Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December 2003, pp. 85–89.
Submitted: May 18, 2003. Published: December 31 2003.

ON THE LARGE PROPER SUBLATTICES OF FINITE LATTICES

Zhang Kunlun, Song Lixia and Sun Yikang

Abstract. In this present note, We study and prove some properties of the large proper sublattices of finite lattices. It is shown that every finite lattice L with $|L| > 4$ contains a proper sublattice S with $|S| \geq [2(|L|-2)]^{1/3} + 2 > (2|L|)^{1/3}$.

A.M.S. (MOS) Subject Classification Codes. 06B05

Key Words and Phrases. Finite lattice, Irreducible , Convex poset, Sublattice

1 Introduction

In [2], Tom Whaley proved the following classic result about sublattices of lattices.

Theorem 1.1. *If L is a lattice with $k = |L|$ infinite and regular, then either*
(1) there is a proper principal ideal of L of size k,or
(2) there is a proper principal filter of L of size k, or
(3) M_k, the modular lattice of height 2 and size k, is a 0, 1-sublattice of L.

Corollary 1.2. *If L is infinite and regular, then L has a proper sublattice of cardinality $|L|$.*

In [4], Ralph Freese, Jennifer Hyndman, and J. B. Nation proved the following classic result about sublattices of finite ordered set and finite lattices.

Theorem 1.3. *Let P be a finite ordered set with $|P| = n$. Let $\gamma = \lceil n^{1/3} \rceil$. Then either*
(1) there is a principal ideal I of P with $|I| \geq \gamma$, or
(2) there is a principal filter F of P with $|F| \geq \gamma$, or
(3) P contains a super-antichain A with $|A| \geq \gamma$.

Department of Mathematics, Zhanjiang Normal College, Guangdong, P. R. China
E-mail Address: zhangkunl@sina.com; songlx100@sina.com sunyk100@sina.com

Research was supported by the National Natural Science Foundation of China (19801016) (10261003)

Theorem 1.4. *Let L be a finite lattice with $|L| = n+2$. Then one of the following must hold.*
(1) There exists $x < 1$ with $|(x]| \geq n^{1/3}$;
(2) There exists $y > 0$ with $|[y)| \geq n^{1/3}$;
(3) M_β is a 0, 1-sublattice of L, where $\beta = \lceil n^{1/3} \rceil$.

The above result gives some lower bound of the large proper sublattices of finite ordered set and finite lattices, but they are not so good. In this present note, We give a new lower bound of the large proper sublattices of finite lattices. Our first main result is that:

Theorem 1.5. *Let L be a finite lattice with $|L| = n > 4$. Then there exists proper sublattice $S \subset L$ with $|S| \geq [2(n-2)]^{1/3} + 2 > (2|L|)^{1/3}$.*

2 Definitions and Lemmas

Let $(P, \leq)$ be a poset and $H \subset P$, $a \in P$. The a is an *upper bound* of H if and only if $h \leq a$ for all $h \in H$. An upper bound a of H is the *least upper bound* of H if and only if , for any upper bound b of H, we have $a \leq b$. We shall write $a = \sup H$. The concepts of *lower bound* and *greatest lower bound* are similarly defined; the latter is denoted by $\inf H$. Set
$M(P) = \{(a, b) \in P \times P| \ \sup\{a, b\}\text{and}\inf\{a, b\}\text{exist in } P\}$

$$(x] = \{a \in P | a \leq x\}; \quad [x) = \{a \in P | x \leq a\}; \quad [x]_P = (x] \cup [x).$$

$$N_x = \bigcup_{a \geq x} (a] \cup \bigcup_{b \leq x} [b).$$

where $x \in P$.

