

## MODELLING THE DURABILITY OF MULTIBODY TOTAL HIP JOINT PROSTHESIS

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**Abstract.** The Total Hip Prosthesis (THP) is one of the biggest successes of the 20th century in the field of orthopaedic biomechanical engineering. The loss of stability THP are scheduled failure. The percentage of good functioning after 10 years of operation for classically THP is 94%. According to the data from the Swedish Prosthesis Register, the working time of a prosthesis reaches an average of 15 years. It is well known that the premature failure of the THP implant, in which there is frictional slip between the acetabular cup and the femoral head, also depends on the surgical accuracy which is required to provide functional angles. One solution is to promote hip prosthesis with multibody rolling. Predicting their function and the analytical determination of the lastingness of multibody prostheses is a challenge. The purpose of this paper is to provide a reliable solution to make a prediction on the durability of a THP multibody implant with the use of computing resources available to any user. This may also be the basis for subsequent risk analyzes of implant failure depending on the physical aspects of the patient and the features of the prosthesis.

### 1 Introduction

Total hip prosthesis arthroplasty is considered to be the most widely used procedure for hip reconstruction and, as a result of more than one million implants/year, represents the quality therapeutic solution for a host of degenerative afflictions. Unfortunately, up until now, their reliability couldn't be improved above the 15th year limit this solution being considered as a programmed failure type intervention. The researchers in the field are approaching the subject of developing this solution along two major routes, namely: through the nature of the materials being used and through the nature of the constructive solutions [11].

One of the most modern trends is represented by replacing the sliding motion between the components of the prosthesis with a rolling motion. This changes the tribological behaviour of the prosthesis and has direct consequences on the working life of the implant. But to develop an analytical model for dynamic formulations for multibody systems is a challenge [26, 16, 27, 14].

Multibody systems analysis in relation to the contact constraints, with or without friction, was the main objectives of many scientific papers [1, 3, 15, 12, 10, 9, 19, 20, 21, 23]. An important concern was the calculation of joint reaction forces in rigid body mechanisms with dependent constraints [6, 7, 30, 31]. Several numerical methods have been proposed [2, 5, 4, 8, 13, 18, 28, 24, 25, 29].

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Developing such prosthesis requires complex theoretical and laboratory approaches. Although the existence of computer systems allows for the complex modelling of such systems, the practical work requires the discovery of simple theoretical models with a high degree of confidence for the results. As a consequence, the high number of balls needed to accomplish the functional role of rolling friction prosthesis and, implicitly, the high number of degrees of freedom implies, from the standpoint of mathematically approaching the system, the existence of a high number of differential equations.

The current article presents a way to apply sparse matrices for make a prediction on the durability of a THP multibody implant by the determination of angular frequencies and frequencies for a high number of rolling bodies. . The proposed method has the advantage of using low resources computing and an acceptable errors rate.

## 2 Description of the Problem

The Romanian Academy's Institute for Solid Mechanics together with the Faculty of Technological Equipment from the University of Civil Engineering Bucharest proposed and developed a constructive solution for the total hip prosthesis (MOM) with rolling friction [11]. This entails the interposition between the head and the acetabular cup of a layer of 2.5 mm diameter balls, without a cage. (Fig. 1)



Figure 1: The MOM total hip prosthesis with rolling friction [11]

The model used for the studies is a spatial model that contains 199 balls arranged on 12 rows.

The first identified requirement is the Analytical Determination of the natural frequency of the balls for the planar model.

### 2.1 Establishing the model and determining the differential motion equations

To begin with, it was decided to test a planar model.

The planar model proposed in Fig. 2 contains 91 balls arranged on 5 rows, as follows:

- a) First row at the upper edge                   – 31 balls
- b) Second row   – 25 balls

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- c) Third row – 18 balls
- d) Fourth row – 12 balls
- e) The last row located in the centre – 5 balls

This layered balls structure is considered at time  $t = 0$ ; for any other time  $t_1 > t$  there's a different ball distribution.

