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ON THE REPRESENTABILITY OF ACTIONS IN A SEMI-ABELIAN CATEGORY

F. BORCEUX, G. JANELIDZE, AND G.M. KELLY

Abstract. We consider a semi-abelian category $\mathcal{V}$ and we write $\mathsf{Act}(G,X)$ for the set of actions of the object G on the object X, in the sense of the theory of semi-direct products in $\mathcal{V}$. We investigate the representability of the functor $\mathsf{Act}(-,X)$ in the case where $\mathcal{V}$ is locally presentable, with finite limits commuting with filtered colimits. This contains all categories of models of a semi-abelian theory in a Grothendieck topos, thus in particular all semi-abelian varieties of universal algebra. For such categories, we prove first that the representability of $\mathsf{Act}(-,X)$ reduces to the preservation of binary coproducts. Next we give both a very simple necessary condition and a very simple sufficient condition, in terms of amalgamation properties, for the preservation of binary coproducts by the functor $\mathsf{Act}(-,X)$ in a general semi-abelian category. Finally, we exhibit the precise form of the more involved "if and only if" amalgamation property corresponding to the representability of actions: this condition is in particular related to a new notion of "normalization of a morphism". We provide also a wide supply of algebraic examples and counter-examples, giving in particular evidence of the relevance of the object representing $\mathsf{Act}(-,X)$, when it turns out to exist.

1. Actions and split exact sequences

A semi-abelian category is a Barr-exact, Bourn-protomodular, finitely complete and finitely cocomplete category with a zero object $\mathbf{0}$. The existence of finite limits and a zero object implies that Bourn-protomodularity is equivalent to, and so can be replaced with, the following split version of the short five lemma:

$$\begin{array}{ccccc} K & \xrightarrow{k_1} & A_1 & \underset{q_1}{\overset{s_1}{\leftrightarrows}} & Q \\ \| & & \downarrow{\scriptstyle\alpha} & & \| \\ K & \xrightarrow{k_2} & A_2 & \underset{q_2}{\overset{s_2}{\leftrightarrows}} & Q \end{array}$$

given a commutative diagram of "kernels of split epimorphisms"

$$q_i s_i = 1_Q, \quad k_i = \mathsf{Ker}\, q_i, \quad i = 1, 2$$

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the morphism α is an isomorphism (see [25], originally from [13]).

This implies the more precise formulation of the short five lemma, where as usual a sequence of morphisms is called exact when the image of each morphism is the kernel of the next one.

1.1. LEMMA. [Short five lemma] *In a semi-abelian category, let us consider a commutative diagram of short exact sequences.*

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & Y & \xrightarrow{l} & B & \xrightarrow{q} & H & \longrightarrow & \mathbf{0} \\
 & & {\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & (*) & \downarrow{\scriptstyle g} & & \\
\mathbf{0} & \longrightarrow & X & \xrightarrow[k]{} & A & \xrightarrow[p]{} & G & \longrightarrow & \mathbf{0}
\end{array}$$

1. *One has always $p = \mathsf{Coker}\, k$ (and analogously, $q = \mathsf{Coker}\, l$).*
2. *If g and h are isomorphisms, f is an isomorphism.*
3. *If g and h are monomorphisms, f is a monomorphism.*
4. *If g and h are regular epimorphisms, f is a regular epimorphism.*
5. *h is an isomorphism if and only if the square (*) is a pullback.*

PROOF. See e.g. [7] 4.6 and [8], 4.2.4 and 4.2.5.[1] ■

The algebraic theories $\mathbb{T}$ giving rise to a semi-abelian variety $\mathsf{Set}^{\mathbb{T}}$ of set-theoretical models have been characterized in [14]: they are the theories containing, for some natural number $n \in \mathbb{N}$

- exactly one constant 0;
- n binary operations α_i satisfying $\alpha_i(x, x) = 0$;
- a $(n+1)$-ary operation θ satisfying $\theta\big(\alpha_1(x, y), \ldots, \alpha_n(x, y), y\big) = x$.

For example, a theory $\mathbb{T}$ with a unique constant 0 and binary operations $+$ and $-$ satisfying the group axioms is semi-abelian: simply put

$$n = 1, \quad \alpha(x, y) = x - y, \quad \theta(x, y) = x + y.$$

Now let $\mathcal{V}$ be an arbitrary semi-abelian category. A *point* over an object G of $\mathcal{V}$ is a triple (A, p, s), where $p\colon A \longrightarrow G$ and $s\colon G \longrightarrow A$ are morphisms in $\mathcal{V}$ with $ps = 1_G$. The points over G form a category $\mathsf{Pt}(G)$ when we define a morphism $f\colon (A, p, s) \longrightarrow (B, q, t)$

[1] For the facility of the reader, we refer often to [8] with precise references, instead of sending him back to a wide number of original papers.

to be a morphism $f\colon A \longrightarrow B$ in $\mathcal{V}$ for which $qf = p$ and $fs = t$ (see [10]). Upon choosing for each point (A, p, s) a definite kernel $\kappa\colon \mathsf{Ker}\, p \longrightarrow A$ of p, we get a functor $K\colon \mathsf{Pt}(G) \longrightarrow \mathcal{V}$ sending (A, p, s) to $\mathsf{Ker}\, p$; this functor has the left adjoint sending X to

$$\big(G + X, (1, 0)\colon G + X \longrightarrow G, i\colon G \longrightarrow G + X\big)$$

(where i is the coprojection), and it is monadic (see [13]). The corresponding monad on $\mathcal{V}$ is written as $G\flat-$, its value at X being the (chosen) kernel $G\flat X$ of $(1, 0)\colon G+X \longrightarrow G$. It is shown in [9] that $G \mapsto G\flat-$ is a functor from $\mathcal{V}$ to the category of monads on $\mathcal{V}$.

Given a $(G\flat-)$-algebra (X, ξ), the corresponding *action* $\xi\colon G\flat X \longrightarrow X$ of the monad $G\flat-$ on the object X of $\mathcal{V}$ will also be called an *action of the object G on X*, or simply a *G-action on X*; we write $\mathsf{Act}(G, X)$ for the set of such actions. A morphism $f\colon G \longrightarrow H$ in $\mathcal{V}$ gives a morphism $f\flat-\colon G\flat - \longrightarrow H\flat-$ of monads, composition with which gives a morphism $\mathsf{Act}(f, X)\colon \mathsf{Act}(H, X) \longrightarrow \mathsf{Act}(G, X)$ of sets; so that $\mathsf{Act}(-, X)$ constitutes a contravariant functor from $\mathcal{V}$ to the category Set of sets. Our concern in this paper is with the representability of this functor; that is, with the existence of an object $[X]$ of $\mathcal{V}$ and a natural isomorphism $\mathsf{Act}(G, X) \cong \mathcal{V}\big(G, [X]\big)$.

We first need an alternative description of $\mathsf{Act}(G, X)$ in terms of split extensions. This description, given in lemma 1.3 below, goes back to [13] and was given in more details in [9], although as part of wider calculations; so as to keep the present paper self-contained, we give here the following direct argument.

Let us call an algebra (X, ξ) for the monad $G\flat-$ simply a *G-algebra*, writing G-Alg for the category of these, with $U\colon G\text{-}\mathsf{Alg} \longrightarrow \mathcal{V}$ for the forgetful functor sending (X, ξ) to X, and with $W\colon \mathsf{Pt}(G) \longrightarrow G\text{-}\mathsf{Alg}$ for the canonical comparison functor having $UW = K\colon \mathsf{Pt}(G) \longrightarrow \mathcal{V}$. To say that K is monadic is to say that W is an equivalence. We may of course denote a G-algebra (X, ξ) by a single letter such as C.

Given a G-algebra (Y, η) and an isomorphism $f\colon X \longrightarrow Y$ in $\mathcal{V}$, there is a unique action ξ of G on X for which $f\colon (X, \xi) \longrightarrow (Y, \eta)$ is a morphism – in fact an isomorphism – of G-algebras; we are forced to take for ξ the composite

$$G\flat X \xrightarrow{\;G\flat f\;} G\flat Y \xrightarrow{\;\eta\;} Y \xrightarrow{\;f^{-1}\;} X.$$

We say that the G-action ξ – the G-structure of the algebra (X, ξ) – has been obtained by transporting along the isomorphism f the G-structure on (Y, η).

$\mathsf{Act}(G, X)$ is in effect the set of G-algebras with underlying object X. Write $\mathsf{ACT}(G, X)$ for the set whose elements are pairs (C, c) consisting of a G-algebra C together with an isomorphism $c\colon X \longrightarrow UC$ in $\mathcal{V}$. There is a function $\mathsf{ACT}(G, X) \longrightarrow \mathsf{Act}(G, X)$ sending (C, c) to the G-action on X obtained by transporting along c the action of G on C; and clearly $\mathsf{Act}(G, X)$ is isomorphic to the quotient of $\mathsf{ACT}(G, X)$ by the equivalence relation $\sim$, where $(C, c) \sim (D, d)$ whenever $dc^{-1}\colon UC \longrightarrow UD$ is a morphism $C \longrightarrow D$ of G-algebras – that is, whenever $dc^{-1}\colon UC \longrightarrow UD$ is Uf for some $f\colon C \longrightarrow D$ (necessarily unique, and necessarily invertible) in G-Alg.

We can imitate the formation of $\mathsf{ACT}(G,X)$, of the equivalence relation $\sim$, and of the quotient set $\mathsf{Act}(G,X) = \mathsf{ACT}(G,X)/\sim$, with any faithful and conservative functor into $\mathcal{V}$ in place of U. In particular, write $\mathsf{SPLEXT}(G,X)$ for the analogue of $\mathsf{ACT}(G,X)$ when U is replaced by $K\colon \mathsf{Pt}(G)\longrightarrow\mathcal{V}$. An object (E,e) of $\mathsf{SPLEXT}(G,X)$ is an object E of $\mathsf{Pt}(G)$ together with an isomorphism $e\colon X\longrightarrow KE$; we have $(E,e)\sim(H,h)$ when $he^{-1}\colon KE\longrightarrow KH$ is Kg for some (necessarily unique and invertible) $g\colon E\longrightarrow H$ in $\mathsf{Pt}(G)$; and we define $\mathsf{SplExt}(G,X)$ as the quotient set $\mathsf{SPLEXT}(G,X)/\sim$. Since $UW = K$, there is a function $\mathsf{SPLEXT}(G,X)\longrightarrow\mathsf{ACT}(G,X)$ sending (E,e) to (WE,e), which respects the equivalence relations $\sim$, and hence induces a function $\mathsf{SplExt}(G,X)\longrightarrow\mathsf{Act}(G,X$; which is easily seen to be a bijection because W is an equivalence.

An object of $\mathsf{SPLEXT}(G,X)$ consists of an object $E=(A,p,s)$ of $\mathsf{Pt}(G)$ and an isomorphism $e\colon X\longrightarrow KE=\mathsf{Ker}\,p$; equivalently, it consists of a short exact sequence

$$\mathbf{0}\longrightarrow X\xrightarrow{\;k\;}A\underset{p}{\overset{s}{\leftrightarrows}}G\longrightarrow\mathbf{0} \tag{1}$$

where $ps=1$ and where k $(=\kappa e)$ is some kernel of p (as distinct from the chosen kernel $\kappa\colon \mathsf{Ker}\,p\rightarrowtail A$).

1.2. Definition. *In a semi-abelian category a short exact sequence with split quotient part as in (1) is said to be a split exact sequence, and to constitute a split extension of G by X. We call a monomorphism $k\colon X\rightarrowtail A$* protosplit *if it forms the kernel part of such a sequence.* (Note that in the abelian case, “protosplit” reduces to “split”.)

It is immediate that the elements of $\mathsf{SPLEXT}(G,X)$ corresponding to two such sequences (k,A,p,s) and (k',A',p',s') are equivalent under the relation $\sim$ precisely when there is a morphism $f\colon A\longrightarrow A'$ of $\mathcal{V}$ (necessarily invertible by Lemma 1.1) satisfying $fk=k'$, $p'f=p$, and $fs=s'$. When this is so, the two split extensions are said to be isomorphic; thus $\mathsf{SplExt}(G,X)$ is the set of isomorphism classes of split extensions of G by X. Summing up, we have established:

1.3. Lemma. *For objects G and X in a semi-abelian category, the comparison functor $W\colon \mathsf{Pt}(G)\longrightarrow G\text{-}\mathsf{Alg}$ induces a bijection*

$$\tau_G\colon \mathsf{SplExt}(G,X)\cong\mathsf{Act}(G,X) \tag{2}$$

between the set of isomorphism classes of split extensions of G by X and the set of G-actions on X. ∎

The right side here is a contravariant functor of G; we now make the left side into such a functor. Given a split extension (k,A,p,s) of G by X as in (1) and a morphism $g\colon H\longrightarrow G$, let the pullback of p and g be given by $q\colon B\longrightarrow H$ and $f\colon B\longrightarrow A$, let $t\colon H\longrightarrow B$ be the unique morphism with $ft=sg$ and $qt=1$, and let $l\colon X\longrightarrow B$ be the unique morphism with $fl=k$ and $ql=0$. In fact the monomorphism l is a kernel of q; for if $qx=0$ we have $pfx=gqx=0$, so that $fx=ky$ for some y; whereupon $fx=ky=fly$ while $qx=0=qly$, giving $x=ly$. Thus (l,B,q,t) is a split extension of H by X.

