Theory and Applications of Categories, Vol. 16, No. 10, 2006, pp. 206–217.

A CHARACTERIZATION OF QUANTIC QUANTIFIERS IN ORTHOMODULAR LATTICES

Dedicated to Professor Humberto Cárdenas on the occasion of his 80th Birthday

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Abstract. Let L be an arbitrary orthomodular lattice. There is a one to one correspondence between orthomodular sublattices of L satisfying an extra condition and quantic quantifiers. The category of orthomodular lattices is equivalent to the category of posets having two families of endofunctors satisfying six conditions.

Introduction

The purpose of this paper is to give some new results concerning quantic quantifiers on orthomodular lattices. As is well known, quantifiers have their main source in the theory of Algebraic Logic and in the theory of orthomodular lattices. More recently, quantifiers became important in the theory of idempotent, right- sided quantales.

In section 1 we deal with the notion of a quantic quantifier and characterize such quantic quantifiers in orthomodular lattices.

In section 2 we apply the results of section 1 for the algebraic foundations of quantum mechanics. We also show the following: the category of orthomodular lattices is equivalent to the category of posets having two families of endofunctors satisfying some conditions.

Part of this work was done when the author was a research visitor at Louisiana Tech University. Many thanks to Prof. R. Greechie for his invitation and many conversations. Also many thanks to Prof. M.F. Janowitz for the electronic mails he sent me and the comments he made about quantifiers. This work was partially supported by a Beca Sabática de la DGAPA de la UNAM.

1. Quantic Quantifiers

The classic notion of a quantifier was introduced in [6] where P. Halmos gave a characterization of quantifiers for Boolean Algebras. Latter, M.F. Janowitz generalized this concept for orthomodular lattices, see [7] for details. There is another concept, namely,

Received by the editors 2005-12-15 and, in revised form, 2006-03-23.
Transmitted by F. W. Lawvere. Published on 2006-04-15.
2000 Mathematics Subject Classification: 03G12; 06C99.
Key words and phrases: Quantum logic; Orthomodular lattice.

the notion of a nucleus for Heyting Algebras. Nuclei and quantifiers have a close relation as we shall see.

1.1. Definition. *A bounded lattice $L = (L, \vee, \wedge, 0, 1)$ is an ortholattice if there exists a unary operation $\perp : L \to L$ satisfying the conditions:*

1. $a^{\perp\perp} = a$.
2. $a, b \in L : \ (a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$.
3. $a \vee a^{\perp} = 1$.
4. $a \wedge a^{\perp} = 0$.

a, b being arbitrary elements of L.

If L is an ortholattice, we shall say L is an orthomodular lattice if it satisfies the following weak modularity property:

Given any $a, b \in L$ with $a \leq b$ we have: $b = a \vee (a^{\perp} \wedge b)$ (equivalently, $a = (a \vee b^{\perp}) \wedge b$).

If L and M are orthomodular lattices, a function $f : L \to M$ is said to be a morphism of orthomodular lattices iff the following properties hold:

1. $f(1) = 1$.
2. $f(a \wedge b) = f(a) \wedge f(b)$, for all $a, b \in L$.
3. $f(a^{\perp}) = f(a)^{\perp}$, for all $a \in L$.

The composition of morphisms is defined in the usual way and clearly, we have a category, denoted by OML.

If L is a bounded lattice with bounds 0,1 and $F : L \to L$ is a function, F will be called a *quantifier* on L in case F satisfies:

1. $F(0) = 0$.
2. For any $a \in L$, $a \leq F(a)$.
3. $F(a \wedge F(b)) = F(a) \wedge F(b)$, for all $a, b \in L$.

If we write $F(a \wedge b) = F(a) \wedge F(b)$ in 3 and we do not assume condition 1 then we get the notion of a *nucleus*. The theory of nuclei is given in the context of Heyting Algebras or Locales, the reader can see [8] where there is a study of nuclei for Heyting Algebras. In [11] the author and Beatriz Rumbos gave a characterization of nuclei and quantic nuclei for orthomodular lattices.