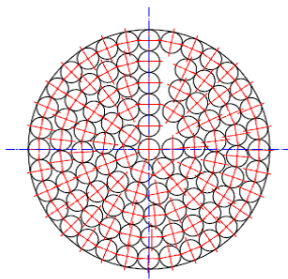


Figure 2: The planar model with 91 balls

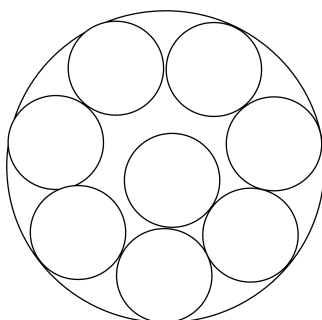


Figure 3: Simplified planar model with 8 balls

A symmetric distribution for the 91 balls is considered, comprising 8 mass with 11 balls each having identical motions and 3 balls which do not take part in these movements and can be ignored. The simplified model is showed in Fig. 3.

Unfolding the circular trajectory on which the 8 balls are rolling allows us to obtain the computational scheme as showed in Fig. 4.

In order to obtain the differential equation set for the motion of this mechanical system with 8 degrees of freedom we begin by using Lagrange's second degree equations:

$$\frac{d}{dt} \left( \frac{\partial E}{\partial \dot{q}_j} \right) - \frac{\partial E}{\partial q_j} = Q_{jF} + Q_{jD} + Q_{jP} \quad (2.1)$$

where  $E$  is the kinetic energy of the mechanical system,  $q_j$  is the generalized coordinate,  $\dot{q}_j$  is the generalized speed,  $Q_{jF}$  is the generalized perturbing force,  $Q_{jD}$  is the dissipative force, and  $Q_{jP}$  is the force created by replacing the active spring-type links.

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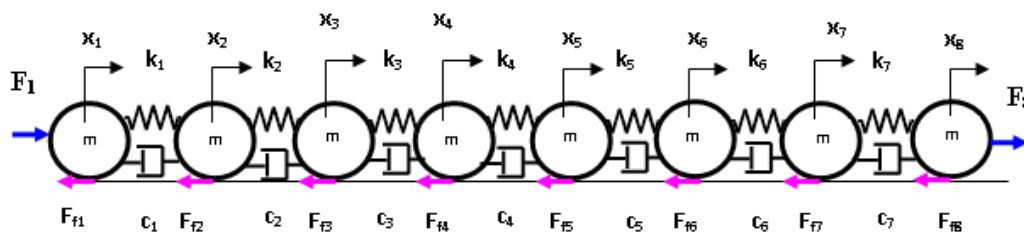


Figure 4: Computational scheme for a mechanical system with 8 degrees of freedom

Through integration, with consideration of the initial conditions, we will obtain the laws of motion for the balls' centres of mass.

The differential equation system is presented below:

$$\begin{cases} m\ddot{x}_1 + 2c\dot{x}_1 + 2kx_1 - c\dot{x}_2 - kx_2 = F_1 - F_f \\ m\ddot{x}_2 + 2c\dot{x}_2 + 2kx_2 - c\dot{x}_1 - c\dot{x}_3 - kx_1 - kx_3 = -F_f \\ m\ddot{x}_3 + 2c\dot{x}_3 + 2kx_3 - c\dot{x}_2 - c\dot{x}_5 - kx_2 - kx_4 = -F_f \\ m\ddot{x}_4 + 2c\dot{x}_4 + 2kx_4 - c\dot{x}_3 - c\dot{x}_5 - kx_3 - kx_5 = -F_f \\ m\ddot{x}_5 + 2c\dot{x}_5 + 2kx_5 - c\dot{x}_4 - c\dot{x}_6 - kx_4 - kx_6 = -F_f \\ m\ddot{x}_6 + 2c\dot{x}_6 + 2kx_6 - c\dot{x}_5 - c\dot{x}_7 - kx_5 - kx_7 = -F_f \\ m\ddot{x}_7 + 2c\dot{x}_7 + 2kx_7 - c\dot{x}_6 - c\dot{x}_8 - kx_6 - kx_7 = -F_f \\ m\ddot{x}_8 + 2c\dot{x}_8 + 2kx_8 - c\dot{x}_7 - k = F_2 - F_f \end{cases}$$