The isomorphism class of the split extension (l, B, q, t) is independent of the choice of the pullback, and depends only on the isomorphism class of (k, A, p, s); so the process gives a function

$$\mathsf{SplExt}(g, X) \colon \mathsf{SplExt}(G, X) \longrightarrow \mathsf{SplExt}(H, X),$$

which clearly makes $\mathsf{SplExt}(-, X)$ into a contravariant functor from $\mathcal{V}$ to Set.

In proving the following proposition, we use the explicit description of the equivalence $W \colon \mathsf{Pt}(G) \longrightarrow G\text{-}\mathsf{Alg}$, as given in Section 6 of [9]: $W(A, p, s)$ is $K(A, p, s) = \mathsf{Ker}\, p$ with the G-action $\zeta \colon G \flat (\mathsf{Ker}\, p) \longrightarrow \mathsf{Ker}\, p$ where ζ is the unique morphism with $k\zeta$ equal to the composite

$$G \flat (\mathsf{Ker}\, p) \xrightarrow{\quad \lambda \quad} G + (\mathsf{Ker}\, p) \xrightarrow{(s, k)} A,$$

in which λ is the (chosen) kernel of $(1, 0) \colon G + (\mathsf{Ker}\, p) \longrightarrow G$.

1.4. Proposition. *The bijection τ_G of (2) above extends to an isomorphism*

$$\tau \colon \mathsf{SplExt}(-, X) \cong \mathsf{Act}(-, X) \tag{3}$$

of functors.

Proof. The function $\mathsf{SPLEXT}(G, X) \longrightarrow \mathsf{ACT}(G, X)$ sends (E, e) to (WE, e), and the surjection $\mathsf{ACT}(G, X) \longrightarrow \mathsf{Act}(G, X)$ transports the structure of WF along e to obtain an action on X. Accordingly the bijection τ_G takes the isomorphism class of (k, A, p, s) to the action $\xi \colon G \flat X \longrightarrow X$, where $k\xi$ is the composite

$$G \flat X \xrightarrow{\quad \lambda \quad} G + X \xrightarrow{(s, k)} A,$$

where λ is the kernel of $(1, 0) \colon G + X \longrightarrow G$. Now let $g \colon H \longrightarrow G$, and let $\mathsf{SplExt}(g, X)$ take the isomorphism class of (k, A, p, s) to that of (l, B, q, t); as above, the image of this under τ_H is the action $\eta \colon H \flat X \longrightarrow X$ where $l\eta$ is the composite

$$H \flat X \xrightarrow{\quad \lambda \quad} H + X \xrightarrow{(t, l)} B.$$

It follows that η is the composite

$$H \flat X \xrightarrow{g \flat X} G \flat X \xrightarrow{\quad \xi \quad} X;$$

for

$$\begin{aligned} k\xi(g \flat X) &= (s, k)\lambda(g \flat X) \\ &= (s, k)(g + X)\lambda \qquad \text{by the naturality of } \lambda \\ &= f(t, l)\lambda = fl\eta = k\eta. \end{aligned}$$

where $f \colon A \longrightarrow B$ is the morphism used above when describing the functoriality of $\mathsf{SplExt}(-, X)$. That is to say, $\eta = \mathsf{Act}(g, X)\xi$, as desired. ∎

As we said, our concern in this paper is with the representability of the functor $\mathsf{Act}(-,X)$; that is, with the existence of an object $[X]$ of $\mathcal{V}$ and a natural isomorphism

$$\mathsf{Act}\big(G,X\big) \cong \mathcal{V}\big(G,[X]\big);$$

this is a very strong property. In fact, from now on, we shall always work with the isomorphic – but more handy – functor $\mathsf{SplExt}(-,X)$ (see proposition 1.4).

Let us give at once examples of such situations.

1.5. Proposition.

1. *When $\mathcal{V}$ is the semi-abelian category of groups, each functor $\mathsf{Act}(-,X)$ is representable by the group $\mathsf{Aut}(-,X)$ of automorphisms of X.*
2. *When $\mathcal{V}$ is the semi-abelian category of Lie algebras on a ring R, each functor $\mathsf{Act}(-,X)$ is representable by the Lie algebra $\mathsf{Der}(X)$ of derivations of X.*
3. *When $\mathcal{E}$ is a cartesian closed category and $\mathcal{V}$ is the corresponding category of internal groups (respectively, internal Lie algebras), each functor $\mathsf{SplExt}(-,X)$ is still representable.*
4. *When $\mathcal{E}$ is a topos with Natural Number Object and $\mathcal{V}$ is the corresponding category of internal groups (respectively, internal Lie rings), $\mathcal{V}$ is semi-abelian and each functor $\mathsf{Act}(-,X)$ is representable.* ∎

Proof. Statements 1, 2, 3 are reformulations of well-known results, as explained in [9]. Notice that in condition 3 of proposition 1.5, the category $\mathcal{V}$ is generally not semi-abelian: thus the functor $\mathsf{Act}(-,X)$ does not exist in general, while the functor $\mathsf{SplExt}(-,X)$ still makes sense.

In statement 4, the theory $\mathbb{T}$ of internal groups (resp. internal Lie rings) admits a finite presentation. Therefore, the corresponding category $\mathcal{E}^{\mathbb{T}}$ of models in a topos $\mathcal{E}$ with Natural Number Object is finitely cocomplete (see [31]). Trivially, $\mathcal{E}^{\mathbb{T}}$ is pointed. It is exact since so is $\mathcal{E}$ (see [2] 5.11). It is protomodular by [8] 3.1.16. It is thus semi-abelian. One concludes by statement 3 and proposition 1.4. ∎

In this paper, we consider first a certain number of other basic examples, where the functor $\mathsf{Act}(-,X)$ is representable by an easily describable object. And next we switch to the main concern of the paper, namely, the proof of a general representability theorem for $\mathsf{Act}(-,X)$.

2. Associative algebras

The developments in this section have non-trivial intersections with several considerations in [32] and [3].

We fix once for all a base ring R, which is commutative and unital. Every "algebra" considered in this section is an associative R-algebra, not necessarily commutative, not necessarily unital; every morphism is a morphism of such algebras. Analogously, given such an algebra A, the term " left A-module" will always mean an A-R-bimodule, and analogously on the right.

We write simply Alg for the category of R-algebras. This category is semi-abelian (see [14]), thus it is equipped with a notion of semi-direct product and a notion of action of an algebra G on an algebra X. Of course when $R = \mathbb{Z}$, the category Alg reduces to the category Rg of rings.

2.1. Proposition. *For a fixed algebra G, there is an equivalence of categories between*

1. *the category of G-bialgebras;*

2. *the category $\mathsf{Pt}(G)$ of points over G in Alg.*

Proof. By a G-bialgebra X, we mean an algebra X equipped with the structure of a G-bimodule and satisfying the additional algebra axioms

$$g(xx') = (gx)x', \quad (xg)x' = x(gx'), \quad (xx')g = x(x'g)$$

for $g \in G$ and $x, x' \in X$.

Given a G-bialgebra X, define A to be the semi-direct product $G \ltimes X$, which is the cartesian product of the corresponding R-algebras, equipped with the multiplication

$$(g,x)(g',x') = (gg', gx' + xg' + xx').$$

We obtain a point $p, s\colon G \ltimes X \overset{\longleftarrow}{\longrightarrow} G$ by defining

$$p(g,x) = g, \quad s(g) = (g,0).$$

Notice that $X = \mathsf{Ker}\, p$.

Conversely, a split epimorphism $p, s\colon A \overset{\longleftarrow}{\longrightarrow} G$ of R-algebras is in particular a split epimorphism of R-modules, thus $A \cong G \times X$ as an R-module, with $X = \mathsf{Ker}\, p$. Notice that given $x \in X$ and $g \in G$,

$$p\big(s(g)x\big) = ps(g)p(x) = g0 = 0$$

thus $s(g)x \in X = \mathsf{Ker}\, p$. Analogously, $xs(g) \in X$. The actions of G on X are then given by

$$gx = s(g) \cdot x, \quad xg = x \cdot s(g). \qquad \blacksquare$$

Proposition 2.1 shows thus that the notion of algebra action, in the sense of the theory of semi-abelian categories, is exactly given by the notion of G-bialgebra structure on an algebra X. In order to study the representability of the functor $\mathsf{Act}(-, X)$ for an algebra X, we prove first the following lemma:

2.2. LEMMA. *Let X be an algebra. Write $\mathsf{LEnd}(X)$ and $\mathsf{REnd}(X)$ for, respectively, the algebras of left-X-linear and right-X-linear endomorphisms of X, with the composition as multiplication. Then*

$$[X] = \big\{(f,g)\big|\forall x, x' \in X \;\; f(x)\cdot x' = x \cdot g(x')\big\} \subseteq \mathsf{LEnd}(X)^{\mathsf{op}} \times \mathsf{REnd}(X)$$

is a subalgebra of the product.

PROOF. This is routine calculation. ∎

2.3. PROPOSITION. *Given an algebra X, the functor*

$$\mathsf{Act}(-,X)\colon \mathsf{Alg} \longrightarrow \mathsf{Set}$$

is representable by the algebra $[X]$ of lemma 2.2 as soon as

$$\forall f \in \mathsf{LEnd}(X) \; \forall g \in \mathsf{REnd}(X) \quad fg = gf.$$

PROOF. It is immediate to observe that a G-bialgebra structure on X is the same thing as two algebra homomorphisms

$$\lambda\colon G \longrightarrow \mathsf{LEnd}(X,X)^{\mathsf{op}}, \quad \rho\colon G \longrightarrow \mathsf{REnd}(X,X)$$

satisfying the additional conditions

1. $\forall g \in G \; \forall x, x' \in X \quad \lambda(g)(x)\cdot x' = x \cdot \rho(g)(x')$;
2. $\forall g, g' \in G \; \forall x \in X \quad \lambda(g)\big(\rho(g')(x)\big) = \rho(g')\big(\lambda(g)(x)\big)$.

The first condition means simply that the pair (λ, ρ) factors through the subalgebra $[X]$. The second condition holds by assumption on X. ∎

Writing IJ for the usual multiplication of two-sided ideals in an algebra, we get an easy sufficient condition for the representability of the functor $\mathsf{Act}(-,X)$:

2.4. PROPOSITION. *Let X be an algebra such that $XX = X$. Then the functor*

$$\mathsf{Act}(-,X)\colon \mathsf{Alg} \longrightarrow \mathsf{Set}$$

is representable.

PROOF. The assumption means that every element $x \in X$ can be written as $x = \sum_{i=1}^n y_i z_i$, with $n \in \mathbb{N}$ and $y_i, z_i \in X$. Then given $f \in \mathsf{LEnd}(X)$ and $g \in \mathsf{REnd}(X)$, we get

$$\begin{aligned} f\big(g(x)\big) &= f\left(g\left(\sum_{i=1}^n y_i z_i\right)\right) = f\left(\sum_{i=1}^n g(y_i) z_i\right) = \sum_{i=1}^n g(y_i) f(z_i) \\ &= g\left(\sum_{i=1}^n y_i f(z_i)\right) = g\left(f\left(\sum_{i=1}^n y_i z_i\right)\right) = g\big(f(x)\big) \end{aligned}$$

thus $fg = gf$ and one concludes by proposition 2.3. ∎

Things become more simple in the category ComAlg of commutative algebras (without necessarily a unit). The results could be deduced from those for arbitrary algebras, but a direct argument is almost as short and more enlightening.

2.5. Proposition. *For a fixed commutative algebra G, there is an equivalence of categories between*

1. *the category of commutative G-algebras;*
2. *the category* $\mathsf{Pt}(G)$ *of points over G in* ComAlg.

Proof. Analogous to that of proposition 2.1. ∎

Proposition 2.5 shows thus that the notion of commutative algebra action, in the sense of the theory of semi-abelian categories, is exactly given by the notion of G-algebra structure on a commutative algebra X.

2.6. Theorem. *Given a commutative algebra X, the following conditions are equivalent:*

1. *the functor*
$$\mathsf{Act}(-,X)\colon \mathsf{ComAlg} \longrightarrow \mathsf{Set}$$
is representable;
2. *the algebra* $\mathsf{End}(X)$ *of X-linear endomorphisms of X is commutative.*

In these conditions, the functor $\mathsf{Act}(-,X)$ *is represented by* $\mathsf{End}(X)$.

Proof. ($2 \Rightarrow 1$). It is immediate to observe that a G-algebra structure on X is the same thing as an algebra homomorphism $G \longrightarrow \mathsf{End}(X)$, where $\mathsf{End}(X)$ is equipped with the pointwise R-module structure and the composition as multiplication. By proposition 2.5, the algebra $\mathsf{End}(X)$ represents the functor $\mathsf{Act}(-,X)$ as soon as this algebra is commutative.

Conversely, suppose that the functor $\mathsf{Act}(-,X)$ is representable by a commutative algebra $[X]$. Proposition 2.5 and the observation at the beginning of this proof show now the existence of natural isomorphisms of functors

$$\mathsf{ComAlg}\big(-,[X]\big) \cong \mathsf{Act}\big(-,X\big) \cong \mathsf{Alg}\big(-,\mathsf{End}(X)\big).$$

In particular, the identity on $[X]$ corresponds by these bijections to an algebra homomorphism $u\colon [X] \longrightarrow \mathsf{End}(X)$. For every commutative algebra G, composition with u induces thus a bijection

$$\mathsf{Alg}\big(G,[X]\big) = \mathsf{ComAlg}\big(G,[X]\big) \xrightarrow{\cong} \mathsf{Alg}\big(G,\mathsf{End}(X)\big).$$

The free non necessarily commutative algebra on one generator is the algebra $R^*[t]$ of polynomials with coefficients in R and a zero constant term. But this algebra is commutative, thus can be chosen as algebra G in the bijection above. And since it is a strong generator in the category Alg of all algebras, u is an isomorphism. ∎

Again we deduce:

2.7. Proposition. *Let X be a commutative algebra such that $XX = X$. Then the algebra $\mathsf{End}(X)$ is commutative and represents the functor*

$$\mathsf{Act}(-, X)\colon \mathsf{ComAlg} \longrightarrow \mathsf{Set}.$$

Proof. See the proof of proposition 2.4. ■

Let us now consider some straightforward examples of interest.