There are always two special quantifiers on L:

1. The *discrete* quantifier = the identity map.

2. The *indiscrete quantifier* quantifier: $F(a) = 1$ for $a \neq 0, \quad F(0) = 0$.

Every nucleus is a quantifier but not conversely. If we take the indiscrete quantifier it is not hard to show that it is not a nucleus whenever L is a Heyting algebra or an orthomodular lattice.

Now, for the notion of a quantic quantifier we need to introduce a binary connective & for an arbitrary orthomodular lattice.

1.2. Definition. *Let L be an orthomodular lattice, we define two binary operations as follows: If a, b are arbitrary elements of L*

1. $a \,\&\, b = (a \vee b^{\perp}) \wedge b$.

2. $a \to b = a^{\perp} \vee (a \wedge b)$.

It is not hard to show the following:

$$a \,\&\, b \leq c \quad \text{iff} \quad a \leq b \to c.$$

The last claim has two equivalent meanings. We can say, the function: $F(-b) : L \to L$ given by: $F(-, b)(a) = a \,\&\, b$ is a residuated map or the functor $F(-, b) : L \to L$ given by the same rule has a right adjoint. So, is just a question of terminology; the important idea here is the last inequality and the connective &. To our knowledge, P.D. Finch was the first person to consider & as a binary connective. See [5] for more details. Also, the reader can consult [2] for a detailed account of Residuation Theory.

We just finish with another comment: $F(-, b)$ is called the Sasaki projection and the right adjoint $H(b, -)$ is known in physics as the Sasaki hook; but remember, we shall view this projection as a binary connective, replacing the classical connective $\wedge$. For a Heyting algebra A, it is well known the functor $a \wedge - : A \to A$, has a right adjoint. In fact, whenever A is a boolean algebra the right adjoint is given by: $a \to b = a^{\perp} \vee b.$, a, b are elements of A.

For orthomodular lattices, the situation is quite different. Indeed, one might ask if the classical connective has a right adjoint: When this property is satisfied, then L is a boolean algebra as the reader can show easily. We shall present now, some properties of &.

1.3. Lemma. *Let L be an orthomodular lattice. Given $a, b \in L$, the binary operation & satisfies:*

1. $a \wedge b \leq a \,\&\, b$.

2. $a \,\&\, b \leq b$.

3. $a \,\&\, a = a$.

4. $1 \,\&\, a = a \,\&\, 1 = a$.

5. $a \,\&\, 0 = 0 \,\&\, a = 0$.

6. $a \,\&\, a^{\perp} = a^{\perp} \,\&\, a = 0$.

The operation & is very strict in the follwing sense:

1.4. Proposition. *Let L be an orthomodular lattice; then L is a boolean algebra iff any one of the following holds:*

1. *& is commutative, i.e., $a \,\&\, b = b \,\&\, a \quad \forall a, b \in L$.*
2. *& is associative, i.e., $a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c \quad \forall a, b, c \in L$.*
3. *For any $a \in L$, $a \,\&\, -$ is functorial.*

The reader can consult [10] for a proof of the lemma and the proposition. Also, in [10] there is a physical interpretation of the binary connective &. We shall introduce now the concept of a quantic quantifier.

1.5. Definition. *Let L be an arbitrary orthomodular lattice. By a quantic quantifier F on L we understand a function $F : L \to L$ satisfying the following conditions:*

1. *$F(0) = 0$.*
2. *For any $a \in L$, we have: $a \leq F(a)$.*
3. *If $a, b \in L$ then $F(a \,\&\, F(b)) = F(a) \,\&\, F(b)$.*

Clearly, the discrete and indiscrete quantifiers are in fact quantic quantifiers. Instead of looking for more examples we shall give the characterization of the quantic quantifiers. It is not hard to show: any quantifier F, induces a quantic quantifier. Indeed, if we restrict F to the fixed points of F then F is in fact a quantic quantifier. Moreover, if L is an orthomodular lattice and we denote by $M(L)$ the semigroup (under function composition) of all endofunctors $\phi : L \to L$, then two endofunctors ϕ and ψ are mutually adjoint in case $\psi((\phi(a))^{\perp}) \leq a^{\perp}$ and $\phi((\psi(b))^{\perp})) \leq b^{\perp}$.

