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RADIAL MINIMIZER OF A P-GINZBURG-LANDAU TYPE FUNCTIONAL WITH NORMAL IMPURITY INCLUSION

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Abstract. The author proves the $W^{1,p}$ and $C^{1,\alpha}$ convergence of the radial minimizers u_ε of an Ginzburg-Landau type functional as $\varepsilon \to 0$. The zeros of the radial minimizer are located and the convergent rate of the module of the minimizer is estimated.

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§1. Introduction

Let $n \geq 2, B_r = \{x \in R^n; |x| < r\}, g(x) = x$ on ∂B_1. Recall the Ginzburg-Landau type functional

$$E_\varepsilon(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{B_1\setminus B_\Gamma} (1-|u|^2)^2 + \frac{1}{4\varepsilon^2}\int_{B_\Gamma} |u|^2,$$

on the class functions $H_g^1(B_1, R^n)$. The functional $E_\varepsilon(u)$ is related to the Ginzburg-Landau model of superconductivity with normal impurity inclusion such as superconducting normal junctions (cf. [5]) if $n = 2$. $B_1 \setminus \overline{B_\Gamma}$ and B_Γ represent the domains occupied by superconducting materials and normal conducting materials, respectively. The minimizer u_ε is the order parameter. Zeros of u_ε are known as Ginzburg-Landau vortices which are of significance in the theory of superconductivity(cf. [1]). The paper [7] studied the asymptotic behaviors of the minimizer of $E_\varepsilon(u, B_1)$ on the function class $H_g^1(B_1, R^2)$ and discussed the vortex-pinning effect. For the simplified Ginzburg-Landau functional, many papers stated the asymptotic behavior of the minimizer u_ε as $\varepsilon \to 0$. When $n = 2$, the asymptotics of u_ε were well-studied by [1]. In the case of higher dimension, for the radial minimizer u_ε of $E_\varepsilon(u, B_1)$, some results on the convergence had been shown in [14] as $\varepsilon \to 0$. There

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were many works for the radial minimizer in [12]. Other related works can be seen in [2] [3] [8] and [17] etc.

Assume $p > n$. Consider the minimizers of the p-Ginzburg-Landau type functional

$$E_\varepsilon(u, B_1) = \frac{1}{p}\int_{B_1} |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_{B_1 \setminus B_\Gamma} (1-|u|^2)^2 + \frac{1}{4\varepsilon^p}\int_{B_\Gamma} |u|^4,$$

on the class functions

$$W = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B_1, R^n); f(1) = 1, r = |x|\}.$$

By the direct method in the calculus of variations we can see that the minimizer u_ε exists and it will be called *radial minimizer*. In this paper, we suppose that $\Gamma \in (0, \varepsilon]$. The conclusion of the case of $\Gamma = O(\varepsilon)$ as $\varepsilon \to 0$ is still true by the same argument. we will discuss he location of the zeros of the radial minimizer. Based on the result, we shall establish the uniqueness of the radial minimizer. The asymptotic behavior of the radial minimizer be concerned with as $\varepsilon \to 0$. The estimates of the rate of the convergence for the module of minimizer will be presented.

We will prove the following theorems.

Theorem 1.1. *Assume $u_\varepsilon$is a radial minimizer of $E_\varepsilon(u, B_1)$. Then for any given $\eta \in (0, 1/2)$ there exists a constant $h = h(\eta) > 0$ such that*

$$Z_\varepsilon = \{x \in B_1; |u_\varepsilon(x)| < 1 - 2\eta\} \subset B(0, h\varepsilon) \cup B_\Gamma.$$

Moreover, the zeros of the radial minimizer are contained in $B_{h\varepsilon}$ as $\Gamma \in (0, h\varepsilon]$. When $\Gamma \in (h\varepsilon, \varepsilon]$, the zeros are contained in $B_\Gamma \setminus B(0, h\varepsilon)$.

Theorem 1.2. *For any given $\varepsilon \in (0, 1)$, the radial minimizers of $E_\varepsilon(u, B_1)$ are unique on W.*

Theorem 1.3. *Assume u_ε is the radial minimizer of $E_\varepsilon(u, B_1)$. Then as $\varepsilon \to 0$,*

$$u_\varepsilon \to \frac{x}{|x|}, \quad in \quad W^{1,p}_{loc}(\overline{B_1} \setminus \{0\}, R^n); \tag{1.1}$$

$$u_\varepsilon \to \frac{x}{|x|}, \quad in \quad C^{1,\beta}_{loc}(B_1 \setminus \{0\}, R^n), \tag{1.2}$$

for some $\beta \in (0, 1)$.

Theorem 1.4. *Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be the radial minimizer of $E_\varepsilon(u, B_1)$. Then for any $T > 0$, there exist $C, \varepsilon_0 > 0$ such that as $\varepsilon \in (0, \varepsilon_0)$,*

$$\int_T^1 r^{n-1}[(f'_\varepsilon)^p + \frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]dr \le C\varepsilon^p.$$

$$\sup_{r\in[T,1]}(1 - f_\varepsilon(r)) \le C\varepsilon^{p-\frac{n}{2}}.$$

Some basic properties of minimizers are given in §2. The main purpose of §3 is to prove Theorem 1.1. In §4 and §5 we present the proof of (1.1). The proof of Theorem 1.2 is given in §6. §7 gives the proof of (1.2). Theorem 1.4 is derived in §8.

§2. Preliminaries

In polar coordinates, for $u(x) = f(r)\frac{x}{|x|}$ we have

$$|\nabla u| = (f_r^2 + (n-1)r^{-2}f^2)^{1/2}, \quad \int_{B_1} |u|^p = |S^{n-1}| \int_0^1 r^{n-1}|f|^p \, dr,$$

$$\int_{B_1} |\nabla u|^p = |S^{n-1}| \int_0^1 r^{n-1}(f_r^2 + (n-1)r^{-2}f^2)^{p/2} \, dr.$$

It is easily seen that $f(r)\frac{x}{|x|} \in W^{1,p}(B_1, R^n)$ implies $f(r)r^{\frac{n-1}{p}-1}, f_r(r)r^{\frac{n-1}{p}} \in L^p(0,1)$. Conversely, if $f(r) \in W^{1,p}_{loc}(0,1], f(r)r^{\frac{n-1}{p}-1}, f_r(r)r^{\frac{n-1}{p}} \in L^p(0,1)$, then $f(r)\frac{x}{|x|} \in W^{1,p}(B_1, R^n)$. Thus if we denote

$$V = \{f \in W^{1,p}_{loc}(0,1]; r^{\frac{n-1}{p}} f_r, r^{(n-1-p)/p} f \in L^p(0,1), f(r) \geq 0, f(1) = 1\},$$

then $V = \{f(r); u(x) = f(r)\frac{x}{|x|} \in W\}$.

Substituting $u(x) = f(r)\frac{x}{|x|} \in W$ into $E_\varepsilon(u, B_1)$, we obtain

$$E_\varepsilon(u, B_1) = |S^{n-1}|E_\varepsilon(f)$$

where

$$E_\varepsilon(f) = \frac{1}{p}\int_0^1 (f_r^2 + (n-1)r^{-2}f^2)^{p/2} r^{n-1} dr + \frac{1}{4\varepsilon^p}\int_\Gamma^1 (1-f^2)^2] r^{n-1} \, dr + \frac{1}{4\varepsilon^p}\int_0^\Gamma f^4 r^{n-1} \, dr.$$

This implies that $u = f(r)\frac{x}{|x|} \in W$ is the minimizer of $E_\varepsilon(u, B_1)$ if and only if $f(r) \in V$ is the minimizer of $E_\varepsilon(f)$.

Proposition 2.1. *The set V defined above is a subset of $\{f \in C[0,1]; f(0) = 0\}$.*

Proof. Let $f \in V$ and $h(r) = f(r^{\frac{p-1}{p-n}})$.Then

$$\int_0^1 |h'(r)|^p \, dr = (\frac{p-1}{p-n})^p \int_0^1 |f'(r^{\frac{p-1}{p-n}})|^p r^{\frac{p(n-1)}{p-n}} \, dr = (\frac{p-1}{p-n})^{p-1} \int_0^1 s^{n-1}|f'(s)|^p \, ds < \infty$$

by noting $f_s(s)s^{(n-1)/p} \in L^p(0,1)$. Using interpolation inequality and Young inequality, we have that for some $y > 1$,

$$\|h\|_{W^{1,y}((0,1),R)} < \infty,$$

which implies that $h(r) \in C[0,1]$ and hence $f(r) \in C[0,1]$.

Suppose $f(0) > 0$, then $f(r) \geq s > 0$ for $r \in [0,t)$ with $t > 0$ small enough since $f \in C[0,1]$. We have

$$\int_0^1 r^{n-1-p} f^p \, dr \geq s^p \int_0^t r^{n-1-p} \, dr = \infty,$$

which contradicts $r^{(n-1)/p-1} f \in L^p(0,1)$. Therefore $f(0) = 0$ and the proof is complete.

It is not difficult to prove the following

Proposition 2.2. *The functional $E_\varepsilon(u, B_1)$ achieves its minimum on W by a function $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$.*

Proposition 2.3. *The minimizer u_ε satisfies the equality*

$$\int_{B_1} |\nabla u|^{p-2}\nabla u\nabla\phi dx - \frac{1}{\varepsilon^p}\int_{B_1\setminus B_\Gamma} u\phi(1-|u|^2)dx + \frac{1}{\varepsilon^p}\int_{B_\Gamma} u\phi|u|^2 dx = 0, \tag{2.1}$$

$$\forall\phi = f(r)\frac{x}{|x|} \in C_0^\infty(B_1, R^n). \quad u|_{\partial B_1} = x. \tag{2.2}$$

Proof. Denote u_ε by u. For any $t \in [0,1)$and $\phi = f(r)\frac{x}{|x|} \in C_0^\infty(B_1, R^n)$, we have $u + t\phi \in W$ as long as t is small sufficiently. Since u is a minimizer we obtain

$$\frac{dE_\varepsilon(u + t\phi, B_1)}{dt}|_{t=0} = 0,$$

namely,

$$\begin{aligned}
0 &= \frac{d}{dt}|_{t=0}\int_{B_1}\frac{1}{p}|\nabla(u+t\phi)|^p + \frac{1}{4\varepsilon^p}\int_{B_1\setminus B_\Gamma}(1-|u+t\phi|^2)^2 dx \\
&\quad + \frac{1}{4\varepsilon^p}\int_{B_\Gamma}|u+t\phi|^4 dx \\
&= \int_{B_1}|\nabla u|^{p-2}\nabla u\nabla\phi dx - \frac{1}{\varepsilon^p}\int_{B_1\setminus B_\Gamma}u\phi(1-|u|^2)dx + \frac{1}{\varepsilon^p}\int_{B_\Gamma}u\phi|u|^2 dx.
\end{aligned}$$

By a limit process we see that the test function ϕ can be any member of $\{\phi = f(r)\frac{x}{|x|} \in W^{1,p}(B_1, R^n); \phi|_{\partial B_1} = 0\}$.

