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# FIXED POINTS FOR NEAR-CONTRACTIVE TYPE MULTIVALUED MAPPINGS

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Abstract. In the present paper we prove some fixed point theorems for near-contractive type multivalued mappings in complete metric spaces. these theorems extend some results in [1], [5], [6] and others

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**Key Words and Phrases.** Fixed points, multivalued mapping, near-contractive conditions, $\delta$-compatible mappings.

## 1 Basic Preliminaries

Let (X, d) be a metric space we put:

$$CB = \{A: \text{ A is a nonempty closed and bounded subset of X }\}$$
$$BN = \{A: \text{ A is a nonempty bounded subset of X }\}$$

If $A, B$ are any nonempty subsets of $X$ we put:

$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\},$$
$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$
$$H(A,B) = \max\{\ \{\sup\{D(a,B) : a \in A\}, \sup\{D(b,A) : b \in B\}\ \}.$$

If follows immediately from the definitoin that

$$\delta(A,B) = 0 \text{ iff } A = B = \{a\},$$
$$H(a,B) = \delta(a,B),$$
$$\delta(A,A) = diam A,$$
$$\delta(A,B) \leq \delta(A,C) + \delta(A,C),$$
$$D(a,A) = 0 \text{ if } a \in A,$$

for all $A, B, C$ in $BN(X)$ and $a$ in $X$.

In general both $H$ and $\delta$ may be infinite. But on $BN(X)$ they are finite. Moreover, on $CB(X)$ $H$ is actually a metric ( the Hansdorff metric).

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**Definition 1.1.** [2] *A sequence* $\{A_n\}$ *of subsets of* $X$ *is said to be convergent to a subset* $A$ *of* $X$ *if*

(i) *given* $a \in A$, *there is a sequence* $\{a_n\}$ *in* $X$ *such that* $a_n \in A_n$ *for* $n = 1, 2, ...,$ *and* $\{a_n\}$ *converges to* $a$

(ii) *given* $\varepsilon > 0$ *there exists a positive integer* $N$ *such that* $A_n \subseteq A_\varepsilon$ *for* $n > N$ *where* $A_\varepsilon$ *is the union of all open spheres with centers in* $A$ *and radius* $\varepsilon$

**Lemma 1.1.** [2,3].*If* $\{A_n\}$ *and* $\{B_n\}$ *are seqences in* $BN(X)$ *converging to* $A$ *and* $B$ *in* $BN(X)$ *respectively, then the sequence* $\{\delta(A_n, B_n)\}$ *converges to* $\delta(A, B)$.

**Lemma 1.2.** [3] *Let* $\{A_n\}$ *be a sequence in* $BN(X)$ *and* $x$ *be a point of* $X$ *such that* $\delta(A_n, x) \to 0$. *Then the sequence* $\{A_n\}$ *converges to the set* $\{x\}$ *in* $BN(X)$.

**Definition 1.2.** [3] *A set-valued mapping* $F$ *of* $X$ *into* $BN(X)$ *is said to be continuous at* $x \in X$ *if the sequence* $\{Fx_n\}$ *in* $BN(X)$ *converges to* $Fx$ *whenever* $\{x_n\}$ *is a sequence in* $X$ *converging to* $x$ *in* $X$. $F$ *is said continuous on* $X$ *if it is continuous at every point of* $X$.

The following Lemma was proved in [3]

**Lemma 1.3.** *Let* $\{A_n\}$ *be a sequence in* $BN(X)$ *and* $x$ *be a point of* $X$ *such that*

$$\lim_{n\to\infty} a_n = x,$$

$x$ *being independent of the particular choice of* $a_n \in A_n$. *If a selfmap* $I$ *of* $X$ *is continuous, then* $Ix$ *is the limit of the sequence* $\{IA_n\}$.

**Definition 1.3.** [4]. *The mappings* $I : X \to X$ *and* $F : X \to BN(X)$ *are* $\delta$-*compatible if* $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in* $X$ *such that* $IFx_n \in BN(X)$,

$$Fx_n \to t \quad \text{and} \quad Ix_n \to t$$

*for some* $t$ *in* $X$.