The differential equation system may be written in matrix form as:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} \tag{2.2}$$

where  $[M]$  is the inertia matrix,  $[K]$  is the stiffness matrix,  $[C]$  is the damping matrix and  $\{F\}$  is the column vector for the perturbing and friction forces. In this particular example:

$$[M] = 3/2m[I]_8 \tag{2.3}$$

$$[K] = \begin{pmatrix} 2k & -k & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 2k & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & -2k & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 2k & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 2k & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & 0 & 0 & 0 & -k & k \end{pmatrix} \tag{2.4}$$

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$$[C] = \begin{vmatrix} 2c & -c & 0 & 0 & 0 & 0 & 0 & 0 \\ -c & 2c & -c & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 2c & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 2c & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 2c & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & -c & 2c & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & -c & 2c & -c \\ 0 & 0 & 0 & 0 & 0 & 0 & -c & c \end{vmatrix} \tag{2.5}$$

$$\begin{aligned} \{F\} &= [(F_1 - F_f) - F_f - F_f - F_f - F_f - F_f - F_f - F_f(F_2 - F_f)]^T \\ \{x\} &= [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8]^T \end{aligned} \tag{2.6}$$

order to determine the angular frequencies and natural frequencies for the 8 balls it is necessary to compute the equivalent mass  $m_e$  and the equivalent stiffness coefficient  $k_e$ .

### 2.2 Determining the angular and natural frequencies for the 8 balls

The determination of the angular frequencies and natural frequencies for the 8 balls is performed using the MATLAB software package [17]. The instructions and the results of the execution are shown in Table 1

Table 1: Angular frequencies and natural frequencies for the 8 balls

No.	Angular frequencies [rad/s]	Natural frequencies [Hz]
1	3.5919	0.5717
2	10.6535	1.6956
3	17.3522	2.7617
4	23.4601	3.7338
5	28.7690	4.5787
6	33.0982	5.2677
7	36.3004	5.7774
8	38.2663	6.0903

Let’s consider a different stiffness matrix which differs from the previous one through its last term, this having the value of  $2k$  instead of  $k$ .

The new matrix is shown below:

$$[K_1] = \begin{vmatrix} 2k & -k & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 2k & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 2k & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 2k & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 2k & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & 0 & 0 & 0 & -k & 2k \end{vmatrix} \tag{2.7}$$

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If we now calculate the angular frequencies and frequencies for the 8 balls using the  $K_1$  matrix and keeping the  $M$  matrix unchanged, we'll get the following values:

angular frequencies:

p1 = 3.5919 [rad/s]  
 p2 = 10.6535 [rad/s]  
 p3 = 17.3522 [rad/s]  
 p4 = 23.4601 [rad/s]  
 p5 = 28.769 [rad/s]  
 p6 = 33.0982 [rad/s]  
 p7 = 36.3004 [rad/s]  
 p8 = 38.2663 [rad/s]

angular frequencies:

p1 = 6.76 [rad/s]  
 p2 = 13.3146 [rad/s]  
 p3 = 19.4646 [rad/s]  
 p4 = 25.0232 [rad/s]  
 p5 = 29.8215 [rad/s]  
 p6 = 33.7136 [rad/s]  
 p7 = 36.5815 [rad/s]  
 p8 = 38.3377 [rad/s]

natural frequencies:

f1 = 0.57167 [Hz]  
 f2 = 1.6956 [Hz]  
 f3 = 2.7617 [Hz]  
 f4 = 3.7338 [Hz]  
 f5 = 4.5787 [Hz]  
 f6 = 5.2677 [Hz]  
 f7 = 5.7774 [Hz]  
 f8 = 6.0903 [Hz]

natural frequencies

f1 = 1.0759 [Hz]  
 f2 = 2.1191 [Hz]  
 f3 = 3.0979 [Hz]  
 f4 = 3.9826 [Hz]  
 f5 = 4.7462 [Hz]  
 f6 = 5.3657 [Hz]  
 f7 = 5.8221 [Hz]  
 f8 = 6.1016 [Hz]

older values

Analysing the newly obtained values in comparison to the old ones we can notice a 3–5% change which is acceptable.

This result opens up a new route which will in the end lead to the easier determination of the angular frequencies and natural frequencies for the planar model with 88 active balls and 3 inactive balls and for the spatial model with 199 balls.

This new approach is performed using sparse matrices which can be easily written and manipulated, regardless of the size of the analysed mechanical system.