1.6. Remark. We borrow this definition from Janowitz,see [7] p. 1242 ; any pair of endofunctors ϕ, ψ which are mutually adjoint induce a pair of endofunctors which are adjoints (in the categorical sense). Indeed, it is not hard to show, ϕ has a right adjoint. Namely, $h = \perp \circ \psi \circ \perp$, as the reader can check easily.

If $S(L)$ denotes the subset of all $\phi : L \to L$ of $M(L)$ having a $\psi : L \to L$ wich are mutually adjoint, then $S(L)$ is a Baer $\star$-semigroup (under function composition) and every element of $S(L)$ preserves 0 and arbitrary suprema whenever they exist in L. See [2] for details. The next theorem can be viewed as a non-commutative, non-associative version of Janowitz' Theorem stated in [7].

1.7. Theorem. *Let L be an arbitrary orthomodular lattice and F a quantic quantifier on L. Then F and $F(L)$ satisfy:*

1. *F(1)= 1.*
2. *F is idempotent.*
3. *F is a functor.*
4. *$F((F(a))^{\perp}) = F(a)^{\perp}$.*
5. *F is a projection in $S(L)$.*
6. *For every $a \in L$ the set $[a, -) \cap F(L)$ has a least element. Namely, $F(a)$. In particular, $F(L)$ is reflective in L; i.e., the inclusion $I : F(L) \to L$ has a left adjoint, denoted by R.*
7. *F(L), the set of the fixed points of L is a suborthomodular lattice of L.*

Proof. Since for any $a \in L$, $a \leq F(a)$ then in particular $1 \leq F(1)$. From this, $F(1) = 1$. Now, if $a \in L$ then $F(1 \,\&\, F(a)) = F(1) \,\&\, F(a) = F(a)$ and the LHS is equal to $F^2(a)$.

If $a \leq b$ then $a \leq F(b)$ and $a = a \,\&\, F(b)$. Hence,$F(a) = F(a \,\&\, F(b)) = F(a) \,\&\, F(b)$. Therefore, $F(a) \leq F(b)$.

We only need to check: $F((F(a))^{\perp}) \leq F(a)^{\perp}$. Now, $F(a) \,\&\, F((F(a))^{\perp}) = F(F(a) \,\&\, F(a)^{\perp}) = F(0) = 0$. From this we have: $F((F(a))^{\perp}) \,\&\, F(a) = 0$, as the reader can check easily. For any $b \in L$ $- \,\&\, b$ has a right adjoint, therefore we get: $F((F(a))^{\perp}) \leq F(a)^{\perp}$ and $F((F(a)^{\perp})) = F(a)^{\perp}$.

F is a projection by the previous results.

Suppose a is an arbitrary element of L then clearly, $F(a) \in [a, -) \cap F(L)$. If $x \in [a, -) \cap F(L)$ then in particular, $a \leq x$ and $F(a) \leq F(x) = x$, since x belongs to $F(L)$.

The functor $R : L \to F(L)$ is defined by: given any element a of L, $R(a) = F(a)$.