## 2. Our Results

We establish the following:

### 2. 1. A Coincidence Point Theorem

**Theorem 2.1.** *Let* $I : X \to X$ *and* $T : X \to BN(X)$ *be two mappings such that* $FX \subset IX$ *and*

$$(C.1) \quad \phi(\delta(Tx, Ty)) \le a\phi(d(Ix, Iy)) + b[\phi(H(Ix, Tx)) + \phi(H(Iy, Ty))] \\ + c\min\{\phi(D(Iy, Tx)), \phi(D(Ix, Ty))\},$$

*where* $x, y \in X$, $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is continuous and strictly increasing such that* $\phi(0) = 0$. $a, b, c$ *are nonnegative,* $a + 2b < 1$ *and* $a + c < 1$. *Suppose in addition that* $\{F, I\}$ *are* $\delta$-*compatible and* $F$ *or* $I$ *is continuous. Then* $I$ *and* $T$ *have a unique common fixed point* $z$ *in* $X$ *and further* $Tz = \{z\}$.

**Proof.** Let $x_0 \in X$ be an arbitrary point in $X$. Since $TX \subset IX$ we choose a point $x_1$ in $X$ such that $Ix_1 \in Tx_0 = Y_0$ and for this point $x_1$ there exists a point $x_2$ in $X$ such that $Ix_2 \in Tx_1 = Y_1$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows:

$$Ix_{n+1} \in Tx_n = Y_n$$

For sinplicity, we can put $V_n = \delta(Y_n, Y_{n+1})$, for $n = 0, 1, 2, ....$ By $(C, 1)$ we have

$$\begin{aligned}\phi(V_n) &= \phi(\delta(Y_n, Y_{n+1})) = \phi(\delta(Tx_n, Tx_{n+1})) \\ &\leq a\phi(d(Ix_n, Ix_{n+1})) + b[\phi(H(Ix_n, Tx_n)) + \phi(H(Ix_{n+1}, Tx_{n+1}))] \\ &+ c \min\{\phi(D(Ix_{n+1}, Tx_n)), \phi(D(Ix_n, Tx_{n+1}))\} \\ &\leq A_1 + A_2 + A_3\end{aligned}$$

Where

$$\begin{aligned}A_1 &= a\phi(\delta(Y_{n-1}, Y_n)) \\ A_2 &= b[\phi(\delta(Y_{n-1}, Y_n)) + \phi(\delta(Y_n, Y_{n+1}))], \\ A_3 &= c\phi(D(Ix_{n+1}, Y_n)).\end{aligned}$$

So

$$\phi(V_n) \leq a\phi(V_{n-1}) + b[\phi(V_{n-1}) + \phi(V_n)]$$

Hence we have

$$\phi(V_n) \leq \frac{a+b}{1-b}\phi(V_{n-1}) < \phi(V_{n-1}) \quad (1)$$

Since $\phi$ is increasing, $\{V_n\}$ is a decreasing sequence. Let $lim_n V_n = V$, assume that $V > 0$. By letting $n \to \infty$ in (1), Since $\phi$ is continuous , we have:

$$\phi(V) \leq \frac{a+b}{1-b}\phi(V) < \phi(V),$$

which is contradiction , hence $V = 0$.

Let $y_n$ be an arbitrary point in $Y_n$ for $n = 0, 1, 2, ....$ Then

$$\lim_{n\to\infty} d(y_n, y_{n+1}) \leq \lim_{n\to\infty} \delta(Y_n, Y_{n+1}) = 0.$$

Now, we wish to show that $\{y_n\}$ is a Cauchy sequence, we proceed by contradiction. Then there exist $\varepsilon > 0$ and two sequences of natural numbers $\{m(i)\}$, $\{n(i)\}$, $m(i) > n(i)$, $n(i) \to \infty$ as $i \to \infty$ such taht

$$\delta(Y_{n(i)}, Y_{m(i)}) > \varepsilon \quad \text{while} \quad \delta(Y_{n(i)}, Y_{m(i)-1}) \leq \varepsilon$$

Then we have

$$\begin{aligned}\varepsilon < \delta(Y_{n(i)}, Y_{m(i)}) &\leq \delta(Y_{n(i)}, Y_{m(i)-1}) + \delta(Y_{m(i)-1}, Y_{m(i)}) \\ &\leq \varepsilon + V_{m(i)-1},\end{aligned}$$

since $\{V_n\}$ converges to 0, $\delta(Y_{n(i)}, Y_{m(i)}) \to \varepsilon$. Futhermore, by triangular inequality, it follows that

$$| \, \delta(Y_{n(i)+1}, Y_{m(i)+1}) - \delta(Y_{n(i)}, Y_{m(i)}) \, | \leq V_{n(i)} + V_{m(i)},$$

and therefore the sequence $\{\delta(Y_{n(i)+1}, Y_{m(i)+1})\}$ converges to $\varepsilon$

