Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December 2003, pp. 1–5.
Submitted: April 15, 2003. Published: December 31 2003.

τ-DISTANCE IN A GENERAL TOPOLOGICAL SPACE (X, τ) WITH APPLICATION TO FIXED POINT THEORY

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ABSTRACT. The main purpose of this paper is to define the notion of a τ-distance function in a general topological space (X, τ). As application, we get a generalization of the well known Banach's fixed point theorem.

A.M.S. (MOS) Subject Classification Codes. 54A05, 47H10, 54H25, 54E70

Key Words and Phrases. Hausdorff topological spaces, Topological spaces of type F, symmetrizable topological spaces, Fixed points of contractive maps

1. INTRODUCTION

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended In many different directions ([2],[3],[4],[6],[9]). On the other hand, it has been observed ([3],[5]) that the distance function used in metric theorems proofs need not satisfy the triangular inequality nor $d(x, x) = 0$ for all x. Motivated by this fact, we define the concept of a τ-distance function in a general topological space (X, τ) and we prove that symmetrizable topological spaces ([5]) and F-type topological spaces introduced in 1996 by Fang [4] (recall that metric spaces, Hausdorff topological vector spaces and Menger probabilistic metric space are all a special case of F-type topological spaces) possess such functions. finally, we give a fixed point theorem for contractive maps in a general topological space (X, τ) with a τ-distance which gives the Banach's fixed point theorem in a new setting and also gives a generalization of jachymski's fixed point result [3] established in a semi-metric case.

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2. τ-DISTANCE

Let (X, τ) be a topological space and $p : X \times X \longrightarrow IR^+$ be a function. For any $\epsilon > 0$ and any $x \in X$, let $B_p(x, \epsilon) = \{y \in X : p(x, y) < \epsilon\}$.

Definition 2.1. The function p is said to be a τ-distance if for each $x \in X$ and any neighborhood V of x, there exists $\epsilon > 0$ with $B_p(x, \epsilon) \subset V$.

Example 2.1. Let $X = \{0; 1; 3\}$ and $\tau = \{\emptyset; X; \{0; 1\}\}$. Consider the function $p : X \times X \longrightarrow IR^+$ defined by

$$p(x, y) = \begin{cases} y, \ x \neq 1 \\ \dfrac{1}{2}y, \ x = 1. \end{cases}$$

We have, $p(1; 3) = \frac{3}{2} \neq p(3; 1) = 1$. Thus, p us not symmetric. Moreover, we have

$$p(0; 3) = 3 > p(0; 1) + p(1; 3) = \frac{5}{2}$$

which implies that p fails the triangular inequality. However, the function p is a τ-distance.

Example 2.2. Let $X = IR^+$ and $\tau = \{X, \emptyset\}$. It is well known that the space (X, τ) is not metrizable. Consider the function p defined on $X \times X$ by $p(x, y) = x$ for all $x, y \in X$. It is easy to see that the function p is a τ-distance.

Example 2.3. In [5], Hicks established several important common fixed point theorems for general contractive selfmappings of a symmetrizable (resp. semi-metrizable) topological spaces. Recall that a symmetric on a set X is a nonnegative real valued function d defined on $X \times X$ by

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$

A symmetric function d on a set X is a semi-metric if for each $x \in X$ and each $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ is a neighborhood of x in the topology τ_d defined as follows

$$\tau_d = \{U \subseteq X / \ \forall x \in U, \ B_d(x, \epsilon) \subset U, \ for some\ \epsilon > 0\}$$

A topological space X is said to be symmetrizable (semi-metrizable) if its topology is induced by a symmetric (semi-metric) on X. Moreover, Hicks [5] proved that very general probabilistic structures admit a compatible symmetric or semi-metric. For further details on semi-metric spaces (resp. probabilistic metric spaces), see, for example, [8] (resp. [7]). Each symmetric function d on a nonempty set X is a τ_d-distance on X where the topology τ_d is defined as follows: $U \in \tau_d$ if $\forall x \in U$, $B_d(x, \epsilon) \subset U$, for some $\epsilon > 0$.

Example 2.4. Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Consider the function $p : X \times X \longrightarrow IR^+$ defined by

$$p(x, y) = e^{|x-y|}, \ \forall x, y \in X$$

It is easy to see that the function p is a τ-distance on X where τ is the usual topology since $\forall x \in X$, $B_p(x, \epsilon) \subset B_d(x, \epsilon)$, $\epsilon > 0$. Moreover, (X, p) is not a symmetric space since for all $x \in X$, $p(x, x) = 1$.

Example 2.5 - Topological spaces of type (EL).