2.8. Proposition. *When the commutative algebra X has one of the following properties:*

- *X is Taylor-regular;*
- *X is pure;*
- *X is von Neumann-regular;*
- *X is Boolean;*
- *X is unital;*

the algebra $\mathsf{End}(X)$ is commutative and represents the functor

$$\mathsf{Act}(-, X)\colon \mathsf{ComAlg} \longrightarrow \mathsf{Set}.$$

Proof. An X-module M is Taylor-regular (see [42]) when the scalar multiplication $X \otimes_X M \longrightarrow M$ is an isomorphism. Putting $M = X$, one concludes by proposition 2.7 since the image of the multiplication is precisely the ideal XX.

An ideal $I \lhd X$ is pure (see [5]) when

$$\forall x \in I \ \exists \varepsilon \in I \ x = x\varepsilon.$$

Putting $I = X$, we get a special case of a Taylor-regular algebra.

The algebra X is von Neumann regular when

$$\forall x \in X \ \exists y \in X \ x = xyx.$$

This is a special case of a pure algebra: put $\varepsilon = yx$.

The algebra X is Boolean when

$$\forall x \in X \ xx = x.$$

This is a special case of a von Neumann regular algebra: put $y = x$.

Finally every unital algebra is pure: simply choose $\varepsilon = 1$. ■

3. Boolean rings

This section studies the case of the category BooRg of Boolean rings. We know already that plain ring actions on a Boolean ring are representable (see proposition 2.8). One has also:

3.1. Proposition. *Given a Boolean ring X, the ring $\mathsf{End}(X)$ is Boolean and still represents the functor*

$$\mathsf{Act}(-, X)\colon \mathsf{BooRg} \longrightarrow \mathsf{Set}.$$

Proof. The ring $\mathsf{End}(X)$ is Boolean, simply because its multiplication coincides with the pointwise multiplication. Indeed, given $f, h \in \mathsf{End}(X)$

$$f\big(h(x)\big) = f\big(h(xx)\big) = f\big(xh(x)\big) = f(x)h(x).$$

Notice moreover that computing $(x+x)^2$ yields at once $x + x = 0$.

We know by proposition 2.5 that every split exact sequence of rings has the form

$$\mathbf{0} \longrightarrow X \xrightarrow{\;k\;} G \ltimes X \underset{p}{\overset{s}{\leftrightarrows}} G \longrightarrow \mathbf{0}$$

for some G-algebra structure on X. When G and X are Boolean, then $G \ltimes X$ is Boolean as well since

$$(g, x) = (g, x)^2 = (g^2, gx + gx + x^2) = (g, gx + gx + x) = (g, 0 + x) = (g, x).$$

Thus every split exact sequence in ComRg with G and X Boolean is a split exact sequence in BooRg. This proves that the functor $\mathsf{Act}(-, X)$ on BooRg is the restriction of the functor $\mathsf{Act}(-, X)$ on ComRg. One concludes by proposition 2.6, since $\mathsf{End}(X)$ is Boolean. ∎

Let us also mention here a useful result, which will turn out to have close connections with our general representability theorem (see proposition 6.2).

3.2. Proposition. *The category BooRg of Boolean rings satisfies the amalgamation property (see definition 6.1).*

Proof. A Boolean algebra can be defined as a Boolean ring with unit (see [4]); the correspondence between the various operations is given by

$$x \vee y = x + xy + y, \quad x \wedge y = xy, \quad x + y = (x \wedge \neg y) \vee (\neg x \wedge y).$$

The category Bool of Boolean algebras is thus a subcategory of BooRg: the subcategory of Boolean rings with a unit and morphisms preserving that unit.

Notice further that writing $\mathbf{2} = \{0, 1\}$ for the two-element Boolean algebra, BooRg is equivalent to the slice category $\mathsf{Bool}/\mathbf{2}$. The morphisms $f\colon B \longrightarrow \mathbf{2}$ of Boolean algebras correspond bijectively with the maximal ideals $f^{-1}(0) \subseteq B$; and each (maximal) ideal is a Boolean ring. Conversely, given a Boolean ring R without necessarily a unit, R is a maximal ideal in the following Boolean ring $\overline{R} = R \times \{0, 1\}$, admitting $(0, 1)$ as a unit:

$$(r, n) + (s, m) = (r + s, n + m), \quad (r, n) \times (s, m) = (rs + nr + ms, nm).$$

The category Bool of Boolean algebras satisfies the amalgamation property (see [20] or [30]), from which each slice category Bool/B does, thus in particular the category BooRg of Boolean rings. ■

4. Commutative von Neumann regular rings

A ring X (without necessarily a unit) is von Neumann regular when

$$\forall x \in X \ \exists y \in X \quad xyx = x.$$

Putting $x^* = yxy$ one gets further

$$x = xx^*x, \quad x^* = x^*xx^*.$$

In the commutative case, an element x^* with those properties is necessarily unique (see for example [27], V.2.6). Indeed if x' is another such element

$$x^* = x^*xx^* = x^*xx'xx^* = x^*xx^*x'x = x^*x'x$$

and analogously starting from x'. This proves that the theory of commutative von Neumann regular rings is algebraic and can be obtained from the theory of commutative rings by adding a unary operation $(\ )^*$ satisfying the two axioms above. This is of course a semi-abelian theory, since it contains a group operation and has a unique constant 0 (see [14]).

The uniqueness of x^* forces every ring homomorphism between commutative regular rings to preserves the operation $(\ )^*$. Thus the category $\mathsf{ComRegRg}$ of commutative regular rings is a full subcategory of the category ComRg of commutative rings.

Let us first summarize several well-known facts (see, e.g.[38]).

4.1. Lemma. *Writing a, a_i, b, c, e for elements of a commutative von Neumann regular ring R:*

1. $(ab)^* = a^*b^*$;
2. $\forall a \ \exists e \ e^2 = e, \ e = e^*, \ a = ae$;
3. $\forall a_1, \ldots, a_n \ \exists e \ e^2 = e, \ e = e^*, \ a_1e = a_1, \ldots, a_ne = a_n$;
4. $(a = b) \Leftrightarrow (\forall c \ ac = bc)$;

Proof. (1) follows at once from the uniqueness of $(ab)^*$. For (2), choose $e = a^*a$. Given a, b with corresponding idempotents e, e' as in condition 2, put $e = e + e' - ee'$; iterate the process to get condition 3; put $c = e$ to get condition 4. ■

4.2. Corollary. *Every finitely generated object R of the category of commutative von Neumann regular rings is a unital ring.*

Proof. Write $a_1, \ldots, a_n$ for a family of generators; choose e as in condition 3 of lemma 4.1. It suffices to prove that e remains a unit for every element constructed form the generators a_i and the operations $+$, $-$, $\times$ and $(\ )^*$. Only the last case requires a comment: if $xe = e$, then

$$x^*e = x^*xx^*e = x^*xex^* = x^*xx^* = x^*.$$ ∎

Proposition 2.8 gives us a first bit of information concerning the representability of actions for von Neumann regular rings. Let us observe further that:

4.3. Lemma. *When X is a commutative von Neumann regular ring, the ring $\mathsf{End}(X)$ of X-linear endomorphisms of X is still a commutative von Neumann regular ring.*

Proof. By proposition 2.8, $\mathsf{End}(X)$ is commutative. Given $f \in \mathsf{End}(X)$, define $f^*(x) = \big(f(x^*)\big)^*$; let us prove that this makes $\mathsf{End}(X)$ a von Neumann regular ring.

First,

$$f^*(ab) = \Big(f\big((ab)^*\big)\Big)^* = \big(f(a^*b^*)\big)^* = \big(a^*f(b^*)\big)^* = a^{**}\big(f(b^*)\big)^* = af^*(b).$$

Next

$$\begin{aligned} f^*(a+b)c &= f^*\big((a+b)c\big) = (a+b)f^*(c) = af^*(c) + bf^*(c) \\ &= f^*(ac) + f^*(bc) = f^*(a)c + f^*(b)c = \big(f^*(a) + f^*(b)\big)c \end{aligned}$$

and so by lemma 4.1.4, $f^* \in \mathsf{End}(X)$.

It remains to observe that $ff^*f = f$. Given $a \in R$ and e as in lemma 4.1.2, we have

$$f^*(e) = \big(f(e^*)\big)^* = \big(f(e)\big)^*.$$

Given two endomorphisms $f, g \in \mathsf{End}(X)$ we have also

$$fg(e) = fg(ee) = f\big(eg(e)\big) = f(e)g(e)$$

and therefore

$$ff^*f(e) = f(e)f^*(e)f(e) = f(e)\big(f(e)\big)^*f(e) = f(e).$$

Finally

$$ff^*f(a) = ff^*f(ae) = aff^*f(e) = af(e) = f(ae) = f(a).$$ ∎

4.4. Lemma. *Consider a split exact sequence*

$$\mathbf{0} \longrightarrow X \xrightarrow{\ k\ } A \underset{q}{\overset{s}{\leftrightarrows}} G \longrightarrow \mathbf{0}$$

in the category of commutative rings. When X and G are von Neumann regular rings, A is a von Neumann regular ring as well.

Proof. In the locally finitely presentable category $\mathsf{ComRegRg}$ of commutative von Neumann regular rings, every object G is the filtered colimit $(\sigma_i\colon G_i \rightarrowtail G)_{i\in I}$ of its finitely generated subobjects. But filtered colimits are computed as in the category of sets, thus are also filtered colimits in the category of all commutative rings.

Pulling back the split exact sequence of the statement along each morphism σ_i yields a filtered diagram of split exact sequences, still with the kernel X (see lemma 1.1.5).

$$\mathbf{0} \longrightarrow X \xrightarrow{k_i} A_i \underset{q_i}{\overset{s_i}{\rightleftarrows}} G_i \longrightarrow \mathbf{0}.$$

By universality of filtered colimits, $A \cong \mathsf{colim}_{i\in I} A_i$. If we can prove that each A_i is a von Neumann regular ring, the same will hold for A, as a filtered colimit of von Neumann regular rings. But each ring G_i is unital by corollary 4.2. So it suffices to prove the statement in the special case where the ring G is unital.

Let us prove next that we can reduce further the problem to the case where both G and X are unital. So we suppose already that G is unital and, for simplicity, we view both s and k as canonical inclusions.

Write Xe for the ideal of X generated by an idempotent element e. If e' is another idempotent element such that $ee' = e$, then $Xe \subseteq Xe'$. By lemma 4.1.3, the family of ideals Xe, with $e = e^2 \in R$, is thus filtered. But still by lemma 4.1.2, the ring X is generated by its idempotent elements. Thus finally X is the filtered union of its principal ideals Xe with e idempotent. Of course, each of these ideals is a unital ring: the unit is simply e.

For each $e = e^2 \in X$, we consider further

$$A_e = \{xe + g | x \in X,\ g \in G\} \subseteq A.$$

We observe that:

1. Since X is an ideal in A, each A_e is a subring of A.
2. Each ring A_e still contains G, so that the pair (q, s) restricts as a split epimorphism

$$q_e, s_e\colon A_e \rightleftarrows G.$$

3. Since $q(xe + g) = g$,

$$\mathsf{Ker}\, q_e = Xe = \{xe | x \in X\} = \{x | x \in X,\ xe = x\}.$$

4. Since $A \cong G \times X$ as abelian groups (see propositions 2.5, 2.1) and X is the filtered union of the various Xe, A is the filtered union of the various $A_e \cong G \times Xe$.

By this last observation, it suffices to prove that each A_e is a von Neumann regular ring. And this time we have a split exact sequence

$$\mathbf{0} \longrightarrow Xe \longrightarrow A_e \underset{q_e}{\overset{s_e}{\rightleftarrows}} G \longrightarrow \mathbf{0}$$

with both Xe and G commutative von Neumann regular rings with a unit.

So we have reduced the problem to the case where both X and G admit a unit. In that case we shall prove that the ring A is isomorphic to the ring $G \times X$. And since the product of two von Neumann regular rings is trivially a von Neumann regular ring, the proof will be complete.