The last claim follows easily since F is a projection in $S(L)$ and F preserves orthocomplements in $F(L)$. ∎

1.8. Corollary. *Let L be an orthomodular lattice and consider a quantic quantifier F on L then if the fixed points of F form a boolean subalgebra of L then the notions of a quantifier and a quantic quantifier are equivalent.*

1.9. Remark. Condition 6, will allow to give the characterization of the quantic quantifiers. If we take any quantifier in an orthomodular lattice, it is true that a similar result holds. However, when we assume we have a pair of orthomodular lattices L, K satisfying, the inclusion $I : K \to L$ has a left adjoint, $R : L \to K$, we cannot prove the composition $I \circ R$ is a quantifier. See the example given after theorem 2.

We can actually generate a quantic quantifier if we start with a complete orthomodular lattice L and with a preclosure operator. $k_o : L \to L$ is a preclosure operator if it satisfies:

$k_o(0) = 0$ and for any $a \in L$, $a \leq k_(a)$. The fixed points of k_o induces a closure operator $k : L \to L$ if we define:

$$k(a) = \bigwedge \{x \in L | a \leq x, \quad k_o(x) = x\}.$$

Now, by a quantic prequantifier $k_o : L \to L$ in a complete orthomodular lattice L, we understand a preclosure operator satisfying:

$$k_o(a) \,\&\, b \leq k_o(a \,\&\, k_o(b)).$$

We shall see: a prequantifier induces a quantic quantifier.

1.10. Lemma. *If L is a complete orthomodular lattice and $k_o : L \to L$ is a quantic prequantifier then the closure operator $k : L \to L$, induced by k_o is in fact a quantic quantifier.*

Proof. Given any $a, b \in L$ we define,

$$W = \{x \in L | a \leq x \leq k(a), \quad k(a) \,\&\, b \leq k(a \,\&\, k(b)), \quad k_o(x) = x\}.$$

Clearly, if $x \in W$ then $k_o(x) \in W$ by the definition of k_o. Moreover, if $\{x_i\}_{i \in I}$ is an arbitrary family of elements of W then the supremum $\bigvee_{i \in I} x_i$ belongs also to W since the map $- \,\&\, b : L \to L$ preserves arbitrary suprema. In particular, if $s = \vee W$ then $k_o(s) \in W$. This means, $s \leq k(a)$, by the definition of $k(a)$ and also $k(a) \leq k(s)$. Hence, $s = k(a)$.

Therefore, $k(a) \,\&\, k(b) \leq k(a \,\&\, k(b))$. Clearly, $a \,\&\, k(b) \leq k(a) \,\&\, k(b)$ and since k is idempotent and preserves order we have: $k(a) \,\&\, k(b) \leq k(k(a) \,\&\, k(b)) \leq k^2(a \,\&\, k(b)) = k(a \,\&\, k(b))$.

Hence, $k(a \,\&\, k(b)) = k(a) \,\&\, k(b)$; i.e., k is a quantic quantifier. ∎

The last theorem has a converse as the next result claims.

1.11. Proposition. *Let L be an orthomodular lattice and K be a suborthomodular lattice of L satisfying the following condition:*

1. *For any $a \in L$ the set $[a, -) \cap K$ has a least element.*

The function given by the rule: $F(a)$ is the least element of $[a, -) \cap K$ is a quantic quantifier.

Proof. We must show three properties. $F(0) = 0$ since in the set $[0, -) \cap K$ the least element is clearly 0. Also, by the definition of F, for any $a \in L$, $a \leq F(a)$. We only need to check: $F(a) \,\&\, F(b) = F(a \,\&\, F(b))$.

First of all, notice F is idempotent and preserves order. Indeed, the least element of $[F(a), -) \cap K$ is $F(a)$. Since $F(a) \in K$, F is idempotent.

If $a \leq b$ then $a \leq F(b)$. Since $F(b) \in K$ we get: $F(a) \leq F^2(b) = F(b)$.

From these two facts, we know: $a \& F(b) \leq F(a) \& F(b) \in K$. Since $- \& F(b)$ preserves order and K is a suborthomodular lattice. Therefore, $F(a \& F(b)) \leq F(a) \& F(b)$ by the definition of F.