Definition 2.2. A topological space (X, τ) is said to be of type (EL) if for each $x \in X$, there exists a neighborhood base $F_x = \{U_x(\lambda, t)/\lambda \in D, t > 0\}$, where $D = (D, \prec)$ denotes a directed set, such that $X = \cup_{t>0} U_x(\lambda, t)$, $\forall \lambda \in D$, $\forall x \in X$.

remark 2.1. In [4], Fang introduced the concept of F-type topological space and gave a characterization of the kind of spaces. The usual metric spaces, Hausdorff topological vector spaces, and Menger probabilistic metric spaces are all the special cases of F-type topological Spaces. Furtheremore, Fang established a fixed point theorem in F-type topological spaces which extends Caristi's theorem [2]. We recall the concept of this space as given in [4]

Definition [4]. A topological space (X, θ) is said to be F-type topological space if it is Hausdorff and for each $x \in X$, there exists a neighborhood base $F_x = \{U_x(\lambda, t)/\lambda \in D, t > 0\}$, where $D = (D, \prec)$ denotes a directed set, such that

(1) If $y \in U_x(\lambda, t)$, then $x \in U_y(\lambda, t)$,
(2) $U_x(\lambda, t) \subset U_x(\mu, s)$ for $\mu \prec \lambda, t \leq s$,
(3) $\forall \lambda \in D$, $\exists \mu \in D$ such that $\lambda \prec \mu$ and $U_x(\mu, t_1) \cap U_y(\mu, t_2) \neq \emptyset$, implies $y \in U_x(\lambda, t_1 + t_2)$,
(4) $X = \cup_{t>0} U_x(\lambda, t)$, $\forall \lambda \in D$, $\forall x \in X$.

It is clear that a topological space of type F is a Hausdorff topological space of type (EL). Therefore The usual metric spaces, Hausdorff topological vector spaces, and Menger probabilistic metric spaces are special cases of a Hausdorff topological Space of type (EL).

proposition 2.1. *Let (X, τ) be a topological space of type (EL). Then, for each $\lambda \in D$, there exists a τ-distance function p_λ.*

Proof. Let $x \in X$ and $\lambda \in D$. Consider the set $E_x = \{U_x(\lambda, t) | \lambda \in D, t > 0\}$ of neighborhoods of x such that $X = \cup_{t>0} U_x(\lambda, t)$. Then for each $y \in X$, there exists $t^* > 0$ such that $y \in U_x(\lambda, t^*)$. Therefore, for each $\lambda \in D$, we can define a function $p_\lambda : X \times X \longrightarrow IR^+$ as follows

$$p_\lambda(x, y) = inf\{t > 0, y \in U_x(\lambda, t)\}.$$

set $B_\lambda(x, t) = \{y \in X | p_\lambda(x, y) < t\}$. let $x \in X$ and V_x a neighborhood of x. Then the exists $(\lambda, t) \in D \times IR^+$, such that $U_x(\lambda, t) \subset V_x$. We show that $B_\lambda(x, t) \subset U_x(\lambda, t)$. Indeed, consider $y \in B_\lambda(x, t)$ and suppose that $y \notin U_x(\lambda, t)$. It follows that $p_\lambda(x, y) \geq t$, which implies that $y \notin B_\lambda(x, t)$. A contradiction. Thus $B_\lambda(x, t) \subset V_x$. Therefore p_λ is a τ-distance function.

remark 2.2. As a consequence of proposition 3.1, we claim that each topological space of type (EL) has a familly of τ-distances $M = \{p_\lambda | \lambda \in D\}$.

3. Some properties of τ-distances

lemma 3.1. *Let (X, τ) be a topological space with a τ-distance p.*

(1) *Let (x_n) be arbitrary sequence in X and (α_n) be a sequence in IR^+ converging to 0 such that $p(x, x_n) \leq \alpha_n$ for all $n \in IN$. Then (x_n) converges to x with respect to the topology τ.*

(2) *If* (X,τ) *is a Hausdorff topological space, then* (2.1) $p(x,y) = 0$ *implies* $x = y$. (2.2) *Given* (x_n) *in* X,

$$\lim_{n\to\infty} p(x,x_n) = 0 \textit{ and } \lim_{n\to\infty} p(y,x_n) = 0$$

imply $x = y$.

Proof.

(1) Let V be a neighborhood of x. Since $\lim p(x,x_n) = 0$, there exists $N \in IN$ such that $\forall n \geq N,\ x_n \in V$. Therefore $\lim x_n = x$ with respect to τ.
(2) (2.1) Since $p(x,y) = 0$, then $p(x,y) < \epsilon$ for all $\epsilon > 0$. Let V be a neighborhood of x. Then there exists $\epsilon > 0$ such that $B_p(x,\epsilon) \subset V$, which implies that $y \in V$. Since V is arbitrary, we conclude $y = x$. (2.2) From (2.1), $\lim p(x,x_n) = 0$ and $\lim p(y,x_n) = 0$ imply $\lim x_n = x$ and $\lim x_n = y$ with respect to the topology τ which is Hausdorff. Thus $x = y$.