Similarly, we also derive

The minimizer $f_\varepsilon(r)$ of the functional $E_\varepsilon(f)$ satisfies

$$\begin{aligned}
&\int_0^1 r^{n-1}(f_r^2 + (n-1)r^{-2}f^2)^{(p-2)/2}(f_r\phi_r + (n-1)r^{-2}f\phi)\,dr \\
&= \frac{1}{\varepsilon^p}\int_\Gamma^1 r^{n-1}(1-f^2)f\phi\,dr - \frac{1}{\varepsilon^p}\int_0^\Gamma r^{n-1}f^3\phi\,dr, \quad \forall\phi \in C_0^\infty(0,1).
\end{aligned} \tag{2.3}$$

By a limit process we see that the test function ϕ in (2.3) can be any member of

$$X = \{\phi(r) \in W^{1,p}_{loc}(0,1]; \phi(0) = \phi(1) = 0, \phi(r) \geq 0, r^{\frac{n-1}{p}}\phi', r^{\frac{n-p-1}{p}}\phi \in L^p(0,1)\}$$

Proposition 2.4. *Let f_ε satisfies (2.3) and $f(1) = 1$. Then $f_\varepsilon \leq 1$ on [0,1].*

Proof. Denote $f = f_\varepsilon$ in (2.3) and set $\phi = f(f^2-1)_+$. Then

$$\begin{aligned}
&\int_0^1 r^{n-1}(f_r^2 + (n-1)r^{-2}f^2)^{(p-2)/2}[f_r^2(f^2-1)_+ + ff_r[(f^2-1)_+]_r \\
&+ (n-1)r^{n-3}f^2(f^2-1)_+]\,dr + \frac{1}{\varepsilon^p}\int_\Gamma^1 r^{n-1}f^2(f^2-1)_+^2\,dr \\
&+ \frac{1}{\varepsilon^p}\int_0^\Gamma r^{n-1}f^4(f^2-1)_+\,dr = 0
\end{aligned}$$

from which it follows that

$$\frac{1}{\varepsilon^p}\int_\Gamma^1 r^{n-1}f^2(f^2-1)_+^2\,dr + \frac{1}{\varepsilon^p}\int_0^\Gamma r^{n-1}f^4(f^2-1)_+\,dr = 0$$

Thus $f = 0$ or $(f^2-1)_+ = 0$ on $[0,1]$ and hence $f = f_\varepsilon \leq 1$ on [0,1].

Proposition 2.5. *Assume u_ε is a weak radial solution of (2.1)(2.2). Then there exist positive constants C_1, ρ which are both independent of ε such that*

$$\|\nabla u_\varepsilon(x)\|_{L^\infty(B(x,\rho\varepsilon/8))} \le C_1\varepsilon^{-1}, \quad if \quad x \in B(0, 1-\rho\varepsilon), \tag{2.4}$$

$$|u_\varepsilon(x)| \ge \frac{10}{11}, \quad if \quad x \in \overline{B_1} \setminus B(0, 1-2\rho\varepsilon). \tag{2.5}$$

Proof. Let $y = x\varepsilon^{-1}$ in (2.1) and denote $v(y) = u(x)$, $B_\varepsilon = B(0, \varepsilon^{-1})$. Then

$$\int_{B_\varepsilon} |\nabla v|^{p-2}\nabla v \nabla\phi dy = \int_{B_\varepsilon \setminus B(0,\Gamma\varepsilon^{-1})} v(1-|v|^2)\phi dy - \int_{B(0,\Gamma\varepsilon^{-1})} v\phi|v|^2 dy \tag{2.6}$$

$\forall \phi \in W_0^{1,p}(B_\varepsilon, R^n)$. This implies that $v(y)$ is a weak solution of (2.6). By using the standard discuss of the Holder continuity of weak solution of (2.6) on the boundary (for example see Theorem 1.1 and Line 19-21 of Page 104 in [4]) we can see that for any $y_0 \in \partial B_\varepsilon$ and $y \in B(y_0, \rho_0)$ (where $\rho_0 > 0$ is a constant independent of ε), there exist positive constants $C = C(\rho_0)$ and $\alpha \in (0,1)$ which are both independent of ε such that

$$|v(y) - v(y_0)| \le C(\rho_0)|y - y_0|^\alpha.$$

Choose $\rho > 0$ sufficiently small such that

$$y \in B(y_0, 2\rho) \subset B(y_0, \rho_0), \quad and \quad C(\rho_0)|y-y_0|^\alpha \le \frac{1}{11}, \tag{2.7}$$

then

$$|v(y)| \ge |v(y_0)| - C(\rho_0)|y-y_0|^\alpha = 1 - C(\rho_0)|y-y_0|^\alpha \ge \frac{10}{11}.$$

Let $x = y\varepsilon$. Thus

$$|u_\varepsilon(x)| \ge \frac{10}{11}, \quad if \quad x \in B(x_0, 2\rho\varepsilon)$$

where $x_0 \in \partial B_1$. This implies (2.5).

Taking $\phi = v\zeta^p, \zeta \in C_0^\infty(B_\varepsilon, R)$ in (2.6), we obtain

$$\int_{B_\varepsilon} |\nabla v|^p \zeta^p dy \le p\int_{B_\varepsilon} |\nabla v|^{p-1}\zeta^{p-1}|\nabla\zeta||v|dy + \int_{B_\varepsilon\setminus B(0,\Gamma\varepsilon^{-1})} |v|^2(1-|v|^2)\zeta^p dy + \int_{B(0,\Gamma\varepsilon^{-1})} |v^4|\zeta^p dy.$$

For the ρ in (2.7), setting $y \in B(0, \varepsilon^{-1} - \rho), B(y, \rho/2) \subset B_\varepsilon$, and

$$\zeta = 1 \ in \ B(y, \rho/4), \zeta = 0 \ in \ B_\varepsilon \setminus B(y, \rho/2), |\nabla\zeta| \le C(\rho),$$

we have

$$\int_{B(y,\rho/2)} |\nabla v|^p\zeta^p \le C(\rho)\int_{B(y,\rho/2)} |\nabla v|^{p-1}\zeta^{p-1} + C(\rho).$$

Using Holder inequality we can derive $\int_{B(y,\rho/4)} |\nabla v|^p \le C(\rho)$. Combining this with the Tolksdroff' theorem in [19] (Page 244 Line 19-23) yields

$$\|\nabla v\|^p_{L^\infty(B(y,\rho/8))} \le C(\rho)\int_{B(y,\rho/4)} (1+|\nabla v|)^p \le C(\rho)$$

which implies

$$\|\nabla u\|_{L^\infty(B(x,\varepsilon\rho/8))} \le C(\rho)\varepsilon^{-1}.$$

Proposition 2.6. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Then*

$$E_\varepsilon(u_\varepsilon, B_1) \le C\varepsilon^{n-p} + C, \tag{2.8}$$

with a constant C independent of $\varepsilon \in (0,1)$.

Proof. Denote

$$I(\varepsilon, R) = Min\{\int_{B(0,R)} [\frac{1}{p}|\nabla u|^p + \frac{1}{\varepsilon^p}(1-|u|^2)^2]; u \in W_R\},$$

where $W_R = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B(0,R), R^n); r = |x|, f(R) = 1\}$. Then

(2.9)

$$\begin{aligned}
&I(\varepsilon, 1) = E_\varepsilon(u_\varepsilon, B_1)\\
&= \frac{1}{p}\int_{B_1}|\nabla u_\varepsilon|^p dx + \frac{1}{4\varepsilon^p}\int_{B_1\setminus B_\Gamma}(1-|u_\varepsilon|^2)^2 dx + \frac{1}{4\varepsilon^p}\int_{B_\Gamma}|u_\varepsilon|^4 dx\\
&= \varepsilon^{n-p}[\frac{1}{p}\int_{B(0,\varepsilon^{-1})}|\nabla u_\varepsilon|^p dy + \frac{1}{4}\int_{B(0,\varepsilon^{-1})\setminus B(0,\Gamma\varepsilon^{-1})}(1-|u_\varepsilon|^2)^2 dy\\
&\quad + \frac{1}{4}\int_{B(0,\Gamma\varepsilon^{-1})}|u_\varepsilon|^4 dy] = \varepsilon^{n-p} I(1, \varepsilon^{-1}).
\end{aligned}$$

Let u_1 be a solution of $I(1,1)$ and define

$$u_2 = u_1, \quad if \quad 0 < |x| < 1; \quad u_2 = \frac{x}{|x|}, \quad if \quad 1 \le |x| \le \varepsilon^{-1}.$$

Thus $u_2 \in W_{\varepsilon^{-1}}$ and,

$$\begin{aligned}
&I(1, \varepsilon^{-1})\\
&\le \frac{1}{p}\int_{B(0,\varepsilon^{-1})}|\nabla u_2|^p + \frac{1}{4}\int_{B(0,\varepsilon^{-1})\setminus B(0,\Gamma\varepsilon^{-1})}(1-|u_2|^2)^2 + \frac{1}{4}\int_{B(0,\Gamma\varepsilon^{-1})}|u_\varepsilon|^4\\
&= \frac{1}{p}\int_{B_1}|\nabla u_1|^p + \frac{1}{4}\int_{B_1}(1-|u_1|^2)^2 + \frac{1}{4}\int_{B_1}|u_1|^4 dx + \frac{1}{p}\int_{B(0,\varepsilon^{-1})\setminus B_1}|\nabla\frac{x}{|x|}|^p\\
&= I(1,1) + \frac{(n-1)^{p/2}|S^{n-1}|}{p}\int_1^{\varepsilon^{-1}} r^{n-p-1} dr\\
&= I(1,1) + \frac{(n-1)^{p/2}|S^{n-1}|}{p(p-n)}(1-\varepsilon^{p-n}) \le C.
\end{aligned}$$

Substituting this into (2.9) yields (2.8).