If $a \& F(b) \leq x$ and $x \in K$ then $a \leq F(b) \to x \in K$. Hence, $F(a) \leq F(b) \to x$ and we get: $F(a) \& F(b) \leq x$. Taking $x = F(a \& F(b))$ we have:

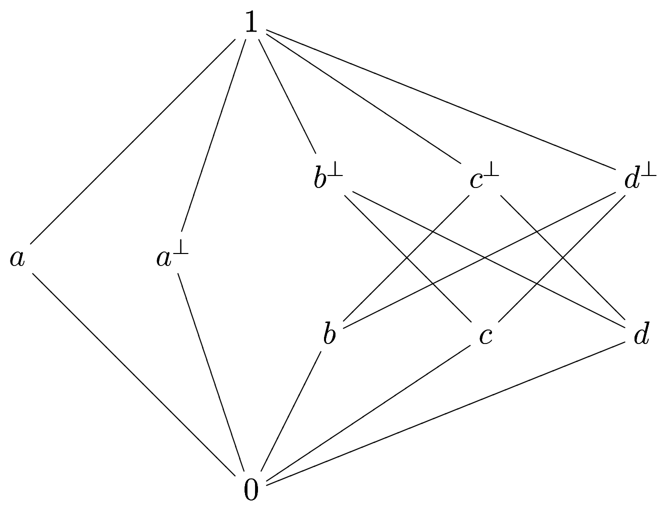
$$F(a) \& F(b) \leq F(a \& F(b)).$$

And the proof is complete. ∎

We summarise these results as follows.

1.12. Theorem. *Let L be an orthomodular lattice. There is a one to one correspondence between quantic quantifiers and pair of orthomodular lattices L, K such that the inclusion $K \to L$ has a left adjoint.*

We shall present now an example of a finite orthomodular lattice L and a quantic quantifier $F : L \to L$ defined on L which is not a quantifier. We shall use the result presented in the last proposition. Consider the following orthomodular lattice L:



The suborthomdular lattice K of L is: $\{0, b, c, d, b^\perp, c^\perp, d^\perp, 1\}$. The map is given by: $F(a) = F(a^\perp) = 1$ and in the rest of the elements of L, $F(x) = x$. Clearly, K satisfies the condition of the last proposition. Hence, F defined as above is a quantic quantifier. However, a simple calculation shows: $F(a \wedge F(d)) \neq F(a) \wedge F(d)$. Hence, F is not a quantifier.

2. On the Algebraic Foundations of Quantum Mechanics

There are many attempts to give some algebraic axioms for quantum mechanics. Why we choose the Sasaki projection and the Sasaki hook as new connectives? The reason is contained in the following Theorem, proved in [12] .

2.1. THEOREM. *Let L be an orthomodular lattice. Let $\to: L\times L \to L$ be a binary operation satisfying the following conditions. Given a, b, c arbitrary elements of L, we have:*

1. $a \leq b$ *iff* $a \to b = 1$.
2. *There exists a binary operation $\oplus : L \times L \to L$ such that:*
   $a \oplus b \leq c$ *iff* $a \leq b \to c$.
3. *Whenever a, b are two compatible elements of L, i.e., $a \,\&\, b = a \wedge b$, then $a \to b = a^{\perp} \vee b$.*

Hence, $\to$ is the Sasaki hook; i.e., $a \to b = a^{\perp} \vee (a \wedge b)$.