¿From (C. 2), we also deduce:

$$\begin{aligned}\phi(\delta(Y_{n(i)+1}, Y_{m(i)+1})) &= \phi(\delta(Tx_{n(i)+1}, Tx_{m(i)+1}))\\ &\le C_1 + C_2 + C_3\\ &\le C_4 + C_5 + C_6 \quad (4)\end{aligned}$$

Where

$$\begin{aligned}&C_1 = a\phi(d(Ix_{n(i)+1}, Ix_{m(i)+1})),\\ &C_2 = b\Big\{\phi(\delta(Ix_{n(i)+1}, Tx_{n(i)+1})) + \phi(\delta(Ix_{m(i)+1}, Tx_{m(i)+1}))\Big\},\\ &C_3 = cmin\{\phi(D(Ix_{n(i)+1}, Y_{m(i)+1}), \phi(D(Ix_{n(i)+1}, Y_{m(i)+1})\},\\ &C_4 = a\phi(\delta(Y_{n(i)}, Y_{m(i)})),\\ &C_5 = [\phi(V_{n(i)}) + \phi(V_{m(i)}],\\ &C_6 = c\phi(\delta(Y_{n(i)}, Y_{m(i)}) + V_{m(i)}).\end{aligned}$$

Letting $i \to \infty$ in (4), we have

$$\phi(\varepsilon) \le (a + c)\phi(\varepsilon) < \phi(\varepsilon)$$

This is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence in $X$ and it has a limit $y$ in $X$. So the sequence $\{Ix_n\}$ converge to $y$ and further, the sequence $\{Tx_n\}$ converge to set $\{y\}$. Now supose that $I$ is continuous. Then

$$I^2x_n \to Iy \quad \text{and} \quad ITx_n \to \{Iy\}$$

by Lemma 1.3. Since $I$ and $T$ are $\delta$-compatible. Therefore $TIx_n \to \{Iy\}$. Using inequality $(C.1)$ , we have

$$\begin{aligned}\phi(\delta(TIx_n, Tx_n)) \le\ & a\phi(d(I^2x_n, Ix_n)) + b[\phi(H(Ix_n, Tx_n)) + \phi(H(I^2x_n, TIx_n))]\\ &+ cmin\{\phi(D(Ix_n, TIx_n)), \phi(D(I^2x_n, Tx_n))\},\end{aligned}$$

for $n \ge 0$. As $n \to \infty$ we obtain by Lemma 1.1

$$\phi(d(Iy, y)) \le a\phi(d(Iy, y)) + c\phi(d(y, Iy)),$$

That is $\phi(d(Iy, y)) = 0$ which implies that $Iy = y$. Further

$$\begin{aligned}\phi(\delta(Ty, Tx_n)) \le\ & a\phi(d(Iy, Ix_n)) + b[\phi(H(Iy, Ty)) + \phi(H(Ix_n, Tx_n))]\\ &+ cmin\{\phi(D(Ix_n, Ty)), \phi(D(Iy, Tx_n))\},\end{aligned}$$

for $n \ge 0$. As $n \to \infty$ we obtain by Lemma 1.1

$$\phi(\delta(Ty, y)) \le b\phi(\delta(Ty, y)),$$

which implies that $Ty = y$. Thus $y$ is a coincidence point for $T$ and $I$. Now suppose that $T$ and $I$ have a second common fixed point $z$ such that $Tz = \{z\} = \{Iz\}$. Then, using inequality $(C.1)$, we obtain

$$\phi(d(y, z)) = \phi(\delta(Ty, Tz)) \le (a + c)\phi(d(z, y)) < \phi(d(z, y))$$

which is a contradiction. This completes the proof of the Theorem.

**Corollary 2.1** ([6.$Theorem$2.1]). *Let* $(X, d)$ *be a complete metric space,* $T : X \longrightarrow CB(X)$ *a multi-valued map satisfying the following condition :*

$$\phi(\delta(Tx, Ty)) \le a\phi(d(x, y)) + b\Big[\phi(\delta(x, Tx)) + \phi(\delta(y, Ty))\Big] + \\ + c\min\Big\{\phi(d(x, Ty)), \phi(d(y, Tx))\Big\} \qquad \forall x, y \in X,$$

*where* $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is continuous and strictly increasing such that* $\phi(0) = 0$ *and* $a, b, c$ *are three positive constants such that* $a + 2b < 1$ *and* $a + c < 1$*, then* $T$ *has a unique fixed point.*

Note that the proof of Theorem 2.1 is another proof of Corollary 2.1 which is of interest in part because it avoids the use of Axiom of choice.

