finite groups*, J. Math. Physics, vol. 6, No. **8** (1965), p.1273-1277. [MR0182388](#). [Zbl 0135.05603](#).
- [25] I. Schur, *Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **127** (1904), p.20-50. [MR1580715](#). [Zbl 35.0155.01](#).
- [26] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. für Math. **139** (1911), p.155-250. [MR1580818](#). [Zbl 42.0154.02](#).
- [27] J. Tappe, *Irreducible projective representations of finite groups*, Manuscripta Math. **22** (1977), p.33-45. [MR0466292](#). [Zbl 0365.20016](#).
- [28] K. Yamazaki, *A note on projective representations of finite groups*, Sci. Papers College Gen. Ed. Univ. Tokyo **14** (1964), p.27-36. [MR0172938](#). [Zbl 0121.03304](#).

- [29] K. Yamazaki, *On projective representations and ring extensions of finite groups*, J. Fac. Sci. Univ. Tokyo Sect. 2, **10** (1964), p.147-195. [MR0180608](#). [Zbl 0125.01601](#).

Tania Luminița Costache
Faculty of Applied Sciences, University "Politehnica" of Bucharest,
Splaiul Independentei 313, Bucharest, Romania.
e-mail: lumycos1@yahoo.com

License

This work is licensed under a [Creative Commons Attribution 4.0 International License](#). 

Surveys in Mathematics and its Applications **15** (2020), 425 – 458
<http://www.utgjiu.ro/math/sma>