2.2. REMARK. In principle, the reader can suspect condition 1 follows from condition 2. If we have $\wedge$, the adjointness written in condition 2, is enough to show condition 1. Something which is true, for instance, for Heyting algebras. We can relax condition 1 and just say: for any $a \in L$, $a \leq a \oplus a$. Assuming this $\oplus$ is an *idempotent* binary operation. If we assume this then condition 1 follows easily by adjointness. However, we wrote the theorem in this form, since for the purposes of the algebraic foundations of quantum mechanics, this presentation is more natural. Therefore, condition 1 is not superfluous. These criteria come from quantum mechanics, see [9] for a discussion of these implicative criteria. For orthomodular lattices there are in principle six implications. These six implications are defined as follows:

2.3. DEFINITION. *Let L be an orthomodular lattice. If a, b are arbitrary elements of L we introduce the following implications:*

1. $a \to_1 b = a^{\perp} \vee b$.
2. $a \to_2 b = a^{\perp}(a \wedge b)$.
3. $a \to_3 b = b \vee (a^{\perp} \wedge b^{\perp})$.
4. $a \to_4 b = (a \wedge b) \vee (a^{\perp} \wedge b) \vee (a^{\perp} \wedge b^{\perp})$.
5. $a \to_5 b = (a \wedge b) \vee (a^{\perp} \wedge b) \vee [(a^{\perp} \vee b^{\perp}) \wedge b]$.
6. $a \to_6 b = (a^{\perp} \wedge b^{\perp}) \vee (a^{\perp} \wedge b) \vee [(a^{\perp} \vee b) \wedge a]$.

Observe that the implications except,of course, for $\to_2$ cannot have a left adjoint. The adjointness written in the last theorem is a weak version of the deduction theorem, stated for instance in classical logic or intuitionistic logic. As is well known, in logic a suitable deduction theorem must be true, if we are really interested in logic or in an algebraic axiomatization of a theory. Therefore, & the Sasaki projection can be viewed as a new logical conjunction. The reader can consult [1] and the recent book [3] where there is a discussion about the logic of quantum mechanics.

Notice, however that all these six implications satisfies conditions 1 and 3 of the last theorem. Hence, condition 2 of the last theorem and the physical reasons given by Piron are crucial if one wants to give an algebraic foundation of quantum mechanics.

The last theorem has a converse. We shall consider first a bounded involution poset. The definition is as follows.

2.4. DEFINITION. *A bounded involution poset* $(L, 0, 1, \perp)$ *is a bounded poset with bounds* $0, 1$ *and a function* $\perp : L \to L$ *satisfying the following two conditions:*

1. *If* $a \leq b$ *then* $b^{\perp} \leq a^{\perp}$.
2. $a^{\perp\perp} = a$.

There are many examples of bounded involutions posets. Clearly, any orthomodular lattice is an example. Any boolean algebra is also an example and if A is a Heyting algebra and we take the double negation on A,the fixed points of the double negation this is also an example. The reader can consult for instance [2] for more examples.

We shall see that under certain conditions we can get an orthomodular lattice. We shall do this in two steps, first of all we shall get from a bounded involution poset a bounded orthoposet under certain conditions. The definition of an orthoposet is as follows:

2.5. DEFINITION. *An orthoposet* $(L, \perp, 0, 1)$ *is a bounded poset satisfying:*

1. *For any* $a \in L$ *we have:* $a^{\perp\perp} = a$.
2. *If* $a \leq b$ *in* L *then* $b^{\perp} \leq a^{\perp}$.
3. *For any* $a \in L$ $\quad a \wedge a^{\perp} = 0$.
4. *For any* $a \in L$ $\quad a \vee a^{\perp} = 1$.

We can formulate now the following:

2.6. THEOREM. *Let* $(L, 0, 1)$ *be a bounded poset. Suppose we have two families of endofunctors* $\{F(-, x) : L \to L\}_{x \in L}$, $\{H(x, -) : L \to L\}_{x \in L}$ *satisfying the following conditions:*

1. *For any* $a \in L$, $F(-, a)$ *is left adjoint to* $H(a, -)$.
2. *If* $x \leq y$ *then* $F(-, x) \circ F(-, y) = F(-, x)$.
3. *For any* $x \in L$, $H(H(x, 0), 0) \leq x$.
4. *For any* $x, y \in L$, $\quad F(H(F(y, x), 0), x) \leq H(y, 0)$.
5. *If* $x \leq y$ *then* $x = F(x, y), y = H(H(x, 0), y)$.
6. *For any* $x \in L$ *we have* $F(1, x) \leq x$.

then L *is an orthomodular lattice. Conversely, any orthomodular lattice, has two families of endofunctors satisfying the last six conditions.*

PROOF. First of all, if we define $x^{\perp} = H(x, 0)$ then by condition 3, we get $x^{\perp\perp} = x$; i.e., $H(x, 0)$ is an involution because $H(x, 0)$ has a left adjoint.