We know already that A and $G \times X$ are isomorphic as abelian groups (see propositions 2.5, 2.1). We still view k and s as canonical inclusions and we write $u \in X$, $v \in G$ for the units of these two rings. Since each element of A can be written as $x + g$ with $x \in X$ and $g \in G$, it follows at once that $e = u + v - uv \in A$ is a unit for the ring A:

$$(x+g)(u+v-uv) = x + xv - xv + gu + g - gu = x + g.$$

Now since u and $e - u$ are idempotent elements of A, the morphism

$$A \longrightarrow A(e-u) \times Au, \quad a \mapsto \big(a(e-u), au\big)$$

is an isomorphism of rings, with inverse

$$A(e-u) \times Au \longrightarrow A, \quad (a,b) \mapsto a+b.$$

It remains to prove that we have ring isomorphisms

$$A(e-u) \cong G, \quad Au \cong X.$$

The second isomorphism is easy: u is the unit of X and X is an ideal of A, thus

$$X = Xu \subseteq Au \subseteq X.$$

To prove the first isomorphism, notice first that $q(u) = 0$ implies $q(e) = q(v) = v$. Consider then the mapping

$$G \longrightarrow A(e-u), \quad g \mapsto g(e-u)$$

which is a ring homomorphism, since $e - u$ is idempotent. This mapping is injective because $g(e-u) = 0$ forces $ge = gu \in X$ and thus

$$g = gv = q(g)q(e) = q(ge) = 0.$$

The mapping is also surjective because every $a \in A$ can be written $a = g + x$, with $g \in G$ and $x \in X$, and

$$a(e-u) = (g+x)(e-u) = g(e-u) + xe - xu = g(e-u) + x - x = g(e-u). \qquad \blacksquare$$

4.5. Proposition. *Let $\mathcal{V}$ be the semi-abelian category of commutative von Neumann regular rings. For every object $X \in \mathcal{V}$, the functor $\mathsf{Act}(-,X)$ is representable by the commutative von Neumann regular ring $\mathsf{End}(X)$ of X-linear endomorphisms of X.*

Proof. By lemma 4.4, the functor $\mathsf{Act}(-,X)$ of the statement is the restriction of the corresponding functor defined on the category of all commutative rings. Since $\mathsf{End}(X)$ is a von Neumann regular ring by lemma 4.3, we conclude by proposition 2.8. $\blacksquare$

As for Boolean rings, let us conclude this section with proving the amalgamation property in the category of commutative von Neumann regular rings.

4.6. Lemma. *The category of (not necessarily unital) commutative von Neumann regular rings satisfies the amalgamation property.*

Proof. In a locally finitely presentable category, every finite diagram can be presented as the filtered colimit of a family of diagrams of the same shape, whose all objects are finitely presentable (see [19], the uniformization lemma). Taking the images of the various canonical morphisms, we conclude that every finite diagram is the filtered colimit of a family of diagrams with the same shape, whose all objects are finitely generated subobjects of the original ones.

Consider now a pushout $\delta\alpha = \gamma\beta$ of commutative von Neumann regular rings, with α and β injective. Applying the argument above to the diagram (α, β), write it as a filtered colimit of diagrams (α_i, β_i), with each X_i a finitely generated subobject of X and analogously for A_i and B_i.

$$\begin{array}{ccc} X & \overset{\alpha}{\rightarrowtail} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\delta} \\ B & \underset{\gamma}{\longrightarrow} & C \end{array} \qquad \begin{array}{ccc} X_i & \overset{\alpha_i}{\rightarrowtail} & A_i \\ {\scriptstyle\beta_i}\downarrow & & \downarrow{\scriptstyle\delta_i} \\ B_i & \underset{\gamma_i}{\longrightarrow} & C_i \end{array}$$

The morphisms α_i and β_i are still injective, as restrictions of α and β; define $(C_i, \delta_i, \gamma_i)$ to be their pushout (which of course is still finitely generated). The pushout $\delta\alpha = \gamma\beta$ is the filtered colimit of the pushouts $\delta_i\alpha_i = \gamma_i\beta_i$; thus if γ_i and δ_i turn out to be monomorphisms, so are δ and γ. So, it suffices to prove the amalgamation property for finitely generated commutative regular rings. By corollary 4.2, we have reduced the problem to the case where the rings are unital.

Let us thus assume that X, A, B have a unit. Of course, α and β have no reason to preserve the unit. Writing Ra for the principal ideal generated by an element a of a commutative ring R, we consider the following squares, where the various morphisms are the restrictions of α, β, γ, δ.

$$\begin{array}{ccc} X & \overset{\alpha_1}{\rightarrowtail} & A\alpha(1) \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\delta_1} \\ B\beta(1) & \underset{\gamma_1}{\longrightarrow} & C\delta\alpha(1) \end{array} \qquad \begin{array}{ccc} 0 & \rightarrowtail & A\big(1-\alpha(1)\big) \\ \downarrow & & \downarrow{\scriptstyle\delta_2} \\ B\big(1-\beta(1)\big) & \underset{\gamma_2}{\longrightarrow} & C\big(1-\delta\alpha(1)\big) \end{array}$$

Now given an idempotent element e in a commutative ring R, the ideal Re is always a unital ring (with unit e) and is also a retract of R, with the multiplication by e as a retraction. Moreover when R is regular, so is every ideal $I \lhd R$, since $i \in I$ implies

$i^* = i^*ii^* \in I$. Thus the two squares above are still pushouts of commutative regular rings, as retracts of the pushout $\delta\alpha = \gamma\beta$.

But the left hand square is now a pushout in the category of unital commutative von Neumann regular rings and morphisms preserving the unit. This category satisfies the amalgamation property (see [18] or [30]), thus δ_1 and γ_1 are injective. And the right hand pushout is in fact a coproduct: therefore δ_2 and γ_2 are injective as well, with retractions $(\mathsf{id}, 0)$ and $(0, \mathsf{id})$.

Finally, if R is a commutative ring with unit and $e \in R$ is idempotent, as already observed in the proof of lemma 4.4, the morphism

$$R \longrightarrow Re \times R(1-e), \quad r \mapsto \big(re, r(1-e)\big)$$

is an isomorphism: it is trivially injective and the pair $\big(ue, v(1-e)\big)$ is the image of $ue + v(1-e)$. Via such isomorphisms, we conclude that $\delta \cong \delta_1 \times \delta_2$ and $\gamma \cong \gamma_1 \times \gamma_2$, thus δ and γ are injective. ∎

5. Locally well-presentable semi-abelian categories

We switch now to the proof of a general representability theorem for the functors $\mathsf{SplExt}(-, X)$. We shall prove such a theorem for a very wide class of semi-abelian categories $\mathcal{V}$: the locally well-presentable ones. For such categories, we reduce first the representability of the functors

$$\mathsf{SplExt}(-, X) \colon \mathcal{V} \longrightarrow \mathsf{Set}$$

to the preservation of binary coproducts.

5.1. Definition. *A category $\mathcal{V}$ is* locally well-presentable *when*

1. *$\mathcal{V}$ is locally presentable;*

2. *in $\mathcal{V}$, finite limits commute with filtered colimits.*

Of course every locally finitely presentable category is locally well-presentable. But also all Grothendieck toposes are locally well-presentable (see [22], or [6] 3.4.16) and these are generally not locally finitely presentable. In fact, the models of a semi-abelian algebraic theory in a Grothendieck topos $\mathcal{E}$ constitute always a semi-abelian locally well-presentable category (see proposition 5.2). Putting $\mathcal{E} = \mathsf{Set}$, this contains in particular the case of the semi-abelian varieties of universal algebra.

5.2. Proposition. *The category $\mathcal{E}^{\mathbb{T}}$ of models of a semi-abelian theory $\mathbb{T}$ in a Grothendieck topos $\mathcal{E}$ is locally well-presentable.*

PROOF. Trivially, $\mathcal{E}^{\mathbb{T}}$ is pointed. It is exact since so is $\mathcal{E}$ (see [2] 5.11). It is protomodular by [8] 3.1.16. It is also semi-abelian locally presentable (see [22] or [6] 3.4.16 and [1] 2.63); thus in particular it is complete and cocomplete. So $\mathcal{E}^{\mathbb{T}}$ is already semi-abelian by [25] 2.5. Finally if $\mathcal{E}$ is the topos of sheaves on a site $(\mathcal{C}, \mathcal{T})$, then $\mathcal{E}^{\mathbb{T}}$ is a localization of the category of $\mathbb{T}$-models in the topos of presheaves $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$. In the case of presheaves, finite limits and filtered colimits of $\mathbb{T}$-models are computed pointwise, thus commute. And the reflection to the category $\mathcal{E}^{\mathbb{T}}$ preserves finite limits and filtered colimits. ■

The following two results are essentially part of the "folklore", but we did not find an explicit reference for them. Of course when we say that a contravariant functor preserves some colimits, we clearly mean that it transforms these colimits in limits.

5.3. PROPOSITION. *Let $\mathcal{V}$ be a locally presentable category. A contravariant functor $F\colon \mathcal{V} \longrightarrow \mathsf{Set}$ is representable if and only if it preserves small colimits.*

PROOF. The category $\mathcal{V}$ is cocomplete and has a generating set; it is also co-well-powered (see [1], 1.58). One concludes by [29], 4.90. ■

Let us recall that a finitely complete category $\mathcal{V}$ is a *Mal'tsev* category when every reflexive relation in $\mathcal{V}$ is at once an equivalence relation (see [15], [16], [17], [35]. Semi-abelian categories are Mal'tsev categories (see [25]).

5.4. PROPOSITION. *Let $\mathcal{V}$ be a finitely cocomplete Barr-exact Mal'tsev category. A contravariant functor $F\colon \mathcal{V} \longrightarrow \mathsf{Set}$ preserves finite colimits if and only if it preserves*

1. *the initial object;*
2. *binary coproducts;*
3. *coequalizers of kernel pairs.*

When the category $\mathcal{V}$ is also locally presentable, the functor F is representable when, moreover, it preserves

4. *filtered colimits.*

PROOF. Conditions 1 and 2 take care of all finite coproducts. But in a category with finite coproducts, every finite colimit can be presented as the coequalizer of a pair of morphisms with a common section (see [33], exercise V-2-1). Given such a pair (u, v) with common section r as in diagram 1, consider the image factorization $(u, v) = \rho\pi$. By assumption, the composite $(u, v)r$ is the diagonal of $B \times B$. Thus R is a reflexive relation on B and by the Mal'tsev property, an equivalence relation. By Barr-exactness of $\mathcal{V}$ (see [2]), R is a kernel pair relation. Since π is an epimorphism, one has still $q = \mathsf{Coker}\,(p_1\rho, p_2\rho)$ and thus $(p_1\rho, p_2\rho)$ is the kernel pair of q.

Consider now for simplicity the covariant functor

$$F\colon \mathcal{V} \longrightarrow \mathsf{Set}^{\mathsf{op}}.$$

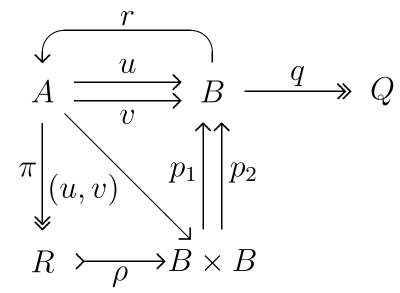


Diagram 1

By condition 4 in the statement, F preserves regular epimorphisms, but also the coequalizer $q = \mathsf{Coker}\,(p_1\rho, p_2\rho)$. Thus $F(\pi)$ is an epimorphism and therefore

$$\begin{aligned} F(q) &= \mathsf{Coker}\,\big(F(p_1\rho), F(p_2\rho)\big) \\ &= \mathsf{Coker}\,\big(F(p_1\rho)F(\pi), F(p_2\rho)F(\pi)\big) \\ &= \mathsf{Coker}\,\big(F(u), F(v)\big). \end{aligned}$$

This concludes the proof that F preserves all finite colimits.

Suppose next that $\mathcal{V}$ is also locally presentable. An arbitrary colimit is the filtered colimit of its finite subcolimits, thus condition 4 forces the preservation of all small colimits. By proposition 5.3, F is then representable. ∎

We now switch back to the functor $\mathsf{SplExt}(-, X)$.

5.5. Lemma. *Let $\mathcal{V}$ be a semi-abelian category. The functor* $\mathsf{SplExt}(-, X)$ *preserves the initial object.*

Proof. When $G = \mathbf{0}$ in the split exact sequence of definition 1.2, k is an isomorphism and thus $\mathsf{SplExt}(\mathbf{0}, X)$ is a singleton. ∎

5.6. Lemma. *Let $\mathcal{V}$ be a semi-abelian category. The functor* $\mathsf{SplExt}(-, X)$ *preserves the coequalizers of kernel pairs.*

Proof. In $\mathcal{V}$, let us consider a regular epimorphism γ and its kernel pair (u, v). We must prove to have an equalizer in Set

$$\mathsf{SplExt}(G, X) \xrightarrow{\mathsf{SplExt}(\gamma, X)} \mathsf{SplExt}(H, X) \underset{\mathsf{SplExt}(v, X)}{\overset{\mathsf{SplExt}(u, X)}{\rightrightarrows}} \mathsf{SplExt}(P, X).$$

We prove first the injectivity of $\mathsf{SplExt}(\gamma, X)$. Given a point (A, p, s) over G, consider its pullback (B, q, t) over H, as in the following diagram.

$$
\begin{array}{ccccc}
C & \overset{u'}{\underset{v'}{\rightrightarrows}} & B & \overset{\gamma'}{\twoheadrightarrow} & A \\
w \downarrow \uparrow r & & q \downarrow \uparrow t & & p \downarrow \uparrow s \\
P & \overset{u}{\underset{v}{\rightrightarrows}} & H & \underset{\gamma}{\twoheadrightarrow} & G
\end{array}
$$

Since $\gamma u = \gamma v$, pulling back (B, q, t) along u or v yields the same point (C, w, r) over P. Now γ' is a strong epimorphism since so is γ, while (u', v') is the kernel pair of γ' since (u, v) is the kernel pair of γ. Therefore $\gamma' = \mathsf{Coker}\,(u', v')$. As a consequence, (A, p, s) is indeed entirely determined via a coequalizer process from the left hand square. Thus if two points over G have the same pullback over H, they are isomorphic.