Definition 2.1. *A poset $(L, \leq)$ is a lattice if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$.*

Theorem 2.2. . *A poset $(P, \leq)$ is a lattice if and only if $M(P) = P$*

Definition 2.3. *If $(A, \leq)$ is a poset, $a, b \in A$, then a and b are comparable if $a \leq b$ or $a \geq b$. Otherwise, a and b are incomparable, in notation $a \| b$. A chain is, therefore, a poset in which there are no incomparable elements. An unorderedposet is one in which $a \| b$ for all $a \neq b$. $(A, \leq)$ is a convex poset if $C(P) = \{a \in P | a \neq \inf P, a \neq \sup P$ and $a \not\| x$ for all $x \in P\} = \varnothing$.*

Definition 2.4. *Let $(A, \leq)$ be a poset and let B be a non-void subset of A. Then there is a natural partial order $\leq_B$ on B induced by $\leq$: for a, $b \in B.a \leq_B b$ if and only if $a \leq b$, we call $(B, \leq_B)$, (or simply, $(B, \leq)$) a subposet of $(A, \leq)$*

Definition 2.5. *Let $(A, \leq)$ be a poset and let B be a subposet of A. If $M(B) = M(A) \cap (B \times B)$ and $sup_B\{a, b\} = sup_A\{a, b\}$, $\inf_B\{a, b\} = \inf_A\{a, b\}$ for all $(a, b) \in M(B)$, then we call $(B, \leq)$ a $semi-sublattice$ of $(A, \leq)$*

Definition 2.6. *A chain C in a poset P is a nonvoid subset which, as a subposet, is a chain. An antichain C in a poset P is a nonvoid subset which, as a subposet, is unordered.*

Definition 2.7. *The length, $l(C)$ of a finite chain is $|C|-1$. A poset P is said to be of length n (in formula $l(P)=n$) where n is a natural number, if and only if there is a chain in P of length n and all chain in P are of length $\leq n$. The width of poset P is m, where m is a natural number, if and only if there is an antichain in P of m elements and all antichain in P have $\leq m$ elements.*

We say that $A \subset P$ is a super-antichain if no pair of distinct elements of A has a common upper bound or a common lower bound. Let $S \subset P$

Lemma 2.8. *. If P is a convex poset , let $x \in P$, then $(x]$, $[x)$ and $[x]_P = (x] \cup [x)$ are proper semi-sublattices of P.*

Lemma 2.9. *If P is a convex poset with $|P| > 1$, let $\eta = \max_{x \in P} |[x]_P|$, then $|N_a| - 1 \leq \frac{1}{2}(\eta-1)^2$ for all $a \in P$.*

Proof. Since η be the largest size of a proper semi-sublattice $[x]_P$ of P, so that $|[x]_P| \leq \eta$ for all $x \in P$. For $a \in P$, if $s = |\{y \in (a] | x \leq y \Rightarrow x = y \text{ for all } x \in P\}|$ and $t = |\{y \in [a) | y \leq x \Rightarrow y = x \text{ for all } x \in P\}|$, then

$$|N_a| \leq t(\eta - s - 1) + s(\eta - t - 1) + 1 = (\eta - 1)(s+t) - 2ts + 1$$

where $0 \leq s+t \leq \eta - 1$. A little calculus shows that this is at most $\frac{1}{2}(\eta-1)(\eta-1)+1$. Then $|N_a| - 1 \leq \frac{1}{2}(\eta-1)^2$ $\square$

Lemma 2.10. *If P is a finite convex poset with $k = (2|P|)^{1/3}$, then either*
(1) there is a proper semi-sublattice $[a]_P$ of P of size $|[a]_P| \geq k$, or
(2) P contains a super-antichain of size k.