§3. Proof of Theorem 1.1

Proposition 3.1. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Then for some constant C independent of $\varepsilon \in (0,1]$*

$$\frac{1}{\varepsilon^n}\int_{B_1\setminus B_\Gamma}(1-|u_\varepsilon|^2)^2 + \frac{1}{\varepsilon^n}\int_{B_\Gamma}|u_\varepsilon|^4 \le C. \tag{3.1}$$

Proof. (3.1) can be derived by multiplying (2.8) by ε^{p-n}.

Proposition 3.2. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Then for any $\eta \in (0, 1/2)$, there exist positive constants λ, μ independent of $\varepsilon \in (0,1)$ such that if*

$$\frac{1}{\varepsilon^n}\int_{A_{\Gamma,1-\rho\varepsilon}\cap B^{2l\varepsilon}}(1-|u_\varepsilon|^2)^2 \le \mu, \tag{3.2}$$

where $A_{\Gamma,1-\rho\varepsilon} = B(0, 1-\rho\varepsilon)\setminus B_\Gamma$, $B^{2l\varepsilon}$ is some ball of radius $2l\varepsilon$ with $l \ge \lambda$, then

$$|u_\varepsilon(x)| \ge 1-\eta, \quad \forall x \in A_{\Gamma,1-\rho\varepsilon}\cap B^{l\varepsilon}. \tag{3.3}$$

Proof. First we observe that there exists a constant $C_2 > 0$ which is independent of ε such that for any $x \in B_1$ and $0 < \rho \le 1$, $|B_1 \cap B(x,r)| \ge |A_{\Gamma,1-\rho\varepsilon}\cap B(x,r)| \ge C_2 r^n$. To prove the proposition, we choose

$$\lambda = \frac{\eta}{2C_1}, \quad \mu = \frac{C_2}{C_1^n}(\frac{\eta}{2})^{n+2}, \tag{3.4}$$

where C_1 is the constant in (2.4). Suppose that there is a point $x_0 \in A_{\Gamma,1-\rho\varepsilon}\cap B^{l\varepsilon}$ such that $|u_\varepsilon(x_0)| < 1-\eta$. Then applying (2.4) we have

$$\begin{aligned}|u_\varepsilon(x) - u_\varepsilon(x_0)| &\le C_1\varepsilon^{-1}|x-x_0| \le C_1\varepsilon^{-1}(\lambda\varepsilon)\\ &= C_1\lambda = \frac{\eta}{2}, \quad \forall x \in B(x_0, \lambda\varepsilon),\end{aligned}$$

hence $(1-|u_\varepsilon(x)|^2)^2 > \frac{\eta^2}{4}$, $\quad \forall x \in B(x_0,\lambda\varepsilon)$. Thus

$$\begin{aligned}\int_{B(x_0,\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}}(1-|u_\varepsilon|^2)^2 &> \frac{\eta^2}{4}|A_{\Gamma,1-\rho\varepsilon}\cap B(x_0,\lambda\varepsilon)|\\ &\ge C_2\frac{\eta^2}{4}(\lambda\varepsilon)^n = C_2\frac{\eta^2}{4}(\frac{\eta}{2C_1})^n\varepsilon^n = \mu\varepsilon^n.\end{aligned} \tag{3.5}$$

Since $x_0 \in B^{l\varepsilon}\cap B_1$, and $(B(x_0,\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}) \subset (B^{2l\varepsilon}\cap A_{\Gamma,1-\rho\varepsilon})$, (3.5) implies

$$\int_{B^{2l\varepsilon}\cap A_{\Gamma,1-\rho\varepsilon}}(1-|u_\varepsilon|^2)^2 > \mu\varepsilon^n,$$

which contradicts (3.2) and thus (3.3) is proved.

Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Given $\eta \in (0,1/2)$. Let λ, μ be constants in Proposition 3.2 corresponding to η. If

$$\frac{1}{\varepsilon^n}\int_{B(x^\varepsilon,2\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}}(1-|u_\varepsilon|^2)^2 \le \mu, \tag{3.6}$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called $\eta-$ good ball, or simply good ball. Otherwise it is called $\eta-$ bad ball or simply bad ball.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of balls satisfying

$$(i): x_i^\varepsilon \in A_{\Gamma,1-\rho\varepsilon}, i \in I; \quad (ii): A_{\Gamma,1-\rho\varepsilon} \subset \cup_{i\in I}B(x_i^\varepsilon,\lambda\varepsilon);$$

$$(iii): B(x_i^\varepsilon, \lambda\varepsilon/4)\cap B(x_j^\varepsilon,\lambda\varepsilon/4) = \emptyset, i \ne j. \tag{3.7}$$

Denote $J_\varepsilon = \{i \in I; B(x_i^\varepsilon, \lambda\varepsilon) \textit{ is a bad ball}\}$.

Proposition 3.3. *There exists a positive integer N such that the number of bad balls*

$$Card\ J_\varepsilon \le N.$$

Proof. Since (3.7) implies that every point in B_1 can be covered by finite, say m (independent of ε) balls, from (3.1)(3.6) and the definition of bad balls,we have

$$\begin{aligned}\mu\varepsilon^n Card J_\varepsilon &\le \sum_{i\in J_\varepsilon}\int_{B(x_i^\varepsilon,2\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}}(1-|u_\varepsilon|^2)^2\\ &\le m\int_{\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,2\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}}(1-|u_\varepsilon|^2)^2\\ &\le m\int_{B_1\setminus B_\Gamma}(1-|u_\varepsilon|^2)^2 \le mC\varepsilon^n\end{aligned}$$

and hence $Card\ J_\varepsilon \le \frac{mC}{\mu} \le N$.

Proposition 3.3 is an important result since the number of bad balls $Card J_\varepsilon$ is always finite as ε turns sufficiently small.

Similar to the argument of Theorem IV.1 in [1], we have

Proposition 3.4. *There exist a subset $J \subset J_\varepsilon$ and a constant $h \ge \lambda$ such that $\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon) \subset \cup_{i\in J}B(x_j^\varepsilon,h\varepsilon)$ and*

$$|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, \quad i,j\in J, \quad i\ne j. \tag{3.8}$$

Proof. If there are two points x_1, x_2 such that (3.8) is not true with $h=\lambda$, we take $h_1 = 9\lambda$ and $J_1 = J_\varepsilon \setminus \{1\}$. In this case, if (3.8) holds we are done. Otherwise we continue to choose a pair points x_3, x_4 which does not satisfy (3.8) and take $h_2 = 9h_1$ and $J_2 = J_\varepsilon\setminus\{1,3\}$. After at most N steps we may choose $\lambda \le h \le \lambda 9^N$ and conclude this proposition.

Applying Proposition 3.4, we may modify the family of bad balls such that the new one, denoted by $\{B(x_i^\varepsilon,h\varepsilon); i\in J\}$, satisfies

$$\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon)\subset\cup_{i\in J}B(x_i^\varepsilon,h\varepsilon), \quad Card\ J \le Card\ J_\varepsilon,$$

$$|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, i,j\in J, i\ne j.$$

The last condition implies that every two balls in the new family are not intersected.

Now we prove our main result of this section.

Theorem 3.5. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B_1)$. Then for any $\eta\in(0,1/2)$, there exists a constant $h=h(\eta)$ independent of $\varepsilon\in(0,1)$ such that $Z_\varepsilon = \{x\in B_1; |u_\varepsilon(x)|<1-\eta\}\subset B(0,h\varepsilon)\cup B_\Gamma$. In particular the zeros of u_ε are contained in $B(0,h\varepsilon)\cup B_\Gamma$.*

Proof. Suppose there exists a point $x_0\in Z_\varepsilon$ such that $x_0 \overline{\in} B(0,h\varepsilon)$. Then all points on the circle $S_0=\{x\in B_1;\ |x|=|x_0|\}$ satisfy $|u_\varepsilon(x)|<1-\eta$ and hence by virtue of Proposition 3.2 and (2.5), all points on S_0 are contained in bad balls. However, since $|x_0|\ge h\varepsilon, S_0$ can not be covered by a single bad ball. S_0 can be covered by at least two bad balls. However this is impossible. Theorem is proved.

Complete the proof of Theorem 1.1.. Using Theorem 3.5 and (2.5), we can see that $|u_\varepsilon(x)| \geq min(\frac{10}{11}, 1-2\eta), \quad x\overline{\in}B(0,h(\eta)\varepsilon)\cup B_\Gamma$. When $\Gamma \in (0, h\varepsilon]$, this means

$$|u_\varepsilon(x)| \geq min(\frac{10}{11}, 1-2\eta), \quad x\overline{\in}B(0,h(\eta)\varepsilon). \tag{3.9}$$

When $\Gamma \in (h\varepsilon, \varepsilon]$, from Theorem 3.5 we know that $|u_\varepsilon| \geq 1-\eta$ on $B_1 \setminus \overline{B_\Gamma}$. Moreover, similar to the proof of Proposition 3.2, we may still obtain: for any given $\eta \in (0, 1/2)$, there are $\lambda = \frac{\eta}{2C_1}, \quad \mu_2 = C_2\lambda^n(\frac{\eta}{2})^{n+2}$, such that if for $l > \lambda$,

$$\frac{1}{\varepsilon^n}\int_{B_\Gamma\cap B^{2l\varepsilon}} |u_\varepsilon|^4 \leq \mu_2 \tag{3.10}$$

holds, then $|u_\varepsilon(x)| \leq \eta, \quad \forall x \in B_\Gamma \cap B^{l\varepsilon}$. We will take (3.10) as the ruler which distinguishes the good and the bad balls. The ball $B(x^\varepsilon, \lambda\varepsilon)$ satisfying

$$\frac{1}{\varepsilon^2}\int_{B_\Gamma\cap B(x^\varepsilon,2\lambda\varepsilon)} |u_\varepsilon|^4 \leq \mu_2$$

is named the bad ball in B_Γ. Otherwise, the ball $B(x^\varepsilon, \lambda\varepsilon)$ is named the good ball in B_Γ. Similar to the proof of Proposition 3.3, from proposition 3.1 we may also conclude that the number of the good balls is finite. Moreover, by the same way to the proof of Theorem 3.5, we obtain that

$$\{x \in B_\Gamma; |u_\varepsilon(x)| > \eta\} \subset B_{h\varepsilon} \quad \textit{and} \quad |u_\varepsilon(x)| \leq \eta \quad \textit{as} \quad x \in B_\Gamma \setminus B_{h\varepsilon}. \tag{3.11}$$

§4. Uniform estimate

Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u, B_1)$, namely f_ε be a minimizer of $E_\varepsilon(f)$ in V. From Proposition 2.6, we have

$$E_\varepsilon(f_\varepsilon) \leq C\varepsilon^{n-p}. \tag{4.1}$$

for some constant C independent of $\varepsilon \in (0, 1)$.