By condition 5, for any $x \in L$, $F(x, x) = x$ and $F(x, 1) = x$. From $x = F(x, x)$ we get $x \leq F(1, x)$ and by condition 6 we have: $F(1, x) = x$.

It is not hard to show $F(a, a^{\perp}) = F(a^{\perp}, a) = 0$. We shall prove first, L is an orthoposet. We define $x \wedge x^{\perp} = F(F(x, x)^{\perp}, x)$ Clearly, the RHS is equal to 0.

Now, since $x \vee x^{\perp} = (x^{\perp} \wedge (x^{\perp})^{\perp})^{\perp}$ then the RHS is equal to $0^{\perp} = 1$ and L is an orthoposet.

We shall see now L is a lattice. If $x, y \in L$ we define $x \wedge y$ as follows:

$$x \wedge y = F(F(y^{\perp}, x)^{\perp}, x).$$

Now, $F(F(y^{\perp}, x)^{\perp}, x) \leq F(1, x) = x$. By condition 4, $F(F(y^{\perp}, x)^{\perp}, x) \leq y$. Hence, $x \wedge y$ is a lower bound.

Suppose $t \leq x, y$. By condition 2, $F(-, t) \circ F(-, x) = F(-, t)$ and also $y^{\perp} \leq t^{\perp}$. Clearly, $F(y^{\perp}, t) = 0$ and therefore $F(F(y^{\perp}, x), t) = F(y^{\perp}, t) = 0$. By condition 1, $F(y^{\perp}, x) \leq t^{\perp}$, hence $t \leq F(y^{\perp}, x)^{\perp}$.

By condition 5, $t = F(t, x) \leq F(F(y^{\perp}, x)^{\perp}, x)$. Therefore, $x \wedge y = F(F(y^{\perp}, x)^{\perp}, x)$ is the greatest lower bound of x, y.

Since $x \vee y = (x^{\perp} \wedge y^{\perp})^{\perp}$ it is now immediate L is a lattice and it is trivial that it is a lattice orthocomplemented by $x \mapsto x^{\perp}$. To prove L is orthomodular we shall see: if $x \wedge y = 0$ and $y^{\perp} \leq x$ then $x = y^{\perp}$.

We want: $x \leq y^{\perp}$. By condition 5, we know $y^{\perp} = H(x^{\perp}, y^{\perp})$. Therefore, $x \leq H(x^{\perp}, y^{\perp})$ is equivalent to prove: $F(x, x^{\perp}) \leq y^{\perp}$ but $F(x, x^{\perp}) = 0$.

We have then $x \leq y^{\perp}$ and $x = y^{\perp}$. This proves, L is an orthomodular lattice.

Finally, it is not hard to show: $F(y, x) = y \,\&\, x$., for any $x, y \in L$.

Conversely, if L is an orthomodular lattice, consider the families, given in definition 2, $F(-, a) = - \,\&\, a : L \to L$ and $H(a, -) = a \to - : L \to L$ for any element $a \in L$. A straightforward calculation shows these families satisfy the six conditions. ■

The posets having two families of endofunctors which are adjoints can be viewed as a category. Indeed, if L, M are two posets having families: $\{F(-, x) : L \to L\}_{x \in L}$, $\{H(x, -) : L \to L\}_{x \in L}$, $\{F'(-, x) : M \to M\}_{x \in M}$, $\{H'(x, -) : M \to M\}_{x \in M}$.