Proof. . Suppose that (1) fail. We will construct a super-antichain by transfinite induction. Let $|P| = n$. For every $a \in P$ set

$$N_a = \bigcup_{x \geq a} (x] \cup \bigcup_{y \leq a} [y).$$

We form a super-antichain A as follows. Choose $a_1 \in P$ arbitrarily. Given $a_1, \cdots, a_m$, choose $a_{m+1} \in P - \bigcup_{1 \leq i \leq m} N_{a_i}$ as long as this last set is nonempty. Thus we obtain a sequence $a_1, \cdots, a_r$ where $r \geq \lceil n/(\frac{1}{2}(\eta-1)^2+1) \rceil \geq n/(\frac{1}{2}\eta^2)$ such that $\{a_1, \cdots, a_r\}$ is a super-antichain. Since $r(\frac{1}{2}\eta^2) \geq n$, either $\eta \geq (2n)^{1/3}$ or $r \geq (2n)^{1/3}$, that is. either $\eta \geq (2|P|)^{1/3}$ or $r \geq (2|P|)^{1/3}$. $\square$

Definition 2.11. *Let $(L, \vee, \wedge)$ is a finite lattice, $a \in L$, it is join-irreducible if $a = b \vee c$ implies that $a = b$ or $a = c$; it is meet-irreducible if $a = b \wedge c$ implies that $a = b$ or $a = c$. An element which is both join- and meet- irreducible is called doubly irreducible, let $Irr(L)$ denote the set of all doubly irreducible elements of L.*

3 Main Theorems

Theorem 3.1. *Let L be a finite lattice with $|L| = n > 4$ and $P = L \setminus \{0,1\}$, $C(P) = \varnothing$. Then one of the following must hold.*
(1) There exists proper sublattice $S = [a]_P \cup \{0\}$ (or $S = [a]_P \cup \{1\}$) $\subset L$ with $|S| \geq (2(n-2))^{1/3} + 1$;
(2) M_k is a 0, 1-sublattice of L, where $k = \lceil (2(n-2))^{1/3} + 2 \rceil$.

Proof. Let $P = L \setminus \{0,1\}$. Note that this makes $|L| = n$ and let $\eta = \max_{x\in P} |[x]_P|$. Then by lemma 2.10 we have :

either there is a semi-sublattice $[a]_P$ of P of size $|[a]_P| \geq (2(n-2))^{1/3}$,

or P contains a super-antichain of size $(2(n-2))^{1/3}$.

Observe that $[a]\cup\{0\}$and $[a]\cup\{1\}$ are proper sublattice of lattice L(for all $a \in P$). Thus either there is a proper sublattice $S = [a]_P \cup \{0\}$(or $S = [a]_P \cup \{1\}$) of L of size $|S| \geq \sqrt{[3]2(n-2)+1}$, or L contains a M_k of size $(2(n-2))^{1/3} + 2$. □

Corollary 3.2. *Let L be a finite lattice with $|L| = n > 4$ and $C(L \setminus \{0,1\}) = \varnothing$. Then there exists proper sublattice $S \subset L$ with $|S| \geq (2(n-2))^{1/3} + 2$;*

Theorem 3.3. *Let L be a finite lattice with $|L| = n > 4$ and $C(L \setminus \{0,1\}) \neq \varnothing$. Then there exists proper sublattice $S \subset L$ with $|S| \geq \frac{1}{2}(n+3)$.*

Proof. Let $a \in C(L \setminus \{0,1\}) \neq \varnothing$. Then either $a \leq x$ or $a \geq x$ for all $x \in L$. Thus we have

$$L = [a]_L = (a] \cup [a).$$

Therefore

(1) If $\min\{|(a]|, |[a)|\} = 2$, then we have: $\max\{|(a]|, |[a)|\} = n - 1 \geq \frac{1}{2}(n+3)$;

(2) If $\min\{|(a]|, |[a)|\} > 2$, then we have: $\max\{|(a]|, |[a)|\} \geq \frac{1}{2}(n-3) + 3 = \frac{1}{2}(n+3)$.

The proof is complete. □

Proof. [Proof of Theorem 1.5] This proof is obvious from Lemma 2.10 and Theorem 3.3. □

4 THE LARGE PROPER SUBLATTICES OF FINITE MODULAR LATTICES

In the construction of the super-antichain in Lemma 2.10 we started with an arbitrary element a_1. We record this stronger fact in the next theorem.