In this section we further prove that for any given $R \in (0, 1)$, there exists a constant $C(R)$ such that

$$E_\varepsilon(f_\varepsilon; R) \leq C(R) \tag{4.2}$$

for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small, where

$$E_\varepsilon(f; R) = \frac{1}{p}\int_R^1 (f_r^2 + (n-1)r^{-2}f^2)^{p/2}r^{n-1}\,dr + \frac{1}{4\varepsilon^p}\int_R^1 (1-f^2)^2r^{n-1}\,dr.$$

Proposition 4.1. *Given $T \in (0, 1)$. There exist constants $T_j \in [\frac{(j-1)T}{N+1}, \frac{jT}{N+1}]$, $(N = [p])$ and C_j, such that*

$$E_\varepsilon(f_\varepsilon; T_j) \leq C_j\varepsilon^{j-p} \tag{4.3}$$

for $j = n, n+1, ..., N$, where $\varepsilon \in (0, \varepsilon_0)$ with ε_0 sufficiently small.

Proof. For $j = n$, the inequality (4.3) can be obtained by (4.1) easily. Suppose that (4.3) holds for all $j \le m$. Then we have, in particular,

$$E_\varepsilon(f_\varepsilon; T_m) \le C_m \varepsilon^{m-p}. \tag{4.4}$$

If $m = N$ then we have done. Suppose $m < N$, we want to prove (4.3) for $j = m+1$.

From (4.4) and integral mean value theorem, we can see that there exists $T_{m+1} \in [\frac{mT}{N+1}, \frac{(m+1)T}{N+1}]$ such that

$$\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2|_{r=T_{m+1}} \le C E_\varepsilon(u_\varepsilon, \partial B(0, T_{m+1})) \le C_m \varepsilon^{m-p}. \tag{4.5}$$

Consider the minimizer ρ_1 of the functional

$$E(\rho, T_{m+1}) = \frac{1}{p}\int_{T_{m+1}}^1 (\rho_r^2+1)^{p/2} dr + \frac{1}{2\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)^2 dr$$

It is easy to prove that the minimizer ρ_ε of $E(\rho, T_{m+1})$ on $W^{1,p}_{f_\varepsilon}((T_{m+1}, 1), R^+)$ exists and satisfies

$$-\varepsilon^p (v^{(p-2)/2}\rho_r)_r = 1-\rho, \quad in\ (T_{m+1}, 1), \tag{4.6}$$

$$\rho|_{r=T_{m+1}} = f_\varepsilon,\ \rho|_{r=1} = f_\varepsilon(1) = 1 \tag{4.7}$$

where $v = \rho_r^2 + 1$. Since $f_\varepsilon \le 1$, it follows from the maximum principle

$$\rho_\varepsilon \le 1. \tag{4.8}$$

Applying (4.1) we see easily that

$$E(\rho_\varepsilon; T_{m+1}) \le E(f_\varepsilon; T_{m+1}) \le C E_\varepsilon(f_\varepsilon; T_{m+1}) \le C\varepsilon^{m-p}. \tag{4.9}$$

Now choosing a smooth function $0 \le \zeta(r) \le 1$ in (0,1] such that $\zeta = 1$ on $(0, T_{m+1}), \zeta = 0$ near $r = 1$ and $|\zeta_r| \le C(T_{m+1})$, multiplying (4.6) by $\zeta\rho_r (\rho = \rho_\varepsilon)$ and integrating over $(T_{m+1}, 1)$ we obtain

$$v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} + \int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})\, dr = \frac{1}{\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)\zeta\rho_r\, dr. \tag{4.10}$$

Using (4.9) we have

$$\begin{aligned} &|\int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})\, dr| \\ &\le \int_{T_{m+1}}^1 v^{(p-2)/2}|\zeta_r|\rho_r^2\, dr + \frac{1}{p}|\int_{T_{m+1}}^1 (v^{p/2}\zeta)_r\, dr - \int_{T_{m+1}}^1 v^{p/2}\zeta_r\, dr| \\ &\le C\int_{T_{m+1}}^1 v^{p/2} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}} + \frac{C}{p}\int_{T_{m+1}}^1 v^{p/2} dr \\ &\le C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}} \end{aligned} \tag{4.11}$$

and using (4.5)(4.7)(4.9) we have

(4.12)

$$|\frac{1}{\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)\zeta\rho_r\,dr| = \frac{1}{2\varepsilon^p}|\int_{T_{m+1}}^1 ((1-\rho)^2\zeta)_r\,dr - \int_{T_{m+1}}^1 (1-\rho)^2\zeta_r\,dr|$$

$$\le \frac{1}{2\varepsilon^p}(1-\rho)^2|_{r=T_{m+1}} + \frac{C}{2\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)^2\,dr| \le C\varepsilon^{m-p}.$$

Combining (4.10) with (4.11)(4.12) yields

$$v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} \le C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}}.$$

Hence for any $\delta \in (0,1)$,

$$v^{p/2}|_{r=T_{m+1}} = v^{(p-2)/2}(\rho_r^2+1)|_{r=T_{m+1}} = v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} + v^{(p-2)/2}|_{r=T_{m+1}}$$

$$\le C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}} + v^{(p-2)/2}|_{r=T_{m+1}}$$

$$= C\varepsilon^{m-p} + (\frac{1}{p}+\delta)v^{p/2}|_{r=T_{m+1}} + C(\delta)$$

from which it follows by choosing $\delta > 0$ small enough that

$$v^{p/2}|_{r=T_{m+1}} \le C\varepsilon^{m-p}. \tag{4.13}$$

Now we multiply both sides of (4.6) by $\rho - 1$ and integrate. Then

$$-\varepsilon^p\int_{T_{m+1}}^1 [v^{(p-2)/2}\rho_r(\rho-1)]_r\,dr + \varepsilon^p\int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r^2\,dr + \int_{T_{m+1}}^1 (\rho-1)^2\,dr = 0.$$

From this, using(4.5)(4.7)(4.13), we obtain

$$E(\rho_\varepsilon;T_{m+1}) \le C|\int_{T_{m+1}}^1 [v^{(p-2)/2}\rho_r(\rho-1)]_r\,dr| \tag{4.14}$$

$$= Cv^{(p-2)/2}|\rho_r||\rho-1|_{r=T_{m+1}} \le Cv^{(p-1)/2}|\rho-1|_{r=T_{m+1}}$$

$$\le (C\varepsilon^{m-p})^{(p-1)/p}(C\varepsilon^m)^{1/2} \le C\varepsilon^{m-p+1}.$$

Define $w_\varepsilon = f_\varepsilon,\ for\ r \in (0,T_{m+1});\quad w_\varepsilon = \rho_\varepsilon,\ for\ r \in [T_{m+1},1]$. Since that f_ε is a minimizer of $E_\varepsilon(f)$, we have $E_\varepsilon(f_\varepsilon) \le E_\varepsilon(w_\varepsilon)$. Thus, it follows that

$$E_\varepsilon(f_\varepsilon;T_{m+1}) \le \frac{1}{n}\int_{T_{m+1}}^1 (\rho_r^2+(n-1)r^{-2}\rho^2)^{p/2}r^{n-1}\,dr + \frac{1}{4\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho^2)^2 r^{n-1}\,dr$$

by virtue of $\Gamma \le \varepsilon < T_{m+1}$ since ε is sufficiently small. Noticing that

$$\int_{T_{m+1}}^1 (\rho_r^2+(n-1)r^{-2}\rho^2)^{p/2}r^{n-1}dr - \int_{T_{m+1}}^1 ((n-1)r^2\rho^2)^{p/2}r^{n-1}dr$$

$$= \frac{p}{2}\int_{T_{m+1}}^1\int_0^1 [\rho_r^2+(n-1)r^{-2}\rho^2)s + (n-1)r^{-2}\rho^2(1-s)]^{(p-2)/2}ds\rho_r^2 r^{n-1}dr$$

$$\le C\int_{T_{m+1}}^1 (\rho_r^2+(n-1)r^{-2}\rho^2)^{(p-2)/2}\rho_r^2 r^{n-1}dr$$

$$+ C\int_{T_{m+1}}^1 ((n-1)r^{-2}\rho^2)^{(p-2)/2}\rho_r^2 r^{n-1}dr$$

$$\le C\int_{T_{m+1}}^1 (\rho_r^p+\rho_r^2)dr$$

and using (4.8) we obtain

$$\begin{aligned}
&E_\varepsilon(f_\varepsilon;T_{m+1})\\
&\le \frac{1}{p}\int_{T_{m+1}}^1((n-1)r^{-2}\rho^2)^{p/2}r^{n-1}\,dr + C\int_{T_{m+1}}^1(\rho_r^p+\rho_r^2)dr\\
&\ +\frac{C}{4\varepsilon^p}\int_{T_{m+1}}^1(1-\rho^2)^2dr\\
&\le \frac{1}{p}\int_{T_{m+1}}^1((n-1)r^{-2})^{p/2}r^{n-1}\,dr + CE(\rho_\varepsilon;T_{m+1}).
\end{aligned}$$

Combining this with (4.14) yields (4.3) for $j=m+1$. It is just (4.3) for $j=m+1$.