Let (X,τ) be a topological space with a τ-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition with respect to p. There are several concepts of completeness in this setting.

Definition 3.1. Let (X,τ) be a topological space with a τ-distance p.

(1) X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim p(x,x_n) = 0$.
(2) X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim x_n = x$ with respect to τ.
(3) X is said to be p-bounded if $sup\{p(x,y)/x,y \in X\} < \infty$.

remark 3.1. Let (X,τ) be a topological space with a τ-distance p and let (x_n) be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim p(x_n,x) = 0$. Lemma 4.1(b) then gives $\lim x_n = x$ with respect to the topology τ. Therefore S-completeness implies p-Cauchy completeness.

4. Fixed point theorem

In what follows, we involve a function $\psi : IR^+ \longrightarrow IR^+$ which satisfies the following conditions

(1) ψ is nondecreasing on IR^+ ,
(2) $\lim \psi^n(t) = 0,\ \forall t \in]0,+\infty[$.

It is easy to see that under the above properties, ψ satisfies also the following condition

$$\psi(t) < t,\ foreach\ t \in]0,+\infty[$$

Theorem 4.1. *Let* (X,τ) *be a Hausdorff topological space with a* τ*-distance* p. *Suppose that* X *is p-bounded and S-complete. Let* f *be a selfmapping of* X *such that*

$$p(fx,fy) \leq \psi(p(x,y)), \quad \forall x,y \in X$$

Then f *has a unique fixed point.*

Proof. Let $x_0 \in X$. Consider the sequence (x_n) defined by

$$\begin{cases} x_0 \in X, \\ x_{n+1} = fx_n \end{cases}$$

We have

$$\begin{aligned} p(x_n, x_{n+m}) =& p(fx_{n-1}, fx_{n+m-1}) \\ \leq& \psi(p(x_{n-1}, x_{n+m-1})) = \psi(p(fx_{n-2}, fx_{n+m-2})) \\ \leq& \psi^2(p(x_{n-2}, x_{n+m-2})) \\ &.... \\ \leq& \psi^n(p(x_0, x_m)) \leq \psi^n(M) \end{aligned}$$

where $M = sup\{p(x,y)/x, y \in X\}$. Since $\lim \psi^n(M) = 0$, we deduce that the sequence (x_n) is a p-cauchy sequence. X is S-complete, then $\lim p(u, x_n) = 0$, for some $u \in X$, and therefore $\lim p(u, x_{n+1}) = 0$ and $\lim p(fu, fx_n) = 0$. Now, we have $\lim p(fu, x_{n+1}) = 0$ and $\lim p(u, x_{n+1}) = 0$. Therefore, lemma 3.1(2.2) then gives $fu = u$. Suppose that there exists $u, v \in X$ such that $fu = u$ and $fv = v$. If $p(u, v) \neq 0$, then

$$p(u,v) = p(fu, fv) \leq \psi(p(u,v)) < p(u,v)$$

a contradiction. Therefore the fixed point is unique. Hence we have the theorem.

When $\psi(t) = kt,\ k \in [0,1[$, we get the following result, which gives a generalization of Banach's fixed point theorem in this new setting

Corollary 4.1. *Let (X,τ) be a Hausdorff topological space with a τ-distance p. Suppose that X is p-bounded and S-complete. Let f be a selfmapping of X such that*

$$p(fx, fy) \leq kp(x,y),\ k \in [0,1[,\ \forall x, y \in X$$

Then f has a unique fixed point.

Since a symmetric space (X,d) admits a τ_d-distance where τ_d is the topology defined earlier in example 2.3, corollary 4.1 gives a genaralization of the following known result (Theorem 1[5] for $f = Id_X$ which generalize Proposition 1[3]). Recall that $(W.3)$ denotes the following axiom given by Wilson [8] in a symmetric space (X,d): $(W.3)$ Given $\{x_n\}, x$ and y in X, $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$ imply $x = y$. It is clear that $(W.3)$ guarantees the uniqueness of limits of sequences.

corollary 4.2. *Let (X,d) be a d-bounded and S-complete symmetric space satisfying $(W.3)$ and f be a selfmapping of X such that*

$$d(fx, fy) \leq kd(x,y), \quad k \in [0,1[, \quad \forall x, y \in X$$

Then f has a fixed point.

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