## 2. 2. A Fixed Point Theorem

**Theorem 2.2.** *Let* $(X, d)$ *be a complete metric space. If* $F : X \to CB(X)$ *is a multi-valued mapping and* $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is continuous and strictly increasing such that* $\phi(0) = 0$*. Furthermore, let* $a, b, c$ *be three functions from* $(0, \infty)$ *into* $[0, 1)$ *such that*
$a + 2b :\ (0, \infty) \to [0, 1)$ *and* $a + c :\ (0, \infty) \to [0, 1)$ *are decreasing functions. Suppose that* $F$ *satisfies the following condition:*

$$(C.3) \quad \phi(\delta(Fx, Fy)) \le a(d(x, y))\phi(d(x, y)) + b(d(x, y))[\phi(H(x, Fx)) + \phi(H(y, Fy))] \\ + c(d(x, y))\ min\{\phi(D(y, Fx)), \phi(D(x, Fy))\},$$

*then* $F$ *has a unique fixed point* $z$ *in* $X$ *such that* $Fz = \{z\}$*.*

**Proof..** First we will establish the existence of a fixed point. Put $p = max\{(a + 2b)^{\frac{1}{2}}, (a+c)^{\frac{1}{2}}\}$, take any $x_o$ in $X$. Since we may assume that $D(x_0, Fx_0)$ is positive, we can choose $x_1 \in Fx_0$ which satisfies $\phi(d(x_0, x_1)) \ge p(D(x_0, Fx_0))\phi(H(x_0, Fx_0))$, we may assume that $p(d(x_0, x_1))$ is positive. Assuming now that $D(x_1, Fx_1)$ is positive, we choose $x_2 \in Fx_1$ such that $\phi(d(x_1, x_2)) \ge p(d(x_0, x_1))\phi(H(x_1, Fx_1))$ and $\phi(d(x_1, x_2)) \ge p(D(x_1, Fx_1))\phi(d(x_1, Fx_1))$, since $d(x_0, x_1) \ge D(x_0, Fx_0)$ and $p$ is deceasing then we have also
$\phi(d(x_0, x_1)) \ge p(d(x_0, x_1))\phi(H(x_0, Fx_0))$. Now

$$\begin{aligned} \phi(d(x_1, x_2)) &\le \phi(\delta(Fx_0, Fx_1)) \\ &\le a(d(x_0, x_1))\phi(d(x_0, x_1)) + b(d(x_0, x_1))[\phi(H(x_0, Fx_0)) + \phi(H(x_1, Fx_1))] \\ &+ c(d(x_0, x_1))\ min\{\phi(D(Fx_0, x_1)), \phi(D(x_0, Fx_1))\} \\ &\le ap^{-1}\phi(d(x_0, x_1)) + bp^{-1}[\phi(d(x_0, x_1)) + \phi(d(x_1, x_2))], \end{aligned}$$

which implies
$$\phi(d(x_1, x_2)) \le q(d(x_0, x_1))\phi(d(x_0, x_1))$$
where
$$q : (0, \infty) \to [0, 1)$$
is defined by
$$q = \frac{a + b}{p - b}.$$

Note that $r \geq t$ implies $q(r) \leq p(t) < 1$. By induction, assumunig that $D(x_i, Fx_i)$ and $p(d(x_{i-1}, x_i))$ are positive, we obtain a sequence $\{x_i\}$ which satisfies $x_i \in Fx_{i-1}$, $\phi(d(x_{i-1}, x_i)) \geq p(d(x_{i-1}, x_i))\phi(H(x_{i-1}, Fx_{i-1}))$,

$$\phi(d(x_i, x_{i+1})) \geq p(d(x_{i-1}, x_i))\phi(H(x_i, Fx_i)),$$

$$\begin{aligned}\phi(d(x_i, x_{i+1})) &\leq q(d(x_{i-1}, x_i))\phi(d(x_{i-1}, x_i))\\ &\leq p(d(x_{i-1}, x_i))\phi(d(x_{i-1}, x_i))\\ &< \phi(d(x_{i-1}, x_i)).\end{aligned}$$

It is not difficult to verify that $lim_i\ d(x_i, x_{i+1}) = 0$. If $\{x_i\}$ is not Cauchy, there exists $\varepsilon > 0$ and two sequences of natural numbers $\{m(i)\}, \{n(i)\}$,