Proposition 4.2. *Given $T\in(0,1)$. There exist constants $T_{N+1}\in[\frac{NT}{N+1},T]$ and C_{N+1} such that*

$$\begin{aligned}
E_\varepsilon(u_\varepsilon;T_{N+1}) \le (n-1)^{p/2}\frac{|S^{n-1}|}{p}\int_{T_{N+1}}^1 r^{n-p-1}dr\\
+C_{N+1}\varepsilon^{N+1-p},\quad N=[p].
\end{aligned}$$

Proof. From (4.3) we can see $E_\varepsilon(u_\varepsilon;T_N)\le C\varepsilon^{N-p}$. Hence by using integral mean value theorem we know that there exists $T_{N+1}\in[\frac{NT}{N+1},T]$ such that

$$\frac{1}{p}\int_{\partial B(0,T_{N+1})}|\nabla u_\varepsilon|^pdx+\frac{1}{4\varepsilon^p}\int_{\partial B(0,T_{N+1})}(1-|u_\varepsilon|^2)^2dx\le C\varepsilon^{N-p}. \tag{4.15}$$

Denote ρ_2 is a minimizer of the functional

$$E(\rho,T_{N+1})=\frac{1}{p}\int_{T_{N+1}}^1(\rho_r^2+1)^{p/2}dr+\frac{1}{2\varepsilon^p}\int_{T_{N+1}}^1(1-\rho)^2dr$$

on $W_{f_\varepsilon}^{1,p}((T_{N+1},1),R^+\cup\{0\})$. It is not difficult to prove by maximum principle that

$$\rho_2\le 1. \tag{4.16}$$

By the same way of the derivation of (4.14), from (4.3) and (4.15) it can be concluded that

$$E(\rho_2,T_{N+1})\le C(T_{N+1})\varepsilon^{N+1-p}. \tag{4.17}$$

Noticing that u_ε is a minimizer and $\rho_2\frac{x}{|x|}\in W_2$, we also have

$$\begin{aligned}
&E_\varepsilon(f_\varepsilon;T_{N+1})\le E_\varepsilon(\rho_2;T_{N+1})\\
&\le\frac{1}{p}\int_{T_{N+1}}^1[\rho_{2r}^2+\rho_2^2(n-1)r^{-2}]^{p/2}r^{n-1}dr+\frac{1}{2\varepsilon^p}\int_{T_{N+1}}^1(1-\rho_2)^2dr.
\end{aligned} \tag{4.18}$$

On the other hand,

$$\begin{aligned}
&\int_{T_{N+1}}^{1}[\rho_r^2+(n-1)r^{-2}\rho^2]^{p/2}r^{n-1}dr-\int_{T_{N+1}}^{1}[(n-1)r^{-2}\rho^2]^{p/2}r^{n-1}dr\\
&=\frac{p}{2}\int_{T_{N+1}}^{1}\int_0^1[\rho_r^2+(n-1)r^{-2}\rho^2]^{(p-2)/2}s+(n-1)r^{-2}\rho^2(1-s)ds\rho_r^2r^{n-1}dr\\
&\le C\int_{T_{N+1}}^{1}[\rho_r^2+(n-1)r^{-2}\rho^2]^{(p-2)/2}\rho_r^2r^{n-1}dr\\
&\quad+C\int_{T_{N+1}}^{1}[(n-1)r^{-2}\rho^2]^{(p-2)/2}\rho_r^2r^{n-1}dr\le C\int_{T_{N+1}}^{1}[\rho_r^p+\rho_r^2]dr.
\end{aligned}$$

Substituting this into (4.18), we have

$$\begin{aligned}
&E_\varepsilon(f_\varepsilon;T_{N+1})\\
&\le\frac{1}{p}\int_{T_{N+1}}^{1}(n-1)^{p/2}\rho_2^pr^{n-p-1}dr+C\int_{T_{N+1}}(\rho_{2r}^p+\rho_{2r}^2)dr\\
&\quad+\frac{1}{2\varepsilon^p}\int_{T_{N+1}}^{1}(1-\rho_2)^2dr\\
&\le\frac{1}{p}\int_{T_{N+1}}^{1}(n-1)^{p/2}\rho_2^pr^{n-p-1}dr+C\varepsilon^{N+1-p}\\
&\le\frac{1}{p}(n-1)^{p/2}\int_{T_{N+1}}^{1}r^{n-p-1}dr+C\varepsilon^{N+1-p},
\end{aligned}$$

by using (4.16) and (4.17). This is the conclusion of Proposition.

§5. $W^{1,p}$ CONVERGENCE

Based on the Proposition 4.2, we may obtain better convergence for radial minimizers.

Theorem 5.1. *Let $u_\varepsilon=f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u,B_1)$. Then*

$$\lim_{\varepsilon\to0}u_\varepsilon=\frac{x}{|x|},\quad \textit{in } W^{1,p}(K,R^n)\tag{5.1}$$

for any compact subset $K\subset\overline{B_1}\setminus\{0\}$.

Proof. Without loss of generality, we may assume $K=\overline{B_1}\setminus B(0,T_{N+1})$. From Proposition 4.2, we have

$$E_\varepsilon(u_\varepsilon,K)=|S^{n-1}|E_\varepsilon(f_\varepsilon;T_{N+1})\le C\tag{5.2}$$

where C is independent of ε. This and $|u_\varepsilon|\le1$ imply the existence of a subsequence u_{ε_k} of u_ε and a function $u_*\in W^{1,p}(K,R^n)$, such that

$$\lim_{\varepsilon_k\to0}u_{\varepsilon_k}=u_*,\quad \textit{weakly in } W^{1,p}(K,R^n),\tag{5.3}$$

$$\lim_{\varepsilon_k \to 0} |u_{\varepsilon_k}| = 1, \quad in\ C^\alpha(K,R),\ \alpha \in (0, 1-n/p). \tag{5.4}$$

(5.4) implies $u_* = \frac{x}{|x|}$. Noticing that any subsequence of u_ε has a convergence subsequence and the limit is always $\frac{x}{|x|}$, we can assert

$$\lim_{\varepsilon \to 0} u_\varepsilon = \frac{x}{|x|}, \quad weakly\ in\ W^{1,p}(K,R^n). \tag{5.5}$$

From this and the weakly lower semicontinuity of $\int_K |\nabla u|^p$, using Proposition 4.2, we know that

$$\begin{aligned}\int_K |\nabla \frac{x}{|x|}|^p &\le \underline{\lim}_{\varepsilon_k \to 0} \int_K |\nabla u_\varepsilon|^p \le \overline{\lim}_{\varepsilon_k \to 0} \int_K |\nabla u_\varepsilon|^p \\ &\le C\varepsilon^{[p]+1-p} + |S^{n-1}| \int_{T_{N+1}}^1 ((n-1)r^{-2})^{p/2} r^{n-1}\, dr\end{aligned}$$

and hence

$$\lim_{\varepsilon \to 0} \int_K |\nabla u_\varepsilon|^p = \int_K |\nabla \frac{x}{|x|}|^p$$

since

$$\int_K |\nabla \frac{x}{|x|}|^p = |S^{n-1}| \int_{T_{N+1}}^1 ((n-1)r^{-2})^{p/2} r^{n-1}\, dr.$$

Combining this with (5.4)(5.5) completes the proof of (5.1).

From (3.5) we also see that the zeroes of the radial minimizer $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ are in B_R for given $R > 0$ if ε is small enough.

§6 Uniqueness and regularized property

Theorem 6.1. *For any given $\varepsilon \in (0,1)$, the radial minimizers of $E_\varepsilon(u, B_1)$ are unique on W.*

Proof. Fix $\varepsilon \in (0,1)$. Suppose $u_1(x) = f_1(r)\frac{x}{|x|}$ and $u_2(x) = f_2(r)\frac{x}{|x|}$ are both radial minimizers of $E_\varepsilon(u, B_1)$ on W, then they are both weak radial solutions of (2.1) (2.2). Thus

$$\begin{aligned}&\int_{B_1} (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla \phi dx \\ =& \frac{1}{\varepsilon^p}\int_{B_1 \setminus B_\Gamma} [(u_1 - u_2) - (u_1|u_1|^2 - u_2|u_2|^2)]\phi dx \\ &- \frac{1}{\varepsilon^p}\int_{B_\Gamma} (u_1|u_1|^2 - u_2|u_2|^2)\phi dx.\end{aligned}$$

Set $\phi = u_1 - u_2 = (f_1 - f_2)\frac{x}{|x|}$. Take η sufficiently small such that $h < 1$.

Case 1. When $\Gamma \leq h\varepsilon$, we have

(6.1)
$$\begin{aligned}
&\int_{B_1} (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1 - u_2)dx \\
&= \frac{1}{\varepsilon^p}\int_{B_1}(f_1 - f_2)^2 dx - \frac{1}{\varepsilon^p}\int_{B_1}(f_1 - f_2)^2(f_1^2 + f_2^2 + f_1 f_2)dx \\
&= \frac{1}{\varepsilon^p}\int_{B_1\setminus B(0,h\varepsilon)}(f_1 - f_2)^2[1 - (f_1^2 + f_2^2 + f_1 f_2)]dx \\
&+ \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2 dx - \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2(f_1^2 + f_2^2 + f_1 f_2)dx.
\end{aligned}$$

Letting $\eta < \frac{1}{2} - \frac{1}{2\sqrt{2}}$ in (3.9), we have $f_1, f_2 \geq 1/\sqrt{2}$ *on* $B_1 \setminus B(0,h\varepsilon)$ for any given $\varepsilon \in (0,1)$. Hence

$$\int_{B_1}(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1 - u_2)dx \leq \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2 dx.$$

Applying (2.11) of [19], we can see that there exists a positive constant γ independent of ε and h such that

$$\gamma\int_{B_1}|\nabla(u_1 - u_2)|^2 dx \leq \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2 dx, \tag{6.2}$$

which implies

$$\int_{B_1}|\nabla(f_1 - f_2)|^2 dx \leq \frac{1}{\gamma\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2 dx. \tag{6.3}$$

Denote $G = B(0,h\varepsilon)$. Applying Theorem 2.1 in Ch II of [16], we have $\|f\|_{\frac{2n}{n-2}} \leq \beta\|\nabla f\|_2$ as $n > 2$, where $\beta = \frac{2(n-1)}{n-2}$. Taking $f = f_1 - f_2$ and applying (6.3), we obtain $f(|x|) = 0$ as $x \in \partial B_1$ and

$$[\int_{B_1}|f|^{\frac{2n}{n-2}}dx]^{\frac{n-2}{n}} \leq \beta^2\int_{B_1}|\nabla f|^2 dx \leq \beta^2\gamma^{-1}\int_G |f|^2 dx\varepsilon^{-p}.$$

Using Holder inequality, we derive

$$\int_G |f|^2 dx \leq |G|^{1-\frac{n-2}{n}}[\int_G |f|^{\frac{2n}{n-2}}dx]^{\frac{n-2}{n}} \leq |B_1|^{1-\frac{n-2}{n}}h^2\varepsilon^{2-p}\frac{\beta^2}{\gamma}\int_G |f|^2 dx.$$

Hence for any given $\varepsilon \in (0,1)$,

$$\int_G |f|^2 dx \leq C(\beta, |B_1|, \gamma, \varepsilon)h^2\int_G |f|^2 dx. \tag{6.4}$$

Denote $F(\eta) = \int_{B(0,h(\eta)\varepsilon)}|f|^2 dx$, then $F(\eta) \geq 0$ and (6.4) implies that

$$F(\eta)(1 - C(\beta, |B_1|, \gamma, \varepsilon)h^2) \leq 0. \tag{6.5}$$

On the other hand, since $C(\beta,|B_1|,\gamma,\varepsilon)$ is independent of η, we may take $0<\eta<\frac{1}{2}-\frac{1}{2\sqrt{2}}$ so small that $h=h(\eta)\le\lambda 9^N=9^N\frac{\eta}{2C_1}$ (which is implied by (3.4)) satisfies $1<C(\beta,|B_1|,\gamma,\varepsilon)h^2$ for the fixed $\varepsilon\in(0,1)$, which and (6.5) imply that $F(\eta)=0$. Namely $f=0$ a.e. on G, or $f_1=f_2$, a.e. on $B(0,h\varepsilon)$. Substituting this into (6.2), we know that $u_1-u_2=C$ a.e. on B_1. Noticing the continuity of u_1,u_2 which is implied by Proposition 2.1, and $u_1=u_2=x$ on ∂B_1, we can see at last that

$$u_1=u_2,\quad on\quad \overline{B_1}.$$

When $n=2$, using

$$\|f\|_6\le\beta\|\nabla f\|_{3/2} \tag{6.6}$$

which implied by Theorem 2.1 in Ch II of [16], and by the same argument above we can also derive $u_1=u_2$ on $\overline{B_1}$.