For the surjectivity, still using the same diagram, consider now a split exact sequence $(l, q, t) \in \mathsf{SplExt}(H, X)$ whose pullbacks along u and v are the same: let us write $(m, w, r) \in \mathsf{SplExt}(P, X)$ for this pullback. We put further $\gamma' = \mathsf{Coker}\,(u', v')$ and p, s are the factorizations of q, t through the cokernels. Trivially $ps = 1_G$.

By a well-known Barr–Kock result (see [2], 6.10), the square $p\gamma' = \gamma q$ is a pullback since so are the squares $qu' = uw$ and $qv' = vw$. By lemma 1.1, the kernel $k = \mathsf{Ker}\, p$ is isomorphic to $l = \mathsf{Ker}\, q$, thus $(k, p, l) \in \mathsf{SplExt}(G, X)$. So (l, q, t) is the image of (k, p, s) under $\mathsf{SplExt}(\gamma, X)$. ∎

5.7. Lemma. *Let $\mathcal{V}$ be a locally well-presentable semi-abelian category. The functor* $\mathsf{SplExt}(-, X)$ *preserves filtered colimits.*

Proof. This is an immediate consequence of the commutation between filtered colimits and kernels in $\mathcal{V}$. Notice that a filtered colimit of a constant diagram on X is again X. ∎

As a conclusion of the various results of this section, we get:

5.8. Theorem. *Let $\mathcal{V}$ be a semi-abelian category. For a fixed object $X \in \mathcal{V}$, the following conditions are equivalent:*

1. *the functor* $\mathsf{SplExt}(-, X)$ *preserves binary coproducts;*
2. *the functor* $\mathsf{SplExt}(-, X)$ *preserves finite colimits.*

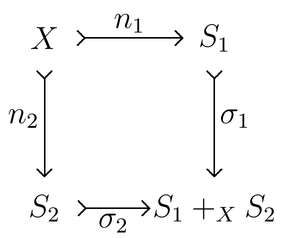
When $\mathcal{V}$ is also locally well-presentable, those conditions are further equivalent to:

3. *the functor* $\mathsf{SplExt}(-, X)$ *is representable.* ∎

6. A necessary condition and some sufficient conditions

Let us first recall the standard form of the *amalgamation property.*

6.1. Definition. *In a category with pushouts, two monomorphisms l, m with the same domain X satisfy the amalgamation property when in the pushout square*



the morphisms σ_1 and σ_2 are still monomorphisms.

Here is at once a necessary condition for the representability of actions.

6.2. Proposition. *Let X be an object of a semi-abelian category $\mathcal{V}$. If the functor $\mathsf{SplExt}(-, X)$ preserves binary coproducts (in particular, when it is representable), the amalgamation property holds in $\mathcal{V}$ for protosplit monomorphisms with domain X (see definition 1.2).*

Proof. Start with two split exact sequences (l, q, t) and (m, r, w) as in diagram 2. Since the functor $\mathsf{SplExt}(-, X)$ preserves binary coproducts, there is a unique split exact

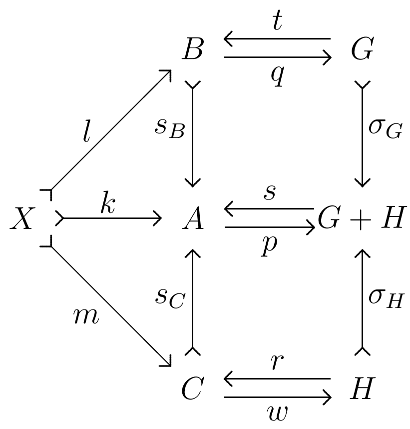


Diagram 2

sequence (k, p, s) such that the squares $\sigma_G q = p s_B$ and $\sigma_H w = p s_C$ are pullbacks. The morphisms σ_G and σ_H are monomorphisms, with retractions $(1_G, 0)$ and $(0, 1_H)$. So s_B and s_C are monomorphisms and the square $s_B l = k = s_C m$ is commutative. Thus s_B and s_C factor through the pushout of l, m and, since s_B, s_C are monomorphisms, so are the morphisms of this pushout. ■

Let us immediately exhibit a sufficient condition for the representability of actions: this result is in fact a corollary of the more general considerations of section 4. We use the notation of definition 6.1.

6.3. Proposition. *Let $\mathcal{V}$ be a semi-abelian category. Suppose that given two protosplit monomorphisms n_1, n_2 with domain X, the diagonal $n = \sigma_i n_i$ of their pushout is a normal monomorphism. Then the functor* $\mathsf{SplExt}(-, X)$ *preserves finite colimits. Moreover, when $\mathcal{V}$ is also locally well-presentable, the functor* $\mathsf{SplExt}(-, X)$ *is representable.*

Proof. Anticipating on definition 7.1 and lemma 7.2, the morphism n is its own universal normalization. Thus the conditions of theorem 8.5 are satisfied and the functors $\mathsf{SplExt}(-, X)$ are representable. ∎

Notice that proposition 6.3 does not require the morphisms σ_i to be monomorphisms. But of course, propositions 6.3 and 6.2 force this to be the case. It is interesting to notice moreover that with the notation of proposition 6.3:

- n being a *monomorphism* is a necessary condition for the representability of $\mathsf{SplExt}(-, X)$ (proposition 6.2);
- n being a *normal monomorphism* is a sufficient condition (proposition 6.3).

Thus the necessary and sufficient condition is "squeezed" between these two properties.

In the applications, it is often more convenient to use the following variation on proposition 6.3.

6.4. Definition. *A semi-abelian category $\mathcal{V}$ satisfies the axiom of* normality of unions *when given a commutative square of subobjects*

$$\begin{array}{ccc} X & \overset{l}{\rightarrowtail} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle s_B} \\ C & \underset{s_C}{\rightarrowtail} & A \end{array}$$

if X is normal in both B and C, then it is also normal in their union $B \vee C$.

6.5. Theorem. *Let $\mathcal{V}$ be a semi-abelian category. If $\mathcal{V}$ satisfies*

1. *the amalgamation property for normal monomorphisms;*
2. *the normality of unions;*

then every functor $\mathsf{SplExt}(-, X)$ *preserves finite limits. When moreover $\mathcal{V}$ is locally well-presentable, every functor* $\mathsf{SplExt}(-, X)$ *is representable.*

Proof. In the pushout of definition 6.1, σ_1 and σ_2 are now monomorphisms, so that the pushout square is a union. One concludes by proposition 6.3. ∎

Let us now exhibit the link between our axiom of *normality of unions* and D. Bourn's axiom of *strong protomodularity*. We thank D. Bourn for fruitful exchanges of messages concerning this.

6.6. Proposition. *In a semi-abelian category $\mathcal{V}$, consider a short exact sequence*

$$\mathbf{0} \longrightarrow K \xrightarrow{k} A \xrightarrow{q} Q \longrightarrow \mathbf{0}.$$

For a subobject $s\colon S \rightarrowtail A$, the union $K \vee S \rightarrowtail A$ is the subobject $q^{-1}\big(q(S)\big)$.

Proof. Trivially,

$$K = q^{-1}(\mathbf{0}) \subseteq q^{-1}\big(q(S)\big), \quad S \subseteq q^{-1}\big(q(S)\big).$$

Next, if $T \subseteq A$ contains K and S, T is saturated for the quotient (see [8], 4.3.8) and thus

$$q^{-1}\big(q(S)\big) \subseteq q^{-1}\big(q(T)\big) = T. \qquad \blacksquare$$

A semi-abelian category is *strongly protomodular* in the sense of Dominique Bourn when the inverse image functors of the fibration of points reflect normality. In terms of short exact sequences this means that given a diagram of split exact sequences,

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & A & \xrightarrow{k} & B & \overset{s}{\underset{p}{\leftrightarrows}} & C & \longrightarrow & \mathbf{0} \\
 & & {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \| & & \\
\mathbf{0} & \longrightarrow & D & \xrightarrow[l]{} & E & \overset{t}{\underset{q}{\leftrightarrows}} & C & \longrightarrow & \mathbf{0}
\end{array}$$

if u is a normal monomorphism, then lu is a normal monomorphism as well. This notion goes back to the work of Gerstenhaber on Moore categories (see [23], [12], [39]).

6.7. Proposition. *If a semi-abelian category $\mathcal{V}$ satisfies the axiom of normality of unions, it is strongly protomodular.*

Proof. In the diagram above, $q(B) = C$ thus by proposition 6.6, $E = D \vee B$. An alternative proof consists in observing that $D \vee C = E$ by [25] 2.4, while $D \vee B \supseteq D \vee C$ follows from the equality $t = vs$. One concludes by the normality of unions. $\blacksquare$

Eventually, here is an abstract and rather general setting forcing the normality of unions.

6.8. Example. Let $\mathbb{T}$ be a one-sorted pointed algebraic theory containing a set N of binary terms such that

1. every normal subalgebra $X \subseteq B$ is such that

$$\forall x \in X \;\; \forall b \in B \;\; \forall t \in \mathsf{N} \;\; t(x, b) \in X;$$

2. a subalgebra $X \subseteq S$ is normal as soon as there exists a subset $U \subseteq S$ that generates S and has the following property

$$\forall t \in \mathsf{N} \;\; \forall x \in X \;\; \forall u \in U \;\; t(x,u) \in X.$$

The category $\mathsf{Set}^{\mathbb{T}}$ of $\mathbb{T}$-algebras satisfies the normality of unions.

Proof. In the conditions of definition 6.4, choose for U the set theoretical union $U = B \cup C$ and take S to be the union of B and C as subobjects of A. Clearly U generates S and condition 1 of the statement implies that $t(x,u) \in X$ for all $x \in X$ and $u \in U$. ■

The reader should observe that conditions 1 and 2 in example 6.8 are inherited by every subvariety (i.e. by all theories obtained from $\mathbb{T}$ by adding axioms). The *categories of interest* (see [34]) are examples of situations as in example 6.8 and these cover the cases of groups, Lie algebras, rings, and so on.

7. The normalization of a morphism

This section introduces a new notion: the *normalization* of a morphism. This will be the key ingredient to express the necessary and sufficient condition forcing the representability of the functors $\mathsf{SplExt}(-,X)$.

7.1. Definition. *Let n be a morphism in a semi-abelian category $\mathcal{V}$. By a* normalization *of n we mean a commutative diagram*

$$\begin{array}{ccccccccc}
 & & X & \xrightarrow{\;n\;} & S & \overset{q}{\twoheadrightarrow} & Q & & \\
 & & \| & & \downarrow{\scriptstyle p} & & \| & & \\
\mathbf{0} & \longrightarrow & X & \underset{\widetilde{n}}{\longrightarrow} & \widetilde{S} & \underset{\widetilde{q}}{\longrightarrow} & Q & \longrightarrow & \mathbf{0}
\end{array}$$

where $q = \mathsf{Coker}\, n$ and the bottom line is a short exact sequence.

Of course, the choice of a specific cokernel of n is unessential in this definition. Let us first make some easy observations.

7.2. Lemma. *In the situation of definition 7.1:*

1. *n is a monomorphism;*

2. *p is a regular (thus normal) epimorphism;*

3. *if q is a split epimorphism, the bottom exact sequence is split;*

4. *n is a normal monomorphism if and only if p is an isomorphism;*

5. *every normal monomorphism n admits a universal normalization (namely, itself together with its cokernel);*

6. *the morphism n admits a universal normalization if and only if n admits a unique normalization (up to isomorphism).*

PROOF. The morphism n is a monomorphism since so is $\widetilde{n}$.

The normal closure of n – the smallest normal subobject of S containing X – is the kernel $\overline{n} = \mathsf{Ker}\, q$ of q. This yields at once the various factorizations as in diagram 3. In

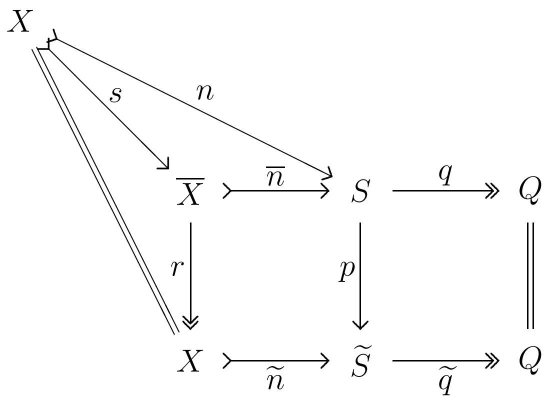


Diagram 3

particular r is a regular epimorphism and by the short five lemma for regular epimorphisms, p is a regular epimorphism as well.

If q admits a section t, $\widetilde{t} = pt$ is a section of $\widetilde{q}$.

When n is a normal monomorphism, (n, q) is a short exact sequence and by the short five lemma for isomorphisms, p is an isomorphism. And of course if p is an isomorphism, n is normal because so is $\widetilde{n}$. This implies trivially condition 5.

Finally by the short five lemma for isomorphisms, every morphism between two normalizations of n is necessarily an isomorphism. This implies at once the last affirmation. ∎

Let us comment a little bit more that notion of "normalization". In the conditions of definition 7.1, n is thus a monomorphism and p is a normal epimorphism (lemma 7.2). Thus $\widetilde{S}$ is the quotient of S by the kernel $m\colon Y \rightarrowtail S$ of p and clearly, the whole situation is entirely determined by the knowledge of n and m. Let us thus translate the notion of "normalization of n" in terms of this normal subobject m of S.