In numerical analysis, a sparse matrix is a matrix in which most of the elements are zero.

### 3 The determination of angular and natural frequencies for the 8 balls using sparse matrices

The matrix  $K_1$  shown below is a special shape matrix, namely tridiagonal.

$$[K_1] = \begin{pmatrix} 2k & -k & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 2k & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 2k & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 2k & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 2k & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & 0 & 0 & 0 & -k & 2k \end{pmatrix} \quad (3.1)$$

Matrix  $K_1$  may be defined through (3.1) as being formed by assembling three diagonal matrices, as follows:

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- An 8<sup>th</sup> degree unit matrix,  $[I_8]$  multiplied by  $2k$
- A superior diagonal matrix  $[K_2]$  formed out of  $-k$  offset by one position from the main diagonal of the unit matrix
- An inferior diagonal matrix  $[K_3]$  formed out of  $-k$  offset by one position from the main diagonal of the unit matrix

Thus

$$[K_1] = 2k [I_8] + [K_2] + [K_3] \quad (3.2)$$

where:

$$[I_8] = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$[K_2] = \begin{pmatrix} 0 & -k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[K_3] = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k & 0 \end{vmatrix}$$

The results obtained running in MATLAB software package are shown below.

Angular frequencies:

$$p1 = 6.76 \text{ [rad/s]}$$

$$p2 = 13.3146 \text{ [rad/s]}$$

$$p3 = 19.4646 \text{ [rad/s]}$$

$$p4 = 25.0232 \text{ [rad/s]}$$

$$p5 = 29.8215 \text{ [rad/s]}$$

$$p6 = 33.7136 \text{ [rad/s]}$$

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$$p7 = 36.5815 \text{ [rad/s]}$$

$$p8 = 38.3377 \text{ [rad/s]}$$

natural frequencies:

$$f1 = 1.0759 \text{ [Hz]}$$

$$f2 = 2.1191 \text{ [Hz]}$$

$$f3 = 3.0979 \text{ [Hz]}$$

$$f4 = 3.9826 \text{ [Hz]}$$

$$f5 = 4.7462 \text{ [Hz]}$$

$$f6 = 5.3657 \text{ [Hz]}$$

$$f7 = 5.8221 \text{ [Hz]}$$

$$f8 = 6.1016 \text{ [Hz]}$$

After analysing the results obtained through sparse matrices and comparing with previous results, it can be seen they are identical.

These results are demonstrating the possibility of determining angular frequencies and natural frequencies for our real mechanical system and for the planar hypothesis respectively:

- a) 88 active balls and 3 inactive balls
- b) all of the 91 balls are active

### 3.1 Determining angular and natural frequencies for the 88 balls through the use of sparse matrices

The results of the execution are shown in Table 2.

Similarly we obtain angular frequencies and natural frequencies for the model with 91 active balls.

Analysing the previous results compared with the ones for the 8 ball model we can see that we can find natural frequencies with good accuracy for a 10 ball periodicity, according to Table 3.

Analysis of the two real cases studied (88 and 91 balls) raised the observation that we have approximately the same frequencies, the fact that there are 3 extra balls not significantly affecting the frequency range.

## 4 Analytical determination of the time in which the wear of the acetabular cup appears

The authors' studies [11] upon total hip prosthesis extracted from the human body after 12-15 years of functioning have clearly pointed out polish traces on acetabular cup.

It will be determined the required time in order to make sure that all those 199 balls of  $r$  ray will cover the inner surface of the cup through areas as against the length of the hypocycloid arch multiplied by a side having an imposed value of the polishing trace such as 0.5 microns.