Case 2. When $h\varepsilon<\Gamma\le\varepsilon$. Similar to (6.1), by taking $\eta<\frac{1}{2}-\frac{1}{\sqrt{2}}$ and using (3.11) we get

$$\begin{aligned}
&\int_{B_1}|\nabla(f_1-f_2)|^p dx\le\int_{B_1}(|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1-u_2)dx\\
&\le\frac{1}{\varepsilon^p}\int_{B_1\setminus B_\Gamma}(f_1-f_2)^2[1-(f_1^2+f_2^2+f_1f_2)]dx\\
&+C(\varepsilon)\eta^2\int_{B_\Gamma\setminus B_{h\varepsilon}}(f_1-f_2)^2dx+C(\varepsilon)\int_{B(0,h\varepsilon)}(f_1-f_2)^2dx\\
&\le C(\varepsilon)\eta^2\int_{B_\Gamma\setminus B_{h\varepsilon}}(f_1-f_2)^2dx+C(\varepsilon)\int_{B(0,h\varepsilon)}(f_1-f_2)^2dx.
\end{aligned} \tag{6.7}$$

Substituting

$$\begin{aligned}
&\eta^2C(\varepsilon)\int_{B_\Gamma\setminus B_{h\varepsilon}}(f_1-f_2)^2dx\le C\eta^2\int_{B_1}(f_1-f_2)^2dx\\
&\le C\eta^2(\int_{B_1}(f_1-f_2)^6dx)^{1/3}\le C\eta^2\int_{B_1}|\nabla(f_1-f_2)|^2dx
\end{aligned}$$

(which implied by (6.6)) into (6.7) and choosing η sufficiently small, we have

$$\int_{B_1}|\nabla(f_1-f_2)|^2dx\le C\int_{B_{h\varepsilon}}(f_1-f_2)^2dx,$$

this is (6.3). The other part of the proof is as same as the Case 1. The theorem is proved.

In the following, we will prove that the radial minimizer u_ε can be obtained as the limit of a subsequence $u_\varepsilon^{\tau_k}$of the radial minimizer u_ε^τ of the regularized functionals

$$E_\varepsilon^\tau(u,B_1)=\frac{1}{p}\int_{B_1}(|\nabla u|^2+\tau)^{p/2}+\frac{1}{4\varepsilon^p}\int_{B_1\setminus\Gamma}(1-|u|^2)^2+\frac{1}{4\varepsilon^p}\int_\Gamma|u|^4,\quad(\tau>0)$$

on W as $\tau_k\to 0$, namely

Theorem 6.2. *Assume that u_ε^τ be the radial minimizer of $E_\varepsilon^\tau(u,B_1)$ in W. Then there exist a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ and $\tilde{u}_\varepsilon \in W$ such that*

$$\lim_{\tau_k\to 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon, \quad in \quad W^{1,p}(B_1,R^n). \tag{6.8}$$

Here $\tilde{u}_\varepsilon$ is just the radial minimizer of $E_\varepsilon(u,B_1)$ in W.

It is not difficult to proof that the minimizer u_ε^τ is a classical solution of the equation

$$-div(v^{(p-2)/2}\nabla u) = \frac{1}{\varepsilon^p}u(1-|u|^2), \quad on \quad B_1\setminus B_\Gamma; \tag{6.9}$$

$$-div(v^{(p-2)/2}\nabla u) = \frac{1}{\varepsilon^p}u|u|^2, \quad on \quad B_\Gamma$$

and also satisfies the maximum principle: $|u_\varepsilon^\tau|\le 1$ on B_1, where $v=|\nabla u|^2+\tau$. By virtue of the uniqueness of the radial minimizer, we know $\tilde{u}_\varepsilon = u_\varepsilon$. Thus the radial minimizer u_ε can be regularized by the radial minimizer u_ε^τ of $E_\varepsilon^\tau(u,B_1)$.

Proof of Theorem 6.2.. First, from (2.8) we have

$$E_\varepsilon^\tau(u_\varepsilon^\tau,B_1)\le E_\varepsilon^\tau(u_\varepsilon,B_1)\le CE_\varepsilon(u_\varepsilon,B_1)\le C\varepsilon^{2-p} \tag{6.10}$$

as $\tau\in(0,1)$, where C does not depend on ε and τ. This and $|u_\varepsilon^\tau|\le 1$ imply that $\|u_\varepsilon^\tau\|_{W^{1,p}(B_1)}\le C(\varepsilon)$. Applying the embedding theorem we see that there exist a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ and $\tilde{u}_\varepsilon\in W^{1,p}(B_1,R^n)$ such that

$$u_\varepsilon^{\tau_k}\to\tilde{u}_\varepsilon, \quad weakly \quad in \quad W^{1,p}(B_1,R^n), \tag{6.11}$$

$$u_\varepsilon^{\tau_k}\longrightarrow\tilde{u}_\varepsilon, \quad in\ C(\overline{B_1},R^n),, \tag{6.12}$$

as $\tau_k\to 0$. Since (6.11) and the weakly low semicontinuity of the functional $\int_{B_1}|\nabla u|^p$, we obtain

$$\int_{B_1}|\nabla\tilde{u}_\varepsilon|^p\le \underline{\lim}_{\tau_k\to 0}\int_{B_1}|\nabla u_\varepsilon^{\tau_k}|^p. \tag{6.13}$$

From (6.12) it follows $\tilde{u}_\varepsilon\in W$. This means $E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k},B_1)\le E_\varepsilon^{\tau_k}(\tilde{u}_\varepsilon,B_1)$, i.e.,

$$\overline{\lim}_{\tau_k\to 0}E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k},B_1)\le\lim_{\tau_k\to 0}E_\varepsilon^{\tau_k}(\tilde{u}_\varepsilon,B_1). \tag{6.14}$$

We can also deduce

$$\int_{B_1\setminus\Gamma}(1-|u_\varepsilon^{\tau_k}|^2)^2+\int_\Gamma|u_\varepsilon^{\tau_k}|^4\to\int_{B_1\setminus\Gamma}(1-|\tilde{u}_\varepsilon|^2)^2+\int_\Gamma|\tilde{u}_\varepsilon|^4$$

from (6.12) as $\tau_k\to 0$. This and (6.14) show

$$\overline{\lim}_{\tau_k\to 0}\int_{B_1}(|\nabla u_\varepsilon^{\tau_k}|^2+\tau_k)^{p/2}\le\lim_{\tau_k\to 0}\int_{B_1}(|\nabla\tilde{u}_\varepsilon|^2+\tau_k)^{p/2}=\int_{B_1}|\nabla\tilde{u}_\varepsilon|^p.$$

Combining this with (6.13) we obtain $\int_{B_1}|\nabla u_\varepsilon^{\tau_k}|^p\to\int_{B_1}|\nabla\tilde{u}_\varepsilon|^p$ as $\tau_k\to 0$, which together with (6.11) implies $\nabla u_\varepsilon^{\tau_k}\to\nabla\tilde{u}_\varepsilon,\ in\ L^p(B_1,R^n)$. Noticing (6.12) we have the conclusion $u_\varepsilon^{\tau_k}\to\tilde{u}_\varepsilon,\ in\ W^{1,p}(B_1,R^n)$ as $\tau_k\to 0$. This is (6.8).

On the other hand, we know

$$E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k},B_1)\le E_\varepsilon^{\tau_k}(u,B_1) \tag{6.15}$$

for all $u\in W$. Noticing the conclusion $\lim_{\tau_k\to 0}E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k},B_1)=E_\varepsilon(\tilde{u}_\varepsilon,B_1)$ which had been proved just now we can say $E_\varepsilon(\tilde{u}_\varepsilon,B_1)\le E_\varepsilon(u,B_1)$ when $\tau_k\to 0$ in (6.15), which implies $\tilde{u}_\varepsilon$ be a minimizer of $E_\varepsilon(u,B_1)$.