7.3. PROPOSITION. *In a semi-abelian category $\mathcal{V}$, consider a monomorphism $n\colon X \rightarrowtail S$ with normal closure $\overline{n}\colon \overline{X} \rightarrowtail S$ (i.e. $\overline{n} = \mathsf{Ker\,Coker}\, n$). Up to isomorphisms, there is a bijection between*

1. *the normalizations of n;*

2. *the normal subobjects $m\colon Y \rightarrowtail S$ such that*

$$Y \subseteq \overline{X}, \quad X \wedge Y = 0, \quad X \vee Y = \overline{X}.$$

Proof. Consider diagram 4 with m a normal subobject. Put $q = \mathsf{Coker}\, n$, $p = \mathsf{Coker}\, m$, $\widetilde{n} = pn$, $\widetilde{q} = \mathsf{Coker}\, \widetilde{n}$. Let χ be the factorization through the cokernels, and s the inclusion of X in $\overline{X}$.

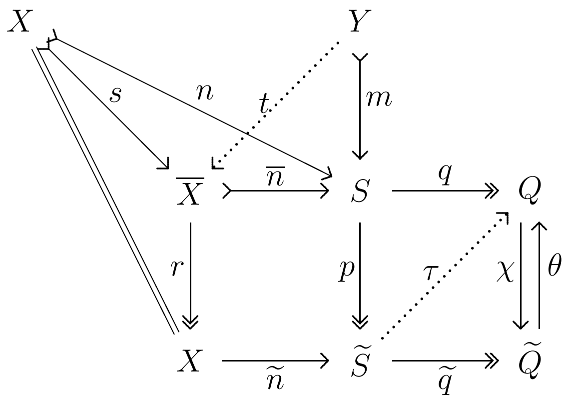


Diagram 4

The construction above determines a normalization of n precisely when

1. there exists a morphism τ such that $q = \tau p$;
2. $\widetilde{n}$ is a monomorphism;
3. that monomorphism $\widetilde{n}$ is normal.

Indeed, in the case of a normalization, χ is an isomorphism and it suffices to put $\tau = \chi^{-1}\widetilde{q}$; conditions 2 and 3 are then trivial. Conversely, by conditions 2 and 3, $\widetilde{n} = \mathsf{Ker}\, \widetilde{q}$, from which we obtain the factorization r through the kernels. Since $\widetilde{n} rs = p\overline{n}s = pn = \widetilde{n}$, we get $rs = 1_X$. Thus $\tau\widetilde{n} = \tau\widetilde{n}rs = \tau p\overline{n}s = qn = 0$, from which we obtain a factorization θ of τ through $\widetilde{q} = \mathsf{Coker}\, \widetilde{n}$. The equalities

$$\theta\chi q = \theta\widetilde{q}p = \tau p = q, \quad \chi\theta\widetilde{q}p = \chi\tau p = \chi q = \widetilde{q}p$$

prove finally that χ and θ are inverse isomorphisms.

The three conditions above are respectively equivalent to:

1. $Y \subseteq \overline{X}$;
2. $X \wedge Y = 0$; indeed $\widetilde{n}$ is a monomorphism if and only if its kernel is zero (see [8] 3.1.21); but the kernel of $\widetilde{n} = pn$ is trivially the kernel Y of p intersected with X;
3. the monomorphism $\widetilde{n}$ is normal;

and it remains to see that we can replace the third condition by $X \vee Y = \overline{X}$.

We know by proposition 6.6 that $Y \vee X = p^{-1}\big(p(X)\big)$. But since $\widetilde{n} = pn$ is a monomorphism by condition 2, $p(X)$ is simply the subobject $\widetilde{n}$. When $\widetilde{n}$ is normal, the factorization

r exists and so $\overline{X}$ factors through $p^{-1}\big(p(X)\big) = Y \vee X$, proving the equality $\overline{X} = X \vee Y$. Conversely when $\overline{X} = X \vee Y = p^{-1}\big(p(X)\big)$, then $\overline{X}$ is the pullback of $\widetilde{n}$ along p, yielding again the existence of the epimorphism r. But then $\widetilde{n}$ is the image of the normal monomorphism $\overline{n}$ along the regular epimorphism p: thus $\widetilde{n}$ is normal (see [7] 3.9.3). ■

Having a normalization is certainly a strong property. For example:

7.4. COUNTEREXAMPLE. In the category of groups, consider a simple group S and a subgroup $n\colon X \rightarrowtail S$; let $q = \mathsf{Coker}\, n$. The morphism n admits a normalization if and only if $X = \mathbf{0}$ or $X = S$.

PROOF. With the notation of definition 7.1, p is a quotient map by lemma 7.2; thus by simplicity of S, $\widetilde{S} = \mathbf{0}$ or $\widetilde{S} = S$. If $\widetilde{S} = \mathbf{0}$, then $X = \mathbf{0}$. If $\widetilde{S} = S$, then $n = \widetilde{n}$ and X is normal in S, thus $X = \mathbf{0}$ or $X = S$. ■

But nevertheless there are highly interesting examples of normalizations:

7.5. EXAMPLE. In a semi-abelian category $\mathcal{V}$, consider an object G and the corresponding monad $G\flat-$ defining the semi-direct product. The G-algebra structures (X, ξ) on a fixed object $X \in \mathcal{V}$ are in bijection with the (isomorphism classes of) normalizations of the canonical morphism $\sigma_X\colon X \rightarrowtail G + X$.

PROOF. It is routine to observe that $(1_G, 0)$ is the cokernel of σ_X.

$$\begin{array}{ccccccccc}
 & & X & \xrightarrow{\ \sigma_X\ } & G + X & \underset{(1_G,0)}{\overset{\sigma_G}{\leftrightarrows}} & G & & \\
 & & \Vert & & \Big\downarrow{\scriptstyle p} & & \Vert & & \\
\mathbf{0} & \longrightarrow & X & \xrightarrow[\ s_X\ ]{} & G \ltimes (X,\xi) & \underset{p}{\overset{s}{\leftrightarrows}} & G & \longrightarrow & \mathbf{0}
\end{array}$$

This cokernel is a split epimorphism with section the canonical inclusion σ_G. The kernel of $(1_G, 0)$ is precisely the object $G\flat X$ (see [13]).

By lemma 7.2, every normalization of σ_X yields a bottom split exact sequence, thus also a factorization $\xi\colon T_G(X) \longrightarrow X$ through the kernels.

Conversely every $(G\flat-)$-algebra structure (X, ξ) on X yields a normalization diagram as above (see [13] again). The equivalence between the category of G-algebras and the category of points over G forces the conclusion. ■

8. The representability theorem

This section presents the central result of this paper: a necessary and sufficient condition for the representability of the functors $\mathsf{SplExt}(-, X)$. Here is that condition:

8.1. Definition. *Let $\mathcal{V}$ be a semi-abelian category. Consider the pushout of two protosplit monomorphisms n_1, n_2 with common domain X and the corresponding diagonal morphism n.*

$$\begin{array}{ccc} X & \overset{n_1}{\rightarrowtail} & S_1 \\ {\scriptstyle n_2}\downarrow & \searrow^{n} & \downarrow{\scriptstyle \sigma_1} \\ S_2 & \underset{\sigma_2}{\longrightarrow} & S_1 +_X S_2 \end{array}$$

We say that the category $\mathcal{V}$ satisfies the protonormalization of pushouts *over X when each such morphism n admits a universal normalization.*

First of all, we recall a well-known fact.

8.2. Lemma. *Let $F\colon \mathcal{C} \longrightarrow \mathsf{Set}$ be a functor defined on a category $\mathcal{C}$ with binary products. The following conditions are equivalent:*

1. *F preserves finite products;*
2. *the category $\mathsf{Elts}(F)$ of elements of F has binary products preserved by the forgetful functor $\phi\colon \mathsf{Elts}(F) \longrightarrow \mathcal{C}$.*

Proof. When F preserves binary products, $(C,x) \times (D,y) \cong \big(C \times D, (x,y)\big)$ in $\mathsf{Elts}(F)$. Conversely if $(C \times D, z) = (C,x) \times (D,y)$, the two elements $x \in F(C)$ and $y \in F(D)$ are the images of $z \in F(C \times D)$ by $F(p_C)$ and $F(p_D)$. If $z' \in F(C \times D)$ is another element mapped on x and y by $F(p_C)$ and $F(p_D)$, the corresponding factorization $(C \times D, z') \longrightarrow (C \times D, z)$ commutes in $\mathcal{C}$ with the projections, thus is the identity on $C \times D$. Therefore $z = z'$. ∎

As an intermediate step in our arguments, we consider the category $\mathsf{SplPsExt}[X]$ with objects the triples (n,q,t) where X is fixed, $q = \mathsf{Coker}\, n$ and $qt = 1_Q$.

$$\begin{array}{ccccc} X & \overset{n}{\longrightarrow} & S & \underset{q}{\overset{t}{\leftrightarrows}} & Q \\ \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ X & \overset{n'}{\longrightarrow} & S' & \underset{q'}{\overset{t'}{\leftrightarrows}} & Q' \end{array}$$

The morphisms are the triples $(1_X, \varphi, \psi)$ making the diagram commutative. This is thus the category of "split-pseudo-right-exact sequences".

8.3. Lemma. *Let $\mathcal{V}$ be a semi-abelian category. The category $\mathsf{SplPsExt}[X]$ admits binary coproducts preserved by the "cokernel part" functor*

$$\mathsf{SplPsExt}[X] \longrightarrow \mathcal{V}, \quad (X \to S \leftrightarrows Q) \mapsto Q.$$

Proof. Given two objects of $\mathsf{SplPsExt}[X]$

$$X \xrightarrow{n_i} S_i \underset{q_i}{\overset{t_i}{\rightleftarrows}} Q_i, \quad i = 1, 2$$

their coproduct is simply

$$X \xrightarrow{(n_1, n_2)} S_1 +_X S_2 \underset{q_1 + q_2}{\overset{t_1 + t_2}{\rightleftarrows}} Q_1 + Q_2.$$ ∎

8.4. Proposition. *Let X be an object of a semi-abelian category $\mathcal{V}$. The following conditions are equivalent:*

1. *the functor* $\mathsf{SplExt}(-, X)$ *preserves binary coproducts;*
2. *the category $\mathcal{V}$ satisfies the protonormalization of pushouts over X.*

Proof. Given $X \in \mathcal{V}$, consider the category of elements of the functor $\mathsf{SplExt}(-, X)$. It is equivalent to the subcategory $\mathsf{SplExt}[X]$ of $\mathsf{SplPsExt}[X]$ whose objects are split exact sequences and whose morphisms yield a pullback diagram at the level of cokernels (see the proof of proposition 1.4). But by lemma 1.1, the pullback requirement holds always since the morphisms have identities in the component X. This shows that we have a full subcategory

$$i\colon \mathsf{SplExt}[X] \hookrightarrow \mathsf{SplPsExt}[X].$$

We use the notation of definition 7.1. Let (n, q, t) be an object of the category $\mathsf{SplPsExt}[X]$ such that n has a universal normalization $(\widetilde{n}, \widetilde{q})$. By lemma 7.2, this normalization is unique and $\widetilde{q}$ admits a unique section $\widetilde{t}$ making the diagram commutative. Let us prove that

$$(1_X, p, 1_Q)\colon (n, q, t) \longrightarrow (\widetilde{n}, \widetilde{q}, \widetilde{t})$$

is the universal reflection of (n, q, t) along the inclusion functor i.

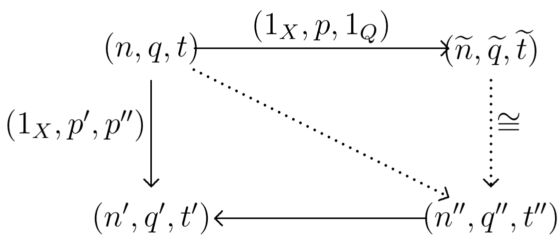
Consider for this another morphism

$$(1_X, p', p'')\colon (n, q, t) \longrightarrow (n', q', t')$$

in $\mathsf{SplPsExt}[X]$, with thus

$$\mathbf{0} \longrightarrow X \xrightarrow{n'} S' \underset{q'}{\overset{t'}{\rightleftarrows}} Q' \longrightarrow \mathbf{0}$$

a split exact sequence. By lemma 1.1, pulling q' back along $p''\colon Q \longrightarrow Q'$ and taking the kernel yields a split exact sequence (n'', q'', t'') with kernel object X; due to the pullback construction, the morphism $(1_X, p', p'')$ factors through this sequence.



Thus (n'', q'') is another normalization of n and by uniqueness of such a normalization, this is simply $(\widetilde{n}, \widetilde{q})$ (up to an isomorphism). So $(1_X, p', p'')$ factors indeed through $(1_X, p, 1_Q)$; this factorization is unique since, by lemma 7.2, p is an epimorphism.

Finally, since $\mathsf{SplPsExt}[X]$ has binary coproducts, a standard argument on universal constructions shows that the coproduct of two objects (n_i, q_i, t_i) of $\mathsf{SplExt}[X]$ exists if and only if the corresponding coproduct in $\mathsf{SplPsExt}[X]$ (which exists by lemma 8.3) admits a universal reflection along the inclusion functor i. By the form of binary coproducts in $\mathsf{SplPsExt}[X]$, this is exactly the protonormalization of pushouts. Moreover, the construction of the universal morphism $(1_X, p, 1_Q)$ above shows that universal reflections preserve the "cokernel part". Thus the "cokernel part functor" on $\mathsf{SplExt}[X]$, which is the forgetful functor of this category of elements, preserves binary coproducts since this is the case for $\mathsf{SplPsExt}[X]$, by lemma 8.3. ∎

And finally, theorem 5.8 and proposition 8.4 yield the expected representability theorem:

8.5. THEOREM. *Let X be an object of a semi-abelian category $\mathcal{V}$. The following conditions are equivalent:*

1. *the category $\mathcal{V}$ satisfies the protonormalization of pushouts over X;*
2. *the functor $\mathsf{SplExt}(-, X)$ preserves finite colimits.*

When moreover, the category $\mathcal{V}$ is locally well-presentable, these conditions are also equivalent to

3. *the functor $\mathsf{SplExt}(-, X)$ is representable.* ∎

It should again be observed that the protonormalization of pushouts in definition 8.1 does not explicitly require σ_1 and σ_2 to be monomorphisms (= the amalgamation property for n_1 and n_2): but we know by theorem 8.5 and proposition 6.2 that this is nevertheless the case.