$m(i) > n(i) > i$ such that $d(x_{m(i)}, x_{n(i)}) > \varepsilon$ while $d(x_{m(i)-1}, x_{n(i)}) \leq \varepsilon$. It is not difficult to verify that

$$d(x_{m(i)}, x_{n(i)}) \to \varepsilon \text{ as } i \to \infty \text{ and } d(x_{m(i)+1}, x_{n(i)+1}) \to \varepsilon \text{ as } i \to \infty.$$

For $i$ sufficiently large $d(x_{m(i)}, x_{m(i)+1}) < \varepsilon$ and $d(x_{n(i)}, x_{n(i)+1}) < \varepsilon$. For these $i$ we have

$$\begin{aligned}\phi(d(x_{m(i)+1}, x_{n(i)+1})) &\leq \phi(\delta(Fx_{m(i)}, Fx_{n(i)}))\\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)}))\\ &+ b(d(x_{m(i)}, x_{n(i)}))[\phi(H(x_{m(i)}, Fx_{m(i)})) + \phi(H(x_{n(i)}, Fx_{n(i)}))]\\ &+ c(d(x_{m(i)}, x_{n(i)}))\ min\{\phi(D(x_{m(i)}, Fx_{n(i)})), \phi(D(x_{n(i)}, Fx_{m(i)})\}\\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)}))\\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{n(i)}, x_{n(i)+1}))\phi(d(x_{n(i)}, x_{n(i)+1}))\\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{m(i)}, x_{m(i)+1}))\phi(d(x_m(i), x_{m(i)+1}))\\ &+ c(d(x_{n(i)}, x_{m(i)})\phi(d(x_{m(i)}, x_{n(i)+1}))\\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)+1}) + d(x_{n(i)+1}, x_{n(i)})\\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{n(i)}, x_{n(i)+1}))\phi(d(x_{n(i)}, x_{n(i)+1}))\\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{m(i)}, x_{m(i)+1}))\phi(d(x_{m(i)}, x_{m(i)+1}))\\ &+ c(d(x_{n(i)}, x_{m(i)})\phi(d(x_{m(i)}, x_{n(i)+1} + d(x_{n(i)+1}, x_{n(i)}))\\ &\leq [a(\varepsilon) + c(\varepsilon)]\phi(d(x_{m(i)}, x_{n(i)}) + d(x_{n(i)}, x_{n(i)+1}))\\ &+ \phi(d(x_{m(i)}, x_{m(i)+1})) + \phi(d(x_{n(i)}, x_{n(i)+1})) \quad (*)\end{aligned}$$

Letting $i \to \infty$ in $(*)$, we have: $\phi(\varepsilon) \leq [a(\varepsilon) + c(\varepsilon)]\phi(\varepsilon) < \phi(\varepsilon)$. This is contradiction. Hence $\{x_i\}$ is cauchy sequence in a complete metric space $X$, then there existe a point $x \in X$ such that $x_n \to x$ as $i \to \infty$. This $x$ is a fixed point of $F$ because

$$\begin{aligned}\phi(H(x_{i+1}, Fx)) = \phi(\delta(x_{i+1}, Fx)) &\leq \phi(\delta(Fx_i, Fx))\\ &\leq a(d(x_i, x))\phi(d(x_i, x))\\ &+ b(d(x_i, x))[\phi(H(x, Fx)) + \phi(H(x_i, Fx_i))]\\ &+ c(d(x_i, x))\ min\{\phi(D(x_i, Fx)), \phi(D(x, Fx_i))\}\\ &\leq a(d(x_i, x))\phi(d(x_i, x))\\ &+ b(d(x_i, x))p^{-1}(d(x_i, x_{i+1})\phi(d(x_i, x_{i+1}))\\ &+ b(d(x_i, x))\phi(H(x, Fx)) + c(d(x_i, x))\phi(d(x, x_{i+1}) \quad (**)\end{aligned}$$

Using $b < \frac{1}{2}$, $p^{-1}(d(x_i, x_{i+1})) < p^{-1}(d(x_0, x_1))$ and letting $i \to \infty$ in $(**)$, we have:

$$\phi(\delta(x, Fx)) \leq \frac{1}{2}\phi(H(x, Fx)).$$

That is $\phi(H(x, Fx)) = 0$ and therefore $H(x, Fx) = 0$ i.e, $Fx = x$. $Fx = \{x\}$. We claim that $x$ is unique fixed point of $F$. For this, we suppose that $y$ $(x \neq y)$ is another fixed point of $F$ such that $Fy = \{y\}$. Then

$$\begin{aligned}\phi(d(y, x)) &\leq \phi(\delta(Fy, Fx)) \\ &\leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Fy))] \\ &+ c\ min\{\phi(D(x, Fy)), \phi(D(y, Fx))\} \\ &\leq [a + c]\phi(d(x, y)) < \phi(d(x, y)),\end{aligned}$$

a contradiction. This completes the proof of the theorem.