§7. Proofs of (1.2)

Proposition 7.1. *Assume* $u_\varepsilon^\tau = u = f(r)\frac{x}{|x|}$. *Then there exists* $C > 0$ *which is independent of* ε, τ *such that*

$$\|f\|_{C^{1,\alpha}(K,R)} \leq C, \quad \forall \alpha \leq 1/2,$$

where $K \subset (0,1)$ *is an arbitrary closed interval.*

Proof. From (6.9) it follows that f solves

$$\begin{aligned} &- (A^{(p-2)/2} f_r)_r - (n-1)r^{-1}A^{(p-2)/2}f_r + r^{-2}A^{(p-2)/2}f \\ &= \frac{1}{\varepsilon^p} f(1-f^2), \quad on \quad (\Gamma, 1) \end{aligned} \tag{7.1}$$

where $A = f_r^2 + (n-1)r^{-2}f^2 + \tau$. Take $R > 0$ sufficiently small such that $K \subset\subset (2R, 1-2R)$. Let $\zeta \in C_0^\infty([0,1],[0,1])$ be a function satisfying $\zeta = 0$ on $[0,R] \cup [1-R,1]$, $\zeta = 1$ on $[2R, 1-2R]$ and $|\nabla\zeta| \leq C(R)$ on $(0,1)$. Differentiating (7.1), multiplying with $f_r\zeta^2$ and integrating, we have

$$\begin{aligned} &-\int_0^1 (A^{(p-2)/2}f_r)_{rr}(f_r\zeta^2)dr - (n-1)\int_0^1 (r^{-1}A^{(p-2)/2}f_r)_r(f_r\zeta^2)dr \\ &+\int_0^1 (r^{-2}A^{(p-2)/2}f)_r(f_r\zeta^2)dr = \frac{1}{\varepsilon^p}\int_0^1 [f(1-f^2)]_r(f_r\zeta^2)dr. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} &\int_0^1 (A^{(p-2)/2}f_r)_r(f_r\zeta^2)_r dr + \int_0^1 A^{(p-2)/2}(f_r\zeta^2)_r[(n-1)r^{-1}f_r \\ &- r^{-2}f]dr \leq \frac{1}{\varepsilon^p}\int_0^1 (1-f^2)f_r^2\zeta^2 dr. \end{aligned}$$

Denote $I = \int_R^{1-R} \zeta^2(A^{(p-2)/2}f_{rr}^2 + (p-2)A^{(p-4)/2}f_r^2 f_{rr}^2)dr$, then for any $\delta \in (0,1)$, there holds

$$I \leq \delta I + C(\delta)\int_R^{1-R} A^{p/2}\zeta_r^2 dr + \frac{1}{\varepsilon^p}\int_R^{1-R} f_r^2(1-f^2)\zeta^2 dr \tag{7.2}$$

by using Young inequality. From (7.1) we can see that

$$\frac{1}{\varepsilon^p}(1-f^2) = f^{-1}[-(A^{(p-2)/2}f_r)_r - (n-1)r^{-1}A^{(p-2)/2}f_r + r^{-2}A^{(p-2)/2}f].$$

Applying Young inequality again we obtain that for any $\delta \in (0,1)$,

$$\frac{1}{\varepsilon^p}\int_0^1 (1-f^2)f_r^2\zeta^2 dr \leq \delta I + C(\delta)\int_R^{1-R} A^{(p+2)/2}\zeta^2 dr.$$

Substituting this into (7.2) and choosing δ sufficiently small, we have

$$I \leq C\int_R^{1-R} A^{p/2}\zeta_r^2 dr + C\int_R^{1-R} A^{(p+2)/2}\zeta^2 dr. \tag{7.3}$$

To estimate the second term of the right hand side of (7.3), we take $\phi = \zeta^{2/q} f_r^{(p+2)/q}$ in the interpolation inequality (Ch II, Theorem 2.1 in [16])

$$\|\phi\|_{L^q} \le C\|\phi_r\|_{L^1}^{1-1/q}\|\phi\|_{L^1}^{1/q}, \quad q \in (1+\frac{2}{p}, 2).$$

We derive by applying Young inequality that for any $\delta \in (0,1)$,

$$\begin{aligned}
(7.4) \qquad & \int_R^{1-R} f_r^{p+2}\zeta^2 dr \le C(\int_R^{1-R} \zeta^{2/q}|f_r|^{(p+2)/q} dr) \\
& \cdot (\int_R^{1-R} \zeta^{2/q-1}|\zeta_r||f_r|^{(p+2)/q} + \zeta^{2/q}|f_r|^{(p+2)/q-1}|f_{rr}|dr)^{q-1} \\
& \le C(\int_R^{1-R} \zeta^{2/q}|f_r|^{(p+2)/q} dr)(\int_R^{1-R} \zeta^{2/q-1}|\zeta_r||f_r|^{(p+2)/q} \\
& + \delta I + C(\delta)\int_R^{1-R} A^{\frac{p+2}{q}-\frac{p}{2}}\zeta^{4/q-2} dr)^{q-1}.
\end{aligned}$$

We may claim

$$(7.5) \qquad \int_R^{1-R} A^{p/2} dr \le C,$$

by the same argument of the proof of Proposition 4.2, where C is independent of ε and τ. In fact, from (6.10) we may also derive (4.17). Noting u_ε^τ is a radial minimizer of $E_\varepsilon^\tau(u, B_1)$, replacing (4.18) we obtain

$$\begin{aligned}
& E_\varepsilon^\tau(f_\varepsilon \frac{x}{|x|}; B_1 \setminus B(0, T_{N+1})) \le CE(\rho_2; T_{N+1}) \\
& \le \frac{C}{p}(n-1)^{p/2}\int_{T_{N+1}}^1 r^{n-p-1} dr + C\varepsilon^{N+1-p}.
\end{aligned}$$

This means that (7.5) holds.

Noting $q \in (1+\frac{2}{p}, 2)$, we may using Holder inequality to the right hand side of (7.4). Thus, by virtue of (7.5),

$$\int_R^{1-R} f_r^{p+2}\zeta^2 dr \le \delta I + C(\delta).$$

Substituting this into (7.3) we obtain

$$\int_R^{1-R} A^{(p-2)/2} f_{rr}^2 \zeta^2 dr \le C,$$

which, together with (7.5), implies that $\|A^{p/4}\zeta\|_{H^1(R,1-R)} \le C$. Noticing $\zeta = 1$ on K, we have $\|A^{p/4}\|_{H^1(K)} \le C$. Using embedding theorem we can see that for any $\alpha \le 1/2$, there holds $\|A^{p/4}\|_{C^\alpha(K)} \le C$. It is not difficult to prove our proposition.

Theorem 7.2. *Let $u_\varepsilon = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u, B_1)$. Then for any compact subset $K \subset B_1 \setminus \{0\}$, we have*

$$\lim_{\varepsilon\to 0} u_\varepsilon = \frac{x}{|x|}, \quad in\ C^{1,\beta}(K, R^n), \quad \beta \in (0,1)$$

Proof. For every compact subset $K \subset B_1 \setminus \{0\}$, applying Proposition 7.1 yields that for some $\beta \in (0, 1/2]$ one has

$$\|u_\varepsilon^\tau\|_{C^{1,\beta}(K)} \le C = C(K), \tag{7.6}$$

where the constant does not depend on ε, τ.

Applying (7.6) and the embedding theorem we know that for any ε and some $\beta_1 < \beta$, there exist $w_\varepsilon \in C^{1,\beta_1}(K, R^n)$ and a subsequence of τ_k of τ such that as $k \to \infty$,

$$u_\varepsilon^{\tau_k} \to w_\varepsilon, \quad in \quad C^{1,\beta_1}(K, R^n). \tag{7.7}$$

Combining this with (6.8) we know that $w_\varepsilon = u_\varepsilon$.

Applying (7.6) and the embedding theorem again we can see that for some $\beta_2 < \beta$, there exist $w \in C^{1,\beta_2}(K, R^n)$ and a subsequence of τ_k which can be denoted by τ_m such that as $m \to \infty$,

$$u_{\varepsilon_m}^{\tau_m} \to w, \quad in \quad C^{1,\beta_2}(K, R^n). \tag{7.8}$$

Denote $\gamma = min(\beta_1, \beta_2)$. Then as $m \to \infty$, we have

$$\begin{aligned} \|u_{\varepsilon_m} - w\|_{C^{1,\beta}(K,R^n)} &\le \|u_{\varepsilon_m} - u_{\varepsilon_m}^{\tau_m}\|_{C^{1,\beta}(K,R^n)} \\ &\quad + \|u_{\varepsilon_m}^{\tau_m} - w\|_{C^{1,\beta}(K,R^n)} \le o(1) \end{aligned} \tag{7.9}$$

by applying (7.7) and (7.8). Noting (1.1) we know that $w = \frac{x}{|x|}$.

Noting the limit $\frac{x}{|x|}$ is unique, we can see that the convergence (7.9) holds not only for some subsquence but for all u_ε. Applying the uniqueness theorem (Theorem 6.1) of the radial minimizers, we know that the regularizable radial minimizer just is the radial minimizer. Theorem is proved.

§8. Proof of Theorem 1.4

First (3.1) shows one rate that the minimizer f_ε converge to 1 as $\varepsilon \to 0$. Moreover, proposition 4.2 implies that for any $T > 0$,

$$\frac{1}{4\varepsilon^p}\int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1} dr \le C. \tag{8.1}$$

In the following we shall give other better estimates of the rate of the convergence for the radial minimizer f_ε than (8.1).

Theorem 8.1. *Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be the radial minimizer of $E_\varepsilon(u, B_1)$. For any $T > 0$, there exists a constant $C > 0$ which is independent of ε such that as ε sufficiently small,*

$$\int_T^1 |f'_\varepsilon|^p r^{n-1} dr + \frac{1}{\varepsilon^p}\int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1} dr \le C\varepsilon^{[p]+1-p}. \tag{8.2}$$

Here $[p]$ is the integer number part of p. Moreover, as $\varepsilon \to 0$,

$$\frac{1}{p}\int_{B_1\setminus B_T} |\nabla u_\varepsilon|^p + \frac{1}{4\varepsilon^p}\int_{B_1\setminus B_T}(1 - |u_\varepsilon|^2)^2 \to \frac{1}{p}\int_{B_1\setminus B_T(0)} |\nabla \frac{x}{|x|}|^p. \tag{8.3}$$

Proof. By proposition 4.2 we have

$$E_\varepsilon(f_\varepsilon; B_T) \le \frac{1}{p}\int_T^1 (n-1)^{p/2} r^{n-p-1} dr + C\varepsilon^{2([p]+1-p)/p}, \tag{8.4}$$

thus,

$$\int_T^1 (1 - f_\varepsilon)^2 dr \le C(T)\varepsilon^p, \tag{8.5}$$

for any $T > 0$. On the other hand, Jensen's inequality implies

$$\begin{aligned} E_\varepsilon(f_\varepsilon; B_T) &\ge \frac{1}{p}\int_T^1 |f'_\varepsilon|^p r^{n-1} dr \\ &+ \frac{1}{p}\int_T^1 ((n-1)\frac{f_\varepsilon^2}{r^2})^{p/2} r^{n-1} dr + \frac{1}{4\varepsilon^p}\int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1} dr. \end{aligned}$$

Combining this with (8.4) we have

$$\begin{aligned} \frac{1}{p}\int_T^1 ((n-1)\frac{f_\varepsilon^2}{r^2})^{p/2} r^{n-1} dr &\le E_\varepsilon(f_\varepsilon; B_T) \\ &\le C\varepsilon^{2([p]+1-p)/p} + \frac{1}{p}\int_T^1 (n-1)^{p/2} r^{n-p-1} dr. \end{aligned} \tag{8.6}$$

Applying (8.5) and Hölder's inequality we obtain

$$\begin{aligned} &\int_T^1 ((n-1)r^{-2})^{p/2} r^{n-1} dr - \int_T^1 ((n-1)r^{-2} f_\varepsilon^2)^{p/2} r^{n-1} dr \\ &= \int_T^1 (n-1)^{p/2} r^{n-p-1}(1 - f_\varepsilon^p) dr \le C(T)\int_T^1 (1 - f_\varepsilon) dr \\ &\le C(\int_T^1 (1 - f_\varepsilon)^2 dr)^{1/2} \le C\varepsilon^{p/2}. \end{aligned}$$

Substituting this into (8.6) we obtain

$$\begin{aligned} -C\varepsilon^{p/2} &\le E_\varepsilon(f_\varepsilon; B_T) \\ &- \frac{1}{p}\int_T^1 ((n-1)r^{-2})^{p/2} r^{n-1} dr \le C\varepsilon^{[p]+1-p}. \end{aligned} \tag{8.7}$$

Noticing

$$\frac{1}{p}\int_{B_1\setminus B_T(0)} |\nabla \frac{x}{|x|}|^p = \frac{|S^{n-1}|}{p}\int_T^1 ((n-1)r^{-2})^{p/2} r^{n-1} dr,$$

from (8.7) we can see that both (8.2) and (8.3) hold.