Theorem 4.1. *Let L be a finite lattice with $|L| = n > 4$ and $C(L \setminus \{0,1\}) = \varnothing$, let $P = L \setminus \{0,1\}$ and $\eta = \max_{x\in P} |[x]_P \cup \{0,1\}|$. Then every element of L is contained in a 0, 1-sublattice M_k of L with $k(\frac{1}{2}(\eta-2)^2) \geq n-2$. In particular, if $n - 2 > \frac{1}{2}(\eta-2)^2$, then L is complemented.*

Proof. This proof is obvious from Lemma 2.8 and Theorem 3.3. □

A simple application is:

Theorem 4.2. *Let L be a finite modular lattice with $|L| = n > 4$. Then L has a proper sublattice S with $|S| \geq \sqrt{2n}$.*

Proof. Let $P = L \setminus \{0,1\}$.

Case 1. When $C(P) = C(L \setminus \{0,1\}) \neq \varnothing$. the proof is trivial(by Theorem 3.3).

Case 2. When $C(P) = C(L \setminus \{0,1\}) = \varnothing$. If $n - 2 \leq \frac{1}{2}(\eta-2)^2$, then L has a sublattice $S = [a]_P \cup \{0,1\}$ with $|S| = \eta \geq \sqrt{2(|L|-2)} + 2 \geq \sqrt{2n}$ (by Theorem 4.1). So we may assume that $n - 2 > \frac{1}{2}(\eta-2)^2$, whence by Theorem 4.1, L is complemented. The result is true for $L \cong M_k$, so we may assume that L has height greater than 2. There is a element $b \in L \setminus \{0,1\}$ with $|[b]_L| = \eta > 3$ (L has height greater than 2), then there exists a element $b' \in [b]_L \setminus \{0,b,1\}$. And we have $N_b \neq P$ by $n - 2 > \frac{1}{2}(\eta-2)^2$. Hence there exists a element $c \in P \setminus N_a$. Since $c \notin N_a$, sublattice $\{0, b, b', c'1\}$ of L is a pentagon, contrary to our assumption. □

Definition 4.3. *For a finite lattice L , let*

$$\lambda(L) = \max_{S \in Sub(L), S \neq L} |S|$$

Theorem 4.4. *Let L be a finite lattice with $|L| = n$. Then $\lambda(L) = n - 1$ if and only if $Irr(L) \setminus \{0, 1\} \neq \varnothing$.*

Proof. . The proof is trivial. $\square$

References

[1] R. Freese, J. Jezek and J. B. Nation, *Lattices with large minimal extension*, Algebra Universalis. 45 (2001), 221-309.

[2] T. P. Whaley, *Large sublattices of a lattice*, Pacific J. Math. 28 (1969), 477-484.

[3] G.Gratzer,*General lattice theory*, Academic Press,New York 1978.

[4] Ralph Freese, Jennifer Hyndman, and J. B. Nation, *Whaley's Theorem for Finite Lattices*, Algebra Universalis, to appear.

[5] G.Birkhoff, *Lattice Theory*, 3rd ed., Am. Math.Soc.,Providence RI, (1967).

[6] M. E. Adams, Ralph Freese, J. B. Nation and Jurg Schmid, *Maximal Sublattices And Frattini Sublattices Of Bounded Lattices* , J. Austral. Math. Soc. Ser. A, 63 (1997), 110-127.

[7] Zsolt Lengvarszky and George F. McNulty, *Covering in the sublattice lattices of finite distributive lattices*, Journal of the Australian Mathematical Society (Series A),65 (1998) 333-353.

[8] I. Rival,*Maximal sublattices of finite distributive lattices, Proc.* Amer. Math. Soc. 37 (1973), 417-420.