9. A topos theoretic example

Examples of the representability of functors $\mathsf{Act}(-, X)$ in a topos theoretic context have already been given in proposition 1.5. And the same type of arguments extends at once to Boolean rings or commutative von Neumann regular rings, in a topos with Natural Number Object.

Let us also observe that the sufficient conditions in theorem 6.5 transfer easily from Set to a Grothendieck topos.

9.1. LEMMA. *Let $\mathbb{T}$ be a semi-abelian category whose category $\mathsf{Set}^{\mathbb{T}}$ of models satisfies the amalgamation property for normal monomorphisms and the normality of unions. The same properties hold in the semi-abelian category $\mathcal{E}^{\mathbb{T}}$ of $\mathbb{T}$-models in a Grothendieck topos $\mathcal{E}$. Then for every object $X \in \mathcal{E}^{\mathbb{T}}$, the functor $\mathsf{Act}(-, X)$ is representable.*

PROOF. If $\mathcal{E}$ is the topos of sheaves on a site $(\mathcal{C}, \mathcal{T})$, the announced properties hold in the category of $\mathbb{T}$-models in the topos $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ of presheaves, simply because they hold pointwise. But $\mathcal{E}^{\mathbb{T}}$ is a localization of this category, thus the corresponding reflection preserves all the ingredients involved in the properties indicated. One concludes by theorem 6.5. ■

Of course, the topos case calls for a more intrinsic approach of the representability theorem. Some routine work shows that the category $\mathcal{E}^{\mathbb{T}}$ and the functor $\mathsf{Act}(-, X)$ can be enriched in $\mathcal{E}$, viewed as a cartesian closed category. One could then ask for the representability of $\mathsf{Act}(-, X)$ as a functor enriched in $\mathcal{E}$. One could even generalize further the situation and consider an internal algebraic theory $\mathbb{T}$ yielding a semi-abelian category of models. But all these considerations escape the scope of the present paper, devoted to the Set-valued representability of actions.

Let us conclude this discussion of the topos theoretic case with an example of highly non-algebraic nature.

9.2. PROPOSITION. *Let $\mathcal{E}_*^{\mathsf{op}}$ be the dual of the category of pointed objects of an elementary topos $\mathcal{E}$. This category is semi-abelian and when $\mathcal{E}$ is boolean, the actions on an object $(X, *)$ are representable by $(X, *)$ itself.*

PROOF. It is known that the dual category of the category of pointed objects of a topos is semi-abelian (see [8] 5.1.8). To avoid any confusion, let us work in the category $\mathcal{E}_*$ of pointed objects, not in its dual.

We consider first a co-split co-exact sequence

$$(\mathbf{1}, *) \longrightarrow (G, *) \underset{s}{\overset{q}{\leftrightarrows}} (A, *) \xrightarrow{\ p\ } (X, *) \longrightarrow (\mathbf{1}, *).$$

Since the topos $\mathcal{E}$ is boolean, the base point $*\colon \mathbf{1} \rightarrowtail X$ is a complemented subobject, thus $X \cong \mathbf{1} \amalg Y$ in $\mathcal{E}$ (to avoid confusion, we write $\amalg$ for the coproduct in $\mathcal{E}$ and $+$ for the coproduct in $\mathcal{E}_*$). Then in $\mathcal{E}$

$$A \cong p^{-1}(\mathbf{1}) \amalg p^{-1}(Y) \cong G \amalg p^{-1}(Y).$$

Since p is an epimorphism in $\mathcal{E}$, its restriction

$$p_Y \colon p^{-1}(Y) \longrightarrow Y$$

is an epimorphism as well. The kernel pair of p_Y is the restriction to $p^{-1}(Y) \times p^{-1}(Y)$ of the kernel pair of p, which is

$$(G \times G) \vee \Delta_A = (G \times G) \vee \Delta_G \vee \Delta_{p^{-1}(Y)} = (G \times G) \vee \Delta_{p^{-1}(Y)}.$$

Since G is disjoint from $p^{-1}(Y)$ in A, the kernel pair of p_Y is thus simply $\Delta_{p^{-1}(Y)}$, proving that p_Y is a monomorphism as well. Thus p_Y is an isomorphism. This yields a corresponding morphism

$$* \amalg p_Y^{-1} \colon \mathbf{1} \amalg Y \longrightarrow G \amalg p^{-1}(Y) \cong A$$

whose composite with q yields a morphism $(X,*)\longrightarrow(G,*)$.

Conversely given a morphism $\tau\colon (X,*)\longrightarrow(G,*)$, we get at once a co-split co-exact sequence

$$(\mathbf{1},*)\longrightarrow(G,*)\underset{s_{(G,*)}}{\overset{(\mathsf{id},\tau)}{\leftrightarrows}}(G,*)+(X,*)\xrightarrow{(0,\mathsf{id})}(X,*)\longrightarrow(\mathbf{1},*).$$

It is routine to check that we have constructed inverse bijections (up to isomorphisms of co-split co-exact sequences). ∎

10. Some counter-examples

This paper introduces three conditions related to the representability of actions: a necessary condition, a sufficient condition, a necessary and sufficient condition. This section presents various counter-examples distinguishing between these conditions, thus proving the pertinence of all of them.

First of all, let us make clear that the representability of actions is by no means a general property of semi-abelian categories, even of semi-abelian varieties. As the following counter-examples will show, this is even a rather exceptional property.

10.1. Counterexample. Actions are generally not representable on the category ComRg of commutative rings.

Proof. By theorem 2.6, it suffices to exhibit a ring X such that $\mathsf{End}(X)$ is not commutative. For example, the additive abelian group $\mathbb{Z}^2$ with the zero multiplication, for which $\mathsf{End}(X)$ is the ring of 2×2-matrices with entries in $\mathbb{Z}$. ∎

Second, let us exhibit a counter-example showing that the necessary condition of proposition 6.2 is not sufficient. By a non-associative ring, we mean clearly an additive abelian group equipped with a multiplication which distributes over the addition in each variable. Analogous arguments can be developed in the commutative case.

10.2. Counterexample. In the variety of non-associative rings, which is semi-abelian:

1. the amalgamation property holds for protosplit monomorphisms;
2. the actions on an object X are representable if and only if X is the zero ring.

Proof. The arguments developed in the proofs of propositions 2.1 and 2.3 apply as such to prove that, for non associative rings, there is a bijection between

- the (isomorphism classes of) split exact sequences of the form

$$\mathbf{0}\longrightarrow X\xrightarrow{k}A\underset{p}{\overset{s}{\leftrightarrows}}G\longrightarrow\mathbf{0};$$

- the pairs of additive abelian group homomorphisms

$$\lambda\colon G \longrightarrow \mathsf{Ab}(X,X), \quad \rho\colon G \longrightarrow \mathsf{Ab}(X,X).$$

This proves thus the isomorphism

$$\mathsf{Act}(G,X) \cong \mathsf{Ab}\big(G, \mathsf{Ab}(X,X) \times \mathsf{Ab}(X,X)\big).$$

Given a second split exact sequence

$$\mathbf{0} \longrightarrow X \xrightarrow{k'} A' \underset{p'}{\overset{s'}{\rightleftarrows}} G' \longrightarrow \mathbf{0};$$

with corresponding pair (λ', ρ'), we get at once abelian group homomorphisms

$$(\lambda,\lambda')\colon G \oplus G' \longrightarrow \mathsf{Ab}(X,X), \quad (\rho,\rho')\colon G \oplus G' \longrightarrow \mathsf{Ab}(X,X)$$

and therefore a corresponding split exact sequence of non-associative rings

$$\mathbf{0} \longrightarrow X \xrightarrow{k''} A'' \underset{p''}{\overset{s''}{\rightleftarrows}} G \times G' \longrightarrow \mathbf{0}.$$

And the two canonical inclusions of G, G' in $G \times G'$ are ring homomorphisms making the following diagram commutative:

$$\begin{array}{ccccc} G & \xrightarrow{(\mathsf{id},0)} & G \times G' & \xleftarrow{(0,\mathsf{id})} & G' \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle(\lambda,\lambda')} & \swarrow{\scriptstyle\lambda'} & \\ & & \mathsf{Ab}(X,X) & & \end{array}$$

and analogously with (ρ, ρ'). But we have

$$A \cong G \ltimes (X, \lambda, \rho), \quad A'' \cong (G \times G') \ltimes \big(X, (\lambda,\lambda'), (\rho,\rho')\big), \quad A' \cong G' \ltimes (X, \lambda', \rho')$$

and therefore the morphisms $(\mathsf{id}, 0)$, $(0, \mathsf{id})$ induce corresponding morphisms of non-associative rings

$$A \xrightarrow{\alpha} A'' \xleftarrow{\alpha'} A'.$$

But at the level of underlying additive abelian groups, the square $\alpha k = \alpha' k'$ is simply

$$\begin{array}{ccc} X & \xrightarrow{k} & G \times X \\ {\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle\alpha} \\ G' \times X & \xrightarrow[\alpha']{} & G \times G' \times X \end{array}$$

$$k(x) = (0, x), \quad k'(x) = (0, x), \quad \alpha(g, x) = (g, 0, x), \quad \alpha'(g', x) = (0, g', x).$$

Thus the ring homomorphisms α, α' are injective and by factorization, also the canonical morphisms of the pushout of k and k' are injective. This proves the amalgamation property for protosplit monomorphisms.

Now suppose that X has representable actions. In particular, the functor $\mathsf{Act}(-, X)$ transforms binary coproducts in binary products, yielding the canonical bijection

$$\mathsf{Act}(G + G, X) \cong \mathsf{Act}(G, X) \times \mathsf{Act}(G, X)$$

for every non-associative ring G. The form of $\mathsf{Act}(-, X)$, established earlier in the proof, yields thus the canonical bijections

$$\mathsf{Ab}\big(G + G, \mathsf{Ab}(X, X)^2\big) \cong \mathsf{Ab}\big(G, \mathsf{Ab}(X, X)^2\big) \times \mathsf{Ab}\big(G', \mathsf{Ab}(X, X)^2\big). \qquad (*)$$

Write now $\mathbb{T}$ for the theory of non-associative rings and $\mathbb{M}$ for the theory of magmas (a set equipped with a binary operation, without any axiom). The forgetful functor $U\colon \mathsf{Set}^{\mathbb{T}} \longrightarrow \mathsf{Set}^{\mathbb{M}}$ mapping a non-associative ring on its underlying multiplicative magma is algebraic, thus has a left adjoint L. Trivially, $L(M, \cdot)$ is the free abelian group on the set M epuipped with the multiplication induced by distributivity from the multiplication on the magma $(M, \cdot)$ of generators. The terminal magma – the singleton $\{x\}$ – is the magma with a single generator x satisfying the relation $xx = x$. The corresponding ring $L\{x\}$ is simply the ring $(\mathbb{Z}, +, \times)$ of integers. Therefore, since L preserves colimits,

$$L\big(\{x\} + \{y\}\big) \cong L\{x\} + L\{y\} \cong \mathbb{Z} + \mathbb{Z}.$$

And the magma $\{x\} + \{y\}$ is the quotient of the magma of all bracketed words in x and y, by the congruence generated by the relations $xx = x$ and $yy = y$. This is thus an infinite countable set and the non-associative ring $\mathbb{Z} + \mathbb{Z}$ admits as underlying additive group a free abelian group on infinitely many generators.

Writing ω for the infinite countable cardinal, the canonical bijection $(*)$ become thus, choosing $G = \mathbb{Z}$,

$$\big(p_{\{x\}}, p_{\{y\}}\big)\colon \big(\mathsf{Ab}(X, X)^2\big)^{\omega} \cong \mathsf{Ab}(X, X)^2 \times \mathsf{Ab}(X, X)^2.$$

But the pair of projections $\big(p_{\{x\}}, p_{\{y\}}\big)$ is a bijection only when $\mathsf{Ab}(X, X)^2$ is the zero group, that is, when $(X, +)$ itself is the zero group. ∎

Finally, let us provide a counter-example showing that the sufficient conditions in proposition 6.3 or theorem 6.5 are not necessary. The theory $\mathbb{T}$ used in this counter-example can already be found in [13] and [12].

10.3. Counterexample. Let $\mathbb{T}$ be the theory having two abelian group operations with the same neutral element. The variety $\mathsf{Set}^{\mathbb{T}}$ is semi-abelian and there exist objects $X \in \mathsf{Set}^{\mathbb{T}}$ such that:

1. the functor $\mathsf{Act}(-, X)$ is representable;
2. the sufficient conditions of proposition 6.3 and theorem 6.5 are not satisfied.