Theorem 8.2. *Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be the radial minimizer of $E_\varepsilon(u, B_1)$ on W. Then there exist $C, \varepsilon_0 > 0$ such that as $\varepsilon \in (0, \varepsilon_0)$,*

$$\int_T^1 r^{n-1}[(f_\varepsilon')^p + \frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]dr \le C\varepsilon^p. \tag{8.8}$$

$$\sup_{r\in[T,1]}(1-f_\varepsilon(r)) \le C\varepsilon^{p-\frac{n}{2}}. \tag{8.9}$$

(8.8) gives the estimate of the rate of f_ε's convergence to 1 in $W^{1,p}[T,1]$ sense, and that in $C^0[T,1]$ sense is showed by (8.9).

Proof. It follows from Jensen's inequality that

$$\begin{aligned} E_\varepsilon(f_\varepsilon;T) &= \frac{1}{p}\int_T^1 [(f_\varepsilon')^2 + \frac{(n-1)}{r^2}f_\varepsilon^2]^{p/2} r^{n-1}dr \\ &+ \frac{1}{4\varepsilon^p}\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1}dr \\ &\ge \frac{1}{p}\int_T^1 (f_\varepsilon')^p r^{n-1}dr + \frac{1}{4\varepsilon^p}\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1}dr \\ &+ \frac{1}{p}\int_T^1 \frac{[n-1]^{p/2}}{r^p} f_\varepsilon^p r^{n-1}dr. \end{aligned}$$

Combining this with Proposition 4.2 yields

$$\begin{aligned} &\frac{1}{p}\int_T^1 (f_\varepsilon')^p r^{n-1}dr + \frac{1}{4\varepsilon^p}\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1}dr \\ \le &\frac{1}{p}\int_T^1 \frac{[n-1]^{p/2}}{r^p}(1-f_\varepsilon^p) r^{n-1}dr + C\varepsilon^{[p]+1-p}. \end{aligned}$$

Noticing (8.1), we obtain

$$\begin{aligned} &\frac{1}{p}\int_T^1 (f_\varepsilon')^p r^{n-1}dr + \frac{1}{4\varepsilon^p}\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1}dr \\ \le &C\int_T^1 \frac{[n-1]^{p/2}}{r^p}(1-f_\varepsilon) r^{n-1}dr + C\varepsilon^{[p]+1-p} \\ \le &C\varepsilon^{p/2} + C\varepsilon^{[p]+1-p} \le C\varepsilon^{[p]+1-p}. \end{aligned} \tag{8.10}$$

Using Proposition 4.2 and (8.10), as well as the integral mean value theorem we can see that there exists

$$T_1 \in [T, T(1+1/2)] \subset [R/2, R]$$

such that

$$[(f_\varepsilon)_r^2 + (n-1)r^{-2}f_\varepsilon^2]_{r=T_1} \le C_1, \tag{8.11}$$

$$[\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]_{r=T_1} \leq C_1\varepsilon^{[p]+1-p}. \tag{8.12}$$

Consider the functional

$$E(\rho,T_1)=\frac{1}{p}\int_{T_1}^1(\rho_r^2+1)^{p/2}dr+\frac{1}{2\varepsilon^p}\int_{T_1}^1(1-\rho)^2dr.$$

It is easy to prove that the minimizer ρ_3 of $E(\rho,T_1)$ in $W_{f_\varepsilon}^{1,p}((T_1,1),R^+\cup\{0\})$ exists.

By the same way to proof of (4.14), using (8.11) and (8.12) we have

$$E(\rho_3,T_1)\leq v^{\frac{p-2}{2}}\rho_{3r}(1-\rho_3)|_{r=T_1}\leq C_1(1-\rho_3(T_1))\leq C\varepsilon^{F[1]},$$

where $F[j]=\frac{[p]+1-p}{2^j}+\frac{(2^j-1)p}{2^j}, j=1,2,\cdots$. Hence, similar to the proof of Proposition 4.2, we obtain

$$E_\varepsilon(f_\varepsilon;T_1)\leq C\varepsilon^{F[1]}+\frac{1}{p}\int_{T_1}^1\frac{[n-1]^{p/2}}{r^{p-1}}dr.$$

Furthermore, similar to the derivation of (8.10), using (8.1) we may get

$$\int_{T_1}^1(f_\varepsilon')^pr^{n-1}dr+\frac{1}{\varepsilon^p}\int_{T_1}^1(1-f_\varepsilon^2)^2r^{n-1}dr\leq C\varepsilon^{F[1]}+C\varepsilon^{p/2}\leq C_2\varepsilon^{F[1]}.$$

Set $T_m=R(1-\frac{1}{2^m})$. Proceeding in the way above (whose idea is improving the exponents of ε from $F[k]$ to $F[k+1]$ step by step), we can see that there exists some $m\in N$ satisfying $F[m-1]\leq\frac{p}{2}\leq F[m]$ such that

$$\begin{aligned}&\int_{T_m}^1(f_\varepsilon')^prdr+\frac{1}{\varepsilon^p}\int_{T_m}^1(1-f_\varepsilon^2)^2r^{n-1}dr\\&\leq C\varepsilon^{\frac{[p]+1-p}{2^m}+\frac{(2^m-1)p}{2^m}}+C\varepsilon^{p/2}\leq C\varepsilon^{p/2}.\end{aligned} \tag{8.13}$$

Similar to the derivation of (8.11) and (8.12), it is known that there exists $T_{m+1}\in[T_m,3T_m/2]$ such that

$$[(f_\varepsilon)_r^2+(n-1)r^{-2}f_\varepsilon^2]_{r=T_{m+1}}\leq C, \tag{8.14}$$

$$[\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]_{r=T_{m+1}}\leq C\varepsilon^{p/2}. \tag{8.15}$$

The minimizer ρ_4 of the functional

$$E(\rho,T_{m+1})=\frac{1}{p}\int_{T_{m+1}}^1(\rho_r^2+1)^{p/2}dr+\frac{1}{2\varepsilon^p}\int_{T_{m+1}}^1(1-\rho)^2dr$$

in $W_{f_\varepsilon}^{1,p}((T_1,1),R^+)$ exists. By the same way to proof of (4.14), using (8.15) and (8.14) we have

$$E(\rho_4,T_{m+1})\leq v^{\frac{p-2}{2}}\rho_{4r}(1-\rho_3)|_{r=T_{m+1}}\leq C(1-\rho_4(T_{m+1}))\leq C\varepsilon^{G[1]},$$

where $G[j] = \frac{p/2}{2^j} + \frac{(2^j-1)p}{2^j}, j = m+1, m+2, \cdots$. By the argument of proof of Proposition 4.2, we obtain

$$E_\varepsilon(f_\varepsilon; T_{m+1}) \leq C\varepsilon^{G[1]} + \frac{1}{p}\int_{T_{m+1}}^1 \frac{[n-1]^{p/2}}{r^{p-1}} dr.$$

Furthermore, similar to the derivation of (8.10), using (8.13) we may get

$$\int_{T_{m+1}}^1 (f'_\varepsilon)^p r^{n-1} dr + \frac{1}{\varepsilon^p}\int_{T_{m+1}}^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \leq C\varepsilon^{G[1]}.$$

Proceeding in the way above (whose idea is improving the exponents of ε from $G[k]$ to $G[k+1]$ step by step), we can see that for any $k \in N$,

$$\int_{T_{m+k}}^1 (f'_\varepsilon)^p r^{n-1} dr + \frac{1}{\varepsilon^p}\int_{T_{m+k}}^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \leq C\varepsilon^{\frac{p/2}{2^k} + \frac{(2^k-1)p}{2^k}}.$$

Letting $k \to \infty$, we derive

$$\int_R^1 (f'_\varepsilon)^p r^{n-1} dr + \frac{1}{\varepsilon^p}\int_R^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \leq C\varepsilon^p.$$

This is (8.8).

From (8.8) we can see that

$$\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \leq C\varepsilon^{2p}. \tag{8.16}$$

On the other hand, from (5.2) and $|u_\varepsilon| \leq 1$ it follows that $\|f_\varepsilon\|_{W^{1,p}((T,1),R)} \leq C$. Applying the embedding theorem we know that for any $r_0 \in [T, 1]$,

$$|f_\varepsilon(r) - f_\varepsilon(r_0)| \leq C|r - r_0|^{1-1/p}, \ \forall r \in (r_0 - \varepsilon, r_0 + \varepsilon).$$

Thus

$$(1 - f_\varepsilon(r))^2 \geq (1 - f_\varepsilon(r_0))^2 - \varepsilon^{1-1/p} \geq \frac{1}{2}(1 - f_\varepsilon(r_0))^2.$$

Substituting this into (8.16) we obtain

$$C\varepsilon^{2p} \geq \int_T^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \geq \int_{r_0-\varepsilon}^{r_0+\varepsilon} (1-f_\varepsilon^2)^2 r^{n-1} dr \geq \frac{1}{2}(1 - f_\varepsilon(r_0))^2 \varepsilon^n$$

which implies $1 - f_\varepsilon(r_0) \leq C\varepsilon^{p-\frac{n}{2}}$. Noting r_0 is an arbitrary point in $[T, 1]$, we have

$$\sup_{r\in[T,1]} (1 - f_\varepsilon(r)) \leq C\varepsilon^{p-\frac{n}{2}}.$$

Thus (8.9) is derived and the proof of Theorem is complete.

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