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Table 2: Angular frequencies

p1=0.68717 [rad/s]	p23=15.3752 [rad/s]	p45=27.7741 [rad/s]	p67=36.038 [rad/s]
p2=1.3741 [rad/s]	p24=16.0041 [rad/s]	p46=28.2514 [rad/s]	p68=36.2926 [rad/s]
p3=2.0607 [rad/s]	p25=16.6281 [rad/s]	p47=28.7199 [rad/s]	p69=36.5358 [rad/s]
p4=2.7465 [rad/s]	p26=17.2468 [rad/s]	p48=29.1794 [rad/s]	p70=36.7677 [rad/s]
p5=3.4316 [rad/s]	p27=17.8602 [rad/s]	p49=29.6298 [rad/s]	p71=36.9881 [rad/s]
p6=4.1155 [rad/s]	p28=18.4681 [rad/s]	p50=30.071 [rad/s]	p72=37.197 [rad/s]
p7=4.7982 [rad/s]	p29=19.0701 [rad/s]	p51=30.5029 [rad/s]	p73=37.3943 [rad/s]
p8=5.4794 [rad/s]	p30=19.6663 [rad/s]	p52=30.9252 [rad/s]	p74=37.5799 [rad/s]
p9=6.1589 [rad/s]	p31=20.2563 [rad/s]	p53=31.3379 [rad/s]	p75=37.7539 [rad/s]
p10=6.8364 [rad/s]	p32=20.84 [rad/s]	p54=31.7409 [rad/s]	p76=37.9161 [rad/s]
p11=7.5119 [rad/s]	p33=21.4172 [rad/s]	p55=32.1339 [rad/s]	p77=38.0665 [rad/s]
p12=8.185 [rad/s]	p34=21.9877 [rad/s]	p56=32.517 [rad/s]	p78=38.205 [rad/s]
p13=8.8555 [rad/s]	p35=22.5514 [rad/s]	p57=32.8899 [rad/s]	p79=38.3316 [rad/s]
p14=9.5233 [rad/s]	p36=23.1081 [rad/s]	p58=33.2525 [rad/s]	p80=38.4463 [rad/s]
p15=10.1881 [rad/s]	p37=23.6576 [rad/s]	p59=33.6049 [rad/s]	p81=38.549 [rad/s]
p16=10.8497 [rad/s]	p38=24.1997 [rad/s]	p60=33.9467 [rad/s]	p82=38.6397 [rad/s]
p17=11.508 [rad/s]	p39=24.7342 [rad/s]	p61=34.278 [rad/s]	p83=38.7184 [rad/s]
p18=12.1627 [rad/s]	p40=25.2611 [rad/s]	p62=34.5986 [rad/s]	p84=38.785 [rad/s]
p19=12.8136 [rad/s]	p41=25.7801 [rad/s]	p63=34.9084 [rad/s]	p85=38.8395 [rad/s]
p20=13.4605 [rad/s]	p42=26.291 [rad/s]	p64=35.2073 [rad/s]	p86=38.8819 [rad/s]
p21=14.1032 [rad/s]	p43=26.7938 [rad/s]	p65=35.4953 [rad/s]	p87=38.9122 [rad/s]
p22=14.7415 [rad/s]	p44=27.2882 [rad/s]	p66=35.7722 [rad/s]	p88=38.9304 [rad/s]

#### 4.1 The determination of the stiffness coefficient $k$

The rolling of a ball of  $r$  ray over a fix disc of  $R$  ray will be considered in order to determine the stiffness coefficient as in Fig. 5.

The disc of  $m$  mass and  $r$  ray is running a plane rolling motion over the fix disc of  $r$  ray. The kinetic energy of the disc has the expression:

$$E = \frac{1}{2} J_C \omega^2 + \frac{1}{2} m v_c^2 \quad (4.1)$$

Considering

$$J_C = \frac{1}{2} m r^2, \quad v_C = (R - r) \omega, \quad \omega = \dot{\theta} \quad (4.2)$$

it results:

$$E = \frac{3}{4} m (R - r)^2 \dot{\theta}^2 \quad (4.3)$$

The force function has the expression :

$$U = mg(R - r) \cos \theta \quad (4.4)$$

The generalized restoring force  $Q_R$  has the expression :

$$Q_R = \frac{\partial U}{\partial \theta} = -mg(R - r) \sin \theta \quad (4.5)$$

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Table 3: Comparison between the 8 ball and the 88 ball and 91 ball models

8 ball model – ball no. [Hz] /natural frequency [Hz]	88 ball model – ball no. /natural frequency [Hz]	91 ball model – ball no. /natural frequency [Hz]
1/1.02	10/1.034	10/1.0006
2/2.009	20/2.0359	21/2.0666
3/2.937	30/2.9745	31/2.9735
4/3.7757	39/3.741	40/3.7165
5/4.4998	49/4.4815	50/4.4387
6/5.087	59/5.0827	60/5.0317
7/5.5198	69/5.526	71/5.5146
8/5.7848	79/5.7976	81/5.7855