A morphism $f : L \to M$ is a functor, preserving everything on the nose; i.e., $f(0) = 0, f(1) = 1$ and $f \circ F(-, x) = F'(-, f(x)), f \circ H(x, -) = H'(f(x), -)$ for any $x \in L$.

The composition of morphisms is defined in the usual way and clearly we have a category. We denote by PosAdj this category.

If we start with a bounded involution poset the last theorem has a different version. The second condition in the next theorem will produce a right adjoint to the family of endofunctors:

2.7. THEOREM. *Let* $(L, 0, 1, \perp)$ *be an involution poset and suppose we have a family of endofunctors* $\{F(-, x) : L \to L\}_{x \in L}$ *satisfying the following three conditions:*

1. *If* $x \leq y$ *then* $F(-,x) \circ F(-,y) = F(-,x)$.
2. *For any* $x, y \in L$ *we have* $F(F(x,y)^{\perp}, y) \leq x^{\perp}$.
3. *For any* $x \in L$, $F(1,x) = x$.

then L *is an orthomodular lattice.*

Conversely, any orthomodular lattice has a family of endofunctors satisfying the last three conditions.

PROOF. First of all, notice $F(-,x)$ is idempotent by condition 1. By condition 2, the endofunctor $F(-,x)$ has a right adjoint, $H(x,-) = \perp \circ F(-,x) \circ \perp$. Moreover, using again condition 2, we can prove the following:

$$F(y,x) = 0 \ \text{ iff } \ y \leq x^{\perp}.$$

We shall prove first, L is an orthoposet. We need to define a meet and a sup for x and $x^{\perp}$, satisfying the conditions of an orthoposet.

$$x \wedge x^{\perp} = F(F(x,x)^{\perp}, x).$$

The RHS of the last equality is equal to 0 since we can write this expression as follows: $F(F(x,x)^{\perp}, x) = F((F(F(1,x),x)^{\perp}, x) = F(F(1,x)^{\perp}, x) = 0$. Since $F(-,x)$ is idempotent and by condition 3.

Now, since $x \vee x^{\perp} = (x^{\perp} \wedge (x^{\perp})^{\perp})^{\perp}$. A simple calculation shows the RHS is equal to $0^{\perp} = 1$. Hence,L is an orthoposet. Instead of proving directly L has binary meets and sups and L satisfies the orthomodular property we shall use a theorem proved by Finch. Nevertheless, we can define for instance the meet of of two elements $x \wedge y$ by the following formula:

$$x \wedge y = F(F((y^{\perp}), x)^{\perp}, x).$$

The endofunctors $F(-,x)$ satisfy the conditions of a theorem proved by Finch in [4], page 321. Hence L is an orthomodular lattice.

Having proven L is an orthomodular lattice, it is not hard to show $F(y,x) = y \mathbin{\&} x$.

Conversely, if L is an arbitrary orthomodular lattice. We take as a family of endofunctors, $F(-,x) = - \mathbin{\&} x : L \to L$. For any element $x \in L$. A straightforward calculation shows this family satisfies the three conditions. ∎

We summarize the results presented in this section with the following:

2.8. THEOREM. *The category of orthomodular lattices, denoted by OML, is equivalent to the category of bounded posets* $(P, 0, 1)$ *with two families of endofunctors* F, H, *parameterized by elements* $a \in P$, *satisfying the conditions of theorem 4.*

From our point of view, the characterization of quantic quantifiers shows that using the connective & instead of $\wedge$ gives us a good control of the algebraic manipulations in an orthomodular lattice, despite the problems of & in the second variable. By proposition

3,if $a \in L$ then $a \,\&\, -$ is not even a functor. Also, notice we do not assume at all any completeness property in an orthomodular lattice to produce this characterization.

A personal comment: I want to thank Prof. F. W. Lawvere for his comments and suggestions.

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