PROOF. A split exact sequence of $\mathbb{T}$-algebras is in particular a split exact sequence of abelian groups for the first abelian group operations, thus has set theoretically the form

$$0 \longrightarrow X \xrightarrow{s_X} G \times X \underset{p_G}{\overset{s_G}{\rightleftarrows}} G \longrightarrow 0$$

$$s_X(x) = (x, 0), \quad p_G(g, x) = g, \quad s_G(g) = (g, 0)$$

with the first abelian group structure on $G \times X$ defined pointwise. And this is a split exact sequence of $\mathbb{T}$-algebras when $G \times X$ is equipped with a second abelian group structure, with neutral element $(0, 0)$, and making s_X, p_G, s_G group homomorphisms for the second abelian group operations. Let us use the symbol $+$ to denote these second abelian group operations.

Since a split exact sequence of $\mathbb{T}$-algebras is also a split exact sequence of abelian groups for the second abelian group operations, we know that

$$\forall (g, x) \in G \times X \quad \exists! g' \in G \quad \exists! x' \in X \quad (g, x) = (g', 0) + (0, x').$$

But p_G is a $+$-homomorphism, thus we have necessarily in $G \times X$

$$(g', 0) + (0, x') = (g', g' * x')$$

for some element $g' * x' \in X$. Therefore $g = g'$ and the condition above can be rephrased as

$$\forall (g, x) \in G \times X \quad \exists! x' \in X \quad x = g * x'.$$

Still in other words, this means that the mapping

$$g * -: X \longrightarrow X, \quad x \mapsto g * x$$

is bijective. For simplicity, we write $(g * -)^{-1}(x) = x_g$ and we have thus

$$g * x_g = x, \quad (g * x)_g = x.$$

Let us also write $\mathsf{Perm}(X)$ for the set of permutations of X. The mapping

$$\xi: (G, 0) \longrightarrow \big(\mathsf{Perm}(X), 1_X\big), \quad g \mapsto g * -$$

is a morphism of pointed sets, since

$$(g, 0) + (0, 0) = (g, 0), \quad (0, 0) + (0, x) = (0, x)$$

that is

$$g * 0 = 0, \quad 0 * x = x.$$

Moreover this mapping ξ determines entirely the second abelian group operation of $G \times X$. Indeed, since s_X and s_G are $+$-homomorphisms, we have

$$\begin{aligned}(g,x)+(h,y) &= (g, g*x_g)+(h,h*y_h)\\ &= (g,0)+(0,x_g)+(h,0)+(0,y_h)\\ &= (g+h,0)+(0,x_g+x_h)\\ &= \big(g+h,(g+h)*(x_g+y_h)\big).\end{aligned}$$

Conversely, given a morphism of pointed sets

$$\xi\colon (G,0) \longrightarrow \big(\mathsf{Perm}(X), 1_X\big), \quad g \mapsto g * -$$

and putting as before $x_g = (g*-)^{-1}(x)$, we define an addition on $G \times X$ by the formula

$$(g,x)+(h,y) = \big(g+h,(g+h)*(x_g+y_h)\big).$$

It is trivial that this addition is commutative. It admits $(0,0)$ as neutral element because ξ is a morphism of pointed sets. It is associative because every sum of (g,x), (h,y) and (k,z) yields the result

$$\big(g+h+k,(g+h+k)*(x_g+y_h+z_k)\big).$$

Finally the opposite of the element (g,x) is given by

$$-(g,x) = \big(-g,(-g)*(-x_g)\big).$$

We conclude that the functor $\mathsf{Act}(-,X)$ is isomorphic to

$$\mathsf{Act}(-,X)\colon \mathsf{Set}^{\mathbb{T}} \longrightarrow \mathsf{Set}, \quad G \mapsto \mathsf{Set}_*\big((G,0),(\mathsf{Perm}(X),1_X\big)$$

where Set_* denotes the category of pointed sets.

Now if $X = \{0,x\}$ has exactly two elements (that is, $X \cong \mathbb{Z}/2\mathbb{Z}$ with both group operations necessarily equal), the only second abelian group operation of $G \times X$ is the pointwise one. Indeed for every $g \in G$, $g*0 = 0$ and therefore $g*x \neq g*0 = 0$, thus $g*x = x$. Thus every permutation $g*-$ is the identity on X and the second abelian group operation of $G \times X$ is pointwise. But then each $\mathsf{Act}(G,X)$ is a singleton and the functor $\mathsf{Act}(-,X)$ is representable by the zero $\mathbb{T}$-algebra.

It remains to show that the sufficient conditions of proposition 6.3 and theorem 6.5 are not satisfied for this $\mathbb{T}$-algebra $X = \mathbb{Z}_2$. For this consider the square of injections

$$\begin{array}{ccc}\mathbb{Z}_2 & \overset{\alpha}{\rightarrowtail} & \mathbb{Z}\times\mathbb{Z}_2\\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\delta}\\ \mathbb{Z}\times\mathbb{Z}_2 & \underset{\gamma}{\rightarrowtail} & \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_2\end{array}$$

$$\alpha(x) = (0, x), \quad \beta(x) = (0, x), \quad \gamma(z, x) = (z, 0, x), \quad \delta(z, x) = (0, z, x).$$

Equip each of the two occurrences of $\mathbb{Z} \times \mathbb{Z}_2$ with twice the pointwise addition inherited form the usual additions of $\mathbb{Z}$ and $\mathbb{Z}_2$. And as first addition on the object $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$, take again the pointwise one.

It is trivial that given an abelian group $(A, +_1)$, every permutation φ of A yields another group operation on the set A, namely

$$a +_2 b = \varphi^{-1}\big(\varphi(a) + \varphi(b)\big).$$

and both operations have the same zero as soon as $\varphi(0) = 0$. Define the second group operation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ in that way, choosing for φ the permutation which interchanges the two elements

$$(1, 1, 0), \quad (-1, 1, 0)$$

and fixes all the other elements. This completes the description of the square above, where α, β, γ, δ are trivially morphisms of $\mathbb{T}$-algebras.

We have a split exact sequence

$$\mathbf{0} \longrightarrow \mathbb{Z}_2 \xrightarrow{\ \alpha\ } \mathbb{Z} \times \mathbb{Z}_2 \underset{p_{\mathbb{Z}}}{\overset{s_{\mathbb{Z}}}{\rightleftarrows}} \mathbb{Z} \longrightarrow \mathbf{0}$$

thus α is a protosplit monomorphism, and analogously for β. Moreover, already for the first additions, every element of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ is the sum of two elements in the two copies of $\mathbb{Z} \times \mathbb{Z}_2$, thus the square $\gamma\alpha = \delta\beta$ is certainly a union of $\mathbb{T}$-algebras. Let us now observe that $\mathbb{Z}_2$ is not a normal subobject of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$.

For this we consider the cokernel q of that inclusion:

$$\mathbb{Z}_2 \overset{\delta\alpha}{\rightarrowtail} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \overset{q}{\twoheadrightarrow} Q.$$

We keep using the notation $+_1$ for the pointwise addition on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ and $+_2$ for the second addition, that is, the addition $+_1$ "twisted" by the permutation φ. Since

$$(1, 1, 0) +_1 (0, 0, 1) = (1, 1, 1)$$

with $(0, 0, 1) \in \mathbb{Z}_2$, we have

$$[1, 1, 0] = [1, 1, 1] \in Q.$$

And since

$$(1, 1, 0) +_2 (0, 0, 1) = \varphi^{-1}\big((-1, 1, 0) +_1 (0, 0, 1)\big) = \varphi^{-1}(-1, 1, 1) = (-1, 1, 1)$$

with again $(0, 0, 1) \in \mathbb{Z}_2$, we have also

$$[1, 1, 0] = [-1, 1, 1] \in Q.$$

By transitivity this implies

$$[1, 1, 1] = [-1, 1, 1] \in Q$$

and thus

$$[2,0,0] = [1,1,1] -_1 [-1,1,1] = 0 \in Q.$$

But $(2,0,0) \notin \mathbb{Z}_2$, thus $\mathbb{Z}_2$ is not a normal subobject of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$. This proves that the sufficient condition in theorem 6.5 is not satisfied.

Finally let us observe that the protosplit monomorphisms α, β do not satisfy either the sufficient condition in proposition 6.3. Writing S for their pushout, the union above (as every union) is a quotient of that pushout and we get a commutative square

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{\;s\;} & S \\ \| & & \downarrow{\scriptstyle p} \\ \mathbb{Z}_2 & \underset{\delta\alpha}{\rightarrowtail} & \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \end{array}$$

where s is the diagonal morphism of the pushout, which is thus a monomorphism. Since p is a quotient morphism, the normality of s would imply that of its image $\delta\alpha$ along p (see [7] 3.9.3), which is not the case. ■

11. Open problems

In this section, $\mathcal{V}$ denotes a semi-abelian variety of universal algebras, although in many cases one could also consider much wider contexts. Let us recall:

(A) Representability of actions in $\mathcal{V}$, for which a necessary and sufficient condition is provided by Theorem 8.5, implies amalgamation property for protosplit monomorphisms (by Proposition 6.2) in $\mathcal{V}$.

(B) Whenever $\mathcal{V}$ satisfies the normality of unions (Definition 6.4), the representability of actions in $\mathcal{V}$ is equivalent to the amalgamation property for protosplit monomorphisms (by Propositions 6.2 and 6.3).

(C) If $\mathcal{V}$ is a category of interest in the sense of M. Barr (see G. Orzech [34]), then it satisfies the normality of unions (see Example 6.8 and the last paragraph of Section 6); therefore – we repeat from (B) – the representability of actions in $\mathcal{V}$ is equivalent to the amalgamation property for protosplit monomorphisms. In particular, this applies to every subvariety of the following varieties:

a. Groups;

b. Associative algebras over an arbitrary fixed commutative unital ring;

c. Lie algebras over an arbitrary fixed commutative unital ring.

d. Any of the structures above equipped with any fixed set of unary linear operators; this includes – up to a category equivalence – crossed modules and other internal categorical structures in the categories of those structures.

Note also that this applies not only to subvarieties, but also to all full subcategories closed under finite limits and quotients that are varieties – such as von Neumann regular rings among all rings.

We would like to propose the following open problems:

Problem 1. Give a syntactical characterization of semi-abelian varieties satisfying all/any of the conditions mentioned in (B).

Problem 2. Using (C), find new examples of categories of interest with representable actions and/or just individual objects in a category of interest with representable actions, and, if possible, describe the representing objects in each case. Note that the amalgamation property of all monomorphisms have been studied by many authors (see [30]), but the amalgamation property of protosplit ones is new, and it is a much weaker condition. For example the amalgamation property of all monomorphisms is known for Lie algebras over a field, but its protosplit version holds for Lie algebras over an arbitrary unital commutative ring – simply because the object actions are representable (by derivations). We expect a kind of negative result for varieties of groups, since the only non-abelian variety of groups where we are able to prove the representability of actions is the variety of all groups.

Problem 3. As we know from Counterexample 10.3, the representability of actions does not imply the normality of unions (even for protosplit monomorphisms). However, we do not have any example of a semi-abelian variety in which every object has representable actions, but the normality of unions does not hold. Does such an example exist?

Before formulating our next problems, let us recall:

a. An object X in, say, a semi-abelian category, is said to be abelian if the codiagonal $X + X \longrightarrow X$ factors through the canonical morphism $X + X \longrightarrow X \times X$. This notion goes back to S.A. Huq [24], who observes that it is a reformulation of S. Mac Lane's observation for groups.

b. An abelian object in a semi-abelian category is the same as a group object; the group structure is unique, as observed by many authors independently in various contexts many years ago.

c. An abelian object in a semi-abelian variety is the same as what universal algebraists call an abelian algebra (in the pointed case).

d. An object X is abelian if and only if its largest commutator $[X, X]$ is trivial; in the case of a semi-abelian variety, all known universal-algebraic and categorical definitions of the largest commutator are equivalent (see e.g. [24], [41], [21], [36], [26]). In particular $[X, X]$ is the usual commutator in the cases of groups and of Lie algebras, and $[X, X] = XX$ for rings (commutative or not).

e. A group is an abelian object in the category of groups if and only if it is abelian in the usual sense. A ring is an abelian object in the category of rings if and only if it has zero multiplication; the same is true for non-associative rings, for commutative rings, for Lie algebras, etc.

Problem 4. Characterize and/or give new examples of semi-abelian varieties in which every abelian object has representable actions.

Problem 5. Characterize and/or give new examples of semi-abelian varieties in which every perfect object has representable actions. Of course we call an object X perfect if it has trivial abelianization, i.e. if $[X, X] = X$. Note that, as follows from Propositions 2.4 and 2.7, in the categories of rings/algebras, commutative or not, every perfect object has representable actions.

Problem 6. For an object X in $\mathcal{V}$, define the conjugation action of X on itself as the action corresponding to the split epimorphism (first projection, diagonal): $X \times X \mathrel{\substack{\longleftarrow \\ \longrightarrow}} X$. If the representing object $[X]$ does exist, this determines a morphism $X \longrightarrow [X]$. Investigate the cases where this morphism is a

a. monomorphism;

b. (normal) epimorphism;

c. isomorphism.

Note (as easily follows from Proposition 2.7; see also proposition 2.8) that $X \longrightarrow [X]$ is an isomorphism for Taylor-regular rings but not, say, for all Boolean rings. For finite Boolean rings, this however follows independently from two results: they are unital and their dual category is equivalent to the category of pointed objects in the topos of finite sets.

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Université Catholique de Louvain, Belgium

Mathematical Institute of the Georgian Academy of Sciences, Tbilisi, Georgia and University of Cape Town, South-Africa

University of Sydney, Australia
Email: borceux@math.ucl.ac.be
janelidg@maths.uct.ac.za
maxk@maths.usyd.edu.au

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