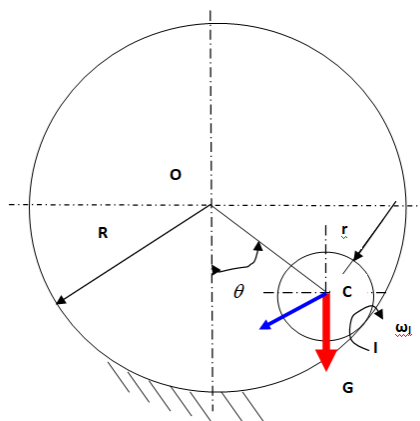


Figure 5: The kinematic model of a r ray ball rolling over a fix disc of R ray

It is calculated :

$$\frac{\partial E}{\partial \theta} = 0 \tag{4.6}$$

$$\frac{\partial E}{\partial \theta} = \frac{3}{2}m(R-r)^2 \dot{\theta} \tag{4.7}$$

$$\frac{d}{dt} \frac{\partial E}{\partial \dot{\theta}} = \frac{3}{2}m(R-r)^2 \ddot{\theta} \tag{4.8}$$

Introducing (4.7) and (4.8) in Lagrange’s second degree equations:

$$\frac{d}{dt} \left( \frac{\partial E}{\partial \dot{\theta}} \right) - \frac{\partial E}{\partial \theta} = Q_R \tag{4.9}$$

A differential equation of motion is obtained under the expression:

$$\frac{3}{2}m(R-r)^2 \ddot{\theta} + mg(R-r) \sin \theta = 0 \tag{4.10}$$

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Considering small angles, approximation  $\sin \theta \approx \theta$  can be done and the equation (4.10) becomes:

$$\dot{\theta} + \frac{2}{3} \frac{g}{(R-r)} \theta = 0 \tag{4.11}$$

Hence:

$$\omega = \frac{2}{3} \frac{g}{(R-r)} = \frac{k}{m} \tag{4.12}$$

Considering  $R = 6.5mm$ ,  $r = 1.25mm$ ,  $g = 9.81m/s^2$  and  $m = 0.06g$ , the result is:

$$k_{ball} = 0.0257 \frac{N}{m} \tag{4.13}$$

Being determined  $m$  and  $k$  one can build matrices  $[M]$  and  $[K]$  given by the relations (2.3) and (2.4). These matrices can be used further to determine the specific pulsations of the balls.

### 4.2 Determination of a trajectory of a point located on a ball of $r$ ray rolling on an acetabular cup of $R$ ray

Let us consider a material point  $M$  located at the edge of a mobile disc of  $r$  ray rolling without sliding inside a fix disc of  $R > r$ , as in Fig. 6.

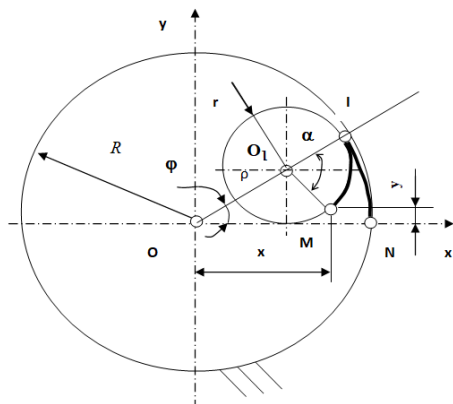


Figure 6: The kinematic generation of hypocycloid

From rolling condition without sliding results:

$$NI = IM \tag{4.14}$$

$$R\varphi = r\alpha \tag{4.15}$$

If there is mandatory an overlapping condition of the trajectory of point  $M$  to the continuous motion of the mobile disc over the fix one, the result is:

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$$2\pi R = 2\pi r\lambda \quad (4.16)$$

where:

$$\lambda = \frac{R}{r} \quad (4.17)$$

Introducing (4.17) in (4.15) we have:

$$\alpha = \lambda\varphi \quad (4.18)$$

The parametric equations of the material point  $M$  motion are as follows:

$$\begin{cases} x = r[(\lambda - 1)\cos\varphi - \cos(\lambda - 1)\varphi] \\ y = r[(\lambda - 1)\sin\varphi - \sin(\lambda - 1)\varphi] \end{cases} \quad (4.19)$$

For different values of  $\lambda$  rays report, it will be obtained hypocycloidal curves.