We may establish a common fixed point theorem for a pair of mappings $F$ and $G$ which stisfying the contractive condition corresponding to $(C.1)$, i.e., for all $x, y \in X$

$$(C.2) \quad \phi(\delta(Fx, Gy)) \leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Gy))] + c\ min\{\phi(D(y, Fx)), \phi(D(x, Gy))\},$$

## 2. 3 A Common Fixed Point Theorem.

**Theorem 2.3.** *Let $(X, d)$ be a metric space. Let $F$ and $G$ be two mappings of $X$ into $BN(X)$ and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$. Furthermore, let a, b, c be three nonnegative constants such that $a + 2b < 1$ and $a + c < 1$. Suppose that $F$ and $G$ satisfy $(C.2)$. Then $F$ and $G$ have a unique common fixed point. This fixed point satisfies $Fx = Gx = \{x\}$.*

**Proof.** Put $p = max\{(a+2b)^{\frac{1}{2}}, c^{\frac{1}{2}}\}$. we may assume that is positive. We define by using the Axiom of choice the two single-valued functions $f, g : X \to X$ by letting $f(x)$ be a point $w_1 \in Fx$ and $g(x)$ be a point $w_2 \in Gx$ such that $\phi(d(x, w_1)) \geq p\phi(H(x, Fx))$ and $\phi(d(x, w_2)) \geq p\phi(H(x, Gx))$. Then for every $x, y \in X$ we have:

$$\begin{aligned}\phi(d(f(x), g(y))) \leq \phi(\delta(Fx, Gy)) &\leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Gy))] \\ &+ c\ min\{\phi(D(y, Fx)), \phi(D(x, Gy))\} \\ &\leq a\phi(d(x, y)) + p^{-1}b[\phi(d(x, fx)) + \phi(d(y, gy))] \\ &+ c\ min\{\phi(d(y, fx)), \phi(d(x, gy))\}.\end{aligned}$$

Since $a+2p^{-1}b \leq p^{-1}(a+2b) \leq p < 1$, from [7, Theorem 2.1] we conclude that $f$ and $g$ has a common fixed point. That is, there exists a point $x$ such that $0 = d(x, fx) = \phi(d(x, fx)) \geq p\phi(H(x, Fx))$ and $0 = d(x, gx) = \phi(d(x, gx)) \geq p\phi(H(x, Gx))$ which implies $\phi(H(x, Fx)) = 0$ and $\phi(H(x, Gx)) = 0$, then $H(x, Fx) = \delta(x, Fx) = 0$ and $H(x, Gx) = \delta(x, Gx) = 0$ i.e. $Fx = Gx = \{x\}$. Hence $F$ and $G$ have a common fixed point $x \in X$. We claim that $x$ is unique common fixed point of $F$ and $G$. For this, we suppose that $y$ $(x \neq y\ )$ is another fixed point of $F$ and $G$. Since $y \in Fy$ and $y \in Gy$, from $(C.2)$ we have

$$\begin{aligned}\max\{\phi(H(y, Fy)), \phi(H(y, Gy))\} &\leq \phi(\delta(Fy, Gy)) \\ &\leq b[\phi(H(y, Fy)) + \phi(H(y, Gy))] \\ &\leq 2b \max\{\phi(H(y, Fy)), \phi(H(y, Gy))\}\end{aligned}$$

which implies $\delta(Fy, Gy) = 0$, that is $Fy = Gy = \{y\}$. Then

$$\begin{aligned}\phi(d(y,x)) &= \phi(\delta(Fy, Gx))\\ &\leq a\phi(d(x,y)) + b[\phi(H(x,Gx)) + \phi(H(y,Fy))]\\ &+ c\min\{\phi(D(x,Fy)), \phi(D(y,Gx))\}\\ &\leq [a+c]\phi(d(x,y)) < \phi(d(x,y)),\end{aligned}$$

a contradiction. This completes the proof of the theorem.

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