The equations written in polar coordinates become:

$$\rho = (R - r)(1 - \cos\theta), \quad -\pi \leq \theta \leq \pi \quad (4.20)$$

The length of the hypocycloidal arch is given by the relation:

$$S = \int_0^{\frac{2\pi}{3}} \sqrt{\dot{\rho}^2 + \rho^2} d\theta \quad (4.21)$$

where  $\dot{\rho}$  is first degree derivative of  $\rho$  polar coordinate of the material point  $M$ .

For a maximal motion of  $120^\circ = \frac{2\pi}{3}$  radians of the femoral tail the course length of the ball is defined with reference to a human's step.

$$S = \int_0^{\frac{2\pi}{3}} \sqrt{\dot{\rho}^2 + \rho^2} d\theta = 4(R - r) \int_0^{\frac{2\pi}{3}} \sin\frac{\theta}{2} d\theta = 4(R - r) \quad (4.22)$$

Performing these calculations, it results the length of the hypocycloidal arch:

$$S = 61 \text{ mm}$$

### 4.3 The analytical determination of the time in which wear appears

The area of acetabular cup (half spherical) of ray  $R = 16.5\text{mm}$  is :

$$A = 2\pi R^2 = 1710\text{mm}^2$$

Considering not all the 199 balls are going in full through the hypocycloidal arch a coefficient of distribution  $k = 16,5\%$  is introduced representing the minimal probability for 32 balls to travel in the same time in the hypocycloidal arch, in other words an equivalent length of the hypocycloidal arch is defined:

$$S_1 = k \bullet s \cong 10\text{mm} \quad (4.23)$$

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The wear area:

$$A_{wear} = S_1 \cdot 0.5 \cdot 10^{-3} mm^2 = 5 \cdot 10^{-3} mm^2 \quad (4.24)$$

The total number of wear areas needed in order to cover the surface of the acetabular cup is:

$$n = \frac{A}{A_{wear}} = 341946 \quad (4.25)$$

In order to determine the time in which appears the total number of wear area it was runned a program in Matlab for the model with 199 balls having matrices  $[M]$  and  $[K]$  given by the relations (2.3) and (2.4) by using sparse matrices method:

1. The 199 self pulsations  $p_i$
2. The 199 translations speed of the centres of the balls  $\nu_i$
3. The 199 beats in which every ball of r ray having  $\nu_i$  speed is travelling through the hyperbole arch  $s_1$
4. The sum of the 199 beats necessary for travelling considering that a ball is running through all positions and surface

Matlab instructions and the related results are not included in this paperwork but can be provided at the request by contacting any of the two authors.

By running the program it was concluded that the sum of the 199 beats of travelling is:

$$T_{199} \cong 389 \text{ s} \quad (4.26)$$

The estimated time in which a wear of 0.5 microns appears on the inner surface of the acetabular cup can be calculated by using the relation :

$$T = \frac{T_{199}n}{3600 \cdot 8 \cdot k_2 \cdot 365} \cong 11 \text{ years} \quad (4.27)$$

It was made an equivalency of 8 working hours during which a motion of  $120^\circ$  of the femoral part. This could correspond to 9.2 hours considering the motion one is having in lunch time and also lifting and sitting motions which will correspond to a coefficient  $k_2 = 1.15$ .

## Conclusions

For people with a high weight or to prevent the risks of implant positionary, the medical rehabilitation by using multibody prostheses can build up the solution.

The proposed approach attempts to identify the risk factors that can lead to the failure of a multibody prosthesis and the prediction of its lifetime.

The use of sparse matrices opens up the perspective of unitary analysis for the spatial mechanical system comprising 199 active balls and a mechanical system having an unlimited number of balls. The advantages of sparse matrices are size and speed. These characteristics allow a faster computation of the durability of a spherical joint, multibody type, setting up a real support for personalised implants.

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